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ABSTRACT. Denote by $\tau_k(n)$, $\omega(n)$ and $\mu_2(n)$ the number of representations of n as product of k natural numbers, the number of distinct prime factors of n and the characteristic function of the square-free integers, respectively. Let $[t]$ be the integral part of real number t. For $f = \omega, 2^\omega, \mu_2, \tau_k$, we prove that

$$\sum_{n \leqslant x} f\left(\left[\frac{x}{n}\right]\right) = x \sum_{d \geqslant 1} \frac{f(d)}{d(d+1)} + O_\varepsilon(x^{\theta_f + \varepsilon})$$

for $x \to \infty$, where $\theta_\omega = \frac{53}{110}$, $\theta_{2^\omega} = \frac{9}{19}$, $\theta_{\mu_2} = \frac{2}{5}$, $\theta_{\tau_k} = \frac{5k-1}{10k-1}$ and $\varepsilon > 0$ is an arbitrarily small positive number. These improve the corresponding results of Bordellès.

1. INTRODUCTION

Denote by $[t]$ the integral part of the real number t. Recently Bordellès, Dai, Heyman, Pan and Shparlinski [3] established an asymptotic formula of

$$S_f(x) := \sum_{n \leqslant x} f\left(\left[\frac{x}{n}\right]\right)$$

under some simple assumptions of f. Subsequently, Wu [9] and Zhai [10] improved their results independently. In particular, if $f(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$ and all $n \geqslant 1$, then [9, Theorem 1.2(i)] or [10, Theorem 1] yields the following asymptotic formula

$$(1.1) \qquad S_f(x) = x \sum_{d \geqslant 1} \frac{f(d)}{d(d+1)} + O_\varepsilon(x^{1/2+\varepsilon})$$

as $x \to \infty$. Ma and Wu [7] observed that if f is factorizable in certain sense, then it is possible to break the $\frac{1}{2}$-barrier for the error term of (1.1). In particular, for the von Mangoldt function $\Lambda(n)$, they proved, with the help of the Vaughan identity and the technique of one-dimensional exponential sum, that

$$(1.2) \qquad S_\Lambda(x) = x \sum_{d \geqslant 1} \frac{\Lambda(d)}{d(d+1)} + O_\varepsilon(x^{35/71+\varepsilon}) \qquad (x \to \infty).$$

Using similar idea to the classical divisor function $\tau(n)$, Ma and Sun [6] showed that

$$(1.3) \qquad S_\tau(x) = x \sum_{d \geqslant 1} \frac{\tau(d)}{d(d+1)} + O_\varepsilon(x^{11/23+\varepsilon}) \qquad (x \to \infty).$$

Subsequently, with the help of a result of Baker on 2-dimensional exponential sums [1, Theorem 6], Bordellès [2] sharpened the exponents in (1.2)-(1.3) and also studied some new examples. Denote by $\tau_k(n)$, $\omega(n)$ and $\mu_2(n)$ the number of representations of n as product

of k natural numbers, the number of distinct prime factors of n and the characteristic function of the square-free integers, respectively. His results can be stated as follows:

$$(1.4) \qquad \sum_{n \leqslant x} f\left(\left[\frac{x}{n}\right]\right) = x \sum_{d \geqslant 1} \frac{f(d)}{d(d+1)} + O_\varepsilon(x^{\vartheta_f + \varepsilon}),$$

with

f	Λ	ω	2^ω	μ_2	τ	τ_3	τ_k $(k \geqslant 4)$
ϑ_f	$\frac{97}{203}$	$\frac{455}{914}$	$\frac{97}{202}$	$\frac{1919}{4268}$	$\frac{19}{40}$	$\frac{283}{574}$	$\frac{4k^3-k-2}{8k^3-2k-2}$

Very recently, Liu, Wu and Yang [5] succeeded to improve Bordellès' exponent $\frac{97}{203}$ to $\frac{9}{19}$ for Λ. Their principal tool is a new estimate on 3-dimensional exponential sums.

The aim of this paper is to propose better exponents than all others cases of the above table.

Theorem 1. *Under the previous notation, the asymptotic formula* (1.4) *holds with*

f	ω	2^ω	μ_2	τ_k $(k \geqslant 2)$
ϑ_f	$\frac{53}{110}$	$\frac{9}{19}$	$\frac{2}{5}$	$\frac{5k-1}{10k-1}$

The key point of proving Theorem 1 is to establish good bounds for

$$\mathfrak{S}_\delta^f(x, D) := \sum_{D < d \leqslant 2D} f(d) \psi\left(\frac{x}{d+\delta}\right).$$

where $\psi(t) := \{t\} - \frac{1}{2}$ and $\{t\}$ means the fractional part of real number t. Let $\mathbb{P}$ be the set of all primes and define

$$(1.5) \qquad \mathbb{1}_\mathbb{P}(n) := \begin{cases} 1 & \text{if } n \in \mathbb{P}, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbb{1}(n) \equiv 1, \qquad \tilde{\mu}(n) := \begin{cases} \mu(d) & \text{if } n = d^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following relations:

$$\omega = \mathbb{1}_\mathbb{P} * \mathbb{1}, \qquad \tau_k = \tau_{k-1} * \mathbb{1}, \qquad 2^\omega = \mu_2 * \tau, \qquad \mu_2 = \tilde{\mu} * \mathbb{1}.$$

Thus when $f = \omega, \tau_k, 2^\omega, \mu_2$, we can use these relations to decompose $\mathfrak{S}_\delta^f(x, D)$ into bilinear forms

$$(1.6) \qquad \mathfrak{S}_\delta^\omega(x, D) := \sum_{D < d\ell \leqslant 2D} \mathbb{1}_\mathbb{P}(d) \psi\left(\frac{x}{d\ell + \delta}\right),$$

$$(1.7) \qquad \mathfrak{S}_\delta^{\tau_k}(x, D) := \sum_{D < d\ell \leqslant 2D} \tau_{k-1}(d) \psi\left(\frac{x}{d\ell + \delta}\right),$$

$$(1.8) \qquad \mathfrak{S}_\delta^{2^\omega}(x, D) := \sum_{D < d^2\ell \leqslant 2D} \mu(d) \tau(\ell) \psi\left(\frac{x}{d^2\ell + \delta}\right),$$

$$(1.9) \qquad \mathfrak{S}_\delta^{\mu_2}(x, D) := \sum_{D < d^2\ell \leqslant 2D} \mu(d) \psi\left(\frac{x}{d^2\ell + \delta}\right).$$

In the third section, we shall use the Fourier analyse and the technique of multiple exponential sums to establish our bounds for these bilinear forms.

2. Preliminary lemmas

In this section, we shall cite three lemmas, which will be needed in the next section. The first one is [5, Proposition 3.1].

