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Transmission eigenvalues for multipoint scatterers*

P.G. Grinevich [†] R.G. Novikov[‡]

Abstract

We study the transmission eigenvalues for the multipoint scatterers of the Bethe-Peierls-Fermi-Zeldovich-Beresin-Faddeev type in dimensions $d = 2$ and $d = 3$. We show that for these scatterers: 1) each positive energy E is a transmission eigenvalue (in the strong sense) of infinite multiplicity; 2) each complex E is an interior transmission eigenvalue of infinite multiplicity. The case of dimension $d = 1$ is also discussed.

Mathematics Subject Classification (2020): 35J10, 47A40, 47A75, 81U40.

Keywords: Schrödinger equation, transparency, transmission eigenvalues, multipoint scatterers.

1 Introduction

Studies of transmission eigenvalues for Helmholtz-type equations have a long history, and many publications are devoted to this problem. In fact, the property that a spectral parameter E is a transmission eigenvalue, is a weakened

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[†]Steklov Mathematical Institute of RAS, 8 Gubkina St. Moscow, 119991, Russia; Landau Institute of Theoretical Physics, pr. Akademika Semanova 1a, Chernogolovka, Moscow region, 142432, Russia; Moscow State University, Moscow, Russia. e-mail: pgg@landau.ac.ru

[‡]CMAP, CNRS, École Polytechnique, Institut Polytechnique de Paris, Palaiseau, France; IEPT RAS, Moscow, Russia. e-mail: novikov@cmap.polytechnique.fr

version of invisibility (transparency) for this E . In the case of the stationary Schrödinger equation the spectral parameter E is interpreted as the energy.

In connection with transparency for the Schrödinger equation, see [11, 14, 13] and references therein. Historically, these studies go back to [23, 20]. In connection with transmission eigenvalues for Helmholtz-type equations, see [7, 21, 8] and references therein. Historically, these studies go back to [18, 10].

In particular, it is known that for sufficiently regular compactly supported scatterers transmission eigenvalues are discrete and have finite multiplicity (see [24, 21, 8] for precise results). On the other hand, in [14] we constructed two-dimensional real-valued potentials from the Schwartz class, which are transparent at one positive energy. As a corollary, this energy is a transmission eigenvalue of infinite multiplicity in the strong sense, i.e., the number 1 is an eigenvalue of infinite multiplicity for the scattering operator at this fixed energy! However, the potentials of [14] are not compactly supported.

In the present work we consider the stationary Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d = 1, 2, 3, \quad (1.1)$$

with multipoint potential (scatterer)

$$v(x) = \sum_{j=1}^n \varepsilon_j \delta(x - y_j). \quad (1.2)$$

It is well-known that point scatterers $\varepsilon\delta(x)$ are well-defined only in dimensions $d = 1, 2, 3$. If $d = 1$, then $\delta(x)$ denotes the standard Dirac δ -function. If $d = 2$ or $d = 3$, then $\varepsilon\delta(x)$ denotes a “renormalization” of the δ -function (depending on a parameter). These “renormalized” δ -functions for $d = 2, 3$ are known as Bethe-Peierls-Fermi-Zeldovich-Beresin-Faddeev-type point scatterers. Exact definitions for the multipoint scatterers in dimensions $d = 2, 3$ can be found in [2, 15, 16]. Historically, mathematical definitions of point scatterers in dimension $d = 3$ go back to [5, 4].

We also consider the equation

$$-\Delta\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d = 1, 2, 3. \quad (1.3)$$

Let \mathcal{U}_j be an open non-empty neighbourhood of y_j such that $y_{j'} \notin \mathcal{U}_j$ if $j' \neq j$. Developing the approach, used in [5] for $d = 3$, a local solution ψ of (1.1), (1.2) in \mathcal{U}_j can be defined as a function such that:

- (i) $\psi(x)$ satisfies (1.3) in $\mathcal{U}_j \setminus y_j$;

- (ii) If $d = 1$, then $\psi(x)$ is continuous at $x = y_j$, and its first derivative has a jump

$$-\alpha_j [\psi'(y_j + 0) - \psi'(y_j - 0)] = \psi(y_j); \quad (1.4)$$

If $d = 2$, then

$$\psi(x) = \psi_{j,-1} \ln |x - y_j| + \psi_{j,0} + O(|x - y_j|) \quad \text{as } x \rightarrow y_j, \quad (1.5)$$

and

$$[-2\pi\alpha_j - \ln 2 + \gamma] \psi_{j,-1} = \psi_{j,0}, \quad (1.6)$$

where $\gamma = 0.577 \dots$ is the Euler's constant.

