Stokes equations under Tresca friction boundary condition: a truncated approach

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Abstract

A priori error analysis of the finite element approximation of Stokes equations under slip boundary condition of friction type has been centered on the interpolation error on the slip zone. In this work, we propose a novel approach based on the approximation of the tangential component of traction force by a truncated (cut off) function. More precisely, we carry out (i) a complete analysis of the truncated formulation from the continuous to discrete level in two and three dimensions. In particular, we show linear convergence rate of the finite element solution by assuming standard regularity of the weak solution. This improves all previous results. (ii) the description of our solution strategy, (iii) a verification of the convergence properties with analytic solution and benchmark tests.

Keywords: Stokes equations; Tresca friction law; cut off function; error estimates; finite element approximation; linear convergence; numerical simulations.

Mathematics subject classification: 65N30, 76M10

Contents

1 Introduction

After a brief introduction in the modelling of a Stokes with Tresca friction boundary condition, Sect 1.2 presents the main findings of this work which is optimal convergence for the finite element solution without imposing extra regularity of the solution on the slip zone.

1.1 Stokes Equations with Tresca friction boundary condition

Let $\Omega \subset \mathbb{R}^d$ (d=2,3) be an open bounded set with boundary $\partial\Omega$ assume to be polygonal or polyhedral. We consider the steady incompressible Stokes equations modeled by the equations

$$-2\mu \operatorname{div} D\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$
$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega\ , \tag{1.2}$$

where $\boldsymbol{u}(\boldsymbol{x})$ is the velocity, the pressure is $p(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x})$ is the external body force applied to the fluid, while μ is the kinematic viscosity and $2D\boldsymbol{u} = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\boldsymbol{T}}$ is the rate of deformation. These equations are complemented by boundary conditions. For that purpose, we assume that $\partial\Omega$ is made of two components S and Γ, such that $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, with $S \cap \Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma\ . \tag{1.3}$$

On the other part of the boundary S, the velocity is decomposed following its normal and tangential part; that is

$$\boldsymbol{u} = u_{\boldsymbol{n}} + u_{\boldsymbol{\tau}} = (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n} + (\boldsymbol{u}\cdot\boldsymbol{\tau})\boldsymbol{\tau},$$

where $\boldsymbol{n}$ is the normal outward unit vector to S and $\boldsymbol{\tau}$ is the tangent vector orthogonal to $\boldsymbol{n}$. We assume the impermeability condition

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \ \text{ on } \ S\,. \tag{1.4}$$

The force within the fluid is the Cauchy stress tensor $\boldsymbol{T}$ given by the relation

$$\boldsymbol{T} = 2\mu D\boldsymbol{u} - p\boldsymbol{I} \text{ on } \Omega\,,$$

$\boldsymbol{I}$ being the identity tensor. Just like the velocity, the traction $\boldsymbol{Tn}$ on S is decomposed following its normal and tangential part; that is

$$\begin{aligned}\boldsymbol{Tn} &= (\boldsymbol{Tn}\cdot\boldsymbol{n})\boldsymbol{n} + (\boldsymbol{Tn}\cdot\boldsymbol{\tau})\boldsymbol{\tau} \\ &= (-p + 2\mu\boldsymbol{n}\cdot D\boldsymbol{u}\boldsymbol{n})\boldsymbol{n} + 2\mu(\boldsymbol{\tau}\cdot D\boldsymbol{u}\boldsymbol{n})\boldsymbol{\tau} \\ &= (\boldsymbol{Tn})_{\boldsymbol{n}} + (\boldsymbol{Tn})_{\boldsymbol{\tau}}\,.\end{aligned}$$

Let $g : S \longrightarrow (0,\infty)$ be the threshold slip function. One considers the following constitutive relation on S (see [1, 2]);

$$-(\boldsymbol{Tn})_{\boldsymbol{\tau}} \in g\partial|\boldsymbol{u}_{\boldsymbol{\tau}}|\,, \tag{1.5}$$

with the pointwise Euclidean norm $|\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v}$. Using the definition of the sub-differential, (1.5) is equivalent to;

$$\text{for all vector } \boldsymbol{v} \quad g|\boldsymbol{v}_{\boldsymbol{\tau}}| - g|\boldsymbol{u}_{\boldsymbol{\tau}}| \geq -(\boldsymbol{Tn})_{\boldsymbol{\tau}}\cdot(\boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}) \quad \text{on} \quad S\,. \tag{1.6}$$

When $\boldsymbol{u}_{\boldsymbol{\tau}}$ is not equal zero, one has

$$-(\boldsymbol{Tn})_{\boldsymbol{\tau}} = g\frac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|}\,.$$

Thus, (1.6) is equivalent to

$$\left.\begin{array}{llll}\text{if} & |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g & \text{then} & \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ \text{if} & |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = g & \text{then} & \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0}\ ,\ \text{and } -(\boldsymbol{Tn})_{\boldsymbol{\tau}} = g\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|}\end{array}\right\} \text{ on } \quad S\,. \tag{1.7}$$

So, when the tangential part of the traction is below the threshold, the velocity of the fluid is zero. No motion is observed on S. But when the tangential part of the traction is the same as the threshold, then the flow takes place in the opposite direction as $(\boldsymbol{Tn})_{\boldsymbol{\tau}}$. From the mathematical point of view the problems reads: Find $(\boldsymbol{u}, p) \in \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\ :\ \boldsymbol{v}|_\Gamma = 0\ ,\ \boldsymbol{v}\cdot\boldsymbol{n}|_S = 0\} \times \{q \in L^2(\Omega)\ ,\ (q,1) = 0\}$ such that

$$\begin{aligned}&J(\boldsymbol{u}, q) \leq J(\boldsymbol{u}, p) \leq J(\boldsymbol{v}, p) \quad \text{for all } (\boldsymbol{v}, q) \text{ admissible functions} \\ &J(\boldsymbol{v}, q) = \mu\int_\Omega |D\boldsymbol{v}|^2 dx - \int_\Omega q \operatorname{div}\boldsymbol{v} dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} dx + \int_S g|\boldsymbol{v}_{\boldsymbol{\tau}}| d\sigma\,,\end{aligned} \tag{1.8}$$

where $d\sigma$ is the surface measure associated to S. This problem has been studied mathematically by many authors [1–6], see also [7–16] for results pertaining to finite element approximations. It is worth mentioning at this juncture that another nonlinear slip boundary condition is formulated and analysed in [17], and numerical investigation of that model is carried out in [18–20].

1.2 Main results

If the velocity and the pressure are approximated by $\mathbb{P}_2^2 \times \mathbb{P}_1$, then the finite element solution associated to the equations (1.1)–(1.5) is convergent with a convergence rate of

$3/4$ provided that the weak solution $(\boldsymbol{u}, p)$ has the regularity $H^2(\Omega)^2 \times H^1(\Omega)$. This reduced rate is mainly due to the presence of the non differentiable function (see [8–10, 16, 20])

$$j(\boldsymbol{v}) = \int_S g|\boldsymbol{v}_{\boldsymbol{\tau}}|d\sigma\,.$$

It is worth noting that optimal convergence rate is obtained if the tangential velocity is required to be H^2 on the slip zone (see [21–23]), or if special procedures are considered on the slip boundary [10, 21–23]. We should also mentioned that if DG-approximations are considered, then the same remedies are required (see [24,25]) if optimal convergence is what one wants to obtain. This work analyses the convergence of the finite element solution associated to the equations (1.1)–(1.5) by introducing a truncated problem whereby $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ is approximated by $-\dfrac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})$ with ε a small positive parameter and $\nu_k : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ the cut off given as follows

$$\nu_k(\boldsymbol{x}) = \begin{cases} \boldsymbol{x} \ \text{ for } \ |\boldsymbol{x}| < k, \\ k\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|} \ \text{ for } \ |\boldsymbol{x}| \geq k\,. \end{cases}$$

With the intermediate (truncated) problem, it is manifest that the error is treated via the triangle's inequality, and beside the velocity and pressure, we also need to estimate the quantity $\dfrac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) - \dfrac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon\boldsymbol{\tau}})$. The treatment of the later term is basically the novelty of this work and has already been analysed by F. Chouly and P. Hild in [26,27] or I. Dione in [28, 29] for different problems. This work does not only differ from the ones presented in [26–29] from the modeling aspect. In the previous analyses, the focus is on the quantity $\boldsymbol{u} - \boldsymbol{u}^h_\varepsilon$, whereas in our work, we want to estimate the quantity $\boldsymbol{u} - \boldsymbol{u}^h$, and $p - p^h$. Finally, the fact that we are analysing a system of equations add more difficulties in our work. Assuming that the velocity and pressure given by the equations (1.1)–(1.5) are such that $\boldsymbol{u} \in \boldsymbol{H}^{3/2+r}(\Omega)$ with $r \in (0, 1/2]$, and taking $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ the solution of the truncated problem (see below), then there exists c independent of ε such that

$$\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_1 + \|p - p_\varepsilon\| + \varepsilon^{1/2}\left\|(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right\|_S \leq c\varepsilon^{1/2+r}\|\boldsymbol{u}\|_{3/2+r}\,. \tag{1.9}$$

(1.9) shows that the truncated problem is a "good" approximation of the continuous problem, and serves as a first step towards the convergence result. Here and throughout this work, h_K stands for the diameter of each element $K \in \mathcal{T}$ and the $h = \max\limits_{K\in\mathcal{T}} h_K$ in the underlying regular triangulation $\mathcal{T}$ into triangles or tetrahedra in the FEM. Just like (1.9), a direct analysis of the finite element problems reveal that for $-(\boldsymbol{T}^h\boldsymbol{n})_{\boldsymbol{\tau}} \in g\partial|\boldsymbol{u}^h_\varepsilon|$,

$$\|\boldsymbol{u}^h - \boldsymbol{u}^h_\varepsilon\|_1 + \|p^h - p^h_\varepsilon\| + \varepsilon^{1/2}\left\|(\boldsymbol{T}^h\boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon\boldsymbol{\tau}})\right\|_S \leq c\varepsilon^{1/2+r}\|g\|_{L^\infty(S)}\,. \tag{1.10}$$

Just like (1.9), (1.10) express the proximity between the finite element solution $(\boldsymbol{u}^h, p^h)$ and its truncated counterpart $(\boldsymbol{u}^h_\varepsilon, p^h_\varepsilon)$. With (1.9) and (1.10) in place, it is now important to estimate the quantities $\boldsymbol{u}_\varepsilon - \boldsymbol{u}^h_\varepsilon$ and $p_\varepsilon - p^h_\varepsilon$. Thus by considering any stable finite element pair for approximating the velocity and pressure, then the surprising result of this work states that

$$\begin{aligned}&\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 + \|p_\varepsilon - p_\varepsilon^h\| + \left(\varepsilon^{1/2} - ch^{1/2}\right)\left\|\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right\|_S \\ &\leq c \inf_{\boldsymbol{v}^h \in \mathbb{V}_h} \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1 + c \inf_{q^h \in M_h} \|p_\varepsilon - q^h\|\,,\end{aligned} \tag{1.11}$$

where $\mathbb{V}_h$ and M_h are finite element spaces approximating the velocity and pressure respectively, and c is a positive constant, independent of both ε and h. It is then manifest that for $\varepsilon = (c+1)^2 h$, and using some interpolation results for the velocity and pressure, one gets a convergence result of order one for $\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 + \|p_\varepsilon - p_\varepsilon^h\|$. Finally, from the triangle's inequality one also obtain a linear convergence for $\|\boldsymbol{u} - \boldsymbol{u}^h\|_1 + \|p - p^h\|$. This surprising result is confirmed in all the numerical experiments of this paper. This work introduces a truncation technique as an approach to obtain a better (an improved) convergence rate for finite element approximation of variational inequalities of second kind. This approach also open the door for future applications to problems in plasticity or Bingham flow.

