Refutation of the Bayer-Diaconis-McGrath conjecture for the riffle shuffle card guessing game with feedback

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Abstract

We consider the following card guessing game with feedback, introduced in [BD92]. An initially ordered deck of cards is shuffled via one or several riffle shuffles (or more generally: one a-shuffle). The player guesses the card on top of the deck, then looks at that card. The player then guesses the next card, looks at that card etc. until there is no card left, and his goal is to get as many correct guesses as possible. The authors detail a simple guessing strategy conjectured to be optimal. We show that this strategy is optimal in the case of a single riffle shuffle but not in general. The present note was sent to Professor Persi Diaconis in June 2018 and is extracted from the Master's thesis [Gal18].

Contents

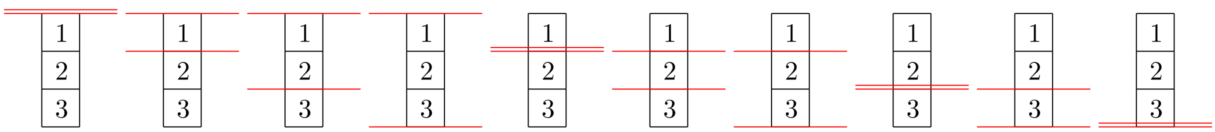
1 Notations used

1.1 General notations for a-shuffles

Notation. We denote by $\mathscr{P}_a(n)$ the set of all possible values $p = (p_1, \ldots, p_a)$ of the multinomial distribution with parameters $(n, \frac{1}{a}, \ldots, \frac{1}{a})$. We see $\mathscr{P}_a(n)$ as the set of all ways to cut a deck of n cards into a packets.

Definition. Let $p \in \mathscr{P}_a(n)$. We call a separation between two packets in p a *cut*. There are $a-1$ cuts in total. The *location* of a cut is the number between 0 and n of the card that is just above the cut (a cut at the very top of the deck is at location 0).

Example. The set $\mathscr{P}_3(3)$ contains 10 elements pictured as follows, with all cuts in red :



Definition. Let $p \in \mathscr{P}_a(n)$. A permutation $\sigma \in \mathfrak{S}_n$ is said to be *p-compatible* if the cards of each packet are in order.

Notation. We denote by $\mathscr{C}_a(n)$ the set of all couples (p, σ) where $p \in \mathscr{P}_a(n)$ and σ is p-compatible.

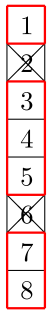
We will exclusively use the maximum entropy description of a-shuffles : let (P, S) be uniform on $\mathscr{C}_a(n)$, so that S is an a-shuffle.

Our probability space is denoted by $(\Omega, \mathscr{F}, \mathbb{P})$, and expectation relative to $\mathbb{P}$ will be denoted by $\mathbb{E}$.

1.2 Specific notations for the card guessing problem with feedback

A deck of n cards is in arrangement S, where S is an a-shuffle. Suppose that m cards have already been revealed, so that we're trying to make the best possible guess for $S(m+1)$. The "Bayer-Diaconis-McGrath strategy", or "BDM strategy" in short, refers to the strategy described in [BD92] where the card guessed is the topmost card of a longest sequence of remaining consecutive cards.

Example. Suppose $n = 8$, with card 6 revealed first and card 2 revealed second ($m = 2$). The BDM strategy chooses card 3 next. This situation is pictured as follows, with revealed cards appearing checked :



Notation. Let $\mathbf{i} = (i_1, \ldots, i_m)$ be the vector of all cards that have already been revealed successively.

Notation. We denote by $\mathscr{C}_a^{\mathbf{i}}(n)$ the set of all couples $(p, \sigma) \in \mathscr{C}_a(n)$ such that $\sigma(1) = i_1, \ldots, \sigma(m) = i_m$, and by $\mathscr{P}_a^{\mathbf{i}}(n)$ the set of all $p \in \mathscr{P}_a(n)$ such that there exists a p-compatible permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma(1) = i_1, \ldots, \sigma(m) = i_m$.

Notation. Let $\mathbb{P}_{\mathbf{i}} := \mathbb{P}(\,\cdot\mid S(1) = i_1,\, \ldots\, , S(m) = i_m)$. We denote by $\mathbb{E}_{\mathbf{i}}$ the corresponding expectation.

Remark. Our goal is therefore to identify a card i_{m+1} that maximizes $\mathbb{P}_{\mathbf{i}}(S(m+1) = i_{m+1})$, where (P, S) is uniform on $\mathscr{C}_a^{\mathbf{i}}(n)$ under $\mathbb{P}_{\mathbf{i}}$.