Lemma 2.1. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\delta \in \mathbb{R}$ be some constants. For $X > 0$, $H \geqslant 1$, $M \geqslant 1$ and $N \geqslant 1$, define*

$$(2.1) \qquad S_\delta = S_\delta(H, M, N) := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_{h,m} b_n \mathrm{e}\left(X \frac{M^\beta N^\gamma}{H^\alpha} \frac{h^\alpha}{m^\beta n^\gamma + \delta} \right),$$

where $\mathrm{e}(t) := \mathrm{e}^{2\pi \mathrm{i} t}$, the $a_{h,m}$ and b_n are complex numbers such that $|a_{h,m}| \leqslant 1$ and $|b_n| \leqslant 1$, and $m \sim M$ means that $M < m \leqslant 2M$. For any $\varepsilon > 0$ we have

$$S_\delta \ll \left((X^\kappa H^{2+\kappa} M^{2+\kappa} N^{1+\kappa+\lambda})^{1/(2+2\kappa)} + HMN^{1/2} + (HM)^{1/2}N + X^{-1/2}HMN \right)X^\varepsilon$$

uniformly for $M \geqslant 1$, $N \geqslant 1$, $H \leqslant M^{\beta-1}N^\gamma$ and $0 \leqslant \delta \leqslant 1/\varepsilon$, where (κ, λ) is an exponent pair and the implied constant depends on $(\alpha, \beta, \gamma, \varepsilon)$ only.

The second one is the Vaughan identity [8, formula (3)].

Lemma 2.2. *There are six real arithmetic functions $\alpha_k(n)$ verifying $\alpha_k(n) \ll \tau(n) \log(2n)$ for $(n \geqslant 1, 1 \leqslant k \leqslant 6)$ such that for $D \geqslant 1$ and any arithmetical function g, we have*

$$(2.2) \qquad \sum_{D < d \leqslant 2D} \Lambda(d) g(d) = S_1 + S_2 + S_3 + S_4,$$

where $\tau(n)$ is the classic divisor function and

$$S_1 := \sum_{m \leqslant D^{1/3}} \alpha_1(m) \sum_{D < \ell mn \leqslant 2D} g(mn),$$

$$S_2 := \sum_{m \leqslant D^{1/3}} \alpha_2(m) \sum_{D < \ell mn \leqslant 2D} g(mn) \log n,$$

$$S_3 := \sum_{\substack{D^{1/3} < m,n \leqslant D^{2/3} \\ D < \ell mn \leqslant 2D}} \alpha_3(m)\alpha_4(n)g(mn),$$

$$S_4 := \sum_{\substack{D^{1/3} < m,n \leqslant D^{2/3} \\ D < \ell mn \leqslant 2D}} \alpha_5(m)\alpha_6(n)g(mn).$$

The same result also holds for the Möbius function μ.

The third one is due to Vaaler (see [4, Theorem A.6]).

Lemma 2.3. *Let $\psi(t) := \{t\} - \frac{1}{2}$, where $\{t\}$ means the fractional part of real number t. For $x \geqslant 1$ and $H \geqslant 1$, we have*

$$\psi(x) = - \sum_{1 \leqslant |h| \leqslant H} \Phi\left(\frac{h}{H+1} \right) \frac{\mathrm{e}(hx)}{2\pi \mathrm{i} h} + R_H(x),$$

where $\Phi(t) := \pi t(1 - |t|) \cot(\pi t) + |t|$ and the second term $R_H(x)$ satisfies

$$(2.3) \qquad |R_H(x)| \leqslant \frac{1}{2H+2} \sum_{0 \leqslant |h| \leqslant H} \left(1 - \frac{|h|}{H+1} \right) \mathrm{e}(hx).$$

3. BILINEAR FORMS

The aim of this section is to establish some non-trivial bounds for the bilinear forms given by (1.6), (1.9), (1.7) and (1.8). For the convenience, we consider

$$\text{(3.1)} \qquad \mathfrak{S}_\delta^{\Lambda*\mathbb{1}}(x, D) := \sum_{D < d\ell \leqslant 2D} \Lambda(d)\psi\Big(\frac{x}{d\ell + \delta}\Big),$$

instead of $\mathfrak{S}_\delta^\omega(x, D)$. The following proposition gives the required estimate for $\mathfrak{S}_\delta^{\Lambda*\mathbb{1}}(x, D)$.

Proposition 3.1. *For any $\varepsilon > 0$, we have*

$$\text{(3.2)} \qquad \mathfrak{S}_\delta^{\Lambda*\mathbb{1}}(x, D) \ll_\varepsilon \big((x^{19}D^{87})^{1/133} + (x^{171}D^{624})^{1/1026}\big)x^\varepsilon$$

uniformly for $x \geqslant 3$, $D \leqslant x^{57/103}$ and $0 \leqslant \delta \leqslant \varepsilon^{-1}$, where the implied constant depends on ε only.

Proof. Applying the Vaughan identity (2.2) with $g(d) = \psi\big(\frac{x}{d\ell+\delta}\big)$, it follows that

$$\text{(3.3)} \qquad \mathfrak{S}_\delta^{\Lambda*\mathbb{1}}(x, D) = \mathfrak{S}_{\delta,1}^{\Lambda*\mathbb{1}} + \mathfrak{S}_{\delta,2}^{\Lambda*\mathbb{1}} + \mathfrak{S}_{\delta,3}^{\Lambda*\mathbb{1}} + \mathfrak{S}_{\delta,4}^{\Lambda*\mathbb{1}},$$

where

$$\mathfrak{S}_{\delta,1}^{\Lambda*\mathbb{1}} := \sum_{\ell \leqslant D} \sum_{m \leqslant (D/\ell)^{1/3}} \alpha_1(m) \sum_{D < \ell mn \leqslant 2D} \psi\Big(\frac{x}{\ell mn + \delta}\Big),$$

$$\mathfrak{S}_{\delta,2}^{\Lambda*\mathbb{1}} := \sum_{\ell \leqslant D} \sum_{m \leqslant (D/\ell)^{1/3}} \alpha_2(m) \sum_{D < \ell mn \leqslant 2D} \psi\Big(\frac{x}{\ell mn + \delta}\Big) \log n,$$

$$\mathfrak{S}_{\delta,3}^{\Lambda*\mathbb{1}} := \sum_{\ell \leqslant D} \sum_{\substack{(D/\ell)^{1/3} < m,n \leqslant (D/\ell)^{2/3} \\ D < \ell mn \leqslant 2D}} \alpha_3(m)\alpha_4(n)\psi\Big(\frac{x}{\ell mn + \delta}\Big),$$

$$\mathfrak{S}_{\delta,4}^{\Lambda*\mathbb{1}} := \sum_{\ell \leqslant D} \sum_{\substack{(D/\ell)^{1/3} < m,n \leqslant (D/\ell)^{2/3} \\ D < \ell mn \leqslant 2D}} \alpha_5(m)\alpha_6(n)\psi\Big(\frac{x}{\ell mn + \delta}\Big).$$