1.3 Outline of the paper

The rest of the paper is organised as follows.

- Section 2 is concerned with the variational formulations of the problem and existence theory.
- Section 3 introduces the truncated problem follow by thorough a priori analysis.
- Section 4 is devoted to the finite element formulation and its a priori analysis.
- Section 5 is about the formulation of the numerical scheme
- Section 6 is concerned with the implementation of the finite element solution, the verification of the theoretical results and some concluding remarks.

2 Variational formulations

We adopt the standard definitions [30] for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and semi-norms $|\cdot|_{s,D}$ for $s \geq 0$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and inner product are denoted as $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. If $D = \Omega$, we drop D.
Throughout this work, boldface characters denote vector quantities, and $\boldsymbol{H}^1(\Omega) = H^1(\Omega)^d$ and $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$.
In order to introduce the functions spaces for the analysis of the boundary value described by the equations (1.1)–(1.5), we take in a naive way the dot product between the equation

(1.1) and $\boldsymbol{u}$ and integrate the resulting equation over Ω. After utilization of the Green's formula and boundary conditions we arrived at

$$2\mu \int_{\Omega} |D\boldsymbol{u}|^2 dx + \int_S g|\boldsymbol{u_\tau}| d\sigma - \int_{\Omega} p \operatorname{div} \boldsymbol{u} dx = \langle \boldsymbol{f}, \boldsymbol{u} \rangle \ , \tag{2.1}$$

with $d\sigma$ being the surface measure associated to S. In this work, we assume once and for all that g is non-negative and $g \in L^\infty(S)$, while $\boldsymbol{f} \in L^2(\Omega)^d$. From (2.1), we introduce the following functions spaces

$$\begin{aligned} &\mathbb{V} = \{\boldsymbol{u} \in H^1(\Omega)^d, \ \ \boldsymbol{u}|_\Gamma = \boldsymbol{0} \ , \ \boldsymbol{u} \cdot \boldsymbol{n}|_S = 0\}, \\ &M = L_0^2(\Omega) = \left\{ v \in L^2(\Omega) \text{ with } \int_\Omega v(\boldsymbol{x}) \mathrm{dx} = 0 \ \right\} \ . \end{aligned}$$

$\mathbb{V}$ is a Hilbert space equipped with the standard H^1-norm on each component, while M is a Banach space equipped with the L^2-norm. With the spaces $\mathbb{V}$ and M, one can introduce the weak formulation for the problem (1.1)–(1.5).
We multiply (1.2) by $q \in L^2(\Omega)$ and integrate over Ω. We take the dot product between (1.1) and $\boldsymbol{v}-\boldsymbol{u}$ with $\boldsymbol{v} \in \mathbb{V}$, integrate the resulting equation over Ω, apply Green's formula, and the boundary conditions (1.3), (1.4) and (1.5) we obtain:

$$\begin{aligned} &\text{Find } (\boldsymbol{u}, p) \in \mathbb{V} \times M \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M \\ &a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, p) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq \ell(\boldsymbol{v} - \boldsymbol{u}) \ , \\ &b(\boldsymbol{u}, q) = 0 \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= 2\mu \int_\Omega D\boldsymbol{u} : D\boldsymbol{v} \, dx \quad , \quad b(\boldsymbol{v}, q) = -\int_\Omega q \operatorname{div} \boldsymbol{v} dx, \\ j(\boldsymbol{v}) &= \int_S g|\boldsymbol{v_\tau}| \, d\sigma \quad , \qquad \ell(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \ , \end{aligned} \tag{2.3}$$

and $\boldsymbol{A} : \boldsymbol{B} = \sum_{1 \leq i,j \leq d} A_{ij} B_{ij}$. One easily proves that

Lemma 2.1 *If $(\boldsymbol{u}, p)$ is the solution of (1.8) then $(\boldsymbol{u}, p)$ solves (2.2) and vice versa.*

Note that the operators defined via (2.3) are well defined for $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{V}$ and $q \in L^2(\Omega)$. One readily verifies that $b(\cdot, \cdot)$ is continuous; that is

$$\text{for all } \ (\boldsymbol{v}, q) \in \mathbb{V} \times L^2(\Omega) \quad , \qquad b(\boldsymbol{v}, q) \leq \|\boldsymbol{v}\|_1 \|q\| \ .$$

One of the crucial property when studying (2.2) is the following inf-sup condition: there exists $\beta > 0$ such that

$$\beta \|q\| \leq \sup_{0 \neq \boldsymbol{v} \in \mathbb{V}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \ \text{ for all } q \in M \, . \tag{2.4}$$

In fact (2.4) is obtained by observing that $\boldsymbol{H}_0^1(\Omega) \subset \mathbb{V}$ and the pair $(\boldsymbol{H}_0^1(\Omega), M)$ is inf-sup stable (see [31, 32]), hence there exists γ such that

$$\text{for all } q \in M \quad , \quad \sup_{0 \neq \boldsymbol{v} \in \mathbb{V}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq \gamma \|q\| \,.$$

At this point we recall that the Korn inequality reads: there is a constant c_K depending only on Ω such that

$$c_K \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \; dx \leq \int_\Omega D\boldsymbol{v} : D\boldsymbol{v} \; dx \quad \text{for all } \; \boldsymbol{v} \in \mathbb{V} \,, \tag{2.5}$$

while the Poincaré-Friedreich inequality states that; there is constant c_{PF} depending only on the domain Ω such that

$$c_{\mathrm{PF}} \int_\Omega |\boldsymbol{v}|^2 \; dx \leq \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \; dx \quad \text{for all } \; \boldsymbol{v} \in \mathbb{V} \,. \tag{2.6}$$

Thus it is manifest that the norms $\|\cdot\|_1$ and $\|\nabla \cdot\|$ are equivalent on $\mathbb{V}$. Thus with (2.6), one deduces that

$$2\mu c_K \|\boldsymbol{v}\|_1^2 \leq a(\boldsymbol{v}, \boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{v} \in \mathbb{V} \,. \tag{2.7}$$

Next, the functional $j(\cdot)$ is convex and l.s.c (in fact even continuous) on $\mathbb{V}$. Thus we claim that

Proposition 2.1 *If $\boldsymbol{f}$ is an element of $L^2(\Omega)^d$, and $g \in L^\infty(S)$, then the variational problem (2.2) has a unique solution $(\boldsymbol{u}, p) \in \mathbb{V} \times M$, and the following estimate hold*

$$\|\boldsymbol{u}\|_1 + \|p\| \leq c(\Omega, \mu) \|\boldsymbol{f}\| \quad .$$

Moreover if S is of class $\mathcal{C}^3$ and Γ of class $\mathcal{C}^2$, with $g \in H^1(S) \cap L^\infty(S)$, then Saito in [3] has shown that $(\boldsymbol{u}, p) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$, and enjoys the a priori estimate

$$\|\boldsymbol{u}\|_2 + \|p\|_1 \leq c(\Omega, \mu)(\|\boldsymbol{f}\| + \|g\|_{1,S}) \,.$$

Another equivalent model, is the three field formulation which reads:

$$\begin{aligned}
&\text{Find } (\boldsymbol{u}, p, (\boldsymbol{Tn})_{\boldsymbol{\tau}}) \in \mathbb{V} \times M \times H^{-1/2}(S) \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M \\
&a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - \langle (\boldsymbol{Tn})_{\boldsymbol{\tau}}, \boldsymbol{v}_{\boldsymbol{\tau}} \rangle = \ell(\boldsymbol{v}) \,, \\
&b(\boldsymbol{u}, q) = 0 \,, \\
&-(\boldsymbol{Tn})_{\boldsymbol{\tau}} \in g \partial |\boldsymbol{u}_{\boldsymbol{\tau}}| \text{ a.e. on } S \,.
\end{aligned} \tag{2.8}$$

3 Truncated Problem

This section introduces the truncated problem and we provide a priori analysis associated to the problem.

3.1 Some preliminaries

With the cut-off function $\nu_k(\cdot)$ introduces, we check easily that for all $\boldsymbol{x}, \boldsymbol{y}$ elements of $\mathbb{R}^d$

$$\begin{aligned}&(\nu_k(\boldsymbol{x}) - \nu_k(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq 0,\\ &|\nu_k(\boldsymbol{x}) - \nu_k(\boldsymbol{y})| \leq |\boldsymbol{x} - \boldsymbol{y}| \ .\end{aligned} \tag{3.1}$$

The truncated problem reads:

$$\begin{cases}\text{Find } (\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \mathbb{V} \times M \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M \ ,\\ a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p_\varepsilon) + \dfrac{1}{\varepsilon}\displaystyle\int_S \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma = \ell(\boldsymbol{v}) \ ,\\ b(\boldsymbol{u}_\varepsilon, q) = 0 \ .\end{cases} \tag{3.2}$$

Using the Green's formula, one can show that the truncated problem (3.2) is associated to the following in the sense of distribution

$$\begin{aligned}&-2\mu \operatorname{div} D\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = \boldsymbol{f} \quad \text{in } \Omega \ ,\\ &\operatorname{div} \boldsymbol{u}_\varepsilon = 0 \quad \text{in } \Omega \ ,\\ &(\boldsymbol{T}(\boldsymbol{u}_\varepsilon, p_\varepsilon)\boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) = 0 \quad \text{on } S \ ,\\ &\boldsymbol{u}_\varepsilon|_\Gamma = 0 \quad , \quad \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}|_S = 0 \,.\end{aligned} \tag{3.3}$$

Remark 3.1 *One observes that* $-\dfrac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \approx (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$.

3.2 Analysis of the truncated problem

Henceforth, c is positive generic constant depending only on Ω unless otherwise stated. c may vary from line to line but are always independent of ϵ. When we need to specify its dependency on other quantity, we write $c(f, g)$, $c(\mu, \varepsilon)$. etc...