Notation. We denote by $\pi = (\pi_1, \ldots, \pi_q\}$ the ordered partition of $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$ consisting of all sequences of remaining consecutive cards from top to bottom. Let $r_k := |\pi_k|$.

Example. In the previous example, we have $\mathbf{i} = (6, 2)$, $\pi_1 = \{1\}$, $\pi_2 = \{3, 4, 5\}$, $\pi_3 = \{7, 8\}$.

Definition. A cut is said to be *mandatory* if P contains a cut at this location $\mathbb{P}_{\mathbf{i}}-$a.s. If P contains several cuts at a mandatory location, **only one of these cuts is considered as mandatory**.

Remark. Note that $\mathscr{P}_a^{\mathbf{i}}(n)$ is the set of all $p \in \mathscr{P}_a(n)$ that contain all mandatory cuts.

Example. Suppose $\mathbf{i} = (6, 1, 5)$, then the mandatory cuts are at locations 5 and 4.

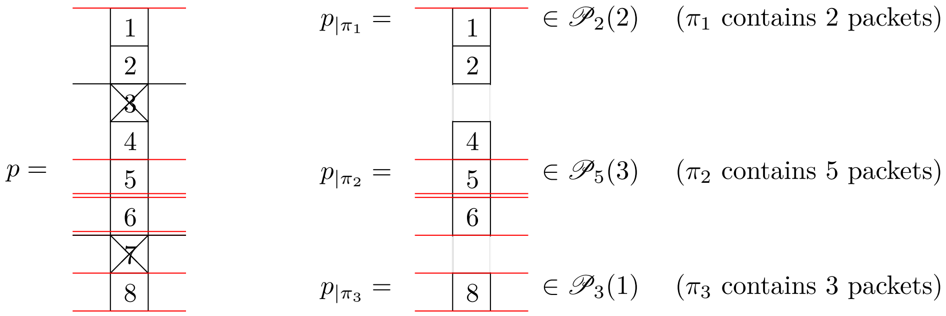
Definition. Let $p \in \mathscr{P}_a^{\mathbf{i}}(n)$. A cut is said to be *in* π_k if it is adjacent to π_k and **non-mandatory**. This means that if $\pi_k = \{j, j+1, \ldots, j+r_k-1\}$, then the cuts in π_k are all the cuts at locations $j-1, j, \ldots, j+r_k-1$ except for the one mandatory cut at location $j+r_k-1$ if there is one. Denoting by $b-1$ the number of cuts in π_k, we say that π_k *contains* b packets.

Notation. Let $p \in \mathscr{P}_a^{\mathbf{i}}(n)$. We denote by $p_{|\pi_k}$ the restriction of p to π_k, i.e. the element of $\mathscr{P}_b(r_k)$ obtained from p by keeping only π_k and the $b-1$ cuts in π_k. Cuts that are not in π_k (k fixed) will be denoted by $p \setminus \pi_k$, and cuts that are in none of the π_k will be denoted by $p \setminus \pi$.

Remark. We will occasionally identify p with an $(a-1)$-tuple of numbers between 0 and n representing the locations of all cuts in p. The same goes for $p \setminus \pi_k$ or $p \setminus \pi$.

Notation. Let $(p, \sigma) \in \mathscr{C}_a^{\mathbf{i}}(n)$. The relative order in σ of the elements of π_k, renumbered from 1 to r_k in increasing order, will be denoted by $\sigma_{\pi_k} \in \mathfrak{S}_{r_k}$.

Example. Suppose $n = 8$, $a = 9$, $\mathbf{i} = (3, 7)$, hence $\pi_1 = \{1, 2\}$, $\pi_2 = \{4, 5, 6\}$, $\pi_3 = \{8\}$. Let p be as follows, where both mandatory cuts appear in black :



The lone p-compatible permutation for π_1 is 1 2. Any relative order of the cards 4,5,6 is possible, therefore the p-compatible permutations for π_2 are 1 2 3, 1 3 2, 2 1 3, 2 3 1, 3 1 2, 3 2 1.
For example, let $\sigma = 3\ 7\ 6\ 1\ 4\ 5\ 2\ 8$, note that $(p, \sigma) \in \mathscr{C}_9^{\mathbf{i}}(8)$. The cards of π_1 are in order "1 2", so that $\sigma_{\pi_1} =$1 2. The cards of π_2 are in order "6 4 5", so that $\sigma_{\pi_2} =$3 1 2.

