(ω, c)-periodic and asymptotically (ω, c)-periodic mild solutions to fractional Cauchy problems

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(ω, c)-PERIODIC AND ASYMPTOTICALLY (ω, c)-PERIODIC MILD SOLUTIONS TO FRACTIONAL CAUCHY PROBLEMS.

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Abstract. In this paper we establish some new properties of (ω, c)-periodic and asymptotically (ω, c)-periodic functions, then we apply them to study the existence and uniqueness of mild solutions of these types to the following semilinear fractional differential equations:

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + {}^cD_t^{\alpha-1} f(t, u(t)),\ 1 < \alpha < 2\ ,\ t \in \mathbb{R}, \\ \quad u(0) = 0 \end{cases} \tag{1}$$

and

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + {}^cD_t^{\alpha-1} f(t, u(t-h)),\ 1 < \alpha < 2\ ,\ t, h \in \mathbb{R}_+, \\ \quad u(0) = 0 \end{cases} \tag{2}$$

where ${}^cD_t^\alpha(\cdot)$ $(1 < \alpha < 2)$ stands for the Caputo derivative and A is a linear densely defined operator of sectorial type on a complex Banach space $\mathbb{X}$ and the function $f(t, x)$ is (ω, c)-periodic or asymptotically (ω, c)-periodic with respect to the first variable. Our results are obtained using the Leray-Schauder alternative theorem, the Banach fixed point principle and the Schauder theorem. Then we illustrate our main results with an application to fractional diffusion-wave equations.

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1. Introduction

In their 2018 pioneering work, Alvarez *et al.* [1] introduced the class of (ω, c)-periodic functions which contains the spaces of periodic, antiperiodic and Bloch periodic functions among others. It is motived by the so-called Mathieu's equation

$$y''(t) + [a - 2q\cos(2t)]\, y(t) = 0, \tag{3}$$

arising in seasonally forced population dynamics modelling. The solution is of the form $y(t + \omega) = cy(t)$ where c is a complex number. The theory has rapidly attracted several authors including Abadias *et al.* [2], Alvarez *et al.* [3], Mophou and N'Guérékata [4], Kéré *et al.* [5], Li et *et al.* [6], Khalladi *al.* [7,8].

In their paper [3], Alvarez, Castillo and Pinto extended the concept to the one of asymptotically (ω, c)-periodic functions, that is functions which can be decomposed uniquely as the sum of a (ω, c)-periodic function and a function that vanishes at infinity.

The aim of this paper is to keep on investigating properties of (ω, c)-periodic and asymptotically (ω, c)-periodic functions and their applications to the following equations:

$$
\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + {}^c D_t^{\alpha-1} f(t,u(t)),\ 1<\alpha<2\ ,\ t\in\mathbb{R}, \\ \quad u(0)=0 \end{cases} \tag{4}
$$

and

$$
\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + {}^c D_t^{\alpha-1} f(t,u(t-h)),\ 1<\alpha<2\ ,\ t,h\in\mathbb{R}_+, \\ \quad u(0)=0 \end{cases} \tag{5}
$$

were ${}^cD_t^\alpha(\cdot)$ $(1<\alpha<2)$ stands for the Caputo derivative and A is a linear densely defined operator of sectorial type on a complex Banach space $\mathbb{X}$, and the function $f(t,x)$ is (ω,c)-periodic or asymptotically (ω,c)-periodic with respect to the first variable. Our main results are Theorems 2.2, 2.5 and 2.7.

In order to illustrate our main results, we propose an application to Mainardi's concept of fractional diffusion-wave equations (see [9–11] for more details). Mainardi and Paradisi have shown in [11] that this class of fractional equations with $(1<\alpha<2)$ is that which governs the propagation of stress waves in viscoelastic media which, by exhibiting a power law creep, are of relevance in acoustics and seismology since their quality factor turns out to be independent of frequency.

2. Preliminaries

Throughout this work, we assume that $(\mathbb{X},\|\cdot\|)$ is a complex Banach space and we will denote by $\mathbf{C}(\mathbb{R},\mathbb{X})$ the collection of all continuous functions from $\mathbb{R}$ into $\mathbb{X}$, and $\mathbf{B}C(\mathbb{R},\mathbb{X})$ the collection of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$. The space $\mathbf{B}C(\mathbb{R},\mathbb{X})$ equipped with the sup norm defined by $\|f\|_\infty := \sup\limits_{t\in\mathbb{R}}\|f\|$ is a Banach space. The notation $\mathbf{B}(\mathbb{X})$ stands for the space of bounded linear operators from $\mathbb{X}$ into itself endowed with the uniform operator topology.

First, we recall some definitions and properties about sectorial linear operators and their associated solution operators.

Definition 2.1. *A closed and linear operator A is said to be of sectorial type $\tilde\omega$ and angle θ, if there exists $(\theta,M,\tilde\omega)\in\left(0,\frac{\pi}{2}\right)\times\mathbb{R}_+^\star\times\mathbb{R}$ such that both following assertions holds true :*

(1) its resolvent exists outside the sector $\tilde\omega+S_\theta := \{\tilde\omega+\lambda, \lambda\in\mathbb{C}, |\arg(-\lambda)|<\theta\}$.
(2) $\|(\lambda-A)^{-1}\| \le \frac{M}{|\lambda-\tilde\omega|}$, $\lambda\notin\tilde\omega+S_\theta$.

Definition 2.2. *Let $\alpha\in(1,2)$ and A be a closed and linear operator with domain $D(A)$ defined on $\mathbb{X}$. The operator A is called a generator of a solution operator if there exists $\tilde\omega\in\mathbb{R}$ and a strongly continuous function $\mathbf{S}_\alpha:\mathbb{R}_+\to\mathbf{B}(\mathbb{X})$ such that $\{\lambda^\alpha, Re(\lambda)>\tilde\omega\}\subseteq\rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}x = \int_0^\infty e^{-\lambda t}\mathbf{S}_\alpha(t)x\,dt$, $Re(\lambda)>\tilde\omega, x\in\mathbb{X}$. In this case, $\mathbf{S}_\alpha(t)$ is called the solution operator generated by A.*

Definition 2.3 ([12])**.** *A family $\{\mathbf{S}_\alpha(t)\}_{t\ge0}\subset\mathbf{B}(\mathbb{X})$ is said to be uniformly integrable if*

$$
\int_0^\infty \|\mathbf{S}_\alpha(t)\|dt<\infty
$$

We note that, if A is a sectorial type of $\tilde{\omega}$ with $\theta \in \left[0, \pi(1-\frac{\alpha}{2})\right)$ then A is the generator of a solution operator given by

$$\mathbf{S}_\alpha(t) := \frac{1}{2i\pi}\int_\xi e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\, d\lambda$$

where ξ is a suitable path lying outside the sector $\tilde{\omega} + S_\theta$.

Lemma 2.4 ([13]). *Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator of type $\tilde{\omega} < 0$ and angle θ in a complex Banach space $\mathbb{X}$, then there exists $C_{\alpha,\theta} > 0$ depending solely on α and θ such that :*

$$\|\mathbf{S}_\alpha(t)\|_{\mathbf{B}(\mathbb{X})} \le \frac{C_{\alpha,\theta}M}{1+|\tilde{\omega}|t^\alpha} \ \ , \ t \ge 0$$

Definition 2.5 ([14,15]). *The derivative of order α of a function $f : \mathbb{R}_+ \to \mathbb{R}$ in the sense of Caputo is defined as*

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds$$

for $n-1 \le \alpha < n$, $n \in \mathbb{N}$. If $1 < \alpha < 2$, then

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}f^{(2)}(s)ds$$

In the following sections, we will recall some properties of both (ω, c)-periodic and asymptotically (ω, c)-periodic functions.

2.1. **On (ω, c)-periodicity.**

We first recall this fundamental definition :

Definition 2.6 ([1]). *Let $\omega > 0$ and c a non-zero complex number. A function $f \in \mathbf{C}(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)-periodic if $f(t+\omega) = cf(t)$, $\forall t \in \mathbb{R}$.*
In this case ω is called a c-period of the function f.

We denote by $\mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$ the set of all (ω, c)-periodic functions from $\mathbb{R}$ to $\mathbb{X}$. When $c = 1$, we write $\mathbf{P}_\omega(\mathbb{R}, \mathbb{X})$ instead of $\mathbf{P}_{\omega,1}(\mathbb{R}, \mathbb{X})$ and we say that f is ω-periodic. Using the principal branch of the complex Logarithm, $c^{\frac{t}{\omega}}$ is defined as $c^{\frac{t}{\omega}} := \exp(\frac{t}{\omega}Log(c)) = c^\wedge(t)$ and we will use the notation $|c|^\wedge(t) := |c^\wedge(t)| = |c|^{\frac{t}{\omega}}$.

In [1], Alvarez *et al.* gave a useful description of the space $\mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$. That is $\mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$ is a translation-invariant subspace over $\mathbb{C}$ of $\mathbf{C}(\mathbb{R}, \mathbb{X})$. Then for any fixed $h > 0$ and $u \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$, we have $u_h(\cdot) := u(\cdot - h) \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$.

Proposition 2.7 ([1]). *Let $f \in \mathbf{C}(\mathbb{R}, \mathbb{X})$. Then, $f \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$ if and only if $f(t) = c^{\frac{t}{\omega}}u(t)$, $u(t) \in \mathbf{P}_\omega(\mathbb{R}, \mathbb{X})$.*

Theorem 2.8 ([4]). *Let $f \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$ and $A \in \mathbf{B}(\mathbb{X})$. Then $Af \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$.*

We state and prove this basic property which follows naturally from the (ω, c)-periodicity definition. It will be very useful in the sequel.