About the qualitative analysis of (3.2), we state that

Proposition 3.1 *The truncated problem (3.2) has a unique solution* $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \mathbb{V} \times M$ *and the following hold*

$$\|\boldsymbol{u}_\varepsilon\|_1 + \|p_\varepsilon\| \leq c(\mu)\|\boldsymbol{f}\| \quad .$$

Proof. The variational problem (3.2) is nonlinear monotone elliptic problem, and its existence theory is well known in the literature (see [34], Theorem 2.1, p. 171). The a priori estimate is a consequence of the coercivity of $a(\cdot, \cdot)$ and the inf-sup condition on $b(\cdot, \cdot)$. □

The next result is important and prepare for the convergence result between the solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ and $(\boldsymbol{u}, p)$. It is a result similar to Lemma 3.9 in [26], or theorem 2.1 in [28]. We claim that

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a bounded domain with polygonal boundary. let $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ be the solution of (3.2) and $(\boldsymbol{u}, p)$ the solution of (2.2). Assume that $\boldsymbol{u}$ is in $\boldsymbol{H}^{3/2+r}(\Omega)$ with $r \in (0, 1/2]$, then the following holds*

$$\left\| (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_{-r,S} \leq c\left(\varepsilon^r \left\| (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_S + \mu\varepsilon^{r-\frac{1}{2}}\left(1 + \frac{1}{\beta}\right)\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_1 \right),$$

with c a positive constant independent of both ε and $\boldsymbol{u}$.

Before starting its proof, it is important to recall some preliminaries introduce by [26]. Since the domain Ω has a polygonal boundary, we have

$$\overline{\Omega} = \bigcup_{i=1}^{N} K_i \,.$$

We denote by $T_\varepsilon = \{K\}$, a regular triangulation of Ω with the mesh size ε. Next, we define the fictitious space $\mathbb{V}_\varepsilon$ given by

$$\mathbb{V}_\varepsilon = \{\boldsymbol{v}_\varepsilon \in \mathcal{C}(\overline{\Omega})^d : \ \boldsymbol{v}_\varepsilon|_T \in \mathbb{P}_2(T)^d \ , \ \ \forall\, T \in \mathcal{T}_\varepsilon, \ \ \boldsymbol{v}_\varepsilon|_\Gamma = \boldsymbol{0} \ \ , \ \ \boldsymbol{v}_\varepsilon \cdot \boldsymbol{n}|_S = 0\} \ .$$

The space of tangential traces on S for the elements in $\mathbb{V}_\varepsilon$ is

$$\mathbb{W}_\varepsilon(S) = \left\{ w_\varepsilon \in \mathcal{C}(\overline{S}) : \ \ \exists\, \boldsymbol{v}_\varepsilon \in \mathbb{V}_\varepsilon \ , \ \ \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\tau} = \boldsymbol{w}_\varepsilon \ \text{ on } S \right\} \ .$$

We also introduce the $L^2(S)$ projection, $\mathcal{P}_\varepsilon : L^2(S) \longrightarrow \mathbb{W}_\varepsilon(S)$ and we assume that the triangulation $\mathcal{T}_\varepsilon$ on $\overline{S}$ is quasi-uniform (in the sense defined in the context of finite element). Then for $s \in [0, 1]$ and $\boldsymbol{v} \in \boldsymbol{H}^s(S)$, there exists c independent of ε such that

$$\begin{aligned} &\|\mathcal{P}_\varepsilon \boldsymbol{v}\|_{s,S} \leq c\|\boldsymbol{v}\|_{s,S} \ , \\ &\|\mathcal{P}_\varepsilon \boldsymbol{v} - \boldsymbol{v}\|_S \leq c\varepsilon^s\|\boldsymbol{v}\|_{s,S} \ . \end{aligned} \tag{3.4}$$

We also recall the following inverse inequality since the mesh on S is quasi-uniform.

$$\text{for all } \boldsymbol{v} \in H^r(S) \ \ \|\mathcal{P}\boldsymbol{v}\|_{\frac{1}{2},S} \leq c\varepsilon^{r-\frac{1}{2}} \|\mathcal{P}\boldsymbol{v}\|_{r,S} \ . \tag{3.5}$$

Finally assuming that the mesh on $\overline{S}$ is quasi-uniform, then there exists an extension operator $\mathcal{E}_\varepsilon : \mathbb{W}_\varepsilon(S) \longrightarrow \mathbb{V}_\varepsilon$ and c independent of ε such that

$$\mathcal{E}_\varepsilon(\boldsymbol{v}_\varepsilon)|_S \cdot \boldsymbol{\tau} = \boldsymbol{v}_\varepsilon, \quad \|\mathcal{E}_\varepsilon \boldsymbol{v}_\varepsilon\|_{1,\Omega} \leq c\|\boldsymbol{v}\|_{\frac{1}{2},S} \ , \ \ \text{for all } \ \boldsymbol{v}_\varepsilon \in \mathbb{W}_\varepsilon(S) \ . \tag{3.6}$$

Proof of Lemma 3.1. By definition and $\boldsymbol{w} = \boldsymbol{w} - \mathcal{P}_\varepsilon\boldsymbol{w} + \mathcal{P}_\varepsilon\boldsymbol{w}$, one has

$$\begin{aligned} \left\| (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_{-r,S} &= \sup_{\boldsymbol{w} \in H^r(S)} \frac{\langle (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}), \boldsymbol{w}\rangle_S}{\|\boldsymbol{w}\|_{r,S}} \\ &= \sup_{\boldsymbol{w} \in H^r(S)} \frac{\langle (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}), \boldsymbol{w} - \mathcal{P}_\varepsilon\boldsymbol{w} + \mathcal{P}_\varepsilon\boldsymbol{w}\rangle_S}{\|\boldsymbol{w}\|_{r,S}} \ . \end{aligned} \tag{3.7}$$

We now estimate the term on the right hand side of (3.7). Using the triangle's inequality, Holder's inequality and (3.4), one obtains the following

$$\begin{aligned}
\left\|(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right\|_{-r,S} &\leq \sup_{\boldsymbol{w}\in H^r(S)} \frac{\left|\langle(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}), \boldsymbol{w} - \mathcal{P}_\varepsilon\boldsymbol{w}\rangle_S\right|}{\|\boldsymbol{w}\|_{r,S}} \\
&\quad + \sup_{\boldsymbol{w}\in H^r(S)} \frac{\left|\langle(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}), \mathcal{P}_\varepsilon\boldsymbol{w}\rangle_S\right|}{\|\boldsymbol{w}\|_{r,S}} \\
&\leq \left\|(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right\|_S \sup_{\boldsymbol{w}\in H^r(S)} \frac{\|\boldsymbol{w} - \mathcal{P}_\varepsilon\boldsymbol{w}\|_S}{\|\boldsymbol{w}\|_{r,S}} \\
&\quad + \sup_{\boldsymbol{w}\in H^r(S)} \frac{\left|\langle(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}), \mathcal{P}_\varepsilon\boldsymbol{w}\rangle_S\right|}{\|\boldsymbol{w}\|_{r,S}} \\
&\leq c\varepsilon^r \left\|(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right\|_S \\
&\quad + \sup_{\boldsymbol{w}\in H^r(S)} \frac{\left|\langle(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}), \mathcal{P}_\varepsilon\boldsymbol{w}\rangle_S\right|}{\|\boldsymbol{w}\|_{r,S}} \ . \qquad (3.8)
\end{aligned}$$

We estimate next the expression $\sup_{\boldsymbol{w}\in H^r(S)} \frac{\left|\langle(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}), \mathcal{P}_\varepsilon\boldsymbol{w}\rangle_S\right|}{\|\boldsymbol{w}\|_{r,S}}$. For that purpose, for $\boldsymbol{v} \in \mathbb{V}$ and $\boldsymbol{u} \in \boldsymbol{H}^{3/2+r}(\Omega)$, one has $(\boldsymbol{Tn})_{\boldsymbol{\tau}} \in \boldsymbol{H}^r(S)$, and $(\boldsymbol{u}, \boldsymbol{u}_\varepsilon)$ is solution of:

$$\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - \int_S (\boldsymbol{Tn})_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma = \ell(\boldsymbol{v}) \ , \\
a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p_\varepsilon) + \frac{1}{\varepsilon}\int_S \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma = \ell(\boldsymbol{v}) \ .
\end{aligned} \qquad (3.9)$$

Putting together the equations in (3.9), one gets

$$\int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right) \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma = a(\boldsymbol{u} - \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p - p_\varepsilon) \ . \qquad (3.10)$$

Thus for $\boldsymbol{v}|_S = \mathcal{P}_\varepsilon\boldsymbol{w}$ in (3.10), using Holder's inequality, (3.6), (3.5) and (3.4), one obtains

$$\begin{aligned}
\int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right) \cdot \mathcal{P}_\varepsilon\boldsymbol{w} d\sigma &= a(\boldsymbol{u} - \boldsymbol{u}_\varepsilon, \mathcal{E}_\varepsilon\mathcal{P}_\varepsilon\boldsymbol{w}) + b(\mathcal{E}_\varepsilon\mathcal{P}_\varepsilon\boldsymbol{w}, p - p_\varepsilon) \\
&\leq 2\mu\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_1 \|\mathcal{E}_\varepsilon\mathcal{P}_\varepsilon\boldsymbol{w}\|_1 + \|p - p_\varepsilon\| \, \|\mathcal{E}_\varepsilon\mathcal{P}_\varepsilon\boldsymbol{w}\|_1 \\
&\leq c\left(\mu\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_1 + \|p - p_\varepsilon\|\right) \|\mathcal{P}_\varepsilon\boldsymbol{w}\|_{\frac{1}{2},S} \\
&\leq c\varepsilon^{r-\frac{1}{2}}\left(\mu\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_1 + \|p - p_\varepsilon\|\right) \|\mathcal{P}_\varepsilon\boldsymbol{w}\|_{r,S} \\
&\leq c\varepsilon^{r-\frac{1}{2}}\left(\mu\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_1 + \|p - p_\varepsilon\|\right) \|\boldsymbol{w}\|_{r,S} \, . \qquad (3.11)
\end{aligned}$$

It is manifest that to close the inequality in (3.8), we need to estimate $\|p-p_\varepsilon\|$. For that purpose, we take $\boldsymbol{v}$ such that $\boldsymbol{v}|_{\partial\Omega}=0$. Then from (3.9) one finds

$$b(\boldsymbol{v}, p_\varepsilon - p) = a(\boldsymbol{u}_\varepsilon - \boldsymbol{u}, \boldsymbol{v}) \ ,$$

which together with (2.4) and $a(\boldsymbol{u}-\boldsymbol{u}_\varepsilon, \boldsymbol{v}) \le 2\mu\|\boldsymbol{u}-\boldsymbol{u}_\varepsilon\|_1\|\boldsymbol{v}\|_1$ gives

$$\|p_\varepsilon - p\| \le \frac{2\mu}{\beta}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_1 \ . \tag{3.12}$$