If $d = 3$, then

$$\psi(x) = \frac{\psi_{j,-1}}{|x - y_j|} + \psi_{j,0} + O(|x - y_j|) \quad \text{as } x \rightarrow y_j, \quad (1.7)$$

and

$$4\pi\alpha_j\psi_{j,-1} = \psi_{j,0}. \quad (1.8)$$

Here, we assume that $\alpha_j \in \mathbb{R} \cup \infty$. In addition, for each $j = 1, \dots, n$, the strength ε_j of the point scatterer $\varepsilon_j \delta(x - y_j)$ in (1.2) is encoded by α_j , see, for example, [2, 16]. If $\alpha_j = \infty$, then $\varepsilon_j = 0$ (and the corresponding single point scatterer vanishes). If $d = 1$, then the renormalized δ -function coincides with the standard one, and we have that $\varepsilon_j = -1/\alpha_j$.

In the present work, we use, in particular, that the scattering functions for our multipoint scatterers are given by formulas (2.6)–(2.10) recalled in Section 2.

In the present work we show that for the Schrödinger equation (1.1) with multipoint potentials (1.2), for $d = 2, 3$, each positive energy E is a transmission eigenvalue of infinite multiplicity in the strong sense, and each complex E is an interior transmission eigenvalue of infinite multiplicity. The related definitions of transmission eigenvalues are recalled in Section 2. One can see that these multipoint potentials (1.2) are compactly supported, but they are singular. We also give proper analogs of these results for $d = 1$ for single-point potentials.

In Section 2 we also recall some basic facts from the scattering theory. In connection with inverse scattering for the multipoint scatterers of Bethe-Peierls-Fermi-Zeldovich-Beresin-Faddeev type see, for example, [1, 3, 12, 17, 22] and references therein.

The main results of the present work are presented in details in Section 3. The proofs are given in Section 4.

2 Preliminaries

For equation (1.1) with multipoint potentials as in (1.2) we consider the scattering eigenfunctions ψ^+ specified by the following asymptotics as $|x| \rightarrow \infty$:

$$\psi^+ = e^{ikx} + f^+ \left(k, |k| \frac{x}{|x|} \right) \frac{e^{i|k||x|}}{|x|^{(d-1)/2}} + O \left(\frac{1}{|x|^{(d+1)/2}} \right), \quad (2.1)$$

$k \in \mathbb{R}^d$, $k^2 = E > 0$, where $f^+ = f^+(k, l)$, $k, l \in \mathbb{R}^d$, $k^2 = l^2 = E$, is a priori unknown. The function f^+ arising in (2.1) is the scattering amplitude, or the far-field pattern. It is also convenient to present f^+ as follows:

$$f^+(k, l) = c(d, |k|)f(k, l), \quad (2.2)$$

$$c(d, |k|) = -\pi i (-2\pi i)^{(d-1)/2} |k|^{(d-3)/2}, \quad \text{for } \sqrt{-2\pi i} = \sqrt{2\pi} e^{-i\pi/4}. \quad (2.3)$$

It is important that for the multipoint scatterers the scattering eigenfunctions and scattering amplitudes are calculated explicitly (see, for example, [2, 15, 16]).

Let

$$G^+(x, E) := -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 - E - i \cdot 0} \quad (2.4)$$

where $x \in \mathbb{R}^d$, $E \in \mathbb{R}$, $E > 0$. We recall that $G^+(x, E)$ is the Green function with the Sommerfeld radiation condition for the operator $\Delta + E$. Note also that

$$G^+(x, E) = \frac{e^{i|k||x|}}{2i|k|}, \quad d = 1,$$

$$G^+(x, E) = -\frac{i}{4} H_0^1(|x||k|),$$

where H_0^1 is the Hankel function of the first type, $d = 2$, and

$$G^+(x, E) = \frac{1}{2\pi} \left[\ln |x| + \ln |k| - \ln 2 + \gamma - \frac{\pi i}{2} \right] + O(|x|^2 \ln |x|), \quad \text{as } |x| \rightarrow 0,$$

where $\gamma = 0.577 \dots$ is the Euler's constant, $d = 2$,

$$G^+(x, k) = -\frac{e^{i|k||x|}}{4\pi|x|}, \quad d = 3, \quad (2.5)$$

where $|k| = \sqrt{E} > 0$.