Notation. We denote by j_k the topmost card of π_k. Let $p \in \mathscr{P}_a(n)$: we denote by $h_p(j_k)$ the size of the packet containing j_k once the first m cards are removed.

Example. In the previous example, we have $h_p(j_2) = 1$. Indeed, the initial packet containing card 4 was $\{3, 4\}$, but card 3 has been revealed already.

Remark. We have $\mathbb{P}_{\mathbf{i}}(S(m+1) = j_k) = \dfrac{\mathbb{E}_{\mathbf{i}}(h_P(j_k))}{n - m}$ for all k.

2 Refutation of the BDM conjecture

It is possible to prove the following :

(i) The optimal card is always the topmost card of one of the π_k. In particular, the BDM strategy is optimal for $a = 2$.

(ii) Choosing the topmost card of a **largest** π_k is not always optimal, as shown by a counterexample with exact computation for $a \in \{3, 4\}$.

We only detail the proof of (ii) here. More precisely, we prove the following result :

Theorem. *For $a = 3$ (resp. $a = 4$), and for all $n \geq 12$ (resp. $n \geq 13$), the BDM strategy is not optimal.*

2.1 Explicit formulas

The following theorem — apart from the independence part which we don't use — is key to both (i) and (ii) : fixing the number a_k of packets in each π_k as well as all cuts outside π leads to each π_k being arranged following an a_k-shuffle.

Theorem 2.1. *Let p_0 be a tuple of numbers representing the locations of all cuts that are in none of the π_k, and $\mathbf{a} = (a_1, \dots, a_q)$ be a vector of integers. Let $A_{p_0,\mathbf{a}} := \{P \setminus \pi = p_0\} \cap \bigcap_{k=1}^{q} \{\pi_k \text{ contains } a_k \text{ packets}\}$. Then, under $\mathbb{P}_\mathbf{i}$ and given $A_{p_0,\mathbf{a}}$: $(P_{|\pi_1}, S_{\pi_1}), \dots, (P_{|\pi_q}, S_{\pi_q})$ are uniform on $\mathscr{C}_{a_1}(r_1), \dots, \mathscr{C}_{a_q}(r_q)$ respectively, and independent. In particular, under $\mathbb{P}_\mathbf{i}$ and given $A_{p_0,\mathbf{a}}$, the S_{π_k} are independent a_k-shuffles.*

Proof.
Let $E_{p_0,\mathbf{a}} := \{(p, \sigma) \in \mathscr{C}_a^\mathbf{i}(n) \mid p \setminus \pi = p_0 \text{ and } \pi_k \text{ contains } a_k \text{ packets } \forall\, k\}$ and $F_\mathbf{a} := \prod_{k=1}^{q} \mathscr{C}_{a_k}(r_k)$.
Let $f : \left| \begin{array}{lcl} E_{p_0,\mathbf{a}} & \longrightarrow & F_\mathbf{a} \\ (p,\sigma) & \longmapsto & ((p_{|\pi_1}, \sigma_{\pi_1}), \dots, (p_{|\pi_q}, \sigma_{\pi_q})) \end{array} \right.$: under $\mathbb{P}_\mathbf{i}$ and given $A_{p_0,\mathbf{a}}$, (P, S) is uniform on $E_{p_0,\mathbf{a}}$, therefore it suffices to show that all elements of $F_\mathbf{a}$ have the same number of inverse images by f. Let $y = ((p_1, \sigma_1), \dots, (p_q, \sigma_q))$ be in $F_\mathbf{a}$ and $x = (p, \sigma) \in E_{p_0,\mathbf{a}}$ be an inverse image of y. The choice of p is unique because it is forced by $p_0, p_1, \dots, p_q$. Moreover, $\sigma(1) = i_1, \dots, \sigma(m) = i_m$, and the relative order of the cards in π_k is forced by σ_k for all k : this means that choosing σ comes down to choosing the r_k positions of the cards in π_k for all k. In conclusion, the number of choices for x is $\binom{n-m}{r_1, \dots, r_q}$, which does not depend on y. $\square$

The idea is to compute $\mathbb{P}_\mathbf{i}(S(m+1) = j_k)$ for all k, using the complete system of events $(A_{p_0,\mathbf{a}})_{p_0,\mathbf{a}}$:

$$\mathbb{P}_\mathbf{i}(S(m+1) = j_k) = \frac{\mathbb{E}_\mathbf{i}(h_P(j_k))}{n-m} = \frac{1}{n-m} \sum_{p_0,\mathbf{a}} \mathbb{E}_\mathbf{i}(h_P(j_k) \mid A_{p_0,\mathbf{a}}) \mathbb{P}_\mathbf{i}(A_{p_0,\mathbf{a}}) \tag{1}$$

Proposition 2.3 gives a formula for $\mathbb{E}_\mathbf{i}(h_P(j_k) \mid A_{p_0,\mathbf{a}})$, whereas Proposition 2.4 deals with $\mathbb{P}_\mathbf{i}(A_{p_0,\mathbf{a}})$.