Returning to (3.11) with (3.12) one has

$$\int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right)\cdot \mathcal{P}_\varepsilon \boldsymbol{w} d\sigma \le c\varepsilon^{r-\frac{1}{2}}\left(\mu + \frac{\mu}{\beta}\right)\|\boldsymbol{u}-\boldsymbol{u}_\varepsilon\|_1\|\boldsymbol{w}\|_{r,S} \, . \tag{3.13}$$

Finally (3.13) and (3.8) lead to the desired inequality and the proof is complete. $\square$

Remark 3.2 *If $r=1/2$, then from (3.10)*

$$\left\|(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right\|_{-1/2,S} \le 2\mu\|\boldsymbol{u}-\boldsymbol{u}_\varepsilon\|_1 + \|p-p_\varepsilon\| \, .$$

The solution $(\boldsymbol{u},p)$ is closer to $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ in the following way

Proposition 3.2 *Let $(\boldsymbol{u},p)$ be the weak solution of problem (1.1)–(1.5) such that $\boldsymbol{u}\in \boldsymbol{H}^{3/2+r}(\Omega)$ with $r\in(0,1/2]$. Let $(\boldsymbol{u}_\varepsilon,p_\varepsilon)$ be the truncated solution defined via (3.2). Then there exists a positive constant c independent of ε such that*

$$\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_1 + \|p-p_\varepsilon\| + \varepsilon^{1/2}\left\|(\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{g\varepsilon}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\right\|_S \le c\varepsilon^{\frac{1+2r}{2}}\|\boldsymbol{u}\|_{r+3/2} \, .$$

Proof. Let $\boldsymbol{u}\in\boldsymbol{H}^{r+3/2}(\Omega)$ with $r\in(0,1/2]$. Then $(\boldsymbol{Tn})_{\boldsymbol{\tau}}\in\boldsymbol{H}^r(S)$. So for $(\boldsymbol{v},q)\in \mathbb{V}\times M$, one gets

$$\begin{aligned}
&a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - \langle(\boldsymbol{Tn})_{\boldsymbol{\tau}}, \boldsymbol{v}_{\boldsymbol{\tau}}\rangle_S = \ell(\boldsymbol{v})\\
&a(\boldsymbol{u}_\varepsilon,\boldsymbol{v}) + b(\boldsymbol{v},p_\varepsilon) + \frac{1}{\varepsilon}\int_S \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})\cdot\boldsymbol{v}_{\boldsymbol{\tau}} d\sigma = \ell(\boldsymbol{v}) \quad ,\\
&b(\boldsymbol{u}_\varepsilon,q) = 0 \quad , \quad b(\boldsymbol{u},q) = 0 \, .
\end{aligned} \tag{3.14}$$

Using the coercivity and linearity of $a(\cdot,\cdot)$, one has

$$\begin{aligned}
2\mu c_K\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_1^2 &\le a(\boldsymbol{u}_\varepsilon-\boldsymbol{u}, \boldsymbol{u}_\varepsilon-\boldsymbol{u})\\
&= a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon-\boldsymbol{u}) - a(\boldsymbol{u}, \boldsymbol{u}_\varepsilon-\boldsymbol{u}) \, .
\end{aligned} \tag{3.15}$$

We then need to determine $a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) - a(\boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u})$. From (3.14), one obtains

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) - a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) = b(\boldsymbol{u}_\varepsilon - \boldsymbol{u}, p_\varepsilon - p) + \displaystyle\int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot (\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}})\, d\sigma \\ b(\boldsymbol{u}_\varepsilon - \boldsymbol{u}, q) = 0 \quad \text{for all } q \in M\,. \end{cases}$$

Because $p_\varepsilon - p \in M$, $b(\boldsymbol{u}_\varepsilon - \boldsymbol{u}, p_\varepsilon - p) = 0$ and one deduces that

$$a(\boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) - a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) = \int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot (\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}})\, d\sigma\,. \tag{3.16}$$

We return to (3.15) with (3.16) and obtain;

$$2\mu c_K \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_1^2 \le \int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma - \int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}_{\varepsilon\boldsymbol{\tau}} d\sigma\,. \tag{3.17}$$

From the slip boundary condition (1.7), we deduce that

$$\begin{aligned} \int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma &= \int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \cdot \boldsymbol{u}_{\boldsymbol{\tau}} \right) d\sigma \\ &= \int_S \left(-g|\boldsymbol{u}_{\boldsymbol{\tau}}| + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \cdot \boldsymbol{u}_{\boldsymbol{\tau}} \right) d\sigma\,. \end{aligned}$$

From the definition of $\nu_{\varepsilon g}$ and Cauchy-Shawrz's inequality, one has

$$\int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma \le \int_S \left(-g|\boldsymbol{u}_{\boldsymbol{\tau}}| + \frac{1}{\varepsilon}|\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}})|\,|\boldsymbol{u}_{\boldsymbol{\tau}}| \right) d\sigma \le 0\,.$$

Thus (3.17) and the definition of $\nu_{\varepsilon g}(\cdot)$ imply that

$$\begin{aligned} 2\mu c_K \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_1^2 &\le -\int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}_{\varepsilon\boldsymbol{\tau}} d\sigma \\ &\le -\int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) d\sigma \\ &\le -\varepsilon \int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot \left(-(\boldsymbol{Tn})_{\boldsymbol{\tau}} + (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) d\sigma \\ &\le -\varepsilon \left\| (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_S^2 + \varepsilon \int_S \left((\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right) \cdot (\boldsymbol{Tn})_{\boldsymbol{\tau}} d\sigma\,, \end{aligned}$$

which with the help of Cauchy-Shawrz's, Holder's and Young's inequality leads to

$$\begin{aligned} 2\mu c_K \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_1^2 + \varepsilon \left\| (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_S^2 &\le \varepsilon^a \left\| (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_{-r,S} \varepsilon^{1-a} \|(\boldsymbol{Tn})_{\boldsymbol{\tau}}\|_{r,S} \\ &\le \frac{\varepsilon^{2a}}{2\alpha} \left\| (\boldsymbol{Tn})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_{-r,S}^2 \\ &\quad + c\mu^2 \varepsilon^{2-2a} \frac{\alpha}{2} \|\boldsymbol{u}\|_{r+3/2}^2, \end{aligned} \tag{3.18}$$

with a and α positive constants that will be made precise later. We Insert in (3.18), Lemma 3.1 and obtain the following

$$
\begin{aligned}
2\mu c_K \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_1^2 + \varepsilon \left\| (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_S^2 &\leq c\mu^2 \varepsilon^{2-2a} \frac{\alpha}{2} \|\boldsymbol{u}\|_{r+3/2}^2 \\
+ c\frac{\varepsilon^{2a}}{2\alpha} \left(\varepsilon^{2r} \left\| (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_S^2 + \mu^2 \varepsilon^{2r-1} \left(1 + \frac{1}{\beta}\right)^2 \|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_1^2 \right) ,
\end{aligned}
$$

which for $a = \frac{1}{2} - r$, gives

$$
\begin{aligned}
\mu \left[c_K - \frac{c\mu}{\alpha}\left(1 + \frac{1}{\beta}\right)^2 \right] \|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_1^2 + \varepsilon \left(1 - \frac{c}{\alpha}\right) \left\| (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_S^2 \\
\leq c\mu^2 \varepsilon^{1+2r} \frac{\alpha}{2} \|\boldsymbol{u}\|_{r+3/2}^2 .
\end{aligned}
\tag{3.19}
$$

Finally for $\alpha \geq \max\left(c, \dfrac{c\mu(1 + 1/\beta)^2}{c_K}\right)$, we deduce that there is a positive constant c independent of ε such that

$$
\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_1^2 + \varepsilon \left\| (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_S^2 \leq c\varepsilon^{1+2r} \|\boldsymbol{u}\|_{r+3/2}^2 ,
$$

which is the desired result. □

Remark 3.3 *From lemma 3.1 and Proposition 3.2, we deduce that*

$$
\left\| (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \right\|_{-r,S} \leq c\varepsilon^{2r} \|\boldsymbol{u}\|_{r+3/2} . \tag{3.20}
$$

4 Finite element Analysis

This section introduces notation pertinent to finite element discretization. We prove (1.10) and (1.11), which together with (1.9) gives an estimation for $\|\boldsymbol{u} - \boldsymbol{u}^h\|_1 + \|p - p^h\|$.

4.1 Preliminaries

We recall that Ω is a polygon when $d = 2$ or polyhedron when $d = 3$, so it can be completely meshed. Now, we describe the approximation spaces. A regular (see [35]) family of triangulations $(\mathcal{T}_h)_h$ of Ω, is a set of closed non-degenerate triangles or tetrahedra called elements, satisfying,

(a) $\overline{\Omega} = \bigcup_{1 \leq n \leq N} K_n$.

(b) the intersection of two different elements is either empty, a corner, or a whole edge of both elements.

(c) The ratio of the diameter h_K of an element K in $\mathcal{T}$ to the diameter of its inscribed circle or sphere is bounded by a constant independent of K and h.

As standard, h stands for the maximal diameter of all elements of $\mathcal{T}_h$. For each non-negative integer l and any K in $\mathcal{T}_h$, $\mathbb{P}_l(K)$ is the space of restrictions to K of polynomials in d variables and total degree less than or equal to l.
In what follows, c stand for generic constants, positive which may vary from line to line but are always independent of both h and ϵ.
We have chosen to work with the Taylor-Hood finite elements [31, 32], thus the discrete spaces of velocities and pressures are defined by

$$\mathbb{V}_h = \{\boldsymbol{v}^h \in \mathcal{C}(\overline{\Omega})^d \cap \mathbb{V} : \text{ for all } K \in \mathcal{T},\ \boldsymbol{v}^h|_K \in \mathbb{P}_2(K)^d\} \ ,$$
$$M_h = \{q^h \in M \cap \mathcal{C}(\overline{\Omega}), \text{ for all } K \in \mathcal{T},\ \ q^h|_K \in \mathbb{P}_1(K)\} \, .$$

It is noted that the couple velcity/pressure is inf-sup stable that is; there exists a positive constant $\widetilde{\beta}$ independent of h such that

$$\widetilde{\beta}\|q^h\| \le \sup_{0\neq \boldsymbol{v}^h\in\mathbb{V}_h} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_1} \quad \text{for all } q^h \in M_h \, . \tag{4.1}$$

Remark 4.1 *It should be made clear that other choice of elements for the couple velocity/pressure can be adopted as long as the compatibility condition (4.1) is satisfied. The reader may consult [31, 32] for a thorough mathematical discussion of the inf-sup condition (4.1), its implications and elements that satisfied the "test".*