Then the following formulas hold (see [2, 15, 16]):

$$\psi^+(x, k) = e^{ikx} + \sum_{j=1}^n q_j(k) G^+(x - y_j, |k|^2), \quad (2.6)$$

$$f(k, l) = \frac{1}{(2\pi)^d} \sum_{j=1}^n q_j(k) e^{-ily_j}. \quad (2.7)$$

Here $q(k) = (q_1(k), \dots, q_n(k))^t$ satisfies the system of linear equations

$$A(|k|)q(k) = b(k), \quad (2.8)$$

where A is the $n \times n$ matrix with the elements

$$\begin{aligned} A_{j,j}(|k|) &= \alpha_j - i(4\pi)^{-1}|k|, & d = 3, \\ A_{j,j}(|k|) &= \alpha_j - (4\pi)^{-1}(\pi i - 2 \ln(|k|)), & d = 2, \\ A_{j,j}(|k|) &= \alpha_j + (2i|k|)^{-1}, & d = 1, \\ A_{j,j'}(|k|) &= G^+(y_j - y_{j'}, |k|^2), & m \neq j, \end{aligned} \quad (2.9)$$

and $b(k) = (b_1(k), \dots, b_n(k))^t$ is defined by

$$b_j(k) = -e^{iky_j}. \quad (2.10)$$

In connection with (1.4)–(1.8) and (2.6) note also that the following formulas hold:

$$\begin{aligned} -\frac{1}{4\pi}q_j &= \psi_{j,-1}, & d = 3, \\ \frac{1}{2\pi}q_j &= \psi_{j,-1}, & d = 2, \\ q_j &= \psi'(y_j + 0) - \psi'(y_j - 0), & d = 1. \end{aligned} \quad (2.11)$$

As we mentioned in Introduction, for each $j = 1, \dots, n$, the strength ε_j of the point scatterer $\varepsilon_j \delta(x - y_j)$ in (1.2) is encoded by a real parameter α_j ; see, also, for example, [2, 16].

The scattering operator $\hat{S} = \hat{S}_E$ at fixed energy $E = |k|^2$ can be defined as follows:

$$(\hat{S}_E u)(\theta) = u(\theta) - i\pi |k|^{d-2} \int_{\mathbb{S}^{d-1}} f(|k|\theta', |k|\theta) u(\theta') d\theta', \quad (2.12)$$

where \mathbb{S}^{d-1} is the unit sphere in \mathbb{R}^d , $\theta, \theta' \in \mathbb{S}^{d-1}$, $d\theta'$ denotes the standard volume element at \mathbb{S}^{d-1} , see, for example, [19], Chapter XVII, § 125, see also [6].

Consider the equation

$$(\hat{S}_E u)(\theta) = u(\theta), \quad \theta \in \mathbb{S}^{d-1}, \quad (2.13)$$

where $u \in L^2(\mathbb{S}^{d-1})$. We say that energy E is a **transmission eigenvalue in the strong sense** if equation (2.13) has a non-trivial solution. Dimension of space of these solutions is the **multiplicity** of this transmission eigenvalue.

Assume that

$$\text{supp } v \subset \mathcal{D}. \quad (2.14)$$

where \mathcal{D} is a connected bounded domain in \mathbb{R}^d with C^2 boundary, where $\mathbb{R}^d \setminus \overline{\mathcal{D}}$ is also connected. Let

$$\psi(x) = \int_{\mathbb{S}^{d-1}} \psi^+(x, |k|\theta') u(\theta') d\theta', \quad (2.15)$$

$$\phi(x) = \int_{\mathbb{S}^{d-1}} e^{i|k|\theta'x} u(\theta') d\theta', \quad (2.16)$$

where u satisfies (2.13), $|k|^2 = E$. Then

$$-\Delta\psi(x) + v(x)\psi(x) = E\psi(x), \quad x \in \mathcal{D}, \quad (2.17)$$

$$-\Delta\phi(x) = E\phi(x), \quad x \in \mathcal{D}, \quad (2.18)$$

and

$$\psi(x) \equiv \phi(x), \quad \frac{\partial}{\partial\nu}\psi(x) \equiv \frac{\partial}{\partial\nu}\phi(x) \quad \text{for all } x \in \partial\mathcal{D}, \quad (2.19)$$

where $\frac{\partial}{\partial\nu}$ denotes the normal derivative, see, for example, [9].