Lemma 2.2. *Let $(X_1, \ldots, X_a)$ follow the multinomial distribution with parameters $(n, \frac{1}{a}, \ldots, \frac{1}{a})$. Let $T := \min\{s \in \{1, \ldots, a\} \mid X_s > 0\}$. Then $\mathbb{E}(X_T) = \dfrac{n}{a^n} \displaystyle\sum_{s=1}^{a} s^{n-1}$.*

Proof.
$\mathbb{E}(X_T) = \displaystyle\sum_{s=1}^{a} \mathbb{E}(X_s \mathbf{1}_{\{X_1 = \ldots = X_{s-1} = 0\}}) = \sum_{s=1}^{a} \mathbb{E}(X_s \mid X_1 = \ldots = X_{s-1} = 0) \mathbb{P}(X_1 = \ldots = X_{s-1} = 0)$, where $\mathbb{E}(X_s \mid X_1 = \ldots = X_{s-1} = 0) = \frac{n}{a-s+1}$ and $\mathbb{P}(X_1 = \ldots = X_{s-1} = 0) = \left(\frac{a-s+1}{a}\right)^n$. □

Proposition 2.3. *With notations as in Theorem 2.1, for all k : $\mathbb{E}_{\mathbf{i}}(h_P(j_k) \mid A_{p_0, \mathbf{a}}) = \dfrac{r_k}{a_k^{r_k}} \displaystyle\sum_{s=1}^{a_k} s^{r_k - 1}$.*

Proof.
This follows immediately from Theorem 2.1 and Lemma 2.2, since the packet containing j_k in P after the first m cards are revealed is precisely the first non-empty packet in $P_{|\pi_k}$. □

Proposition 2.4. *With notations as in the proof of Theorem 2.1 : $|E_{p_0, \mathbf{a}}| = \dbinom{n-m}{r_1\,,\;\ldots\;,\,r_q} a_1^{r_1} \cdots a_q^{r_q}$.*

Proof.
Since $p \setminus \pi$ is forced (equal to p_0), choosing p comes down to choosing how each π_k is cut into a_k packets. Once p is fixed, choosing σ with $\sigma(1) = i_1, \ldots, \sigma(m) = i_m$ comes down to choosing positions for the cards of each of the $a_1 + \ldots + a_q$ packets. This yields :

$$\begin{aligned}
|E_{p_0, \mathbf{a}}| &= \sum_{x_{1,1} + \ldots + x_{1,a_1} = r_1} \cdots \sum_{x_{q,1} + \ldots + x_{q,a_q} = r_q} \binom{n-m}{x_{1,1}, \ldots, x_{1,a_1}\,,\;\ldots\;,\,x_{q,1}, \ldots, x_{q,a_q}} \\
&= \sum_{x_{1,1} + \ldots + x_{1,a_1} = r_1} \cdots \sum_{x_{q,1} + \ldots + x_{q,a_q} = r_q} \binom{r_1}{x_{1,1}, \ldots, x_{1,a_1}} \cdots \binom{r_q}{x_{q,1}, \ldots, x_{q,a_q}} \binom{n-m}{r_1, \ldots, r_q} \\
&= \binom{n-m}{r_1, \ldots, r_q} \left(\sum_{x_{1,1} + \ldots + x_{1,a_1} = r_1} \binom{r_1}{x_{1,1}, \ldots, x_{1,a_1}} \right) \cdots \left(\sum_{x_{q,1} + \ldots + x_{q,a_q} = r_q} \binom{r_q}{x_{q,1}, \ldots, x_{q,a_q}} \right) \\
&= \binom{n-m}{r_1\,,\;\ldots\;,\,r_q} a_1^{r_1} \cdots a_q^{r_q}\,.
\end{aligned}$$
□

2.2 Our counterexample for $a \in \{3, 4\}$

Theorem 2.5. *For $a = 3$, and for all $n \geq 12$, the BDM strategy is not optimal.*

Proof.
The first counterexample appears at $m = 1$. Let $i := i_1$ to simplify notations. If $i \in \{1, n\}$ then (i) shows that the strategy chooses optimally, so suppose $2 \leq i \leq n - 1$. We get $q = 2$, $\pi_1 = \{1, \ldots, i-1\}$, $r_1 = i - 1$, $\pi_2 = \{i+1, \ldots, n\}$, $r_2 = n - i$. Necessarily $P \setminus \pi = (i-1)$: this is the mandatory cut. Therefore, there are only two terms in the sum (1), corresponding to whether the second and final cut is in π_1 or in π_2.