Using Lemma 2.3 and splitting the intervals of summation into the dyadic intervals, we can write

$$\text{(3.4)} \qquad \begin{aligned} \mathfrak{S}_{\delta,j}^{\Lambda*\mathbb{1}} = &-\frac{1}{2\pi \mathrm{i}} \sum_{H'} \sum_L \sum_M \sum_N \big(\mathfrak{S}_{\delta,j,\flat}^{\Lambda*\mathbb{1}}(H', L, M, N) + \overline{\mathfrak{S}_{\delta,j,\flat}^{\Lambda*\mathbb{1}}(H', L, M, N)}\big) \\ &+ \sum_L \sum_M \sum_N \mathfrak{S}_{\delta,j,\dagger}^{\Lambda*\mathbb{1}}(L, M, N), \end{aligned}$$

where $1 \leqslant H' \leqslant H \leqslant D$, $a_h := \frac{H'}{h}\Phi\big(\frac{h}{H+1}\big) \ll 1$ and

$$\mathfrak{S}_{\delta,1,\flat}^{\Lambda*\mathbb{1}} = \mathfrak{S}_{\delta,1,\flat}^{\Lambda*\mathbb{1}}(H', L, M, N) := \frac{1}{H'} \sum_{h \sim H'} \sum_{\ell \sim L} \sum_{m \sim M} \sum_{\substack{n \sim N \\ D < \ell mn \leqslant 2D}} a_h \alpha_1(m)\mathrm{e}\Big(\frac{hx}{\ell mn + \delta}\Big),$$

$$\mathfrak{S}_{\delta,1,\dagger}^{\Lambda*\mathbb{1}} = \mathfrak{S}_{\delta,1,\dagger}^{\Lambda*\mathbb{1}}(L, M, N) := \sum_{\ell \sim L} \sum_{m \sim M} \sum_{\substack{n \sim N \\ D < \ell mn \leqslant 2D}} \alpha_1(m) R_H\Big(\frac{x}{\ell mn + \delta}\Big),$$

and $\mathfrak{S}_{\delta,j,\flat}^{\Lambda*\mathbb{1}}(H', L, M, N)$, $\mathfrak{S}_{\delta,j,\dagger}^{\Lambda*\mathbb{1}}(L, M, N)$ can be defined similarly for $j = 2, 3, 4$. We have

$$\text{(3.5)} \qquad LMN \asymp D, \qquad M \leqslant (D/L)^{1/3} \quad \text{for} \quad j = 1, 2$$

and

$$(3.6) \qquad LMN \asymp D, \quad (D/L)^{1/3} \leqslant M \leqslant (D/L)^{1/2} \leqslant N \leqslant (D/L)^{2/3} \ \text{ for } \ j = 3, 4.$$

In the last case, we have considered the symmetry of m and n.

A. *Bounds of* $\mathfrak{S}_{\delta,j}^{\Lambda*\mathbb{1}}$ *for* $j = 1, 2$.

Since we can remove the factor $\log n$ by a simple partial integration, we only treat $\mathfrak{S}_{\delta,1}^{\Lambda*\mathbb{1}}$. If $L \leqslant D^{1/5}$, applying the exponent pair $(\frac{1}{6}, \frac{4}{6})$ to the sum over n, we have

$$\mathfrak{S}_{\delta,1,\flat}^{\Lambda*\mathbb{1}} \ll x^{\varepsilon} \frac{1}{H'} \sum_{h \sim H'} \sum_{\ell \sim L} \sum_{m \sim M} \left(\left(\frac{xH'}{LMN^2} \right)^{1/6} N^{4/6} + \left(\frac{xH'}{LMN^2} \right)^{-1} \right)$$
$$\ll x^{\varepsilon} \big((xHL^5M^5N^2)^{1/6} + x^{-1}(LMN)^2 \big).$$

In view of $L \leqslant D^{1/5}$ and (3.5), we derive that

$$(3.7) \qquad \qquad \mathfrak{S}_{\delta,1,\flat}^{\Lambda*\mathbb{1}} \ll \big((x^5 D^{17} H^5)^{1/30} + x^{-1} D^2 \big) x^{\varepsilon}$$

for $H' \leqslant H \leqslant D$ and (L, M, N) verifying (3.5).

Next we suppose that $D^{1/5} \leqslant L \leqslant D^{9/19}$. Firstly we remove the extra multiplicative condition $D < \ell mn \leqslant 2D$ at the cost of a factor $(\log x)^2$ and then apply Lemma 2.1 with $\alpha = \beta = \gamma = 1$, $(X, H, M, N) = (xH'/LMN, H', LM, N)$ and $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$ to get

$$(3.8) \qquad \begin{aligned} \mathfrak{S}_{\delta,1,\flat}^{\Lambda*\mathbb{1}} &\ll \big((xL^4M^4N^3)^{1/6} + LMN^{1/2} + (LM)^{1/2}N + x^{-1/2}(LMN)^{3/2} \big) x^{\varepsilon} \\ &\ll \big((x^3 D^{10} L^2)^{1/18} + (D^2 L)^{1/3} + DL^{-1/2} + (x^{-1} D^3)^{1/2} \big) x^{\varepsilon} \\ &\ll \big((x^{171} D^{624})^{1/1026} + D^{9/10} + (x^{-1} D^3)^{1/2} \big) x^{\varepsilon} \end{aligned}$$

for $H' \leqslant H \leqslant D^{20/57}$ $(\leqslant (D/L)^{2/3} \leqslant N)$ and (L, M, N) verifying (3.5) and $D^{1/5} \leqslant L \leqslant D^{9/19}$.

When $D^{9/19} \leqslant L \leqslant D$, we apply the exponent pair $(\frac{1}{6}, \frac{4}{6})$ to the sum over ℓ to get

$$\mathfrak{S}_{\delta,1,\flat}^{\Lambda*\mathbb{1}} \ll x^{\varepsilon} \frac{1}{H'} \sum_{h \sim H'} \sum_{m \sim M} \sum_{n \sim N} \left(\left(\frac{xH'}{L^2MN} \right)^{1/6} L^{4/6} + \left(\frac{xH'}{L^2MN} \right)^{-1} \right)$$
$$\ll x^{\varepsilon} \big((xHL^2M^5N^5)^{1/6} + x^{-1}H'^{-1}(LMN)^2 \big).$$

From this we deduce that

$$(3.9) \qquad \qquad \mathfrak{S}_{\delta,1,\flat}^{\Lambda*\mathbb{1}} \ll \big((x^{19} D^{68} H^{19})^{1/114} + x^{-1} D^2 \big) x^{\varepsilon}$$

for $H' \leqslant H \leqslant D$ and (L, M, N) verifying (3.5) and $D^{9/19} \leqslant L \leqslant D$.

Combining (3.7), (3.8) and (3.9), we obtain

$$(3.10) \qquad \begin{aligned} \mathfrak{S}_{\delta,1,\flat}^{\Lambda*\mathbb{1}} \ll \big(&(x^{171} D^{624})^{1/1026} + D^{9/10} + (x^{-1} D^3)^{1/2} \\ &+ (x^{19} D^{68} H^{19})^{1/114} + (x^5 D^{17} H^5)^{1/30} \big) x^{\varepsilon} \end{aligned}$$

for $H' \leqslant H \leqslant D^{20/57}$ and (L, M, N) verifying (3.5), where we have used the fact that $x^{-1} D^2 \leqslant (x^{-1} D^3)^{1/2}$. The same bound holds for $\mathfrak{S}_{\delta,2}^{\Lambda,\dagger}$. Therefore for $j = 1, 2$, we have

$$(3.11) \qquad \begin{aligned} \mathfrak{S}_{\delta,j}^{\Lambda*\mathbb{1}} \ll \big(&(x^{171} D^{624})^{1/1026} + D^{9/10} + (x^{-1} D^3)^{1/2} \\ &+ (x^{19} D^{68} H^{19})^{1/114} + (x^5 D^{17} H^5)^{1/30} \big) x^{\varepsilon} \end{aligned}$$

for $H \leqslant D^{20/57}$ and (L, M, N) verifying (3.5).