Lemma 2.9. *Let $f \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$. Then, $f \in \mathbf{P}_{-\omega,c^{-1}}(\mathbb{R}, \mathbb{X})$.*

Proof. Let $f \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$. According to Proposition (2.7), $f(t) = c^{\frac{t}{\omega}}u(t)$ with $u(t) \in \mathbf{P}_\omega(\mathbb{R}, \mathbb{X})$. Then, we have $f(t-\omega) = c^{\frac{t-\omega}{\omega}}u(t-\omega) = c^{-1}c^{\frac{t}{\omega}}u(t) = c^{-1}f(t)$. So $f \in \mathbf{P}_{-\omega,c^{-1}}(\mathbb{R}, \mathbb{X})$. $\square$

Now we establish the following theorem :

Theorem 2.10. *Let* $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$. *Then,* ${}^cD_t^\alpha f(t) \notin \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$.

Proof. We have

$$\begin{aligned}
{}^cD_t^\alpha f(t+\omega) &= \frac{1}{\Gamma(n-\alpha)}\int_0^{t+\omega}(t+\omega-s)^{n-\alpha-1}f^{(n)}(s)ds \\
&= c({}^cD_t^\alpha f(t)) + \frac{c}{\Gamma(n-\alpha)}\int_{-\omega}^{0}(t-s)^{n-\alpha-1}f^{(n)}(s)ds \\
&= c({}^cD_t^\alpha f(t)) + \frac{1}{\Gamma(n-\alpha)}\int_0^{\omega}(t+\omega-s)^{n-\alpha-1}f^{(n)}(s)ds
\end{aligned}$$

Keeping in mind that $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ we know that f is not constant. It comes that $\displaystyle\int_0^{\omega}(t+\omega-s)^{n-\alpha-1}f^{(n)}(s)ds \neq 0$. Finally, ${}^cD_t^\alpha f(t) \notin \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$. ☐

Theorem 2.11 ([1]). $\mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ *is a Banach space with the norm* $\|f\|_{\omega,c} := \sup_{t\in[0,\omega]} \||c|\wedge(-t)f(t)\|$.

We note that if $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$, then $\|f\|_{\omega,c} < \infty$ and we say that f is c-bounded. The use of $\|f\|_{\omega,c}$ instead of $\|f\|_\infty$ will allow us to handle the (ω,c)-periodicity properties of f (see [1] for more details).

We now prove the following:

Proposition 2.12. *Let* $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ *be a sectorial operator of type* $\tilde{\omega} < 0$ *and angle* θ *in a complex Banach space* $\mathbb{X}$, *then there exists* $C_{\alpha,\theta} > 0$ *depending solely on* α *and* θ *such that :*

$$\|\mathbf{S}_\alpha(t)\|_{\omega,c} \leq \frac{C_{\alpha,\theta}M}{1+|\tilde{\omega}|t^\alpha}\gamma_1 \ \ , \ t \geq 0$$

where $\gamma_1 = \max\{1, |c|^{-1}\}$.

Proof. Using Lemma 2.4, we have

$$\begin{aligned}
\|\mathbf{S}_\alpha(t)\|_{\omega,c} = \sup_{t\in[0;\omega]}\||c|\wedge(-t)\mathbf{S}_\alpha(t)\| &\leq \|\mathbf{S}_\alpha(t)\|_{\mathbf{B}(\mathbb{X})}\cdot\sup_{t\in[0;\omega]}\||c|\wedge(-t)\| \\
&\leq \frac{C_{\alpha,\theta}M}{1+|\tilde{\omega}|t^\alpha}\cdot\sup_{t\in[0;\omega]}\||c|\wedge(-t)\| \ , \ t \geq 0 \\
&\leq \frac{C_{\alpha,\theta}M}{1+|\tilde{\omega}|t^\alpha}\cdot\max\{1,|c|^{-1}\} \ , \ t \geq 0
\end{aligned}$$

☐

Now we will investigate some more general features of (ω,c)-periodic functions linked with their integration.

Proposition 2.13. *Let* $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$. *Then for all* $x_1, x_2 \in \mathbb{R}$,

$$\int_{x_1}^{x_1+\omega} f(s)ds = c^{\lfloor\xi\rfloor}\int_{x_2+\{\xi\}\omega}^{x_2+\{\xi\}\omega+\omega} f(s)ds$$

where $\xi = \frac{x_1-x_2}{\omega}$, $\lfloor\xi\rfloor$ *stands for the integer part of* ξ *and* $\{\xi\}$ *for the fractional part of* ξ.

Proof. Let $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$. We have

$$\begin{aligned}
\int_{x_1}^{x_1+\omega} f(s)ds &= \int_{x_2+(\lfloor\frac{x_1-x_2}{\omega}\rfloor+\{\frac{x_1-x_2}{\omega}\})\omega}^{x_2+(\lfloor\frac{x_1-x_2}{\omega}\rfloor+\{\frac{x_1-x_2}{\omega}\})\omega+\omega} f(s)ds \\
&= \int_{x_2+\{\frac{x_1-x_2}{\omega}\}\omega}^{x_2+\{\frac{x_1-x_2}{\omega}\}\omega+\omega} f(s+\lfloor\frac{x_1-x_2}{\omega}\rfloor\omega)ds \\
&= c^{\lfloor\frac{x_1-x_2}{\omega}\rfloor}\int_{x_2+\{\frac{x_1-x_2}{\omega}\}\omega}^{x_2+\{\frac{x_1-x_2}{\omega}\}\omega+\omega} f(s)ds
\end{aligned}$$

□

Theorem 2.14. *Let $(b,\bar{b}) \in [0,\infty)\times[0,\infty]$ and $\Lambda_b, \Lambda_{\bar{b}}^{\sup}, \Lambda_{\bar{b}}^{\inf}$ define as :*

$$\Lambda_{\bar{b}}^{\sup}(t) = \int_t^{t+\bar{b}} f(s)ds \ , \ \Lambda_{\bar{b}}^{\inf}(t) = \int_{t-\bar{b}}^{t} f(s)ds$$
$$\text{and}$$
$$\Lambda_b(t) = (\Lambda_b^{\sup}+\Lambda_b^{\inf})(t) = \int_{t-b}^{t+b} f(s)ds$$

Then $\Lambda_b, \Lambda_{\bar{b}}^{\sup}, \Lambda_{\bar{b}}^{\inf}$ are (ω,c)-periodic functions if and only if $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$.

Proof. We have that :

$$\Lambda_b(t+\omega) = \int_{t+\omega-b}^{t+\omega+b} f(s)ds = \int_{t-b}^{t+b} cf(s)ds = c\Lambda_b(t)$$

The proof is similar for $\Lambda_{\bar{b}}^{\sup}$ and $\Lambda_{\bar{b}}^{\inf}$ case. □

One can note that $\Lambda_\infty^{\sup} = \int_t^\infty f(s)ds$ is well defined when $|c|<1$ and $\Lambda_\infty^{\inf} = \int_{-\infty}^t f(s)ds$ is well defined when $|c|>1$ and both are in $\mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ whenever $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$.

In [3], authors gave following proposition :

Proposition 2.15 ([3])**.** *Assume that f is a (ω,c)-periodic function and $b \in [-\infty,\infty)$. Then $F(t) = \int_b^t f(s)ds$ is a (ω,c)-periodic function if and only if $F(b+\omega)=0$.*

We offer a more precise version of the latter for all $b\in\mathbb{R}$ because the case $b=-\infty$ is already treated in Theorem (2.14).

Theorem 2.16. *Let $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ and $b\in(-\infty,\infty)$. Then $F(t) = \int_b^t f(s)ds$ is a (ω,c)-periodic function if and only if $c=1$ and $f := u(t)$, with $u(t)$ $\frac{\omega}{2}$-antiperiodic.*

Proof. Let $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$. According to Proposition 2.7, we have $f = c^{\frac{t}{\omega}}u(t)$, $u \in \mathbf{P}_{\omega}(\mathbb{R},\mathbb{X})$. If $c = 1$ and $f := u(t)$, with $u(t)$ $\frac{\omega}{2}$-antiperiodic, we have that

$$\begin{aligned} F(t+\omega) = \int_b^{t+\omega} f(s)ds &= \int_b^{b+\omega} f(s)ds + \int_{b+\omega}^{t+\omega} f(s)ds \\ &= \int_b^{b+\frac{\omega}{2}} f(s)ds - \int_b^{b+\frac{\omega}{2}} f(s)ds + cF(t) \\ &= cF(t) \end{aligned}$$

Now, if we have $F(t) \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$. According to Proposition (2.15), for $b \in (-\infty,\infty)$, we have

$$F(b+\omega) = 0 \tag{6}$$

Using 2.13, it comes that $\int_b^{b+\omega} f(s)ds = c^{\lfloor \xi_t \rfloor} \int_{t+\{\xi_t\}\omega}^{t+\{\xi_t\}\omega+\omega} f(s)ds$, for each $t \in \mathbb{R}$, where $\xi_t = \frac{b-t}{\omega}$. Then for all $t \in \mathbb{R}$:

$$\begin{aligned} \int_b^{b+\omega} f(s)ds &= c^{\lfloor \xi_t \rfloor} \int_{t+\{\xi_t\}\omega}^{t+\{\xi_t\}\omega+\omega} f(s)ds \\ &= c^{\lfloor \xi_t \rfloor} \int_{t+\{\xi_t\}\omega}^{b+\omega} f(s)ds + c^{\lfloor \xi_t \rfloor} \int_{b+\omega}^{t+\{\xi_t\}\omega+\omega} f(s)ds \\ &= -c^{\lfloor \xi_t \rfloor-1} \int_b^{t+\{\xi_t\}\omega+\omega} f(s)ds + c^{\lfloor \xi_t \rfloor+1} \int_b^{t+\{\xi_t\}\omega} f(s)ds \\ &= -c^{\lfloor \xi_t \rfloor-1} F(t+\{\xi_t\}\omega+\omega) + c^{\lfloor \xi_t \rfloor+1} F(t+\{\xi_t\}\omega) \end{aligned}$$