In addition, $\phi \not\equiv 0$ on \mathbb{R}^d if $u \neq 0$ in $L^2(\mathbb{S}^{d-1})$, and $|k| \neq 0$.

The energy E such that (2.17), (2.18), (2.19) are fulfilled with non-trivial ϕ, ψ is called an **interior transmission eigenvalue** for equation (2.17) in the domain \mathcal{D} ; see [6, 7, 9, 10, 18, 21, 24].

Remark 2.1. *If $E \in \mathbb{R}$, $E > 0$, is a transmission eigenvalue of multiplicity N in the strong sense for equation (1.1), then E is an interior transmission eigenvalue of multiplicity $\geq N$ for equation (1.1) in any domain \mathcal{D} as in (2.14).*

3 Main results

Our results on transmission eigenvalues in the strong sense (in the sense of equation (2.13)) for multipoint scatterers (1.2) are formulated in Theorem 3.1 and in Proposition 3.1.

Theorem 3.1. *Let v be a multipoint scatterer (1.2) of the Bethe-Peierls-Fermi-Zeldovich-Beresin-Faddeev type in dimension $d = 2$ or $d = 3$.*

Then each energy $E > 0$ is a transmission eigenvalue of infinite multiplicity for equation (1.1) in the strong sense.

Proposition 3.1. *Let v be a single point scatterer in dimension $d = 1$ as in (1.2) with $n = 1$.*

Then each energy $E > 0$ is a transmission eigenvalue for equation (1.1) in the strong sense.

To prove our results on interior transmission eigenvalues for multipoint scatterers (1.2) we use the following lemma, which is also of independent interest.

Lemma 3.1. *Let v be a multipoint scatterer as in (1.2), (2.14), ϕ satisfy (2.18), and $\phi(y_j) = 0$ for all $j = 1, \dots, n$. Then ϕ also satisfies (2.17).*

In Lemma 3.1 we assume that in equations (2.17), (2.18) the energy $E \in \mathbb{C}$.

Our results on interior transmission eigenvalues (in the sense of (2.17)–(2.19)) for multipoint scatterers (1.2) are formulated in Theorem 3.2 and in Proposition 3.2.

Theorem 3.2. *Let v be a multipoint scatterer as in (1.2) of the Bethe-Peierls-Fermi-Zeldovich-Beresin-Faddeev type in dimension $d = 2$ or $d = 3$, satisfying condition (2.14).*

Then each energy $E \in \mathbb{C}$ is an interior transmission eigenvalue of infinite multiplicity for equation (1.1) in the domain \mathcal{D} .

Proposition 3.2. *Let v be a single point scatterer in dimension $d = 1$ as in (1.2) with $n = 1$, satisfying condition (2.14).*

Then each energy $E \in \mathbb{C}$ is an interior transmission eigenvalue for equation (1.1) in the domain \mathcal{D} .

In Theorem 3.2 and Proposition 3.2, \mathcal{D} is the domain of formulas (2.14), (2.17)–(2.19).

4 Proofs of the main results

Proof of Theorem 3.1. It is convenient to use that

$$f(k, l) = f(-l, -k), \quad \text{where } k, l \in \mathbb{R}^d, \quad k^2 = l^2 = E. \quad (4.1)$$

The reciprocity property (4.1) follows from formula (2.7), equation (2.8) and the property that $A = A^t$.