B. *Bounds of* $\mathfrak{S}_{\delta,j}^{\Lambda*\mathbb{1}}$ *for* $j = 3, 4$.

In this case, we should estimate

$$\mathfrak{S}_{\delta,3,\flat}^{\Lambda*\mathbb{1}} = \mathfrak{S}_{\delta,3,\flat}^{\Lambda*\mathbb{1}}(H', L, M, N) := \frac{1}{H'} \sum_{h \sim H'} \sum_{\substack{\ell \sim L \\ D < \ell mn \leqslant 2D}} \sum_{m \sim M} \sum_{n \sim N} a_h \alpha_3(m) \alpha_4(n) \mathrm{e}\Big(\frac{hx}{\ell mn + \delta}\Big),$$

$$\mathfrak{S}_{\delta,3,\dagger}^{\Lambda*\mathbb{1}} = \mathfrak{S}_{\delta,3,\dagger}^{\Lambda*\mathbb{1}}(L, M, N) := \sum_{\substack{\ell \sim L \\ D < \ell mn \leqslant 2D}} \sum_{m \sim M} \sum_{n \sim N} \alpha_3(m) \alpha_4(n) R_H\Big(\frac{x}{\ell mn + \delta}\Big),$$

for (L, M, N) verifying (3.6).

Firstly we bound $\mathfrak{S}_{\delta,3,\flat}^{\Lambda*\mathbb{1}}(H', L, M, N)$. We remove the extra multiplicative condition $D/\ell < mn \leqslant 2D/\ell$ at the cost of a factor $\log x$. By Lemma 2.1 with $\alpha = \beta = \gamma = 1$, $(X, H, M, N) = (xH'/LMN, H', M, LN)$ and $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$, we can derive

$$(3.12) \qquad \begin{aligned} \mathfrak{S}_{\delta,3,\flat}^{\Lambda*\mathbb{1}} &\ll_\varepsilon \big((xL^3 M^4 N^3)^{1/6} + M(LN)^{1/2} + M^{1/2}LN + (x^{-1}DH')^{1/2}\big)x^\varepsilon \\ &\ll_\varepsilon \big((x^2 D^7 L^{-1})^{1/12} + (D^5 L)^{1/6} + (x^{-1}DH')^{1/2}\big)x^\varepsilon, \end{aligned}$$

for $H' \leqslant H \leqslant D^{1/2} (\leqslant (LD)^{1/2} \leqslant LN)$ and (L, M, N) verifying (3.6), where we have removed the term $(D^3 L^{-1})^{1/4} (\leqslant (D^5 L)^{1/6})$. On the other hand, we can apply the exponent pair $(\frac{1}{6}, \frac{4}{6})$ to the sum over ℓ (similar to (3.7)) to get

$$(3.13) \qquad \mathfrak{S}_{\delta,3,\flat}^{\Lambda*\mathbb{1}} \ll \big((xD^5 HL^{-3})^{1/6} + x^{-1}D^2\big)x^\varepsilon$$

for $H' \leqslant H \leqslant D$ and (L, M, N) verifying (3.6).

From (3.12) and (3.13), we deduce that

$$(3.14) \qquad \mathfrak{S}_{\delta,3,\flat}^{\Lambda*\mathbb{1}} \ll_\varepsilon \big((x^2 D^7 L^{-1})^{1/12} + (xD^{20}H)^{1/24} + (x^{-1}DH)^{1/2} + x^{-1}D^2\big)x^\varepsilon,$$

for $H \leqslant D^{1/2}$ and (L, M, N) verifying (3.6), where we have used the following estimate

$$\min\{(D^5 L)^{1/6}, (xD^5 HL^{-3})^{1/6}\} \leqslant ((D^5 L)^{1/6})^{\frac{18}{24}} ((xD^5 HL^{-3})^{1/6})^{\frac{6}{24}} = (xD^{20}H)^{1/24}.$$

Secondly we bound $\mathfrak{S}_{\delta,3,\dagger}^{\Lambda*\mathbb{1}}$. Using (2.3) of Lemma 2.3, we have

$$\begin{aligned} \mathfrak{S}_{\delta,3,\dagger}^{\Lambda*\mathbb{1}} &\ll x^\varepsilon \sum_{\ell \sim L} \sum_{m \sim M} \sum_{n \sim N} \Big|R_H\Big(\frac{x}{\ell mn + \delta}\Big)\Big| \\ &\ll \frac{x^\varepsilon}{H} \sum_{\ell \sim L} \sum_{m \sim M} \sum_{n \sim N} \sum_{0 \leqslant |h| \leqslant H} \Big(1 - \frac{|h|}{H+1}\Big) \mathrm{e}\Big(\frac{hx}{\ell mn + \delta}\Big) \\ &\ll x^\varepsilon \big(DH^{-1} + \max_{1 \leqslant H' \leqslant H} \big|\widetilde{\mathfrak{S}}_{\delta,3,\dagger}^{\Lambda*\mathbb{1}}(H', L, M, N)\big|\big), \end{aligned}$$

where

$$\widetilde{\mathfrak{S}}_{\delta,3,\dagger}^{\Lambda*\mathbb{1}}(H', L, M, N) := \frac{1}{H} \sum_{\ell \sim L} \sum_{m \sim M} \sum_{n \sim N} \sum_{h \sim H'} \Big(1 - \frac{|h|}{H+1}\Big) \mathrm{e}\Big(\frac{hx}{\ell mn + \delta}\Big).$$

Clearly we can bound $\widetilde{\mathfrak{S}}_{\delta,3,\dagger}^{\Lambda*\mathbb{1}}(H', L, M, N)$ in the same way as $\mathfrak{S}_{\delta,3,\flat}^{\Lambda*\mathbb{1}}(H', L, M, N)$ and get

$$(3.15) \quad \mathfrak{S}_{\delta,3,\dagger}^{\Lambda*\mathbb{1}} \ll_\varepsilon \big(DH^{-1} + (x^2 D^7 L^{-1})^{1/12} + (xD^{20}H)^{1/24} + (x^{-1}DH)^{1/2} + x^{-1}D^2\big)x^\varepsilon$$

for $H \leqslant D^{1/2}$ and (L, M, N) verifying (3.6). Thus (3.14) and (3.15) imply that

$$(3.16) \qquad \mathfrak{S}_{\delta,j}^{\Lambda*\mathbb{1}} \ll_\varepsilon \big((x^2 D^7)^{1/12} + x^{-1}D^2 + DH^{-1} + (xD^{20}H)^{1/24} + (x^{-1}DH)^{1/2}\big)x^\varepsilon,$$

for $j = 3, 4$, $H \leqslant D^{1/2}$ and (L, M, N) verifying (3.6).