But by hypothesis, $F \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ then $F(t+\{\xi_t\}\omega+\omega) = cF(t+\{\xi_t\}\omega)$. Therefore, for all $t \in \mathbb{R}$

$$-c^{\lfloor \xi_t \rfloor} F(t+\{\xi_t\}\omega) + c^{\lfloor \xi_t \rfloor+1} F(t+\{\xi_t\}\omega) = 0 \iff c = 1$$

Finally, (6) implies that $\int_b^{b+\frac{\omega}{2}} f(s) + f(s+\frac{\omega}{2})ds = 0 \iff \int_b^{b+\frac{\omega}{2}} u(s) + u(s+\frac{\omega}{2})ds = 0.$ It follows that $c = 1$ and $f(t) = u(t)$ with $u(t)$ $\frac{\omega}{2}$-antiperiodic. □

Corollary 2.17. *Let $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ with $|c| \neq 1$ and $b \in (-\infty,\infty)$. Then $F(t) = \int_b^t f(s)ds$ is not (ω,c)-periodic.*

Let us denote the Nemytskii's operator associated with $f \in \mathbf{B}C(\mathbb{R},\mathbb{X})$ by $\mathcal{N}(\varphi)(\cdot) := f(\cdot,\varphi(\cdot))$. Then we recall the following composition theorem :

Theorem 2.18 ([4]). *Let $f \in \mathbf{C}(\mathbb{R},\mathbb{X})$. Then the following assertions are equivalent :*

(1) For every $\varphi \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$, $\mathcal{N}(\varphi)(\cdot) \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$,
(2) For all $(t,u) \in \mathbb{R} \times \mathbb{X}$, $f(t+\omega, cu) = cf(t,u)$.

We end this section with the new following convolution result:

Theorem 2.19. *Let $\{\mathbf{S}_\alpha(t)\}_{t\geq 0} \subset \mathbf{B}(\mathbb{X})$ be a uniformly integrable and strongly continuous family. If $f \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ then the function $(\mathbf{S}_\alpha \diamond f)$ given by*

$$(\mathbf{S}_\alpha \diamond f)(t) = \int_{-\infty}^t \mathbf{S}_\alpha(t-s)f(s)ds$$

is also in $\mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$.

Proof. We have

$$(\mathbf{S}_\alpha \diamond f)(t+\omega) = \int_{-\infty}^{t+\omega} \mathbf{S}_\alpha(t+\omega-s)f(s)ds = \int_{-\infty}^{t} \mathbf{S}_\alpha(t-s)f(\omega+s)ds = c(\mathbf{S}_\alpha \diamond f)(t)$$

The proof is complete ☐

2.2. **On asymptotically (ω,c)-periodicity.**

We first define the following spaces of functions vanishing at infinity:

$$\mathbf{C}_0(\mathbb{X}) := \Big\{h \in \mathbf{C}(\mathbb{R}_+,\mathbb{X}) \text{ such that } \lim_{t\to\infty} h(t) = 0\Big\}$$

and

$$\mathbf{C}_0(\Omega,\mathbb{X}) := \Big\{h \in \mathbf{C}(\mathbb{R}_+\times\Omega,\mathbb{X}) \text{ such that } \lim_{t\to\infty} h(t,u) = 0 \text{ for all u in any compact subset of } \Omega.\Big\}$$

Definition 2.20 ([3]). *Let $\omega > 0$ and c a complex number. A function $f \in \mathbf{C}(\mathbb{R},\mathbb{X})$ is said to be c-asymptotic if $c^\wedge(-t)h(t) \in \mathbf{C}_0(\mathbb{X})$, that is*

$$\lim_{t\to\infty} c^\wedge(-t)f(t) = 0$$

And we denote by $\mathbf{C}_{0,c}(\mathbb{X})$ this collection of function. Analogously, a function $g \in \mathbf{C}(\mathbb{R}\times\Omega,\mathbb{X})$ is said to be c-asymptotic if $c^\wedge(-t)g(t,u) \in \mathbf{C}_0(\Omega,\mathbb{X})$, that is

$$\lim_{t\to\infty} c^\wedge(-t)g(t,u) = 0$$

for all u in any compact subset of Ω. The collection of all such functions will be denoted by $\mathbf{C}_{0,c}(\Omega,\mathbb{X})$.

Definition 2.21 ([3]). *A function $f \in \mathbf{C}(\mathbb{R},\mathbb{X})$ is said to be asymptotically (ω,c)-periodic if $f = g + h$ where $g \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ and $h \in \mathbf{C}_{0,c}(\mathbb{X})$.*

We denote by $\mathbf{AP}_{\omega,c}(\mathbb{X})$ the collection of all those functions (with the same c-period ω for the first component). Reader should note that the previous decomposition is unique, that is we have $\mathbf{AP}_{\omega,c}(\mathbb{X}) = \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X}) \bigoplus \mathbf{C}_{0,c}(\mathbb{X})$.

Similarly to the previous section, we have these fundamentals results :

Proposition 2.22 ([3]). *Let $f \in \mathbf{C}(\mathbb{R},\mathbb{X})$. Then f is asymptotically (ω,c)-periodic if and only if*

$$f(t) = c^{\frac{t}{\omega}}u(t)\ ,\ u(t) \in \mathbf{AP}_{\omega,1}(\mathbb{X})$$

Lemma 2.23 ([3]). *Following assertions holds true :*

(1) $(f+g) \in \mathbf{AP}_{\omega,c}(\mathbb{X})$ *whenever* $f,g \in \mathbf{AP}_{\omega,c}(\mathbb{X})$.
(2) *Let $g \in \mathbf{P}_{\omega,c}(\mathbb{R},\mathbb{X})$ and $h \in \mathbf{C}_{0,c}(\mathbb{X})$ such that $g,h \in \mathbf{C}^1(\mathbb{R},\mathbb{X})$. Then the derivative of $f = g+h \in \mathbf{AP}_{\omega,c}(\mathbb{X})$ belongs to $\mathbf{AP}_{\omega,c}(\mathbb{X})$.*

Now we state and prove following theorems :

Theorem 2.24. *Let $f \in \mathbf{AP}_{\omega,c}(\mathbb{X})$. Then if $A \in \mathbf{B}(\mathbb{X})$, $Af \in \mathbf{AP}_{\omega,c}(\mathbb{X})$.*

Proof. Let $f = g + h$ where $g \in \mathbf{P}_{\omega,c}(\mathbb{R}_+,\mathbb{X})$ and $h \in \mathbf{C}_{0,c}(\mathbb{X})$. We have that $Af(t) = Ag(t) + Ah(t)$. According to Theorem (2.8), $Ag \in \mathbf{P}_{\omega,c}(\mathbb{R}_+,\mathbb{X})$.

Also we have $\|\lim_{t\to\infty} c^\wedge(-t)Ah(t)\| \le \|A\|_{B(\mathbb{X})}\|\lim_{t\to\infty} c^\wedge(-t)h(t)\| = 0$

So $\lim\limits_{t\to\infty} c^\wedge(-t)Ah(t) = 0$, which proves that $Ah(t) \in \mathbf{C}_{0,c}(\mathbb{X})$. The proof is complete. $\square$

As in [1] with Theorem 2.11, Alvarez et al. prove that there exists a norm suitable for the study of asymptotically (ω, c)-periodic functions with the proposition which follows.

Theorem 2.25 ([3]). $\mathbf{AP}_{\omega,c}([d,\infty) \times \mathbb{X}, \mathbb{X})$ *is a Banach space with the norm* $\|f\|_{a\omega,c} := \sup\limits_{t\geq d} \||c|^\wedge(-t)f(t)\|$.

In the following, we will focus on the case $t \in \mathbb{R}_+$. Consequently, we will use $\|f\|_{a\omega,c}$ as $\|f\|_{a\omega,c} := \sup\limits_{t\geq 0} \||c|^\wedge(-t)f(t)\|$. We propose the two following results :

Proposition 2.26. *Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator of type $\tilde{\omega} < 0$ and angle θ in a complex Banach space $\mathbb{X}$. If $|c| \geq 1$, then there exists $C_{\alpha,\theta} > 0$ depending solely on α and θ such that :*

$$\|\mathbf{S}_\alpha(t)\|_{a\omega,c} \leq \frac{C_{\alpha,\theta}M}{1 + |\tilde{\omega}|t^\alpha}\ ,\ t \geq 0.$$

Proof. Using Lemma 2.4, we have

$$\begin{aligned}
\|\mathbf{S}_\alpha(t)\|_{a\omega,c} = \sup_{t\geq 0} \||c|^\wedge(-t)\mathbf{S}_\alpha(t)\| &\leq \|\mathbf{S}_\alpha(t)\|_{\mathbf{B}(\mathbb{X})} \cdot \sup_{t\geq 0} \||c|^\wedge(-t)\| \\
&\leq \frac{C_{\alpha,\theta}M}{1 + |\tilde{\omega}|t^\alpha} \cdot \sup_{t\geq 0} \||c|^\wedge(-t)\|\ ,\ t \geq 0 \\
&\leq \frac{C_{\alpha,\theta}M}{1 + |\tilde{\omega}|t^\alpha}\ ,\ t \geq 0
\end{aligned}$$

$\square$

Theorem 2.27. *Assume that A is sectorial of type $\tilde{\omega} < 0$. If $f : \mathbb{R}_+ \to \mathbb{X}$ is an asymptotically (ω, c)-periodic function and $(\mathbf{S}_\alpha \diamond f)(t)$ is given by*

$$(\mathbf{S}_\alpha \diamond f)(t) = \int_0^t \mathbf{S}_\alpha(t-s)f(s)ds,\ t \geq 0.$$

Then $(\mathbf{S}_\alpha \diamond f) \in \mathbf{AP}_{\omega,c}(\mathbb{X})$.