- First case : the second cut is in π_1, i.e. $p_0 = (i-1)$ and $\mathbf{a} = (2, 1)$.
  Proposition 2.4 ensures that there are $2^{i-1}\binom{n-1}{i-1}$ such combinations. Over all these, the packet

containing card $i+1$ is always of size $n-i$, and the average size of the packet containing card 1 is $\frac{i-1}{2^{i-1}}(1+2^{i-2})$ according to 2.3.

  - Second case : the second cut is in π_2, i.e. $p_0=(i-1)$ and $\mathbf{a}=(1,2)$.
    Proposition 2.4 ensures that there are $2^{n-i}\binom{n-1}{i-1}$ such combinations. Over all these, the packet containing card 1 is always of size $i-1$, and the average size of the packet containing card $i+1$ is $\frac{n-i}{2^{n-i}}(1+2^{n-i-1})$ according to 2.3.

Thus, we obtain :

$$\mathbb{E}_{\mathbf{i}}(h_P(1)) = \frac{2^{i-1}\binom{n-1}{i-1}\frac{i-1}{2^{i-1}}(1+2^{i-2})+2^{n-i}\binom{n-1}{i-1}(i-1)}{2^{i-1}\binom{n-1}{i-1}+2^{n-i}\binom{n-1}{i-1}} = \frac{(i-1)(2^{i-2}+2^{n-i}+1)}{2^{i-1}+2^{n-i}} \ ,$$
$$\mathbb{E}_{\mathbf{i}}(h_P(i+1)) = \frac{2^{i-1}\binom{n-1}{i-1}(n-i)+2^{n-i}\binom{n-1}{i-1}\frac{n-i}{2^{n-i}}(1+2^{n-i-1})}{2^{i-1}\binom{n-1}{i-1}+2^{n-i}\binom{n-1}{i-1}} = \frac{(n-i)(2^{i-1}+2^{n-i-1}+1)}{2^{i-1}+2^{n-i}} \ .$$

- For n even, take $i=\frac{n}{2}+1$. We get $r:=r_1=\frac{n}{2}$, $r_2=r-1<r_1$, and the previous formulas yield :

$$\mathbb{P}_{\mathbf{i}}(S(2)=1) = \frac{r(2^r+1)}{(n-1)3.2^{r-1}} \ ,$$
$$\mathbb{P}_{\mathbf{i}}(S(2)=\tfrac{n}{2}+2) = \frac{(r-1)(2^r+2^{r-2}+1)}{(n-1)3.2^{r-1}} \ .$$

  It is easily verified that $\mathbb{P}_{\mathbf{i}}(S(2)=1)<\mathbb{P}_{\mathbf{i}}(S(2)=\frac{n}{2}+2)$ if and only if $r\geq 6$ i.e. $n\geq 12$. Since $r_1>r_2$, the BDM strategy chooses card 1 for all n, therefore it is suboptimal if $n\geq 12$.

- For n odd, take $i=\frac{n+3}{2}$. We get $r:=r_1=\frac{n+1}{2}$, $r_2=r-2<r_1$, and the previous formulas yield :