C. *End of proof of* (3.2).

Inserting (3.11) and (3.16) into (3.3) , we find that

$$\mathfrak{S}_\delta^{\Lambda*\mathbb{1}}(x, D) \ll \big((x^{171} D^{624})^{1/1026} + (x^2 D^7)^{1/12} + D^{9/10} + (x^{-1} D^3)^{1/2}$$
$$+ DH^{-1} + (x^{19} D^{68} H^{19})^{1/114} + (x^5 D^{17} H^5)^{1/30} + (x D^{20} H)^{1/24}\big) x^\varepsilon$$

for $H \leqslant D^{20/57}$, where we have removed the term $(x^{-1} DH)^{1/2} \leqslant (x^{-1} D^3)^{1/2}$. Optimising H over $[1, D^{20/57}]$, it follows that

$$\mathfrak{S}_\delta^{\Lambda*\mathbb{1}}(x, D) \ll \big((x^{171} D^{624})^{1/1026} + (x^2 D^7)^{1/12} + D^{9/10} + (x^{-1} D^3)^{1/2}$$
$$+ (x^{19} D^{87})^{1/133} + (x^5 D^{22})^{1/35} + (x D^{21})^{1/25} + D^{9/19}$$
$$+ (x^{19} D^{68})^{1/114} + (x^5 D^{17})^{1/30} + (x D^{20})^{1/24}\big) x^\varepsilon$$

which implies the required inequality, since the 5th term dominates the 7th one provided $D \leqslant x^{171/308}$ and the first term dominates the others ones provided $D \leqslant x^{57/103}$. □

The second proposition gives the required estimate for $\mathfrak{S}_\delta^{\tau_k}(x, D)$ defined as in (1.7).

Proposition 3.2. *Let $k \geqslant 2$ be an integer. For any $\varepsilon > 0$, we have*

$$(3.17) \qquad \mathfrak{S}_\delta^{\tau_k}(x, D) \ll_{k,\varepsilon} (x^k D^{4k-1})^{1/6k} x^\varepsilon$$

uniformly for $x \geqslant 3$, $1 \leqslant D \leqslant x^{\min\{4k/(5k+1), k/(2k-2)\}}$ and $0 \leqslant \delta \leqslant \varepsilon^{-1}$, where the implied constant depends on k and ε only.

Proof. Noticing that $\tau_k = \tau_{k-1}*\mathbb{1}$, using Lemma 2.3 and splitting the interval of summation into the dyadic intervals, we can write

$$(3.18) \qquad \begin{aligned} \mathfrak{S}_\delta^{\tau_k}(x, D) = &-\frac{1}{2\pi \mathrm{i}} \sum_{H'} \sum_M \sum_N \big(\mathfrak{S}_{\delta,\flat}^{\tau_k}(H', M, N) + \overline{\mathfrak{S}_{\delta,\flat}^{\tau_k}(H', M, N)}\big) \\ &+ \sum_M \sum_N \mathfrak{S}_{\delta,\dagger}^{\tau_k}(M, N), \end{aligned}$$

where $H \leqslant D$, $MN \asymp D$ (i.e. $D \ll MN \ll D$), $N \geqslant D^{1/k}$, $a_h := \frac{H'}{h} \Phi\big(\frac{h}{H+1}\big) \ll 1$ and

$$\mathfrak{S}_{\delta,\flat}^{\tau_k}(H', M, N) := \frac{1}{H'} \sum_{h \sim H'} a_h \sum_{\substack{m \sim M \ n \sim N \\ D < mn \leqslant 2D}} \tau_{k-1}(m) \mathrm{e}\Big(\frac{hx}{mn + \delta}\Big),$$

$$\mathfrak{S}_{\delta,\dagger}^{\tau_k}(M, N) := \sum_{\substack{m \sim M \ n \sim N \\ D < mn \leqslant 2D}} \tau_{k-1}(m) R_H\Big(\frac{x}{mn + \delta}\Big).$$

Firstly we bound $\mathfrak{S}_{\delta,\flat}^{\tau_k}(H', M, N)$. Applying the exponent pair $(\frac{1}{2}, \frac{1}{2})$ to the sum over n, it follows that

$$(3.19) \qquad \begin{aligned} \mathfrak{S}_{\delta,\flat}^{\tau_k}(H', M, N) &\ll \frac{x^\varepsilon}{H'} \sum_{h \sim H'} \sum_{m \sim M} \Big\{\Big(\frac{xh}{mN^2}\Big)^{1/2} N^{1/2} + \Big(\frac{xh}{mN^2}\Big)^{-1}\Big\} \\ &\ll \big((xH'MN^{-1})^{1/2} + x^{-1}D^2\big) x^\varepsilon. \end{aligned}$$

On the other hand, we remove the extra multiplicative condition $D < mn \leqslant 2D$ at the cost of a factor $\log x$, and then apply Lemma 2.1 with $\alpha = \beta = \gamma = 1$, $(X, H, M, N) = (xH'/MN, H', M, N)$ and $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$ to get

$$\mathfrak{S}_{\delta,\flat}^{\tau_k}(H', M, N) \ll \big((xM^4 N^3)^{1/6} + MN^{1/2} + H'^{-1/2} M^{1/2} N + (x^{-1} H'^{-1} M^3 N^3)^{1/2}\big) X^\varepsilon$$

provided $H' \leqslant H \leqslant D^{1/k}$ ($\leqslant N$). Using $MN \asymp D$ and $N \geqslant D^{1/k}$, we can derive that

$$
(3.20) \quad
\begin{aligned}
\mathfrak{S}_{\delta,\flat}^{\tau_k}(H', M, N) &\ll \big((x^k D^{4k-1})^{1/6k} + D^{1-1/(2k)} + H'^{-1/2} M^{1/2} N + (x^{-1} D^3)^{1/2}\big) x^\varepsilon \\
&\ll \big((x^k D^{4k-1})^{1/6k} + H'^{-1/2} M^{1/2} N\big) x^\varepsilon
\end{aligned}
$$

for $H' \leqslant H \leqslant D^{1/k}$. In the last inequality, we can remove $D^{1-1/(2k)}$ and $(x^{-1} D^3)^{1/2}$ because they can be absorbed by $(x^k D^{4k-1})^{1/6k}$ for $D \leqslant x^{\min\{4k/(5k+1), k/(2k-2)\}}$. From (3.19) and (3.20), we deduce that

$$
(3.21) \qquad\qquad \mathfrak{S}_{\delta,\flat}^{\tau_k}(H', M, N) \ll (x^k D^{4k-1})^{1/6k} x^\varepsilon
$$

for $H' \leqslant H \leqslant D^{1/k}$ and $MN = D \leqslant x^{\min\{4k/(5k+1), k/(2k-2)\}}$, where we have used the following estimates

$$
\begin{aligned}
\min\{(xH'MN^{-1})^{\frac{1}{2}}, H'^{-\frac{1}{2}} M^{\frac{1}{2}} N\} &\leqslant \big((xH'MN^{-1})^{\frac{1}{2}}\big)^{\frac{1}{3}} \big(H'^{-\frac{1}{2}} M^{\frac{1}{2}} N\big)^{\frac{2}{3}} \\
&= (xH'^{-1} M^3 N^3)^{\frac{1}{6}} \\
&\leqslant (xD^3)^{1/6}
\end{aligned}
$$

and $\max\{(xD^3)^{1/6}, x^{-1} D^2\} \leqslant (x^k D^{4k-1})^{1/6k}$ for $D \leqslant x^{7k/(8k+1)}$.