Proof. If $f = g + h$, where $g \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$ and $h \in \mathbf{C}_{0,c}(\mathbb{X})$, then we have that $(\mathbf{S}_\alpha \diamond f)(t) = (\mathbf{S}_\alpha \odot g)(t) + (\mathbf{S}_\alpha \odot h_{-g})(t)$ where

$$(\mathbf{S}_\alpha \odot g)(t) = \int_{-\infty}^t \mathbf{S}_\alpha(t-s)f(s)ds,\ t \geq 0$$

and

$$(\mathbf{S}_\alpha \odot h_{-g})(t) = \int_0^t \mathbf{S}_\alpha(t-s)h(s)ds - \int_{-\infty}^0 \mathbf{S}_\alpha(t-s)g(s)ds,\ t \geq 0$$

By Theorem (2.19), $(\mathbf{S}_\alpha \odot g) \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$. Now, let us show that $(\mathbf{S}_\alpha \odot h_{-g}) \in \mathbf{C}_{0,c}(\mathbb{X})$. Since $h \in \mathbf{C}_{0,c}(\mathbb{X})$, for each $\varepsilon > 0$ there exists $m > 0$ such that

$\|h(s)\|_{a\omega,c} \leq \varepsilon$ for all $s \geq m$. Then for all $t \geq 2m$, we deduce

$$\begin{aligned}
\|c^\wedge(-t)(\mathbf{S}_\alpha \odot h_{-g})(t)\| &= \|c^\wedge(-t)\bigg[\int_0^{\frac{t}{2}} \frac{h(s)C_{\alpha,\theta}M}{1+|\tilde{\omega}|(t-s)^\alpha}ds \\
&\quad + \int_{\frac{t}{2}}^{t} \frac{h(s)C_{\alpha,\theta}M}{1+|\tilde{\omega}|(t-s)^\alpha}ds - \int_{-\infty}^{0} \frac{g(s)C_{\alpha,\theta}M}{1+|\tilde{\omega}|(t-s)^\alpha}ds\bigg]\| \\
&\leq C_{\alpha,\theta}M\bigg[\|h\|_{a\omega,c}\int_0^{\frac{t}{2}} \frac{1}{1+|\tilde{\omega}|(t-s)^\alpha}ds \\
&\quad + \varepsilon\int_{\frac{t}{2}}^{t} \frac{1}{1+|\tilde{\omega}|(t-s)^\alpha}ds \\
&\quad + \|g\|_{\omega,c}\int_{-\infty}^{0} \frac{1}{1+|\tilde{\omega}|(t-s)^\alpha}ds\bigg] \\
&\leq C_{\alpha,\theta}M\int_t^{\infty} \frac{\|h\|_{a\omega,c}+\|g\|_{\omega,c}}{1+|\tilde{\omega}|s^\alpha}ds + \frac{\varepsilon C_{\alpha,\theta}M|\tilde{\omega}|^{\frac{-1}{\alpha}}\pi}{\alpha\sin(\frac{\pi}{2})}
\end{aligned}$$

It comes that

$$\|c^\wedge(-t)(\mathbf{S}_\alpha \odot h_{-g})(t)\| \leq C_{\alpha,\theta}M\bigg(\|h\|_{a\omega,c}+\|g\|_{\omega,c}\bigg)\int_t^{\infty} \frac{1}{1+|\tilde{\omega}|s^\alpha}ds + \frac{\varepsilon C_{\alpha,\theta}M\pi}{\alpha|\tilde{\omega}|^{\frac{1}{\alpha}}\sin(\frac{\pi}{\alpha})}$$

Therefore, $\lim\limits_{t\to\infty} c^\wedge(-t)(\mathbf{S}_\alpha \odot h_{-g})(t) = 0$, that is, $(\mathbf{S}_\alpha \odot h_{-g}) \in \mathbf{C}_{0,c}(\mathbb{X})$. This completes the proof. □

Now we recall this composition theorem

Theorem 2.28 ([3]). *Let $f(t,x) = g(t,x) + h(t,x)$ where $g(t+\omega, cx) = cg(t,x)$ and $h \in \mathbf{C}_{0,c}(\mathbb{X},\mathbb{X})$. Assume that*

(1) $h_t(z) = c^\wedge(-t)h(t, c^\wedge(-t)z)$ is uniformly continuous for z in any bounded subset of $\mathbb{X}$ uniformly for $t \geq 0$, and $h_t(z) \to 0$ as $t \to \infty$ uniformly in z.

(2) There exists $\nu \in \mathbf{B}C(\mathbb{R}_+,\mathbb{R}_+)$ such that

$$\|f(t,u_1) - f(t,u_2)\| \leq \nu(t)\|u_1 - u_2\|, \forall u_1, u_2 \in \mathbb{X}, \forall t \in \mathbb{R}_+$$

If $\varphi \in \mathbf{AP}_{\omega,c}(\mathbb{X})$, then $f(\cdot,\varphi(\cdot)) \in \mathbf{AP}_{\omega,c}(\mathbb{X})$.

In the sequel, we need the following results :

Let $h^\star : \mathbb{R}_+ \to [1;\infty)$ be a continuous function such that $h^\star(t) \geq 1$ for all $t \in \mathbb{R}_+$ and $h^\star(t) \to \infty$ as $t \to \infty$. Initially we set $\mathbf{C}_{h^\star}(\mathbb{R}_+,\mathbb{X})$ for the space consisting of continuous functions $u : \mathbb{R}_+ \to \mathbb{X}$ such that $\|u\|_{h^\star} = \sup\limits_{t\in\mathbb{R}_+} \dfrac{\|u\|_{a\omega,c}}{h^\star(t)} < \infty$. endowed with the norm $\|u\|_{h^\star} = \sup\limits_{t\in\mathbb{R}_+} \dfrac{\|u\|_{a\omega,c}}{h^\star(t)}$. It turns out to be a Banach space.

We also denote

$$\mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X}) = \left\{u \in \mathbf{C}_{h^\star}(\mathbb{R}_+,\mathbb{X}) : \lim_{t\to\infty} \frac{\|u\|_{a\omega,c}}{h^\star(t)} = 0\right\}$$

Here we adapt with the norm $\|\cdot\|_{a\omega,c}$ and prove an existing lemma in [16] .

Lemma 2.29. *A subset $\mathbf{R} \subseteq \mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X})$ is a relatively compact set if it verifies the following conditions :*

(1) *The set* $\mathbf{R}_b = \{u_{|_{[0,b]}} : u \in \mathbf{R}\}$ *is relatively compact in* $\mathbf{C}([0,b],\mathbb{X})$, $\forall b \in \mathbb{R}_+$.

(2) $\lim_{t\to\infty} \dfrac{\|u\|_{a\omega,c}}{h^\star(t)} = 0$, *uniformly for* $u \in \mathbf{R}$.

Proof. First note that $\mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X})$ is isometrically isomorphic to the space $\mathbf{C}_0(\mathbb{X})$ of functions vanishing at infinity.
Now, let us consider the set $\mathbf{R}_b = \{u_{|_{[0,b]}} : u \in \mathbf{R}\} \subset \mathbf{C}^0_{h^\star}([0,b],\mathbb{X})$. Then, since for all $b \in \mathbb{R}_+$, $\mathbf{C}^0_{h^\star}([0,b],\mathbb{X})$ is isometrically isomorphic to the space $\mathbf{C}_0([0,b],\mathbb{X})$, the set $\mathbf{R}_b$ is isomorphic to a subset of the latter space. Keeping in mind assumption (1), we have that $\forall b \in \mathbb{R}_+$, $\mathbf{R}_b$ is relatively compact in $\mathbf{C}([0,b],\mathbb{X}) \supset \mathbf{C}_0([0,b],\mathbb{X})$ which leads to the fact that $\mathbf{R}_b$ is relatively compact in $\mathbf{C}_0([0,b],\mathbb{X})$.
Then, for any fixed $b \in \mathbb{R}_+$, $\mathbf{R}_b$ admits a finite ε-net $\{\eta_1,\eta_2,...,\eta_p\}$, in $\mathbf{C}^0_{h^\star}([0,b],\mathbb{X})$. This result allows us to exhibit a finite ε-net for $\mathbf{R} \subset \mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X})$ as follows :

$$\{\eta_k(t)\}_{k\in\{1,...,p\}} \text{ for all } t \in \mathbb{R}_+$$

. Hence, according to (2) and latter result, for any $u \in \mathbf{R}$ there is a $k \in \{1,...,p\}$ such that

$$\|u(t) - \eta_k(t)\|_{h^\star} = \sup_{t\in\mathbb{R}_+} \frac{\|u(t)-\eta_k(t)\|_{a\omega,c}}{h^\star(t)} \le \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, this last relation proves that $\mathbf{R}$ is relatively compact in $\mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X})$. ☐

3. Main results

Because of the estimate on $\|\mathbf{S}_\alpha(t)\|_{\mathbf{B}(\mathbb{X})}$, we can make the following assumptions :
(**H1**) : The operator A is sectorial operator of type $\tilde{\omega} < 0$ and angle $\theta \in \big[0, \pi(1-\frac{\alpha}{2})\big)$.
(**H2**) : Let $(\mathbf{S}_\alpha(t))_{t\ge 0} \subset \mathbf{B}(\mathbb{X})$ be a strongly continuous family of linear operators.
(**H3**) : $f \in \mathbf{C}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ such that for all $(t,u) \in \mathbb{R}\times\mathbb{X}$, $f(t+\omega, cu) = cf(t,u)$, and there exist $\delta \in (0,1]$ such that we have

$$\|f(t,u_1) - f(t,u_2)\| \le \delta\|u_1 - u_2\|, \forall u_1, u_2 \in \mathbb{X},$$

(**H3** Bis) : $f \in \mathbf{C}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ such that for all $(t,u) \in \mathbb{R}\times\mathbb{X}$, $f(t+\omega, cu) = cf(t,u)$, and there exist $\kappa \in (0,1]$ such that we have

$$\|f(t,u_h(t))\| \le \kappa\|u_h(t)\|, \forall u \in \mathbb{X},\ \forall t,h \in \mathbb{R}_+$$

With all of this in mind, we can have the following results.