We also introduce $W_h(S)$, the space of traces on S for discrete functions in $\mathbb{V}_h$:

$$W_h(S) = \{\alpha^h \in \mathcal{C}(\overline{S});\ \ \exists\ \boldsymbol{v}^h \in \mathbb{V}_h,\ \ \boldsymbol{v}^h \cdot \boldsymbol{\tau} = \alpha^h \ \text{ on } S\} \ .$$

We assume that the mesh on S is the one induced by $\mathcal{T}$. Hence the mesh of S is locally quasi-uniform. We also introduce the $L^2(S)$ projection, $\mathcal{P}_h : L^2(S) \longrightarrow \mathbb{W}_h(S)$ and because the triangulation $\mathcal{T}$ on $\overline{S}$ is quasi-uniform, then for $s \in [0, 1]$ and $\boldsymbol{v} \in \boldsymbol{H}^s(S)$, there exists c such that [26]

$$\begin{aligned} &\|\mathcal{P}_h\boldsymbol{v}\|_{s,S} \le c\|\boldsymbol{v}\|_{s,S} \ , \\ &\|\mathcal{P}_h\boldsymbol{v} - \boldsymbol{v}\|_S \le ch^s\|\boldsymbol{v}\|_{s,S} \ . \end{aligned} \tag{4.2}$$

Finally assuming that the mesh on $\overline{S}$ is quasi-uniform, then there exists an extension operator $\mathcal{E}_h : \mathbb{W}_h(S) \longrightarrow \mathbb{V}_h$ and c such that [26]

$$\mathcal{E}_h(\boldsymbol{v}^h)|_S \cdot \boldsymbol{\tau} = \boldsymbol{v}^h, \quad \left\|\mathcal{E}_h\boldsymbol{v}^h\right\|_{1,\Omega} \le c\|\boldsymbol{v}^h\|_{\frac{1}{2},S} \ , \quad \text{for all } \ \boldsymbol{v}^h \in \mathbb{W}_h(S) \ . \tag{4.3}$$

The finite element solution associated to the truncated problem (3.2) is defined as follows:

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon^h, p_\varepsilon^h) \in \mathbb{V}_h \times M_h \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V}_h \times M_h \ , \\ a(\boldsymbol{u}_\varepsilon^h, \boldsymbol{v}) + b(\boldsymbol{v}, p_\varepsilon^h) + \dfrac{1}{\varepsilon}\displaystyle\int_S \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}^h) \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma = \ell(\boldsymbol{v}) \ , \\ b(\boldsymbol{u}_\varepsilon^h, q) = 0 \ . \end{cases} \tag{4.4}$$

We recall that $(\boldsymbol{u}^h, p^h)$ is the solution of the finite element problem

$$\begin{cases} \text{Find } (\boldsymbol{u}^h, p^h) \in \mathbb{V}_h \times M_h \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V}_h \times M_h \ , \\ a(\boldsymbol{u}^h, \boldsymbol{v}) + b(\boldsymbol{v}, p^h) - \langle (\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}}, \boldsymbol{v}_{\boldsymbol{\tau}} \rangle = \ell(\boldsymbol{v}) \ , \\ -(\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} \in g \partial |\boldsymbol{u}^h_{\boldsymbol{\tau}}| \quad \text{on } S, \\ b(\boldsymbol{u}^h, q) = 0 \ . \end{cases} \tag{4.5}$$

4.2 Error estimates

The analog of lemma 3.1 is

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^d$, be a bounded domain with polygonal boundary. Let $(\boldsymbol{u}^h_\varepsilon, p^h_\varepsilon)$ be the solution of (4.4) and $(\boldsymbol{u}^h, p^h)$ the solution of (4.5). Then there exists c such that the following holds*

$$\left\| \left(\boldsymbol{T}^h \boldsymbol{n}\right)_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right\|_{-r,S} \leq c \left(\varepsilon^r \left\| \left(\boldsymbol{T}^h \boldsymbol{n}\right)_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right\|_S + \mu \varepsilon^{r - \frac{1}{2}} \left(1 + \frac{1}{\widetilde{\beta}}\right) \|\boldsymbol{u}^h - \boldsymbol{u}^h_\varepsilon\|_1 \right) .$$

The proof is the same as the proof of lemma 3.1.
Next, the analog of proposition 3.2 reads

Proposition 4.1 *Let $(\boldsymbol{u}^h, p^h)$ be the solution of (4.5). Let $(\boldsymbol{u}^h_\varepsilon, p^h_\varepsilon)$ the truncated finite element solution defined via (4.4). Then there exists c such that*

$$\|\boldsymbol{u}^h_\varepsilon - \boldsymbol{u}^h\|_1 + \|p^h - p^h_\varepsilon\| + \varepsilon^{1/2} \left\| \left(\boldsymbol{T}^h \boldsymbol{n}\right)_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{g \varepsilon}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right\|_S \leq c \varepsilon^{\frac{1+2r}{2}} \|g\|_{L^\infty(S)} .$$

Proof. We follow the proof of proposition 3.2.
First the inequality (3.17) in this context reads

$$\begin{aligned} 2\mu c_K \|\boldsymbol{u}^h_\varepsilon - \boldsymbol{u}^h\|_1^2 \leq \quad & \int_S \left((\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}^h_{\boldsymbol{\tau}} d\sigma \\ & - \int_S \left((\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}} d\sigma \, . \end{aligned} \tag{4.6}$$

The second relation in (4.5) together with the definition of $\nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}})$ imply that

$$\begin{aligned} & \int_S \left((\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}^h_{\boldsymbol{\tau}} d\sigma \leq - \int_S g |\boldsymbol{u}^h_{\boldsymbol{\tau}}| d\sigma + \int_S \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \cdot \boldsymbol{u}^h_{\boldsymbol{\tau}} d\sigma \leq 0 \, , \\ & - \int_S \left((\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right) \cdot \boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}} d\sigma \leq - \int_S \left((\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right) \cdot \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) d\sigma \, . \end{aligned}$$

Thus (4.6) imply that (see the derivation of (3.18))

$$
\begin{aligned}
2\mu c_K \|\boldsymbol{u}^h_\varepsilon - \boldsymbol{u}^h\|_1^2 &+ \varepsilon \left\| (\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right\|_S^2 \\
&\leq \frac{\varepsilon^{2a}}{2\alpha} \left\| (\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right\|_{-r,S}^2 + \varepsilon^{2-2a} \frac{\alpha}{2} \|(\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}}\|_{r,S}^2 \,.
\end{aligned}
\tag{4.7}
$$

with a and α positive constants that will be made precise later. We Insert in (4.7), Lemma 4.1 and take $a = \frac{1}{2} - r$ to obtain

$$
\begin{aligned}
\mu \left[c_K - \frac{c\mu}{\alpha} \left(1 + \frac{1}{\widetilde{\beta}} \right)^2 \right] \|\boldsymbol{u}^h - \boldsymbol{u}^h_\varepsilon\|_1^2 + \varepsilon \left(1 - \frac{c}{\alpha} \right) \left\| (\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right\|_S^2 \\
\leq \varepsilon^{2r+1} \frac{\alpha}{2} \|(\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}}\|_{r,S}^2 \,.
\end{aligned}
\tag{4.8}
$$

Finally for $\alpha \geq \max \left(c, \dfrac{c\mu(1 + 1/\widetilde{\beta})^2}{c_K} \right)$, we deduce that there is a constant c such that

$$
\|\boldsymbol{u}^h_\varepsilon - \boldsymbol{u}^h\|_1^2 + \varepsilon \left\| \left(\boldsymbol{T}^h \boldsymbol{n} \right)_{\boldsymbol{\tau}} + \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_{\varepsilon \boldsymbol{\tau}}) \right\|_S^2 \leq c \varepsilon^{2r+1} \|(\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}}\|_{r,S}^2 \,. \tag{4.9}
$$

From the second equation of (4.5) and the definition of sub-differential one has

$$
\text{for all vector } \ \boldsymbol{v}^h \quad , \quad -(\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} \cdot \left(\boldsymbol{v}^h_{\boldsymbol{\tau}} - \boldsymbol{u}^h_{\boldsymbol{\tau}} \right) \leq g |\boldsymbol{v}^h_{\boldsymbol{\tau}}| - g |\boldsymbol{u}^h_{\boldsymbol{\tau}}| \ \text{ on } S \,.
$$

We take $\boldsymbol{v}^h$ such that $\boldsymbol{v}^h_{\boldsymbol{\tau}} = \boldsymbol{0}$ and $\boldsymbol{v}^h_{\boldsymbol{\tau}} = 2\boldsymbol{u}^h_{\boldsymbol{\tau}}$. Comparing the resulting inequalities, one obtains

$$
-(\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}^h_{\boldsymbol{\tau}} = g |\boldsymbol{u}^h_{\boldsymbol{\tau}}| \,.
$$

Application of Cauchy-Shwarz's inequality leads to

$$
\left| (\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}} \right| \leq g \quad \text{on} \quad S \,.
$$

Hence Holder's inequality implies that

$$
\|(\boldsymbol{T}^h \boldsymbol{n})_{\boldsymbol{\tau}}\|_{r,S} \leq c \|g\|_{L^\infty(S)} \,,
$$

which together with (4.9) gives the desired result. □

We discuss next the a priori error estimates for the solution $(\boldsymbol{u}, p)$ of (2.2) with the regularity $\boldsymbol{u} \in \boldsymbol{H}^{3/2+r}(\Omega)$, $r \in (0, 1/2]$, and its finite elements counterpart $(\boldsymbol{u}^h, p^h)$ given by (4.5). The main result on this paragraph is stated as follows

Theorem 4.1 *Let $(\boldsymbol{u}^h_\varepsilon, p^h_\varepsilon)$ be the solution of (4.4), and $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ the solution of (3.2). Then there exists c independent of h, ε such that for all $(\boldsymbol{v}^h, q^h) \in \mathbb{V}_h \times M_h$,*

$$
\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}^h_\varepsilon\|_1 + \|p_\varepsilon - p^h_\varepsilon\| + \left(\varepsilon^{1/2} - c h^{1/2} \right) \left\| \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon} \nu_{\varepsilon g}(\boldsymbol{u}^h_\varepsilon) \right\|_S \leq c \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1 + c \|p_\varepsilon - q^h\| \,.
$$

Proof. We recall that $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ solves

$$\begin{cases} \text{for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M \ , \\ a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p_\varepsilon) + \dfrac{1}{\varepsilon} \displaystyle\int_S \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon \boldsymbol{\tau}}) \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma = \ell(\boldsymbol{v}) \ , \\ b(\boldsymbol{u}_\varepsilon, q) = 0 \ , \end{cases}$$

and $(\boldsymbol{u}_\varepsilon^h, p_\varepsilon^h)$ solves

$$\begin{cases} \text{for all } (\boldsymbol{v}^h, q^h) \in \mathbb{V}_h \times M_h \ , \\ a(\boldsymbol{u}_\varepsilon^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_\varepsilon^h) + \dfrac{1}{\varepsilon} \displaystyle\int_S \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon \boldsymbol{\tau}}^h) \cdot \boldsymbol{v}_{\boldsymbol{\tau}}^h d\sigma = \ell(\boldsymbol{v}^h) \ , \\ b(\boldsymbol{u}_\varepsilon^h, q^h) = 0 \ . \end{cases}$$

