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Abstract

We consider a viscous incompressible fluid interacting with an elastic structure located on a part of its boundary. The fluid motion is modeled by the bi-dimensional Navier-Stokes system and the structure follows the linear wave equation in dimension 1 in space. Our aim is to study the linearized system coupling the Stokes system with a wave equation and to show that the corresponding semigroup is analytic. In particular the linear system satisfies a maximal regularity property that allows us to deduce the existence and uniqueness of strong solutions for the nonlinear system. This result can be compared to the case where the elastic structure is a beam equation for which the corresponding semigroup is only of Gevrey class.

Keywords: fluid-structure, Navier-Stokes system
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Contents

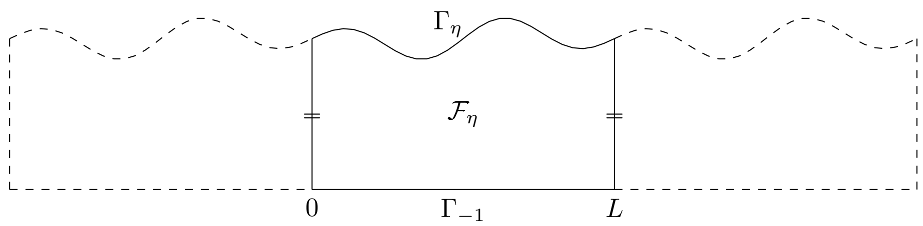

Figure 1: Our geometry

1 Introduction

We consider a viscous incompressible fluid modeled by the Navier-Stokes system in interaction with a deformable boundary whose dynamics is governed by the wave equation. More precisely, we consider that the reference spatial domain for the fluid is $(0,L)\times(0,1)\subset\mathbb{R}^2$, $L>0$ with periodic boundary conditions on the lateral boundaries $\{0\}\times(0,1)$ and $\{L\}\times(0,1)$. To simplify the notation, we set

$$\mathcal{I}\stackrel{\text{def}}{=}\mathbb{R}/L\mathbb{Z}, \tag{1.1}$$

so that the reference fluid domain writes $\mathcal{F}_0\stackrel{\text{def}}{=}\mathcal{I}\times(0,1)$. The elastic deformation is a function $\eta:\mathcal{I}\to(-1,\infty)$, and the corresponding fluid domain writes as follows:

$$\mathcal{F}_\eta\stackrel{\text{def}}{=}\{(x_1,x_2)\in\mathcal{I}\times\mathbb{R}\ ;\ x_2\in(0,1+\eta(x_1))\}\,. \tag{1.2}$$

Note that the boundary of $\mathcal{F}_\eta$ is the disjoint union of the elastic structure:

$$\Gamma_\eta\stackrel{\text{def}}{=}\{(s,1+\eta(s)),\ s\in\mathcal{I}\}\,,$$

and a fixed bottom:

$$\Gamma_{-1}\stackrel{\text{def}}{=}\mathcal{I}\times\{0\}.$$

We recall the geometry in Figure 1.

We denote by v and p the velocity and the pressure of the fluid and our fluid-structure interaction system writes as follows

$$\left\{\begin{array}{c}\partial_t v+(v\cdot\nabla)v-\operatorname{div}\mathbb{T}(v,p)=0,\quad t>0,\ x\in\mathcal{F}_{\eta(t)},\\ \operatorname{div}v=0,\quad t>0,\ x\in\mathcal{F}_{\eta(t)},\\ v(t,s,1+\eta(t,s))=(\partial_t\eta)(t,s)e_2,\quad t>0,\ s\in\mathcal{I},\\ v=0,\quad t>0,\ x\in\Gamma_{-1},\\ \partial_{tt}\eta-\tau\partial_{ss}\eta=-\widetilde{\mathbb{H}}_\eta(v,p),\quad t>0,\ s\in\mathcal{I},\end{array}\right. \tag{1.3}$$

with the initial conditions

$$\eta(0,\cdot)=\eta_1^0,\quad \partial_t\eta(0,\cdot)=\eta_2^0\quad\text{and}\quad v(0,\cdot)=v^0\text{ in }\mathcal{F}_{\eta_1^0}. \tag{1.4}$$

In system (1.3), the first two equations are the Navier-Stokes system, the last one is the wave equation. The other equations are the boundary conditions obtained by assuming the no-slip conditions for the fluid.