3.1. (ω,c)**-periodic case.** In this section, we mainly deal with the existence and uniqueness result of (ω,c)-periodic mild solutions to the following fractional Cauchy problem :

$$\begin{cases} {}^cD^\alpha_t u(t) = Au(t) +^c D^{\alpha-1}_t f(t,u(t)),\ 1 < \alpha < 2\ ,\ t \in \mathbb{R}, \\ \quad u(0) = 0 \end{cases} \tag{7}$$

Definition 3.1. *Assume that* A *is of sectorial type* $\tilde{\omega} < 0$ *and angle* $\theta \in \big[0, \pi(1-\frac{\alpha}{2})\big)$. *A function continuous function* $u : \mathbb{R} \to \mathbb{X}$ *is called a mild solution to Equation (7) on* $\mathbb{R}$, *if the function* $s \to \mathbf{S}_\alpha(t-s)f(s,u(s))$ *is integrable on* $[0,t)$ *for each* $t \in \mathbb{R}$ *and*

$$u(t) = \int_{-\infty}^t \mathbf{S}_\alpha(t-s)f(s,u(s))\ ds, \text{ for any } t \in \mathbb{R}.$$

Theorem 3.2. *Under previous assumptions, if we assume that (**H1**) – (**H3**) hold, then there exist a unique (ω, c)-periodic mild solution to Equation (7), provided that there is a constant $\eta_{\alpha,\theta,\delta} \in (0,1)$ such that $\eta_{\alpha,\theta} \geq \dfrac{C_{\alpha,\theta} M \gamma_1 \delta |\tilde{\omega}|^{\frac{-1}{\alpha}} \pi}{\alpha \sin(\frac{\pi}{\alpha})}$ where* $\gamma_p = \max\{1, |c|^{-p}\}, \forall p \in (0, \infty)$.

Proof. Consider the operator $\Gamma : \mathbf{P}_{\omega,c}(\mathbb{R}_+, \mathbb{X}) \to \mathbf{P}_{\omega,c}(\mathbb{R}_+, \mathbb{X})$ such that :

$$(\Gamma u)(t) := \int_{-\infty}^{t} \mathbf{S}_\alpha(t-s) f(s, u(s)) ds \ \ , \ t \geq 0$$

then Γ is well defined. In fact, let $u \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$ using Theorem (2.18), $s \to f(s, u(s))$ belong to $\mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$. Then by theorem (2.19) with $f \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$, we have $\Gamma u \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$. Thus we infer that Γ maps $\mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$ into itself. For $u, v \in \mathbf{P}_{\omega,c}(\mathbb{R}, \mathbb{X})$, we get

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\|_{\omega,c} &= \left\| \int_{-\infty}^{t} \mathbf{S}_\alpha(t-s) \left[f(s, u(s)) - f(s, v(s)) \right] \ ds \right\|_{\omega,c} \\
&\leq \sup_{t \in [0,\omega]} \left\{ \int_{-\infty}^{t} \| \ |c|^\wedge(-(t-s)) \mathbf{S}_\alpha(t-s) \| \cdot \delta \cdot |c|^\wedge(-s) \|u(s) - v(s)\| \ ds \right\} \\
&\leq \delta \sup_{t \in [0,\omega]} \left\{ \int_{-\infty}^{t} \| \ |c|^\wedge(-(t-s)) \mathbf{S}_\alpha(t-s) \| \ ds \right\} \|v - u\|_{\omega,c} \\
&\leq \delta \sup_{t \in [0,\omega]} \left\{ \int_{-\infty}^{t} \|\mathbf{S}_\alpha(t-s)\|_{\omega,c} \ ds \right\} \|v - u\|_{\omega,c} \\
&\leq C_{\alpha,\theta} M \gamma_1 \delta \sup_{t \in [0,\omega]} \left\{ \int_{-\infty}^{t} \frac{1}{1 + |\tilde{\omega}|(t-s)^\alpha} \ ds \right\} \|v - u\|_{\omega,c} \\
&\leq C_{\alpha,\theta} M \gamma_1 \delta \sup_{t \in [0,\omega]} \left\{ \int_{0}^{\infty} \frac{1}{1 + |\tilde{\omega}| s^\alpha} \ ds \right\} \|v - u\|_{\omega,c} \\
&\leq C_{\alpha,\theta} M \gamma_1 \delta \sup_{t \in [0,\omega]} \left\{ \frac{|\tilde{\omega}|^{\frac{-1}{\alpha}} \pi}{\alpha \sin(\frac{\pi}{\alpha})} \right\} \|v - u\|_{\omega,c} \\
&\leq \frac{C_{\alpha,\theta} M \gamma_1 \delta |\tilde{\omega}|^{\frac{-1}{\alpha}} \pi}{\alpha \sin(\frac{\pi}{\alpha})} \|v - u\|_{\omega,c} \\
&\leq \eta_{\alpha,\theta,\delta} \|v - u\|_{\omega,c}
\end{aligned}$$

Finally, Γ is a contraction. So by using the Banach fixed point theorem, there is $u \in \mathbf{P}_{\omega,c}(\mathbb{R}_+, \mathbb{X})$ which is the unique mild solution to Equation (7).
The proof is complete. □

We need the following theorems for the sequel:

Theorem 3.3. *(Arzela-Ascoli) Let X be compact metric space and Y be a metric space. Then $A \subset \mathbf{C}(X, Y)$ is relatively compact if and only if both conditions are satisfied*

(1) *$A(x) := \{f(x), f \in A\}$ is relatively compact in Y for all $x \in X$,*
(2) *A is equicontinuous.*

Theorem 3.4. *(Schauder) Let E be a Banach Space, C a nonempty closed convex set of E and $T : C \to C$ continuous. If $T(C)$ is relatively compact, then T has at least one fixed point.*

Let us assume that :

$(\mathbf{H4})$: $f(t,u)$ is of Caratheodory ; that is, for any $t \in \mathbb{R}$, $f(t,u)$ is continuous with respect to $u \in \mathbb{X}$, and for any $u \in \mathbb{X}$, $f(t,u)$ is strongly measurable with respect to $t \in \mathbb{R}$.

Now, we state and prove this additional existence theorem :

Theorem 3.5. *Assume that $(\boldsymbol{H1}), (\boldsymbol{H2}), (\boldsymbol{H3}\ Bis)$ and $(\boldsymbol{H4})$ hold. Then Equation (7) has at least one (ω, c)-periodic mild solution on $\mathbb{R}_+$.*

Proof. Let $\tau = \dfrac{|\tilde{\omega}|^{\frac{-1}{\alpha}}\pi}{\alpha \sin(\frac{\pi}{\alpha})}$. We recall that $\sup\limits_{t\in[0,\omega]} \left\{ \int_{-\infty}^{t} \frac{1}{1+|\tilde{\omega}|(t-s)^{\alpha}}\, ds \right\} \le \tau$.
Now, choose

$$r \ge 2C_{\alpha,\theta} M \gamma_1 \tau \kappa \|u\|_{\omega,c}$$

and consider $B_r = \{u \in \mathbf{P}_{\omega,c}(\mathbb{R}_+, \mathbb{X}), \|u\|_{\omega,c} \le r\}$. Define the operator N on B_r by :

$$(Nu)(t) = \int_{-\infty}^{t} \mathbf{S}_{\alpha}(t-s) f(s,u(s))\, ds \tag{8}$$

Let us observe that if $u \in B_r$ then $Nu \in B_r$. Indeed, we have

$$\begin{aligned}
\|(Nu)(t)\|_{\omega,c} &= \left\| \int_{-\infty}^{t} \mathbf{S}_{\alpha}(t-s) f(s,u(s))\, ds \right\|_{\omega,c} \\
&\le \sup_{t\in[0,\omega]} \left\{ \int_{-\infty}^{t} \||c|^{\wedge}(-(t-s))\mathbf{S}_{\alpha}(t-s)\| \cdot \kappa \cdot \||c|^{\wedge}(-s)u(s)\|\, ds \right\} \\
&\le \kappa \sup_{t\in[0,\omega]} \left\{ \int_{-\infty}^{t} \|\mathbf{S}_{\alpha}(t-s)\|_{\omega,c} \|u\|_{\omega,c}\, ds \right\} \\
&\le C_{\alpha,\theta} M \gamma_1 \kappa \sup_{t\in[0,\omega]} \left\{ \int_{-\infty}^{t} \frac{1}{1+|\tilde{\omega}|(t-s)^{\alpha}}\, ds \right\} \|u\|_{\omega,c} \\
&\le C_{\alpha,\theta} M \gamma_1 \tau \kappa \|u\|_{\omega,c} \\
&\le r
\end{aligned}$$

Now let us prove that N is continuous and relatively compact. We show the continuity first.
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in B_r such that $u_n \to u$ in B_r.
According to $(\mathbf{H4})$ f is continuous on $\mathbb{R}\times\mathbb{X}$. It comes that $f(s,u_n(s)) \to f(s,u(s))$ whenever $n \to \infty$ that is for $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that for all $n \ge N$,

$\|f(s,u_n(s))-f(s,u(s))\|<\varepsilon$. Then, for any $t\in\mathbb{R}$, we have

$$\begin{aligned}\|(Nu_n)(t)-(Nu)(t)\|_{\omega,c} &\leq \sup_{t\in[0,\omega]}\left\{\int_{-\infty}^{t}\|\mathbf{S}_\alpha(t-s)\|_{\mathbf{B}(\mathbb{X})}\||c|^{\wedge}(-s)f(s,u_n(s))-f(s,u(s))\|\ ds\right\}\\ &\leq C_{\alpha,\theta}M\gamma_1\varepsilon\sup_{t\in[0,\omega]}\left\{\int_{-\infty}^{t}\frac{1}{1+|\tilde{\omega}|(t-s)^\alpha}\ ds\right\}\\ &\leq (C_{\alpha,\theta}M\gamma_1\tau)\varepsilon\end{aligned}$$