Since $\mathbb{V}_h \subset \mathbb{V}$ and $M_h \subset M$, we deduce that

$$\begin{cases} \text{for all } (\boldsymbol{v}^h, q^h) \in \mathbb{V}_h \times M_h \ , \\ a(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_\varepsilon - p_\varepsilon^h) + \dfrac{1}{\varepsilon} \displaystyle\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon \boldsymbol{\tau}}) - \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon \boldsymbol{\tau}}^h) \Big) \cdot \boldsymbol{v}_{\boldsymbol{\tau}}^h d\sigma = 0 \ , \\ b(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h, q^h) = 0 \ . \end{cases} \tag{4.10}$$

<u>Step1: Inf-sup condition on $b(\cdot,\cdot)$.</u> Let $q^h \in M_h$, the inf-sup condition on $b(\cdot,\cdot)$ and the first equation in (4.10) imply the existence of $\widetilde{\beta}$ independent of h and ε such that

$$\begin{aligned} \widetilde{\beta} \| p_\varepsilon^h - q^h \| \leq & \sup_{\boldsymbol{v}^h \in H_0^1(\Omega)} \frac{b(\boldsymbol{v}^h, p_\varepsilon^h - q^h)}{\|\boldsymbol{v}^h\|_1} \\ \leq & \sup_{\boldsymbol{v}^h \in H_0^1(\Omega)} \frac{b(\boldsymbol{v}^h, p_\varepsilon^h - p_\varepsilon) + b(\boldsymbol{v}^h, p_\varepsilon - q^h)}{\|\boldsymbol{v}^h\|_1} \\ \leq & \sup_{\boldsymbol{v}^h \in H_0^1(\Omega)} \frac{a(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_\varepsilon - q^h)}{\|\boldsymbol{v}^h\|_1} \\ \leq & 2\mu \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 + \|p_\varepsilon - q^h\| \ . \end{aligned}$$

Thus,

$$\|p_\varepsilon - p_\varepsilon^h\| \leq \left(1 + \frac{1}{\widetilde{\beta}}\right) \|p_\varepsilon - q^h\| + \frac{2\mu}{\widetilde{\beta}} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 \ . \tag{4.11}$$

<u>Step2: coercivity on $a(\cdot,\cdot)$.</u> Let $\boldsymbol{v}^h \in \mathbb{V}_h$

$$\begin{aligned} 2\mu c_K \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1^2 \leq & a(\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h) \\ = & a(\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h) + a(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h) \ . \end{aligned}$$

In (4.10) we replace $\boldsymbol{v}^h$ by $\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h$, compute $a(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h)$. Then one obtains the following

$$
\begin{aligned}
2\mu c_K \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1^2 \leq\ & a(\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h) + b(\boldsymbol{u}_\varepsilon^h - \boldsymbol{v}^h, p_\varepsilon - p_\varepsilon^h) \\
& + \frac{1}{\varepsilon}\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\Big) \cdot \Big(\boldsymbol{u}_\varepsilon^h - \boldsymbol{v}^h\Big)\, d\sigma \\
=\ & a(\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h) + b(\boldsymbol{u}_\varepsilon^h - \boldsymbol{u}_\varepsilon, p_\varepsilon - q^h) + b(\boldsymbol{u}_\varepsilon^h - \boldsymbol{u}_\varepsilon, q^h - p_\varepsilon^h) \\
& + b(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h, p_\varepsilon - p_\varepsilon^h) + \frac{1}{\varepsilon}\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\Big) \cdot \Big(\boldsymbol{u}_\varepsilon^h - \boldsymbol{v}^h\Big)\, d\sigma \\
=\ & a(\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h) + b(\boldsymbol{u}_\varepsilon^h - \boldsymbol{u}_\varepsilon, p_\varepsilon - q^h) \\
& + b(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h, p_\varepsilon - p_\varepsilon^h) + \frac{1}{\varepsilon}\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\Big) \cdot \Big(\boldsymbol{u}_\varepsilon^h - \boldsymbol{v}^h\Big)\, d\sigma
\end{aligned}
\tag{4.12}
$$

where we have used the second relation in (4.10). We re-write (4.12) as follows

$$
\begin{aligned}
& 2\mu c_K \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1^2 + \frac{1}{\varepsilon}\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\Big) \cdot \Big(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\Big)\, d\sigma \\
\leq\ & a(\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h) + b(\boldsymbol{u}_\varepsilon^h - \boldsymbol{u}_\varepsilon, p_\varepsilon - q^h) + b(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h, p_\varepsilon - p_\varepsilon^h) \\
& + \frac{1}{\varepsilon}\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\Big) \cdot \Big(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\Big)\, d\sigma\,.
\end{aligned}
\tag{4.13}
$$

It is manifest that in order to close the inequality (4.13), we need to bound from below $\frac{1}{\varepsilon}\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\Big) \cdot \Big(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\Big)\, d\sigma$ and from above $\frac{1}{\varepsilon}\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\Big) \cdot \Big(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\Big)\, d\sigma$. We follow [29] (theorem 4, or theorem 5).
First, from the definition of $\nu_{\varepsilon g}$, we have

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_S \Big(\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\Big) \cdot \Big(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\Big)\, d\sigma &= \varepsilon \int_S \left(\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right) \cdot \left(\frac{1}{\varepsilon}\boldsymbol{u}_\varepsilon - \frac{1}{\varepsilon}\boldsymbol{u}_\varepsilon^h\right) d\sigma \\
&\geq \varepsilon \left\| \frac{1}{\varepsilon}\boldsymbol{u}_\varepsilon - \frac{1}{\varepsilon}\boldsymbol{u}_\varepsilon^h \right\|_S^2 \\
&\geq \varepsilon \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right\|_S^2 ,
\end{aligned}
\tag{4.14}
$$

where the continuity of $\nu_{\varepsilon g}(\cdot)$ has been used. Next,

$$
\begin{aligned}
I = & \int_S \left(\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right) \cdot \left(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h \right) d\sigma \\
= & \int_S \left(\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right) \cdot \left(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h - \mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h) \right) d\sigma \\
& + \int_S \left(\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right) \cdot \left(\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h) \right) d\sigma \\
\leq & \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right\|_S \left\| \boldsymbol{u}_\varepsilon - \boldsymbol{v}^h - \mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h) \right\|_S \\
& + \int_S \left(\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right) \cdot \left(\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h) \right) d\sigma \\
\leq & \, ch^{1/2} \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right\|_S \left\| \boldsymbol{u}_\varepsilon - \boldsymbol{v}^h \right\|_1 \\
& + \int_S \left(\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right) \cdot \left(\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h) \right) d\sigma \, .
\end{aligned}
\tag{4.15}
$$

From the lighting operator $\mathcal{E}_h$ and the first equation in (4.10), one has

$$
\begin{aligned}
& \int_S \left(\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right) \cdot \left(\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h) \right) d\sigma \\
= & \int_S \left(\frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right) \cdot \left(\mathcal{E}_h\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h) \right) d\sigma \\
= & \; - a(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h, \mathcal{E}_h\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h)) - b(\mathcal{E}_h\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h), p_\varepsilon - p_\varepsilon^h) \\
\leq & \; 2\mu\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 \|\mathcal{E}_h\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h)\|_1 + \|\mathcal{E}_h\mathcal{P}_h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h)\|_1 \|p_\varepsilon - p_\varepsilon^h\| \\
\leq & \; c\mu\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 \|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 + c\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 \|p_\varepsilon - p_\varepsilon^h\| \, .
\end{aligned}
\tag{4.16}
$$

We replace (4.16) in (4.15) and obtain

$$
\begin{aligned}
I = & ch^{1/2} \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right\|_S \left\| \boldsymbol{u}_\varepsilon - \boldsymbol{v}^h \right\|_1 + c\mu\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 \|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 \\
& + c\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 \|p_\varepsilon - p_\varepsilon^h\| \, .
\end{aligned}
\tag{4.17}
$$

We insert (4.17) and (4.14) in (4.13) and obtain

$$
\begin{aligned}
& 2\mu c_K \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1^2 + \varepsilon \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right\|_S^2 \\
& \leq a(\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon, \boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h) + b(\boldsymbol{u}_\varepsilon^h - \boldsymbol{u}_\varepsilon, p_\varepsilon - q^h) + b(\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h, p_\varepsilon - p_\varepsilon^h) \\
& + ch^{1/2} \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right\|_S \left\| \boldsymbol{u}_\varepsilon - \boldsymbol{v}^h \right\|_1 + c\mu\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 \|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 + c\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 \|p_\varepsilon - p_\varepsilon^h\| \\
& \leq 2\mu\|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1 \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1 + \|\boldsymbol{u}_\varepsilon^h - \boldsymbol{v}^h\|_1 \|p_\varepsilon - q^h\| + \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1 \|p_\varepsilon - q^h\| + \|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 \|p_\varepsilon - p_\varepsilon^h\| \\
& + ch^{1/2} \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h) \right\|_S \left\| \boldsymbol{u}_\varepsilon - \boldsymbol{v}^h \right\|_1 + c\mu \left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 + \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1 \right) \|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 \, ,
\end{aligned}
$$

which with (4.11) gives

$$
\begin{aligned}
&2\mu c_K \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1^2 + \varepsilon \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right\|_S^2 \\
&\le 2\mu \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1 \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1 + \|\boldsymbol{u}_\varepsilon^h - \boldsymbol{v}^h\|_1 \|p_\varepsilon - q^h\| + \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1 \|p_\varepsilon - q^h\| \\
&+ \|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 \left(\left(1 + \frac{1}{\beta}\right) \|p_\varepsilon - q^h\| + \frac{2\mu}{\beta}\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 + \frac{2\mu}{\beta}\|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1 \right) \\
&+ ch^{1/2} \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right\|_S \left\| \boldsymbol{u}_\varepsilon - \boldsymbol{v}^h \right\|_1 + c\mu \left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 + \|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1 \right) \|\boldsymbol{u}_\varepsilon - \boldsymbol{v}^h\|_1 .
\end{aligned}
\tag{4.18}
$$

Finally, the application of Young's inequality leads to

$$
\|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1^2 + (\varepsilon - ch) \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right\|_S^2 \le c\|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1^2 + c\|p_\varepsilon - q^h\|^2
$$

which is re-written as follows

$$
\|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon^h\|_1^2 + \varepsilon \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right\|_S^2 \le ch \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right\|_S^2 + c\|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1^2 + c\|p_\varepsilon - q^h\|^2 .
$$

We take the square root on both sides, use triangle's inequality and (4.11) and one obtains the desired result. □