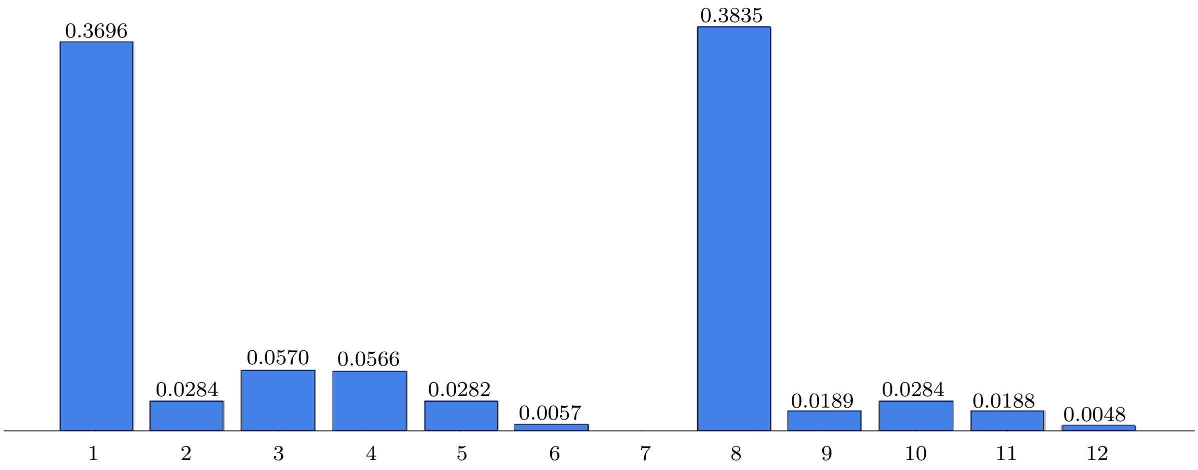
$$\mathbb{P}_{\mathbf{i}}(S(2)=1) = \frac{r(2^{r-1}+2^{r-2}+1)}{(n-1)5.2^{r-2}} \ ,$$
$$\mathbb{P}_{\mathbf{i}}(S(2)=\tfrac{n+5}{2}) = \frac{(r-2)(2^r+2^{r-3}+1)}{(n-1)5.2^{r-2}} \ .$$

  It is easily verified that $\mathbb{P}_{\mathbf{i}}(S(2)=1)<\mathbb{P}_{\mathbf{i}}(S(2)=\frac{n+5}{2})$ if and only if $r\geq 7$ i.e. $n\geq 13$. $\square$

Example. For $n=12$, $a=3$, $\mathbf{i}=(7)$, the BDM strategy opts for card 1 :

1
2
3
4
5
6
$\cancel{7}$
8
9
10
11
12

Nevertheless, we actually have $\mathbb{P}_{\mathbf{i}}(S(2)=1)=\frac{65}{176}\approx 0.3693$ and $\mathbb{P}_{\mathbf{i}}(S(2)=8)=\frac{135}{352}\approx 0.3835$: given that the top card of the deck is card 7, the next one has a 38.35% chance of being card 8 but only a 36.93% chance of being card 1, despite the sequence of cards starting with 1 being longer. These results are comforted by the following histogram, which shows frequencies for the value of the second card based on 1,000,000 trials of a 3-shuffle on 12 cards conditioned on having card 7 as the top card :



Remark. A consequence of this counterexample is that an optimal strategy must depend on a in general, contrary to the BDM strategy. Indeed, if $n = 12$ and $\mathbf{i} = (7)$, then the optimal guess is different if $a = 2$ (card 1) or if $a = 3$ (card 8).

Theorem 2.6. *For $a = 4$, and for all $n \geq 13$, the BDM strategy is not optimal.*

Proof.
The easiest counterexample appears at $m = 2$, but there also exists some for $m = 1$ as will be shown in our last section.

- For n even, take $\mathbf{i} = (\frac{n}{2}+1, n)$. We get $q = 2$, $\pi_1 = \{1, \dots, \frac{n}{2}\}$, $r := r_1 = \frac{n}{2}$, $\pi_2 = \{\frac{n}{2}+2, \dots, n-1\}$, $r_2 = r - 2 < r_1$. There are two mandatory cuts here, at locations $\frac{n}{2}$ and $n-1$. This is almost identical to the case n odd for $a = 3$, except that the sum contains a third term corresponding to the possibility that the non-mandatory cut is at location n. We obtain after computation :

$$\mathbb{E}_{\mathbf{i}}(h_P(1)) = \frac{2^r\binom{n-2}{r}\frac{r}{2^r}(1+2^{r-1}) + 2^{r-2}\binom{n-2}{r}r + \binom{n-2}{r}r}{2^r\binom{n-2}{r} + 2^{r-2}\binom{n-2}{r} + \binom{n-2}{r}} \quad ,$$
$$\mathbb{E}_{\mathbf{i}}(h_P(\tfrac{n}{2}+2)) = \frac{2^r\binom{n-2}{r}(r-2) + 2^{r-2}\binom{n-2}{r}\frac{r-2}{2^{r-2}}(1+2^{r-3}) + \binom{n-2}{r}(r-2)}{2^r\binom{n-2}{r} + 2^{r-2}\binom{n-2}{r} + \binom{n-2}{r}} \quad ,$$

which yields :

$$\mathbb{P}_{\mathbf{i}}(S(2) = 1) = \frac{r(2^{r-1} + 2^{r-2} + 2)}{(n-2)(5.2^{r-2}+1)} \quad ,$$
$$\mathbb{P}_{\mathbf{i}}(S(2) = \tfrac{n}{2}+2) = \frac{(r-2)(2^r + 2^{r-3} + 2)}{(n-2)(5.2^{r-2}+1)} \quad .$$

It is easily verified that $\mathbb{P}_{\mathbf{i}}(S(2) = 1) < \mathbb{P}_{\mathbf{i}}(S(2) = \frac{n}{2}+2)$ if and only if $r \geq 7$ i.e. $n \geq 14$.