Now, we show the compactness of N, using Ascoli's theorem. First, we prove $(Nu)(t):u\in B_r$ is uniformly bounded.

$$\|(Nu)(t)\|_{\omega,c}\leq\sup_{t\in[0,\omega]}\left\{\int_{-\infty}^{t}\|\mathbf{S}_\alpha(t-s)\|_{\mathbf{B}(\mathbb{X})}\||c|^{\wedge}(-s)f(s,u(s))\|\ ds\right\}\leq C_{\alpha,\theta}M\tau\kappa\|u\|_{\omega,c}$$

So, $\|(Nu)(t)\|_{\omega,c}<\infty$ and it is proved. Now, let us prove that $N(B_r)$ is equicontinuous.
Let $t\in(-\infty,b]$, $b\in\mathbb{R}$. For $t_1,t_2\in(-\infty,b]$ such that $t_2<t_1$, we have :

$$\begin{aligned}\|(Nu)(t_1)-(Nu)(t_2)\|_{\omega,c} &= \left\|\int_{-\infty}^{t_1}\mathbf{S}_\alpha(t_1-s)f(s,u(s))\ ds-\int_{-\infty}^{t_2}\mathbf{S}_\alpha(t_2-s)f(s,u(s))\ ds\right\|_{\omega,c}\\ &= \left\|\int_{-\infty}^{t_2}\mathbf{S}_\alpha(t_1-s)f(s,u(s))\ ds+\int_{t_2}^{t_1}\mathbf{S}_\alpha(t_1-s)f(s,u(s))\ ds\right.\\ &\quad\left.-\int_{-\infty}^{t_2}\mathbf{S}_\alpha(t_2-s)\ f(s,u(s))\ ds\right\|_{\omega,c}\\ &\leq\kappa\left(\int_{-\infty}^{t_2}\|\mathbf{S}_\alpha(t_1-s)-\mathbf{S}_\alpha(t_2-s)\|_{\mathbf{B}(\mathbb{X})}\|u\|_{\omega,c}\ ds\right.\\ &\quad\left.+\int_{t_2}^{t_1}\|\mathbf{S}_\alpha(t_1-s)\|_{\mathbf{B}(\mathbb{X})}\|u\|_{\omega,c}\ ds\right)\\ &\leq\kappa\left(\int_{-\infty}^{t_2}\|\mathbf{S}_\alpha(t_1-s)-\mathbf{S}_\alpha(t_2-s)\|_{\mathbf{B}(\mathbb{X})}\|u\|_{\omega,c}\ ds\right.\\ &\quad\left.+\int_{t_2}^{t_1}\|\mathbf{S}_\alpha(t_1-s)\|_{\mathbf{B}(\mathbb{X})}\|u\|_{\omega,c}\ ds\right)\end{aligned}$$

Let $I_1:=\displaystyle\int_{-\infty}^{t_2}\|\mathbf{S}_\alpha(t_1-s)-\mathbf{S}_\alpha(t_2-s)\|_{\mathbf{B}(\mathbb{X})}\|u\|_{\omega,c}\ ds$ and $I_2:=\displaystyle\int_{t_2}^{t_1}\|\mathbf{S}_\alpha(t_1-s)\|_{\mathbf{B}(\mathbb{X})}\|u\|_{\omega,c}\ ds$.
Actually, both I_1 and I_2 tend to 0 independently of $u\in B_r$ when $t_2\to t_1$.
Therefore the continuity of the function $t\to\|\mathbf{S}_\alpha(t)\|_{\mathbf{B}(\mathbb{X})}$ for $t\in(-\infty,b]$ allows us to conclude that $\lim\limits_{t_2\to t_1}I_1=0$. In the other hand, we have :

$$I_2=\int_{t_2}^{t_1}\|\mathbf{S}_\alpha(t_1-s)\|_{\mathbf{B}(\mathbb{X})}\|u\|_{\omega,c}\ ds\leq\left(C_{\alpha,\theta}M\|u\|_{\omega,c}\tau\right)|t_1-t_2|$$

And consequently, it comes that $\lim\limits_{t_2\to t_1}I_2=0$.
So, using Theorem 3.3 we have proved that $N(B_r)$ is relatively compact for $t\in(-\infty,b]$ for all $b\in\mathbb{R}$. Then $Nu:u\in B_r$ is a family of equicontinuous functions. Then Schauder's theorem (see theorem 3.4) allows us to conclude that (7) has at least one (ω,c)-periodic mild solution on $\mathbb{R}$. $\square$

3.2. (ω, c)**-asymptotically periodic case.** In this section, we mainly deal with the existence and uniqueness result of asymptotically (ω, c)-periodic mild solutions to the following fractional Cauchy problem with delay :

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + {}^c D_t^{\alpha-1} f(t, u(t-h)), \ 1 < \alpha < 2 \ , \ t, h \in \mathbb{R}_+, \\ \quad u(0) = 0 \end{cases} \tag{9}$$

Note that in the following we will write $u_h(\cdot)$ as $u(\cdot - h)$.

Now let us assume that :

$(\mathbf{H5})$: f is satisfying Theorem 2.28.

Definition 3.6. *Assume that A is sectorial type of $\tilde{\omega} < 0$ and angle $\theta \in \big[0, \pi(1 - \frac{\alpha}{2})\big)$. A function continuous function $u : \mathbb{R}_+ \to \mathbb{X}$ is called a mild solution to Equation (9) on $\mathbb{R}_+$, if the function $s \to \mathbf{S}_\alpha(t-s) f(s, u_h(s))$ is integrable on $[0, t)$ for each $t, h \in \mathbb{R}_+$ and*

$$u(t) = \int_0^t \mathbf{S}_\alpha(t-s) f(s, u_h(s)) \ ds, \text{ for any } t, h \in \mathbb{R}_+.$$

Theorem 3.7. *Under previous assumptions, if we assume that $|c| > 1$, $(\boldsymbol{H1})$–$(\boldsymbol{H3})$ and $(\boldsymbol{H5})$ hold, then there exists a unique asymptotically (ω, c)-periodic mild solution to Equation (9), provided that there is a constant $\eta_{\alpha,\theta} \in (0, 1)$ such that*

$$\eta_{\alpha,\theta} \geq C_{\alpha,\theta} M \delta \sup_{t \in \mathbb{R}_+} \left\{ \int_0^t \frac{1}{1 + |\tilde{\omega}|(t-s)^\alpha} \ ds \right\}.$$

Proof. Consider the operator $\Pi : \mathbf{AP}_{\omega,c}(\mathbb{X}) \to \mathbf{AP}_{\omega,c}(\mathbb{X})$ such that

$$(\Pi u)(t) := \int_0^t \mathbf{S}_\alpha(t-s) f(s, u_h(s)) ds \ , \ t \geq 0, \text{ and } h \geq 0 \text{ (fixed)}.$$

One can easily see that Π is well defined and continuous. It follows from $(\mathbf{H5})$ that $f \in \mathbf{AP}_{\omega,c}(\mathbb{X})$. By Theorem (2.27), it comes that

$$\int_0^t \mathbf{S}_\alpha(t-s) f(s, u_h(s)) ds, \in \mathbf{AP}_{\omega,c}(\mathbb{X})$$

Thus we infer Π maps $\mathbf{AP}_{\omega,c}(\mathbb{X})$ into itself.
For $u, v \in \mathbf{AP}_{\omega,c}(\mathbb{X})$, we get

$$\begin{aligned} \|\Pi u(t) - \Pi v(t)\|_{a\omega,c} &= \left\| \int_0^t \mathbf{S}_\alpha(t-s) \left[f(s, u_h(s)) - f(s, v_h(s)) \right] \ ds \right\|_{a\omega,c} \\ &\leq \sup_{t \in \mathbb{R}_+} \left\{ \left\| |c|^\wedge(-t) \int_0^t \mathbf{S}_\alpha(t-s) \left[f(s, u_h(s)) - f(s, v_h(s)) \right] \ ds \right\| \right\} \\ &\leq \delta |c|^\wedge(-h) \sup_{t \in \mathbb{R}_+} \left\{ \int_0^t \|\mathbf{S}_\alpha(t-s)\|_{a\omega,c} \ ds \right\} \|v - u\|_{a\omega,c} \\ &\leq C_{\alpha,\theta} M \delta \sup_{t \in \mathbb{R}_+} \left\{ \int_0^t \frac{1}{1 + |\tilde{\omega}|(t-s)^\alpha} \ ds \right\} \|v - u\|_{a\omega,c} \\ &\leq \eta_{\alpha,\theta} \|v - u\|_{a\omega,c} \end{aligned}$$

Finally, Π is a contraction. So by using the Banach fixed point theorem, there is a unique mild solution $u \in \mathbf{AP}_{\omega,c}(\mathbb{X})$. ∎

We give the following assumptions :
(**H6**) : $f(t,u)$ is uniformly continuous on any bounded subset $\Omega \in \mathbb{X}$ uniformly in $t \in \mathbb{R}_+$ and for every bounded subset $\Omega \in \mathbb{X}$, $\{f(\cdot,u) : u \in \Omega\}$ is bounded in $\mathbf{AP}_{\omega,c}(\Omega,\mathbb{X})$.
(**H7**) : There exists a continuous nondecreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$ and $u \in \mathbb{X}$, $\|f(t,u)\|_{a\omega,c} \leq \Psi(\|u\|_{a\omega,c})$.

Now we establish an existence theorem of asymptotically (ω, c)-periodic mild solution to Equation (9).