On the other hand, (2.3) of Lemma 2.3 allows us to derive that

$$
\begin{aligned}
\big|\mathfrak{S}_{\delta,\dagger}^{\tau_k}(M, N)\big| &\leqslant \sum_{m \sim M} \sum_{n \sim N} \Big| R_H\Big(\frac{x}{mn + \delta}\Big)\Big| \\
&\leqslant \frac{1}{2H + 2} \sum_{0 \leqslant |h| \leqslant H} \Big(1 - \frac{|h|}{H + 1}\Big) \sum_{m \sim M} \sum_{n \sim N} \mathrm{e}\Big(\frac{xh}{d^2 mn + \delta}\Big).
\end{aligned}
$$

When $h \neq 0$, we can bound the triple sums as before. Thus

$$
(3.22) \qquad\qquad \mathfrak{S}_{\delta,\dagger}^{\tau_k}(M, N) \ll \big(DH^{-1} + (x^k D^{4k-1})^{1/6k}\big) x^\varepsilon
$$

for $H \leqslant D^{1/k}$ and $MN \asymp D \leqslant x^{\min\{4k/(5k+1), k/(2k-2)\}}$.

Inserting (3.21) and (3.22) into (3.18) and taking $H = D^{1/k}$, we can obtain the required inequality. $\qquad\square$

The third proposition gives the required estimate for $\mathfrak{S}_\delta^{2^\omega}(x, D)$ defined as in (1.8).

Proposition 3.3. *For any $\varepsilon > 0$, we have*

$$
\mathfrak{S}_\delta^{2^\omega}(x, D) \ll_\varepsilon (x^2 D^7)^{1/12} x^\varepsilon
$$

uniformly for $x \geqslant 3$ and $1 \leqslant D \leqslant x^{8/11}$ and $0 \leqslant \delta \leqslant \varepsilon^{-1}$, where the implied constant depends on ε only.

Proof. Firstly we write

$$
\begin{aligned}
\mathfrak{S}_\delta^{2^\omega}(x, D) &= \sum_{d \leqslant \sqrt{2D}} \mu(d) \sum_{D/d^2 < \ell \leqslant 2D/d^2} \tau(\ell) \psi\Big(\frac{x/d^2}{\ell + \delta/d^2}\Big) \\
&= \sum_{d \leqslant \sqrt{2D}} \mu(d) \mathfrak{S}_{\delta/d^2}^\tau(x/d^2, D/d^2).
\end{aligned}
$$

Secondly it is easy to see that $D \leqslant x^{8/11}$ implies $D/d^2 \leqslant (x/d^2)^{8/11}$. Thus we can apply Proposition 3.2 with $k = 2$ to get the required result. $\qquad\square$

The last proposition gives the required estimate for $\mathfrak{S}_\delta^{\mu_2}(x, D)$ defined as in (1.9).

Proposition 3.4. *For any $\varepsilon > 0$, we have*

$$\mathfrak{S}_\delta^{\mu_2}(x, D) \ll_\varepsilon \left((xD^3)^{1/7} + (xD^2)^{1/6}\right)x^\varepsilon$$

uniformly for $x \geqslant 3$, $1 \leqslant D \leqslant x^{8/11}$ and $0 \leqslant \delta \leqslant \varepsilon^{-1}$, where the implied constant depends on ε only.

Proof. Using Lemma 2.3 and splitting the interval of summation $[1, 2D]$ into the dyadic intervals $(L, 2L]$, we can write

$$(3.23) \qquad \mathfrak{S}_\delta^{\mu_2} = -\frac{1}{2\pi i}\sum_L \left(\mathfrak{S}_{\delta,\flat}^{\mu_2}(L) + \overline{\mathfrak{S}_{\delta,\flat}^{\mu_2}(L)}\right) + \sum_L \mathfrak{S}_{\delta,\dagger}^{\mu_2}(L),$$

where $H \leqslant D$ and

$$\mathfrak{S}_{\delta,\flat}^{\mu_2}(L) := \sum_{h \leqslant H}\frac{1}{h}\Phi\left(\frac{h}{H+1}\right)\sum_{\ell \sim L}\sum_{(D/\ell)^{1/2} < d \leqslant (2D/\ell)^{1/2}}\mu(d)e\left(\frac{hx}{d^2\ell + \delta}\right),$$

$$\mathfrak{S}_{\delta,\dagger}^{\mu_2}(L) := \sum_{\ell \sim L}\sum_{(D/\ell)^{1/2} < d \leqslant (2D/\ell)^{1/2}}\mu(d)R_H\left(\frac{x}{d^2\ell + \delta}\right).$$

Inverting the order of summations and applying the exponent pair $(\frac{1}{6}, \frac{4}{6})$ to the sum over ℓ, it follows that

$$(3.24) \qquad \begin{aligned} \mathfrak{S}_{\delta,\flat}^{\mu_2}(L) &\ll \sum_{h \sim H}\frac{1}{h}\sum_{(D/2L)^{1/2} < d \leqslant (2D/L)^{1/2}}\left(\left(\frac{xh}{d^2L^2}\right)^{1/6}L^{4/6} + \left(\frac{xh}{d^2L^2}\right)^{-1}\right) \\ &\ll \left((xD^2H)^{1/6} + x^{-1}D^2\right)x^\varepsilon. \end{aligned}$$

On the other hand, (2.3) of Lemma 2.3 allows us to derive that

$$\begin{aligned} \left|\mathfrak{S}_{\delta,\dagger}^{\mu_2}(L)\right| &\leqslant \sum_{(D/2L)^{1/2} < d \leqslant (2D/L)^{1/2}}\sum_{\ell \sim L}\left|R_H\left(\frac{x}{d^2\ell + \delta}\right)\right| \\ &\leqslant \frac{1}{2H+2}\sum_{0 \leqslant |h| \leqslant H}\left(1 - \frac{|h|}{H+1}\right)\sum_{(D/2L)^{1/2} < d \leqslant (2D/L)^{1/2}}\sum_{\ell \sim L}e\left(\frac{xh}{d^2\ell + \delta}\right). \end{aligned}$$

When $h \neq 0$, as before we apply the exponent pair $(\frac{1}{6}, \frac{4}{6})$ to the sum over ℓ and obtain

$$(3.25) \qquad \mathfrak{S}_{\delta,\dagger}^{\mu_2}(L) \ll \left(DH^{-1} + (xD^2H)^{1/6} + x^{-1}D^2\right)x^\varepsilon.$$