- For n odd, take $\mathbf{i} = (\frac{n+1}{2}, n)$. We get $q = 2$, $\pi_1 = \{1, \dots, \frac{n-1}{2}\}$, $r := r_1 = \frac{n-1}{2}$, $\pi_2 = \{\frac{n+3}{2}, \dots, n-1\}$,

$r_2 = r - 1 < r_1$. A similar reasoning yields :

$$\begin{aligned}
\mathbb{P}_{\mathbf{i}}(S(2) = 1) &= \frac{r(2^r + 2)}{(n-2)(3.2^{r-1} + 1)} \ , \\
\mathbb{P}_{\mathbf{i}}(S(2) = \tfrac{n+3}{2}) &= \frac{(r-1)(2^r + 2^{r-2} + 2)}{(n-2)(3.2^{r-1} + 1)} \ ,
\end{aligned}$$

hence $\mathbb{P}_{\mathbf{i}}(S(2) = 1) < \mathbb{P}_{\mathbf{i}}(S(2) = \frac{n+3}{2})$ if and only if $r \geq 6$ i.e. $n \geq 13$. $\square$

Example. For $n = 13$, $a = 4$, $\mathbf{i} = (7, 13)$, the BDM strategy opts for card 1 :

1
2
3
4
5
6
~~7~~
8
9
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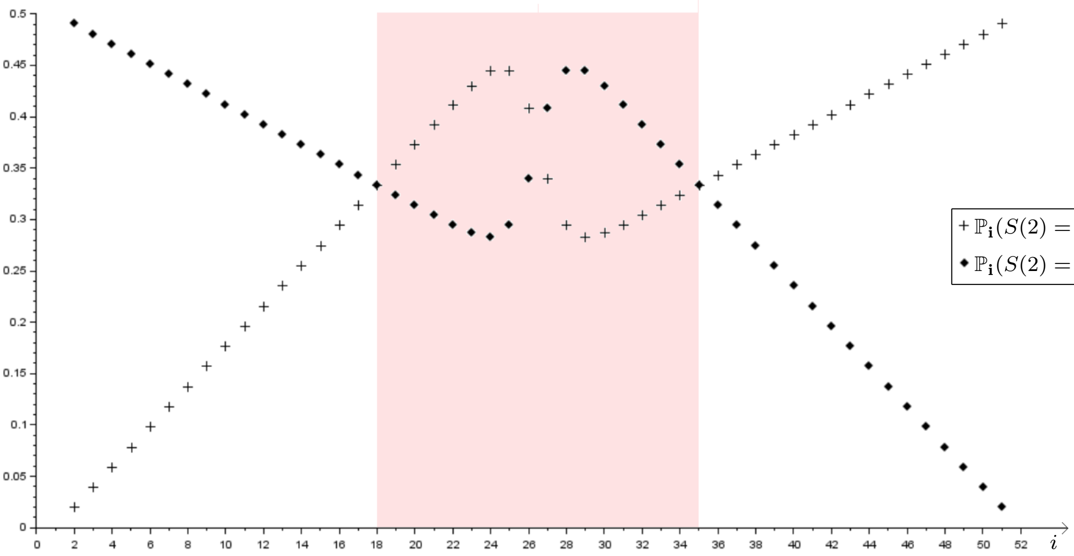
12
~~13~~

Nevertheless, we actually have $\mathbb{P}_{\mathbf{i}}(S(2) = 1) = \frac{36}{97} \approx 0.3711$ and $\mathbb{P}_{\mathbf{i}}(S(2) = 8) = \frac{410}{1067} \approx 0.3843$.

2.3 Some error regions of the BDM strategy

In this section, we take a look at the case $m = 1$ and $a \in \{3, 4\}$ to see which values of the first card lead the BDM strategy to be suboptimal for the second card.