Theorem 3.8. *Assume that $f \in \mathbf{AP}_{\omega,c}(\Omega,\mathbb{X})$ with $|c| \geq 1$, satisfying (**H1**) – (**H2**), (**H5**) – (**H7**) and the following additional conditions :*

(1) For each $r > 0$,

$$\sup_{t\in\mathbb{R}_+}\left\{\int_0^t \frac{\Psi(rh^\star(s))}{1+|\tilde{\omega}|(t-s)^\alpha}ds\right\} < \infty$$

that is

$$\lim_{t\to\infty}\frac{1}{h^\star(t)}\left(\sup_{t\in\mathbb{R}_+}\left\{\int_0^t \frac{\Psi(rh^\star(s))}{1+|\tilde{\omega}|(t-s)^\alpha}ds\right\}\right) = 0$$

where $h^\star$ is the function given in lemma (2.29) and we set

$$\varrho(r) = C_{\alpha,\theta}M\left\|\sup_{t\in\mathbb{R}_+}\left\{\int_0^t \frac{\Psi(rh^\star(s))}{1+|\tilde{\omega}|(t-s)^\alpha}ds\right\}\right\|_{h^\star}$$

(2) For each $\varepsilon > 0$, and any fixed $h \geq 0$ there is $\delta_0 > 0$ such that for every $u, v \in \mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X})$, $\|u-v\|_{h^\star} < \delta_0$ implies that for all $t \in \mathbb{R}_+$,

$$\sup_{t\in\mathbb{R}_+}\left\{\int_0^t \frac{\|f(s,u_h(s)) - f(s,v_h(s))\|_{a\omega,c}}{1+|\tilde{\omega}|(t-s)^\alpha}ds\right\} \leq \frac{\varepsilon}{C_{\alpha,\theta}M},$$

(3) For each $\alpha, \beta \in \mathbb{R}_+$ and $r > 0$, the set $\{f(s,h^\star(s)u) : \alpha \leq s \leq \beta, u \in \mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X}), \|u\|_{h^\star} \leq r\}$ is relatively compact in $\mathbb{X}$.

(4) $\liminf\limits_{\xi\to\infty}\frac{\xi}{\varrho(\xi)} > 1$.

Then Equation (9) admits one mild solution in $\mathbf{AP}_{\omega,c}(\Omega,\mathbb{X})$.

Proof. We define the nonlinear operator $\Lambda : \mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X}) \to \mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X})$ by

$$(\Lambda u)(t) := \int_0^t \mathbf{S}_\alpha(t-s)f(s,u_h(s))ds \ , \ t \geq 0, \text{ and } h \geq 0 \text{ (fixed)}.$$

We will show that Λ has a fixed point in $\mathbf{AP}_{\omega,c}(\Omega,\mathbb{X})$ by the following steps :

(1) For $u \in \mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X})$, we have $\|u\|_{h^*} < \infty$ and

$$\frac{\|\Lambda u\|_{a\omega,c}}{h^\star(t)} \leq \frac{1}{h^\star(t)}\left(C_{\alpha,\theta}M\sup_{t\in\mathbb{R}_+}\left\{\int_0^t \frac{\Psi(\|u\|_{h^\star}h^\star(s))}{1+|\tilde{\omega}|(t-s)^\alpha}ds\right\}\right)$$

It follows from condition 1. that Λ is well defined.

(2) For each $\varepsilon > 0$, and any fixed $h \geq 0$ there is $\delta_0 > 0$ satisfying condition 2. such that for $u, v \in \mathbf{C}^0_{h^\star}(\mathbb{R}_+,\mathbb{X})$, with $\|u-v\|_{h^\star} < \delta_0$ we have

$$\|\Lambda u(t) - \Lambda v(t)\|_{a\omega,c} \leq C_{\alpha,\theta}M\sup_{t\in\mathbb{R}_+}\left\{\int_0^t \frac{\|f(s,u_h(s)) - f(s,v_h(s))\|_{a\omega,c}}{1+|\tilde{\omega}|(t-s)^\alpha}ds\right\} \leq \varepsilon$$

which shows that Λ is continuous.

(3) Next we show that Λ is completely continuous. We set $\mathbf{B}_r(\mathbb{X})$ for the closed unit ball with centre at 0 and radius r in the space $\mathbb{X}$. Let $\vartheta = \Lambda(\mathbf{B}_r(\mathbf{C}^0_{h^\star}(\mathbb{R}_+, \mathbb{X})))$ and $\varsigma = \Lambda(u)$ for $u \in \mathbf{B}_r(\mathbf{C}^0_{h^\star}(\mathbb{R}_+, \mathbb{X}))$. First, we will prove that $\vartheta_b(t)$ is a relatively compact subset of $\mathbb{X}$ for each $t \in [0, b]$. In fact, by the continuity of $\mathbf{S}_\alpha(\cdot)$ and condition 3. of f, we infer that the set $\Sigma = \{\mathbf{S}_\alpha(s) f(\tau, h^\star(\tau)u) : 0 \leq s, \tau \leq t, u \in \mathbf{C}^0_{h^\star}(\mathbb{R}_+, \mathbb{X}), \|u\|_{h^\star} \leq r\}$ is relatively compact. On the other hand, we can get $\vartheta_b(t) \in t \cdot \overline{co(\Sigma)}$, where $\overline{co(\Sigma)}$ denotes the convex hull of Σ, which establishes our assertion.

Second, we show that the set ϑ_b is equicontinuous. In fact, we can decompose

$$\begin{aligned}\varsigma(t+s) - \varsigma(t) &= \int_t^{t+s} \mathbf{S}_\alpha(t-s-\tau) f(\tau, u_h(\tau))\, d\tau \\ &+ \int_0^t \Big[\mathbf{S}_\alpha(t+s) - \mathbf{S}_\alpha(t)\Big] f(t-\tau, u_h(\tau))\, d\tau\end{aligned}$$

Then from (**H7**) and above decomposition of $\varsigma(t+s) - \varsigma(t)$, it follows that the set ϑ_b is equicontinuous.

Finally, applying condition 1., we have

$$\frac{\|\varsigma\|_{a\omega,c}}{h^\star(t)} \leq \frac{C_{\alpha,\theta} M}{h^\star(t)} \sup_{t \in \mathbb{R}_+} \left\{ \int_0^t \frac{\Psi(r h^\star(s))}{1 + |\tilde{\omega}|(t-s)^\alpha} ds \right\} \to 0, \quad \text{as } t \to \infty.$$

and this convergence is independent of $u \in \mathbf{B}_r(\mathbf{C}^0_{h^\star}(\mathbb{R}_+, \mathbb{X}))$. Hence, by Lemma 2.29, ϑ is a relatively compact set in $\mathbf{C}^0_{h^\star}(\mathbb{R}_+, \mathbb{X})$.

(4) Let $u^\lambda(\cdot)$ be a solution of equation $u^\lambda = \lambda \Lambda(u^\lambda)$ for some $\lambda \in (0,1)$. From

$$\|u^\lambda\|_{a\omega,c} \leq C_{\alpha,\theta} M \sup_{t \in \mathbb{R}_+} \left\{ \int_0^t \frac{\Psi(\|u^\lambda\|_{h^\star} h^\star(s))}{1 + |\tilde{\omega}|(t-s)^\alpha} ds \right\} \leq \varrho(\|u^\lambda\|_{h^\star}) h^\star(t)$$

we get

$$\frac{\|u^\lambda\|_{h^\star}}{\varrho(\|u^\lambda\|_{h^\star})} \leq 1$$

and by condition 4., we see that the set $\{u^\lambda : u^\lambda = \lambda \Lambda(u^\lambda), \lambda \in (0,1)\}$ is bounded.

(5) It follows from (**H6**) that $t \to f(t, u_h(t))$ belongs to $\mathbf{AP}_{\omega,c}(\Omega, \mathbb{X})$ when $u \in \mathbf{AP}_{\omega,c}(\Omega, \mathbb{X})$.

Moreover, from Theorem (2.27), we can deduce that $\Lambda(\mathbf{AP}_{\omega,c}(\Omega, \mathbb{X})) \subset \mathbf{AP}_{\omega,c}(\Omega, \mathbb{X})$. We note that $\mathbf{AP}_{\omega,c}(\Omega, \mathbb{X})$ is a closed subspace of $\mathbf{C}^0_{h^\star}(\mathbb{R}_+, \mathbb{X})$, consequently, we can consider $\Lambda : \overline{\mathbf{AP}_{\omega,c}(\Omega, \mathbb{X})} \to \overline{\mathbf{AP}_{\omega,c}(\Omega, \mathbb{X})}$. By propositions 1. – 3., we deduce that this map is completely continuous. Applying the well-known Leray-Schauder alternative theorem (see [17]), we infer that Λ has a fixed point $u \in \mathbf{AP}_{\omega,c}(\mathbb{X})$ which is the asymptotically (ω, c)-periodic mild solution to Equation (9). □

From Theorem (3.8), we can obtain the following interesting corollary.