Inserting (3.24) and (3.25) into (3.23), it follows that

$$\mathfrak{S}_\delta^{\mu_2} \ll \left(DH^{-1} + (xD^2H)^{1/6} + x^{-1}D^2\right)x^\varepsilon$$

for $H \leqslant D$. Optimising H on $[1, D]$, we find that

$$\mathfrak{S}_\delta^{\mu_2} \ll \left((xD^3)^{1/7} + (xD^2)^{1/6} + x^{-1}D^2\right)x^\varepsilon$$

This implies the required result, since $x^{-1}D^2 \leqslant (xD^3)^{1/7}$ for $D \leqslant x^{8/11}$. $\qquad\square$

4. Proof of Theorem 1

Let $f = \omega$ or τ_k or 2^ω or μ_2 and let $N_f \in [1, x^{1/2})$ be a parameter which can be chosen later. First we write

$$S_f(x) = \sum_{n \leqslant x} f\left(\left[\frac{x}{n}\right]\right) = S_f^\dagger(x) + S_f^\sharp(x) \tag{4.1}$$

with

$$S_f^\dagger(x) := \sum_{n \leqslant N_f} f\left(\left[\frac{x}{n}\right]\right), \qquad S_f^\sharp(x) := \sum_{N_f < n \leqslant x} f\left(\left[\frac{x}{n}\right]\right).$$

In our case, we have $f(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$ and all $n \geqslant 1$. Thus

$$S_f^\dagger(x) \ll_\varepsilon N_f x^\varepsilon. \tag{4.2}$$

In order to bound $S_f^\sharp(x)$, we put $d = [x/n]$. Noticing that

$$x/n - 1 < d \leqslant x/n \iff x/(d+1) < n \leqslant x/d,$$

we can derive that

$$\begin{aligned}
S_f^\sharp(x) &= \sum_{d \leqslant x/N_f} f(d) \sum_{x/(d+1) < n \leqslant x/d} 1 \\
&= \sum_{d \leqslant x/N_f} f(d)\left(\frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{x}{d+1} + \psi\left(\frac{x}{d+1}\right)\right) \\
&= x \sum_{d \geqslant 1} \frac{f(d)}{d(d+1)} + \mathcal{R}_1^f(x, N_f) - \mathcal{R}_0^f(x, N_f) + O(N_f),
\end{aligned} \tag{4.3}$$

where we have used the following bounds

$$x \sum_{d > x/N_f} \frac{f(d)}{d(d+1)} \ll_\varepsilon N_f x^\varepsilon, \qquad \sum_{d \leqslant N_f} f(d)\left(\psi\left(\frac{x}{d+1}\right) - \psi\left(\frac{x}{d}\right)\right) \ll_\varepsilon N_f x^\varepsilon$$

and

$$\mathcal{R}_\delta^f(x, N_f) = \sum_{N_f < d \leqslant x/N_f} f(d)\psi\left(\frac{x}{d+\delta}\right).$$

Combining (4.1), (4.2) and (4.3), it follows that

$$S_f(x) = x \sum_{d \geqslant 1} \frac{f(d)}{d(d+1)} + O_\varepsilon\left(|\mathcal{R}_1^f(x, N_f)| + |\mathcal{R}_0^f(x, N_f)| + N_f x^\varepsilon\right).$$

Thus in order to prove Theorem 1, it suffices to show that

$$\mathcal{R}_\delta^f(x, N_f) \ll N_f x^\varepsilon \qquad (x \geqslant 1) \tag{4.4}$$

for

f	ω	τ_k	2^ω	μ_2
N_f	$x^{53/110}$	$x^{(5k-1)/(10k-1)}$	$x^{9/19}$	$x^{2/5}$

4.1. **Proof of** (4.4) **with** $f = \omega$.

We have $N_\omega = x^{53/110}$. In order to apply (3.2) of Proposition 3.1, we must switch from

$$\mathcal{R}_\delta^\omega(x, N_\omega) = \sum_{N_\omega < p\ell \leqslant x/N_\omega} \psi\Big(\frac{x}{p\ell + \delta}\Big) = \sum_{N_\omega < p\ell \leqslant x/N_\omega} \frac{\Lambda(p)}{\log p}\psi\Big(\frac{x}{p\ell + \delta}\Big)$$

to

$$\widetilde{\mathcal{R}}_\delta^\omega(x, N_\omega, t) := \sum_{d \leqslant t} \Lambda(d) \sum_{N_\omega/d < \ell \leqslant (x/N_\omega)/d} \psi\Big(\frac{x}{d\ell + \delta}\Big).$$

For this, we need to estimate the contribution of prime powers:

$$\mathscr{R}_1 := \sum_{\substack{N_\omega < p^\nu \ell \leqslant x/N_\omega \\ \ell \leqslant (x/N_\omega)^{53/114}, \nu \geqslant 2}} \frac{\Lambda(p^\nu)}{\log p^\nu}\psi\Big(\frac{x}{p^\nu\ell + \delta}\Big) = \sum_{\substack{N_\omega < p^\nu \ell \leqslant x/N_\omega \\ \ell \leqslant (x/N_\omega)^{53/114}, \nu \geqslant 2}} \frac{1}{\nu}\psi\Big(\frac{x}{p^\nu\ell + \delta}\Big),$$

$$\mathscr{R}_2 := \sum_{\substack{N_\omega < p^\nu \ell \leqslant x/N_\omega \\ \ell > (x/N_\omega)^{53/114}, \nu \geqslant 2}} \frac{\Lambda(p^\nu)}{\log p^\nu}\psi\Big(\frac{x}{p^\nu\ell + \delta}\Big) = \sum_{\substack{N_\omega < p^\nu \ell \leqslant x/N_\omega \\ \ell > (x/N_\omega)^{53/114}, \nu \geqslant 2}} \frac{1}{\nu}\psi\Big(\frac{x}{p^\nu\ell + \delta}\Big).$$

Firstly we have trivially

$$\mathscr{R}_1 \ll \sum_{\ell \leqslant (x/N_\omega)^{53/114}} (x/N_\omega\ell)^{1/2} \ll (x/N_\omega)^{167/228} \ll x^{167/440}.$$