- The case $m = 1$, $a = 3$.
  We use our formulas from the proof of Theorem 2.5. Denoting $\mathbf{i} = (i)$ where i varies from 2 to $n - 1$, the probabilities $\mathbb{P}_{\mathbf{i}}(S(2) = 1)$ and $\mathbb{P}_{\mathbf{i}}(S(2) = i + 1)$ are represented below for $n = 52$:

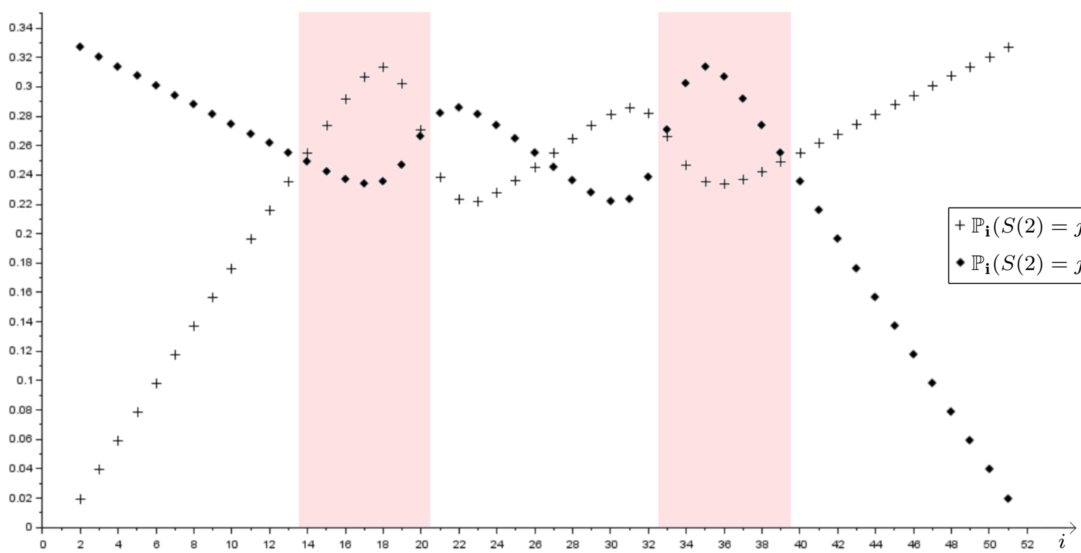


The coloured region corresponds to values of i for which the BDM strategy chooses the second card suboptimally. We see that this region goes from $\frac{n}{3}$ to $\frac{2n}{3}$ ($\lceil \frac{n}{2}+1 \rceil$ is the counterexample we used to prove Theorem 2.5). This seems to hold for any n large enough. For $n=52$, the biggest error happens at $i=24$ or $i=29$: for $i=24$, we have $\mathbb{P}_{\mathbf{i}}(S(2)=1) \approx 0.444$ and $\mathbb{P}_{\mathbf{i}}(S(2)=25) \approx 0.283$ despite the BDM strategy recommending card 25.

- The case $m=1$, $a=4$.
This is not the case we used for our counterexample in Theorem 2.6, but formulas can be computed just as easily from Proposition 2.3 and Proposition 2.4. Denoting $\mathbf{i}=(i)$ again, we obtain :

$$\mathbb{P}_{\mathbf{i}}(S(2)=1) = \frac{3^{i-1}\frac{i-1}{3^{i-1}}(1+2^{i-2}+3^{i-2}) + 2^{i-1}2^{n-i}\frac{i-1}{2^{i-1}}(1+2^{i-2}) + 3^{n-i}(i-1)}{(n-1)(3^{i-1}+2^{i-1}2^{n-i}+3^{n-i})} ,$$
$$\mathbb{P}_{\mathbf{i}}(S(2)=i+1) = \frac{3^{i-1}(n-i) + 2^{i-1}2^{n-i}\frac{n-i}{2^{n-i}}(1+2^{n-i-1}) + 3^{n-i}\frac{n-i}{3^{n-i}}(1+2^{n-i-1}+3^{n-i-1})}{(n-1)(3^{i-1}+2^{i-1}2^{n-i}+3^{n-i})} .$$

For $n=52$, we observe two disjoint error regions this time :



Errors are made between $\frac{n}{4}$ and $\frac{3n}{8}$, as well as between $\frac{5n}{8}$ and $\frac{3n}{4}$ symmetrically. This also seems to hold for any n large enough. For $n = 52$, the biggest error happens at $i = 18$ or $i = 35$: for $i = 18$, we have $\mathbb{P}_{\mathbf{i}}(S(2) = 1) \approx 0.314$ and $\mathbb{P}_{\mathbf{i}}(S(2) = 19) \approx 0.235$ despite the BDM strategy recommending card 19.

References

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