Due to (2.7), (4.1), we have that

$$f(k, l) = \frac{1}{(2\pi)^d} \sum_{j=1}^n q_j(-l) e^{iky_j}. \quad (4.2)$$

Due to (2.12), (4.2), we have that equation (2.13) is fulfilled for each $u \in L^2(\mathbb{S}^{d-1})$ such that

$$\int_{\mathbb{S}^{d-1}} e^{i|k|\theta y_j} u(\theta) d\theta = 0, \quad \text{for all } j = 1, \dots, n. \quad (4.3)$$

One can see that a finite system of homogeneous linear equations (4.3) in $L^2(\mathbb{S}^{d-1})$ has infinite-dimensional space of solutions for $d = 2, 3$.

Therefore E is a transmission eigenvalue of infinite multiplicity in the strong sense for equation (1.1). \square

Proof of Proposition 3.1. It is convenient to use that, for $d = 1$, $n = 1$:

$$f(k, l) = \frac{1}{2\pi} \frac{e^{iky_1} e^{-ily_1}}{\alpha_1 + (2i|k|)^{-1}}, \quad \text{where } k, l \in \mathbb{R}, \quad k^2 = l^2 = E. \quad (4.4)$$

Due to (4.4), we have that equation (2.13) is fulfilled for each $u = (u^-, u^+) \in L^2(\mathbb{S}^0) = \mathbb{C}^2$ such that

$$e^{-i|k|y_1} u^- + e^{i|k|y_1} u^+ = 0. \quad (4.5)$$

One can see that equation (4.5) has one-dimensional space of solutions.

Therefore E is a transmission eigenvalue in the strong sense for equation (1.1). \square

Proof of Lemma 3.1. This result follows from the definition of solutions of equation (1.1); see Introduction. Indeed, equation (1.3) is fulfilled for $\psi = \phi$ in \mathcal{D} containing all y_j , and we have that:

1. If $d = 1$, then

$$\phi(y_j) = 0, \quad \phi'(y_j + 0) = \phi'(y_j - 0),$$

and condition (1.4) is fulfilled;

2. If $d = 2, 3$, then

$$\phi_{j,-1} = \phi_{j,0} = 0,$$

and condition (1.6), for $d = 2$, or condition (1.8), for $d = 3$, is fulfilled. \square

Proof of Theorem 3.2. For $E \in \mathbb{R}$, $E > 0$, the result follows from Remark 2.1 and Theorem 3.1.

For $E \in \mathbb{C}$, the proof is as follows. Let ϕ_l , $l = 1, \dots, \infty$, denote an infinite set of linearly independent smooth solutions of (2.18) in $\mathcal{D} \cup \partial\mathcal{D}$. For any $N \in \mathbb{Z}$, $N > n$, consider the linear system on $z = (z_1, \dots, z_N) \in \mathbb{C}^N$:

$$\sum_{l=1}^N z_l \phi_l(y_j) = 0. \quad (4.6)$$

This system has at least $N - n$ dimensional space of solutions. For each $z = (z_1, \dots, z_N)$ satisfying (4.6) the function

$$\Phi(x) = \sum_{l=1}^N z_l \phi_l(x)$$

satisfies (2.18), and $\Phi(y_j) = 0$ for $j = 1, \dots, n$. Therefore, by Lemma 3.1, $\Phi(x)$ satisfies (2.17) and (2.18) simultaneously! In addition, $\Phi(x) \not\equiv 0$ in \mathcal{D} , due to linear independence of ϕ_l . Thus, E is an interior transmission eigenvalue for equation (2.17) in the sense of (2.17)–(2.19), where $\psi \equiv \phi \equiv \Phi$ in $\mathcal{D} \cup \partial\mathcal{D}$. In addition, the space of such Φ has dimension $\geq N - n$. Therefore, E is an interior transmission eigenvalue of multiplicity, at least, $N - n$ for arbitrary large N , i.e. the multiplicity of E is infinite. \square

Proof of Proposition 3.2. For $E \in \mathbb{R}$, $E > 0$, the result follows from Remark 2.1 and Proposition 3.1.

For $E \in \mathbb{C}$, the proof is as follows. Denote by Φ a non-zero solution of (2.18) such that $\Phi(y_1) = 0$. By Lemma 3.1, $\Phi(x)$ satisfies (2.17) and (2.18) simultaneously. Therefore, E is an interior transmission eigenvalue for equation (2.17) in the sense of (2.17)–(2.19), where $\psi \equiv \phi \equiv \Phi$ in $\mathcal{D} \cup \partial\mathcal{D}$. \square

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