On the other hand, applying [10, Lemma 2.4] with $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$, we can derive that

$$\mathscr{R}_2 = \sum_{\substack{p^\nu \leqslant (x/N_\omega)^{61/114} \\ \nu \geqslant 2}} \frac{1}{\nu} \sum_{(x/N_\omega)^{53/114} < \ell \leqslant x/N_\omega p^\nu} \psi\Big(\frac{x}{p^\nu\ell + \delta}\Big)$$

$$\ll \sum_{\substack{p^\nu \leqslant (x/N_\omega)^{61/114} \\ \nu \geqslant 2}} \max_{\substack{(x/N_\omega)^{53/114} < N_1 \leqslant x/N_\omega p^\nu \\ N_1 < N_2 \leqslant 2N_1}} N_1 \Big|\sum_{N_1 < \ell \leqslant N_2} \frac{1}{\ell}\psi\Big(\frac{x}{p^\nu\ell + \delta}\Big)\Big|$$

$$\ll \sum_{\substack{p^\nu \leqslant (x/N_\omega)^{61/114} \\ \nu \geqslant 2}} \max_{\substack{(x/N_\omega)^{53/114} < N_1 \leqslant x/N_\omega p^\nu \\ N_1 < N_2 \leqslant 2N_1}} \Big(\frac{N_1^2}{(x/p^\nu)} + (x/p^\nu)^{1/3}\Big)$$

$$\ll \sum_{\substack{p^\nu \leqslant (x/N_\omega)^{61/114} \\ \nu \geqslant 2}} \Big(\frac{x}{N_\omega^2 p^\nu} + (x/p^\nu)^{1/3}\Big)$$

$$\ll x/N_\omega^2 + x^{1/3}(x/N_\omega)^{(61/114)/6}$$

$$\ll x^{167/440}$$

Thus we can write

$$\mathcal{R}_\delta^\omega(x, N_\omega) = \sum_{N_\omega < d\ell \leqslant x/N_\omega} \frac{\Lambda(d)}{\log d}\psi\Big(\frac{x}{d\ell + \delta}\Big) + O\big(x^{167/440}\big)$$

(4.5)
$$= \int_{2-}^{x/N_\omega} \frac{1}{\log t} \, d\widetilde{\mathcal{R}}_\delta^\omega(x, N_\omega, t) + O\big(x^{167/440}\big)$$

$$\ll \max_{2 \leqslant t \leqslant x/N_\omega} \big|\widetilde{\mathcal{R}}_\delta^\omega(x, N_\omega, t)\big| + x^{167/440}.$$

Writing $D_j := x/(2^j N_\omega)$, we have $N_\omega \leqslant D_j \leqslant x/N_\omega \leqslant x^{57/110}$ for $0 \leqslant j \leqslant \frac{\log(x/N_\omega^2)}{\log 2}$. Thus we can apply (3.2) of Proposition 3.1 to get

$$
\begin{aligned}
|\widetilde{\mathcal{R}}_\delta^\omega(x, N_\omega, t)| &\leqslant \sum_{0 \leqslant j \leqslant \log(x/N_\omega^2)/\log 2} |\mathfrak{S}_\delta^{\Lambda * \mathbb{1}}(x, D_j)| \\
&\ll \sum_{0 \leqslant j \leqslant \log(x/N_\omega^2)/\log 2} \big((x^{19} D_j^{87})^{1/133} + (x^{171} D_j^{624})^{1/1026}\big) x^\varepsilon \\
&\ll \big((x^{106} N_\omega^{-87})^{1/133} + (x^{795} N_\omega^{-624})^{1/1026}\big) x^\varepsilon
\end{aligned}
$$

for all $2 \leqslant t \leqslant x/N_\omega$. Inserting this into (4.5), we get the required result.

4.2. **Proof of** (4.4) **with** $f = \tau_k$.

We have $N_{\tau_k} = x^{(5k-1)/(10k-1)}$. Let $D_j := x/(2^j N_{\tau_k})$, then we easily see that $N_{\tau_k} \leqslant D_j \leqslant x/N_{\tau_k} \leqslant x^{\min\{4k/(5k+1), k/(2k-2)\}}$ for $0 \leqslant j \leqslant \frac{\log(x/N_{\tau_k}^2)}{\log 2}$. Thus Proposition 3.2 gives us

$$
\begin{aligned}
|\mathcal{R}_\delta^{\tau_k}(x, N_{\tau_k})| &\leqslant \sum_{0 \leqslant j \leqslant \log(x/N_{\tau_k}^2)/\log 2} |\mathfrak{S}_\delta^{\tau_k}(x, D_j)| \\
&\ll \sum_{0 \leqslant j \leqslant \log(x/N_{\tau_k}^2)/\log 2} (x^k D_j^{4k-1})^{1/6k} x^\varepsilon \\
&\ll (x^{5k-1} N_{\tau_k}^{-(4k-1)})^{1/6k} x^\varepsilon,
\end{aligned}
$$

which implies the required result.

4.3. **Proof of** (4.4) **with** $f = 2^\omega$.

We have $N_{2^\omega} = x^{9/19}$. Let $D_j := x/(2^j N_{2^\omega})$, then $N_{2^\omega} \leqslant D_j \leqslant x/N_{2^\omega} \leqslant x^{10/19}$ for $0 \leqslant j \leqslant \frac{\log(x/N_{2^\omega}^2)}{\log 2}$. Noticing that $2^\omega = \tilde{\mu} * \tau$, we can apply Proposition 3.3 to get

$$
\begin{aligned}
|\mathcal{R}_\delta^{2^\omega}(x, N_{2^\omega})| &\leqslant \sum_{0 \leqslant j \leqslant \log(x/N_{2^\omega}^2)/\log 2} |\mathfrak{S}_\delta^{2^\omega}(x, D_j)| \\
&\ll \sum_{0 \leqslant j \leqslant \log(x/N_{2^\omega}^2)/\log 2} (x^2 D_j^7)^{1/12} x^\varepsilon \\
&\ll (x^9 N_{2^\omega}^{-7})^{1/12} x^\varepsilon,
\end{aligned}
$$

which implies the required result.

4.4. **Proof of** (4.4) **with** $f = \mu_2$.

We have $N_{\mu_2} = x^{2/5}$. Let $D_j := x/(2^j N_{\mu_2})$, then $N_{\mu_2} \leqslant D_j \leqslant x/N_{\mu_2} \leqslant x^{3/5}$ for $0 \leqslant j \leqslant \frac{\log(x/N_{\mu_2}^2)}{\log 2}$. Noticing that $\mu_2 = \tilde{\mu} * \mathbb{1}$, we can apply Proposition 3.4 to get

$$
\begin{aligned}
|\mathcal{R}_\delta^{\mu_2}(x, N_{\mu_2})| &\leqslant \sum_{0 \leqslant j \leqslant \log(x/N_{\mu_2}^2)/\log 2} |\mathfrak{S}_\delta^{\mu_2}(x, D_j)| \\
&\ll \sum_{0 \leqslant j \leqslant \log(x/N_{\mu_2}^2)/\log 2} \big((x D_j^3)^{1/7} + (x D_j^2)^{1/6}\big) x^\varepsilon \\
&\ll \big((x^4 N_{\mu_2}^{-3})^{1/7} + (x^3 N_{\mu_2}^{-2})^{1/6}\big) x^\varepsilon,
\end{aligned}
$$

which implies the required result.

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Kui Liu, School of Mathematics and Statistics, Qingdao University, 308 Ningxia Road, Qingdao, Shandong 266071, China
  E-mail address: `liukui@qdu.edu.cn`

Jie Wu, CNRS UMR 8050, Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est Créteil, 94010 Créteil cedex, France
  E-mail address: `jie.wu@u-pec.fr`

Zhishan Yang, School of Mathematics and Statistics, Qingdao University, 308 Ningxia Road, Qingdao, Shandong 266071, China
  E-mail address: `zsyang@qdu.edu.cn`