Remark 4.2 *If $\varepsilon(h) = (c+1)^2 h$, then one has*

$$
\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 + \|p_\varepsilon - p_\varepsilon^h\| + ch^{1/2} \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right\|_S \le c\|\boldsymbol{v}^h - \boldsymbol{u}_\varepsilon\|_1 + c\|p_\varepsilon - q^h\| .
\tag{4.19}
$$

To estimate the right hand side of (4.19), we consider the approximation operators constructed in Girault and Hecht [33], Chapter 5. These are $\Pi_h \in \mathcal{L}(\mathbb{V};\mathbb{V}_h)$ and $r_h \in \mathcal{L}(M;M_h)$ which satisfy; for each real number $\alpha \in [0,1]$

(1) There exists a constant c, independent of h such that

$$
\text{for all } \boldsymbol{v} \in W^{\alpha+1,2}(\Omega)^2 \cap \mathbb{V} \,, \quad \|\nabla(\Pi_h \boldsymbol{v} - \boldsymbol{v})\| \le ch^\alpha \, |\boldsymbol{v}|_{W^{\alpha+1,2}(\Omega)} \,.
\tag{4.20}
$$

(2) There exists a constant c, independent of h such that

$$
\text{for all } q \in W^{\alpha,2}(\Omega)^2 \cap M \,, \quad \|r_h q - q\| \le ch^\alpha \, |q|_{W^{\alpha,2}(\Omega)} \,.
\tag{4.21}
$$

Thus (4.19) together with (4.20) and (4.21) gives

$$
\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 + \|p_\varepsilon - p_\varepsilon^h\| + ch^{1/2} \left\| \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon) - \frac{1}{\varepsilon}\nu_{\varepsilon g}(\boldsymbol{u}_\varepsilon^h)\right\|_S \le ch^\alpha \left(|\boldsymbol{u}_\varepsilon|_{W^{\alpha+1,2}(\Omega)} + |p_\varepsilon|_{W^{\alpha,2}(\Omega)} \right) .
$$

Thus one obtains linear convergence if $\alpha = 1$.

Remark 4.3 *Having in mind proposition 4.1, proposition 3.2 and theorem 4.1, we have for ε given as before and $\alpha = 1$,*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_1 + \|p - p^h\| \leq ch \ .$$

Remark 4.4 *From proposition 4.1, proposition 3.2 and theorem 4.1, it is clear that one only need the interpolation estimate on the entire domain and an appropriate choice of ϵ. We also note that the regularity of the solution on the slip/dissipation zone is not needed at all. Hence this result differ from the ones derived previously (see [9, 10, 20, 22, 23] just to cited a few).*

Remark 4.5 *In this text,we do not dealt with domains with curved boundary, and the readers interested in that direction may consult the work of Ibrahima Dione [28].*

5 Numerical algorithm

This section is devoted to the formulation of the numerical scheme for the Stokes equations under Tresca's boundary condition. We recall that because we are interested in rate of convergence of the finite element solution, and in the previous section it is observed that $\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon^h\|_1 + \|p_\varepsilon - p_\varepsilon^h\|$ and $\|\boldsymbol{u} - \boldsymbol{u}^h\|_1 + \|p - p^h\|$ have the same convergence rate for $\varepsilon = ch$, we will compute the solution for $\varepsilon = ch$ of the truncated finite element problem

$$\begin{cases} \text{for all } (\boldsymbol{v}^h, q^h) \in \mathbb{V}_h \times M_h \ , \\ a(\boldsymbol{u}_\varepsilon^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_\varepsilon^h) + \dfrac{1}{\varepsilon}\displaystyle\int_S \nu_{\varepsilon g}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}^h) \cdot \boldsymbol{v}_{\boldsymbol{\tau}}^h d\sigma = \ell(\boldsymbol{v}^h) \ , \\ b(\boldsymbol{u}_\varepsilon^h, q^h) = 0 \ . \end{cases} \tag{5.1}$$

(5.1) is a nonlinear system of equations, hence for its resolution an incremental or iterative method is needed. In the lines that follows, we propose a Newton type algorithm. The Gateaux-derivative of the projection (cut off) function

$$\nu_k(\boldsymbol{x}) = \begin{cases} \boldsymbol{x} & \text{for } |\boldsymbol{x}| < k, \\ k\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|} & \text{for } |\boldsymbol{x}| \geq k. \end{cases}$$

is given by

$$\nu_k'(\boldsymbol{x})\boldsymbol{y} = \begin{cases} \boldsymbol{y} & \text{for } |\boldsymbol{x}| < k, \\ k\left(\frac{\boldsymbol{y}}{|\boldsymbol{x}|} - \frac{(\boldsymbol{x}\cdot\boldsymbol{y})\boldsymbol{x}}{|\boldsymbol{x}|^3}\right) & \text{for } |\boldsymbol{x}| \geq k. \end{cases}$$

We then propose the following method for solving the nonlinear equation (5.1), described in Algorithm 1.

Remark 5.1 *It should be noted that $(\tilde{\boldsymbol{u}}_\varepsilon^k, \tilde{p}_\varepsilon^k) \in \mathbb{V}_h \times M_h$ given in (5.2) and (5.3) is well defined by a direct application of Babuska-Brezzi's theorem for mixed formulations. Thus, knowing $(\boldsymbol{u}_\varepsilon^k, p_\varepsilon^k)$ one can always compute $(\boldsymbol{u}_\varepsilon^{k+1}, p_\varepsilon^{k+1})$. The convergence analysis of the sequence $(\boldsymbol{u}_\varepsilon^k, p_\varepsilon^k)_k$ may be done following similar analysis in [31, 36].*

Algorithm 1 Newton-Raphson algorithm for (5.1)

$k = 0$. $(\boldsymbol{u}^0_\varepsilon, p^0_\varepsilon) \in \mathbb{V}_h \times M_h$

$k \geq 0$. Assuming $(\boldsymbol{u}^k_\varepsilon, p^k_\varepsilon)$ is known, compute $(\boldsymbol{u}^{k+1}_\varepsilon, p^{k+1}_\varepsilon)$ as follows

1. Compute $(\tilde{\boldsymbol{u}}^k_\varepsilon, \tilde{p}^k_\varepsilon) \in \mathbb{V}_h \times M_h$ such that

$$\begin{aligned} a(\tilde{\boldsymbol{u}}^k_\varepsilon, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, \tilde{p}^k_\varepsilon) + \frac{1}{\varepsilon}\int_S \nu'_{\varepsilon g}(\boldsymbol{u}^k_{\varepsilon\boldsymbol{\tau}})\tilde{\boldsymbol{u}}^k_{\varepsilon\boldsymbol{\tau}} \cdot \boldsymbol{v}^h_{\boldsymbol{\tau}} d\sigma &= \ell(\boldsymbol{v}^h) - \tilde{\ell}(\boldsymbol{v}^h), \quad (5.2) \\ b(\tilde{\boldsymbol{u}}^k_\varepsilon, q^h) &= -b(\boldsymbol{u}^k_\varepsilon, q^h), \quad (5.3) \end{aligned}$$

   where
$$\tilde{\ell}(\boldsymbol{v}^h) = a(\boldsymbol{u}^k_\varepsilon, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^k_\varepsilon) + \frac{1}{\varepsilon}\int_S \nu_{\varepsilon g}(\boldsymbol{u}^k_{\varepsilon\boldsymbol{\tau}}) \cdot \boldsymbol{v}^h_{\boldsymbol{\tau}} d\sigma.$$

2. $\boldsymbol{u}^{k+1}_\varepsilon = \boldsymbol{u}^k_\varepsilon + \tilde{\boldsymbol{u}}^k_\varepsilon$, $p^{k+1}_\varepsilon = p^k_\varepsilon + \tilde{p}^k_\varepsilon$

6 Numerical experiments

We now study the numerical behavior of the Newton-Raphson algorithm described in the previous section. We have implemented Algorithm 1, using vectorized assembling functions and the mesh generator provided in [37–39], on a computer running Linux (Ubuntu 16.04) with 3.00GHz clock frequency and 32GB RAM. We use some classical tests problems to evaluate the behavior of Algorithm 1 with the stoppting criterion

$$\| \tilde{\boldsymbol{u}}^k_\varepsilon \|^2_{L^2(\Omega)} + \| \tilde{p}^k_\varepsilon \|^2_{L^2(\Omega)} < 10^{-10} \left(\| \boldsymbol{u}^k_\varepsilon \|^2_{L^2(\Omega)} + \| p^k_\varepsilon \|^2_{L^2(\Omega)} \right).$$

6.1 2D Driven cavity

This problem has been considered by many authors [8–10,13,16,18–20]. We set $\Omega = (0,1)^2$ and we assume that its boundary consists of two portions Γ_D and S defined as follows

$$\begin{aligned} \Gamma_D &= \{0\} \times (0,\ 1) \cup (0,\ 1) \times \{0\} \\ S &= S_1 \cup S_2, \quad S_1 = (0,\ 1) \times \{1\}, \quad S_2 = \{1\} \times (0,\ 1). \end{aligned}$$

The right-hand side is
$$\boldsymbol{f} = -2\mu \operatorname{div} D\boldsymbol{u} + \nabla p$$

where $(\boldsymbol{u}, p)$ is given by

$$\begin{aligned} u_1(x,y) &= -x^2 y(x-1)(3y-2), \quad (6.1) \\ u_2(x,y) &= xy^2(y-1)(3x-2), \quad (6.2) \\ p(x,y) &= (2x-1)(2y-1). \quad (6.3) \end{aligned}$$

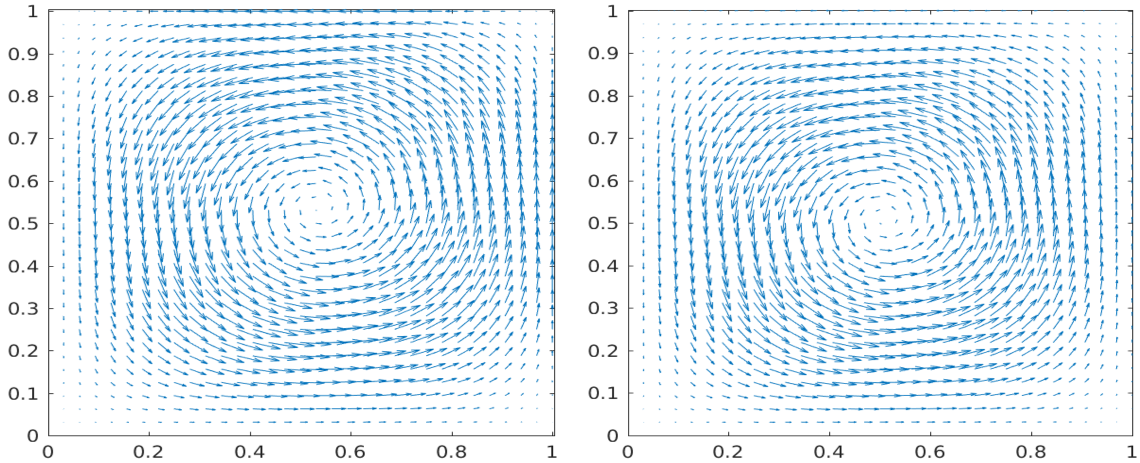
In [19], it is shown that

$$
\begin{aligned}
(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} &= -4\mu x^2(x-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ on } S_1 \\
(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} &= -4\mu y^2(y-1)\begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ on } S_2
\end{aligned}
$$

For $\mu = 1$, a straightforward calculation reveals that

$$\max_S |(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}| = 4\mu \max_{x\in S_1} x^2(x-1) = 4\mu \max_{x\in S_2} y^2(y-1) = 0.59\,.$$

Then for an appropriate choice of g, both, slip and stick zones can appear on S. We first run our code with $h = 1/32$, $\varepsilon = h$ and two values of g. Figures 1-2 show the velocity fields and the streamlines obtained, using Algorithm 1. We can notice that for $g = 0.25 < \max_S|(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}|$ a non-trivial slip occurs, while for $g = 1 > \max_S|(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}|$ the solution is such that $\boldsymbol{u}_{\boldsymbol{\tau}} = 0$ on S.