Corollary 3.9. *Let $f : \mathbb{R}_+ \times \mathbb{X} \longrightarrow \mathbb{X}$ be a function satisfying assumption **(H6)** and the following Hölder-type condition :*

$$\|f(t, u_h) - f(t, v_h)\|_{a\omega,c} \leq \rho \|u - v\|^\vartheta_{a\omega,c} \quad, 0 < \vartheta < 1$$

for all $t \in R_+$ and $u, v \in \mathbb{X}$ where $\rho, h > 0$ are constant. Moreover, assume the following conditions are satisfied :

a) $\sup_{t\in\mathbb{R}_+}\left\{\int_0^t \frac{(h^\star(s))^\vartheta}{1+|\tilde{\omega}|(t-s)^\alpha}ds\right\} = \gamma < \infty,$

b) *For each* $\alpha, \beta \in \mathbb{R}_+$ *and* $r > 0$*, the set* $\{f(s, h^\star(s)u) : \alpha \leq s < \beta, u \in \mathbf{C}^0_{h^\star}(\mathbb{R}_+, \mathbb{X}), \|u\|_{h^\star} \leq r\}$ *is relatively compact in* $\mathbb{X}$*,*

c) $\liminf_{\xi\to\infty} \frac{\xi}{\varrho(\xi)} > 1.$

Then Equation (9) admits at least one asymptotically (ω, c)*-periodic mild solution.*

Proof. Let $\gamma_1 = \rho$ and we take $\Psi(\xi) = \gamma_1\xi^\vartheta$. Then, condition (**H7**) is satisfied. It follows from a), that the function f satisfies (1) in Theorem (3.8). Note that for each $\varepsilon > 0$ there is $0 < \delta^\vartheta < \frac{\varepsilon}{\gamma_1}$ such that for every $u, v \in \mathbf{C}^0_{h^\star}(\mathbb{R}_+, \mathbb{X})$, $\|u-v\|_{h^\star} \leq \delta$ implies that $C_{\alpha,\theta}M\sup_{t\in\mathbb{R}_+}\left\{\int_0^t \frac{\|f(s, u_h(s)) - f(s, v_h(s))\|_{a\omega,c}}{1+|\tilde{\omega}|(t-s)^\alpha}ds\right\} \leq \varepsilon$ for all $t \in \mathbb{R}_+$. The assumption (3) in Theorem (3.8) can be easily verified by the definition of Ψ. So, from Theorem (3.8) we can conclude that Equation (9) admits at least one asymptotically (ω, c)-periodic mild solution. □

4. An application to fractional diffusion-wave equations.

To illustrate Theorem (3.2), we consider the following fractional diffusion-wave equation type :

$$(10)\quad \begin{cases} {}^cD_t^\alpha u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) - \beta u(t,x) + {}^c D_t^{\alpha-1}f(t, u(t,x)),\ t \in \mathbb{R}, x \in [0,\pi] \\ \qquad u(t,0) = u(t,2\pi) = 0,\ t \in \mathbb{R} \end{cases}$$

where $1 < \alpha < 2$ and $f(t, u(t,x)) = a^{\frac{t}{2\pi}}\sin(a^{\frac{-t}{2\pi}}u(t,x))$ with $u(t,x)$ $(2\pi, a)$-periodic with respect to the first variable and $|a| \neq 1$.
We set $(\mathbb{X}, \|\cdot\|_\mathbb{X}) = (L^2([0,\pi]), \|\cdot\|_2)$ and define
$D(A) = \{u \in L^2([0,\pi]), u(0) = u(\pi) = 0)\}$
$Au = \Delta u = u''$, $\forall u \in D(A)$.

It is well known that A is the infinitesimal generator of an analytic semigroup on $L^2([0,\pi])$. Thus, A is sectorial of type $\tilde{\omega} = -\beta < 0$. In addition to that, we have

$$\begin{aligned} f(t+2\pi, u(t+2\pi,\cdot)) &= a^{\frac{t+2\pi}{2\pi}}\sin(a^{\frac{-t-2\pi}{2\pi}}u(t+2\pi,\cdot)) \\ &= aa^{\frac{t}{2\pi}}\sin(a^{\frac{-t}{2\pi}}a^{-1}au(t,\cdot)) \\ &= af(t, u(t,\cdot)) \end{aligned}$$

For all $u(t,\cdot) \in L^2([0,\pi]), t \in \mathbb{R}$. Thus, $f(t, u(t,x)) \in \mathbf{P}_{2\pi,a}(\mathbb{R}\times L^2([0,\pi]), L^2([0,\pi]))$. Furthermore, for $u_1(t,\cdot), u_2(t,\cdot) \in L^2([0,\pi])$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} \|f(t, u_1(t,\cdot)) - f(t, u_2(t,\cdot))\|_2 &\leq \|a^{\frac{t}{2\pi}}\sin(a^{\frac{-t}{2\pi}}u_1(t)) - a^{\frac{t}{2\pi}}\sin(a^{\frac{-t}{2\pi}}u_2(t))\|_2 \\ &\leq |a^{\frac{t}{2\pi}}|\|\sin(a^{\frac{-t}{2\pi}}u_1(t,\cdot)) - \sin(a^{\frac{-t}{2\pi}}u_2(t,\cdot))\|_2 \\ &\leq |a^{\frac{t}{2\pi}}|\|a^{\frac{-t}{2\pi}}u_1(t,\cdot) - a^{\frac{-t}{2\pi}}u_2(t,\cdot)\|_2 \\ &\leq |a^{\frac{t}{2\pi}}||a^{\frac{-t}{2\pi}}|\|u_1(t,\cdot) - u_2(t,\cdot)\|_2 \\ &\leq \|u_1(t,\cdot) - u_2(t,\cdot)\|_2 \end{aligned}$$

Hence choosing β such that

$$\frac{C_{\alpha,\theta} M |\tilde{\omega}|^{\frac{-1}{\alpha}} \pi}{\alpha \sin(\frac{\pi}{\alpha})} < \gamma_1^{-1}$$

where $\gamma_1 = \max\{1, |a|^{-1}\}$, assumptions of theorem (3.2) are satisfied and (10) has a unique solution in $\mathbf{P}_{2\pi,a}(\mathbb{R} \times L^2([0,\pi]), L^2([0,\pi]))$.

We end this section with the study of existence and uniqueness of an asymptotically (ω, c)-periodic mild solution to the following fractional diffusion-wave equation type :

$$(11) \qquad \begin{cases} {}^c D_t^{\alpha} v(t,x) = \dfrac{\partial^2}{\partial x^2} v(t,x) - \mu v(t,x) + {}^c D_t^{\alpha-1}\big(2^{\frac{t}{2\pi}} \xi \cos(2^{\frac{-t}{2\pi}} v(t-h,x)) \\ \qquad\qquad + \varrho e^{-|t|} \sin(v(t-h,x))\big),\ t \in \mathbb{R}_+, x \in [0,\pi] \\ v(t,0) = v(t,2\pi) = 0,\ t \in \mathbb{R}_+ \\ v(t,\zeta) = v_0(\zeta), \zeta \in [0,\pi] \end{cases}$$

where $1 < \alpha \le 2$, h is nonnegative, $v_0(\xi) \in L^2([0,\pi])$ and $v(t,x)$ is asymptotically $(2\pi, 2)$-periodic with respect to the first variable. We assume that $\xi, \varrho \in (0,1)$ such that $\xi + \varrho < 1$.
Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}}) = (L^2([0,\pi]), \|\cdot\|_2)$ and $(A, D(A))$ as in the preceding example. Hence, A is sectorial of type $\tilde{\omega} = -\mu < 0$. Equation (11) can be formulated by the inhomogeneous problem (9), where $v_h(\cdot) = v(t-h,\cdot)$ for any fixed $h \in \mathbb{R}_+$.
Let us consider the nonlinearity $f(t, v_h)(s) = 2^{\frac{t}{2\pi}} \cos(2^{\frac{-t}{2\pi}} v_h(s)) + \varrho e^{-|t|} \sin(v_h(s))$ for all $v_h \in L^2([0,\pi])$, $t \in \mathbb{R}_+$, and $s \in [0,\pi]$ with $v_h \in \mathbf{AP}_{2\pi,2}(L^2([0,\pi]), L^2([0,\pi]))$.
We observe that $f(t, v_h)$ is asymptotically $(2\pi, 2)$-periodic in t for each $v_h \in \mathbf{AP}_{2\pi,2}(L^2([0,\pi]), L^2([0,\pi]))$.
Indeed, let us decompose f as $f(t,v_h)(s) = g(t,v_h)(s) + h(t,v_h)(s)$ with :

$$g(t,v_h)(s) = 2^{\frac{t}{2\pi}} \xi \cos(2^{\frac{-t}{2\pi}} v_h(s)) := \varphi(t) \xi \cos(\varsigma(t) v_h(s))$$

and

$$h(t,v_h)(s) = \varrho e^{-|t|} \sin(v_h(s))$$

We have $\varphi(t) \in \mathbf{P}_{2\pi,2}(\mathbb{R}_+)$, $\varsigma(t) \in \mathbf{P}_{2\pi,2^{-1}}(\mathbb{R}_+)$ and $g(t+2\pi, 2x) = 2g(t,x)$ for all $x \in \mathbb{R}$. In addition to that, we know that (see previous example) :

$$\|g(t, v_{h,1}(s)) - g(t, v_{h,2}(s))\|_2 \le \xi \|v_{h,1}(s) - v_{h,2}(s)\|_2$$

for all $v_h \in L^2([0,\pi]), t \in \mathbb{R}_+$. Otherwise, $h \in \mathbf{C}_{0,c}(L^2([0,\pi]), L^2([0,\pi]))$, and

$$\|h(t, v_{h,2}(s)) - h(t, v_{h,1}(s))\|_2 \le \varrho \|v_{h,1}(s) - v_{h,2}(s)\|_2$$

for all $v_h \in L^2([0,\pi]), t \in \mathbb{R}_+$. Thus

$$\|f(t, v_{h,1}(s)) - f(t, v_{h,2}(s))\|_2 \le (\xi + \varrho)\|v_{h,1}(s) - v_{h,2}(s)\|_2 < \|v_{h,1}(s) - v_{h,2}(s)\|_2$$

Furthermore, for each fixed t in $\mathbb{R}_+$,

$$2\hat{}(-t) h(t, 2\hat{}(-t) v_h)(s) = 2\hat{}(-t) \varrho e^{-|t|} \sin(2\hat{}(-t) v_h(s)) \le \varrho e^{-|t|}$$

Then, h is uniformly continuous for u in any bounded subset of $L^2([0,\pi])$ uniformly for $t \ge 0$, and $h(t, v_h) \to 0$ as $t \to \infty$ uniformly in $v_h(s), s \in [0,\pi]$. Finally, assumptions of Theorem (2.28) are satisfied.

Hence, choosing μ such that $C_{\alpha,\theta} M \sup_{t\in\mathbb{R}_+} \left\{ \int_0^t \frac{1}{1+|\tilde{\omega}|(t-s)^\alpha}\, ds \right\} < 1$, each assumptions of theorem (3.7) are satisfied. Finally, Equation (11) has a unique asymptotically $(2\pi, 2)$-periodic mild solution.

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