(a) (b)

Figure 1: Velocity fields for the driven cavity problem (a): $g = 0.25$, (b): $g = 1$

We evaluate the accuracy and the behavior of Algorithm 1 by calculating the error between approximate solution and the exact solution. Since we do not know the exact solution explicitly, we use an approximate solution on a finer mesh of size $h = 1/124$ as the reference solution. The convergence errors are computed as follows

$$
\begin{aligned}
e_1 := e_h(\boldsymbol{u}) &= \|\, \boldsymbol{u}_h - \boldsymbol{u}_* \,\|_{L^2} \\
e_2 := e_h(\boldsymbol{u}, p) &= \|\, \boldsymbol{u}_h - \boldsymbol{u}_* \,\|_{H^1} + \|\, p_h - p_* \,\|_{L^2}
\end{aligned}
$$

where $(\boldsymbol{u}_*, p_*)$ is the reference solution. Note that Algorithm 1 always converges for $\varepsilon = h$. For $\varepsilon << h$, a suitable initialization is needed for the friction case since large values of

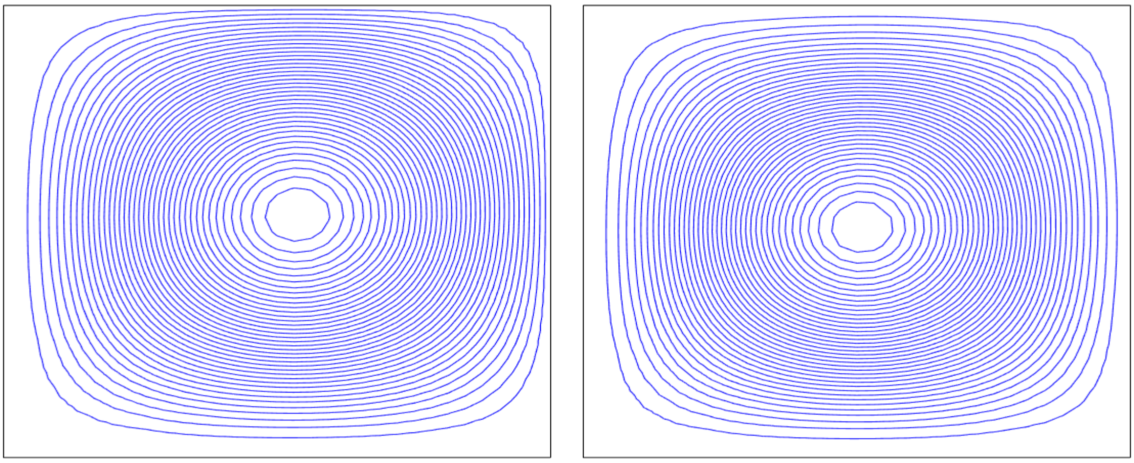


Figure 2: Streamlines for the driven cavity problem (a): $g = 0.25$, (b) : $g = 1$

$1/\varepsilon$ imply $\boldsymbol{u_\tau} \approx 0$. In our code to compute the solution for $\varepsilon = h/10$, we start with $\varepsilon = h$ and after convergence we adjust ε to compute the adequate solution.

We report in Table 1-2 the number of iterations, the CPU times (in Seconds) and the convergence rates for $g = 0.25$ and $g = 1$. For the stick case ($g = 0.25$) the best convergence rates are obtained with $\varepsilon = h/10$. For $\varepsilon = h$ the convergence rates are less good than expected but the convergence of the Newton-Raphson algorithm is very fast. For the stick case ($g = 1$) the convergence rates are less good than expected (for the standard Stokes equation) for both choices of ε. This can be explained by the fact that the penalization term $1/\varepsilon$ in (5.1) is not large enough to enforce $\boldsymbol{u}^k_{\boldsymbol{\tau}} \approx 0$.

	$\varepsilon = h$				$\varepsilon = h/10$			
h	Iter.	CPU	Rate e_1	Rate e_2	Iter.	CPU	Rate e_1	Rate e_2
1/16	7	0.05			22	0.13		
1/32	8	0.18	1.98	1.20	17	0.38	1.96	1.32
1/64	8	0.95	1.78	1.16	17	2.02	2.03	1.31
1/128	8	6.01	1.37	1.15	21	15.54	2.04	1.29
1/256	10	49.99	1.23	1.21	23	113.56	2.10	1.34

Table 1: Performances and accuracy of Algorithm 1 on the 2D driven cavity for $g = 0.25$

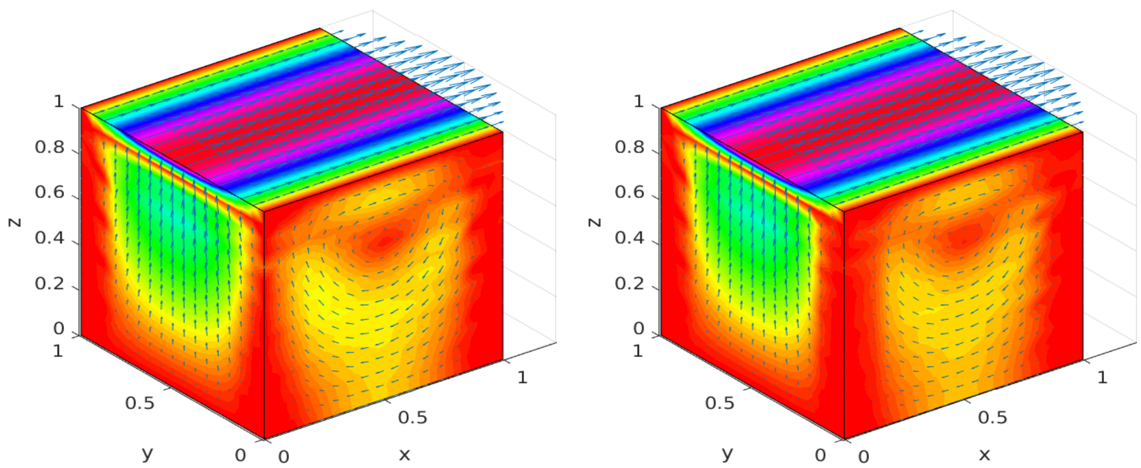
6.2 3D lid-driven cavity

This problem has already been considered in [18] for Stokes and Navier Stokes. The geometry consists of a cubic cavity $\Omega = (0,1)^3$ with a moving wall at $\Gamma = \{z = 1\}$, and $\boldsymbol{u}|_\Gamma = (4x_2(1-x_2), 0, 0)$. The remaining part of the boundary is $S = \partial\Omega\backslash\Gamma$ where the slip takes place and we take $\mu = 1$. Figure 3 shows the velocity fields and the magnitude of its tangential component $\| \boldsymbol{u_\tau} \|$ at the walls for $g = 0.25$ and $g = 2$. In [18], numerical

h	Iter.	CPU	Rate e_1	Rate e_2	Iter.	CPU	Rate e_1	Rate e_2
	$\varepsilon = h$				$\varepsilon = h/10$			
1/16	2	0.02			3	0.02		
1/32	2	0.03	0.76	0.90	3	0.06	1.34	1.43
1/64	2	0.25	0.92	0.96	3	0.35	0.94	1.34
1/128	2	1.5	1.03	1.05	3	2.22	0.99	1.28
1/256	2	9.90	1.19	1.19	3	14.81	1.16	1.30

Table 2: Performances and accuracy of Algorithm 1 on the 2D driven cavity for $g = 1$

experiments show that, for this 3D lid-driven cavity, the friction always occurs. Table 3 summarizes the performances of Algorithm 1 in terms of the number of iterations and the CPU times (in Seconds), with various values of g. We can notice that the proposed Newton-Raphson algorithm is virtually independent of the mesh size, for a fixed parameter ε. Moreover, the convergence of the Newton-Raphson algorithm is fast.



(a) (b)

Figure 3: Velocity field and $\| \boldsymbol{u_\tau} \|$ for the 3D lid-driven cavity (a): $g = 0.25$, (b): $g = 2$

6.3 Concluding remarks

We have studied, theoretically and numerically a new approach based on a cut off function for the numerical approximation of the Stokes equation under Tresca friction boundary condition. Numerical experiments have shown that optimal rates of convergence can be obtained for the friction case. For the 3D lid-driven cavity studied, the proposed method proved to be fast since the friction always occurs. Then, the proposed algorithm can be useful for pure friction problem in 2D or 3D. However, with the proposed approach, the tangential stress must be computed at the end of the algorithm while for the Lagrange multiplier based approach (e.g., [18, 19]), the tangential stress is available as a Lagrange multiplier.

	$g = 0.25$		$g = 1$		$g = 2$	
	$\varepsilon = h$	$\varepsilon = h/10$	$\varepsilon = h$	$\varepsilon = h/10$	$\varepsilon = h$	$\varepsilon = h/10$
h	Iter. / CPU	Iter. / CPU	Iter. / CPU	Iter. / CPU	Iter. / CPU	Iter. / CPU
1/4	4 / 0.07	6 / 0.08	2 / 0.02	5 / 0.04	2 / 0.02	4 / 0.03
1/8	5 / 0.45	6 / 0.52	4 / 0.32	5 / 0.42	3 / 0.24	5 / 0.42
1/16	5 / 13.44	7 / 18.98	4/ 10.75	5 / 13.55	3 / 8.06	5 / 13.51
1/32	6 / 921.05	7 / 1078.2	4 / 616.79	5 / 779.80	3 / 462.35	5 / 769.50

Table 3: Performances a of Algorithm 1 on the 3D lid-driven cavity, for various values of g.

Conflicts of interests. We declare no conflicts of interests

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