For the notation, we have denoted by(e_1, e_2) the canonical basis of $\mathbb{R}^2$, the Cauchy stress for the fluid is

$$\mathbb{T}(v,p) \stackrel{\text{def}}{=} 2\nu\mathbb{D}(v) - pI_2, \quad \mathbb{D}(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^*\right), \tag{1.5}$$

and the force of the fluid acting on the structure is

$$\widetilde{\mathbb{H}}_\eta(v,p)(t,s) \stackrel{\text{def}}{=} \left\{(1+|\partial_s\eta|^2)^{1/2}\left[\mathbb{T}(v,p)n\right](t,s,1+\eta(t,s))\cdot e_2\right\}. \tag{1.6}$$

To simplify, we have assumed that the density of the fluid is constant and equal to 1. For the other physical constants, we suppose that

$$\nu > 0 \text{ (viscosity)}, \quad \tau > 0.$$

In (1.6), the vector fields n is the unit exterior normal to $\mathcal{F}_{\eta(t)}$. We have $n = -e_2$ on Γ_{-1} and on $\Gamma_{\eta(t)}$,

$$n(t,x_1,x_2) = \frac{1}{\sqrt{1+|\partial_s\eta(t,x_1)|^2}}\begin{bmatrix} -\partial_s\eta(t,x_1) \\ 1 \end{bmatrix}. \tag{1.7}$$

Similar systems as (1.3) have been already analyzed in the literature (see below), and an important feature of these systems is that the incompressibility of the fluid and the no-slip boundary conditions yield the following relation

$$\frac{d}{dt}\int_0^L \eta(t,s)\ ds = 0.$$

In what follows, we choose the initial deformation η_1^0 with null mean value so that our solutions satisfy

$$\int_0^L \eta(t,s)\ ds = 0 \quad (t \geqslant 0). \tag{1.8}$$

This leads us to consider the spaces

$$L^2_\#(\mathcal{I}) \stackrel{\text{def}}{=} \left\{f \in L^2(\mathcal{I})\ ;\ \int_0^L f(s)\ ds = 0\right\}, \quad H^r_\#(\mathcal{I}) \stackrel{\text{def}}{=} H^r(\mathcal{I}) \cap L^2_\#(\mathcal{I}) \quad (r \geqslant 0) \tag{1.9}$$

and the orthogonal projection $M : L^2(\mathcal{I}) \to L^2_\#(\mathcal{I})$.

We take the projection of the last equation of (1.3) on $L^2_\#(\mathcal{I})$ and on $L^2_\#(\mathcal{I})^\perp$. The projection on $L^2_\#(\mathcal{I})$ writes

$$\partial_{tt}\eta + A_1\eta = -\mathbb{H}_\eta(v,p), \quad t > 0,\ s \in \mathcal{I}, \tag{1.10}$$

where

$$\mathcal{H}_\mathcal{S} \stackrel{\text{def}}{=} L^2_\#(\mathcal{I}), \quad \mathcal{D}(A_1) \stackrel{\text{def}}{=} H^2_\#(\mathcal{I}), \quad A_1 : \mathcal{D}(A_1) \to \mathcal{H}_\mathcal{S}, \quad \eta \mapsto -\tau\partial_{ss}\eta, \tag{1.11}$$

and

$$\mathbb{H}_\eta(v,p) \stackrel{\text{def}}{=} M\widetilde{\mathbb{H}}_\eta(v,p). \tag{1.12}$$

The projection on $L^2_\#(\mathcal{I})^\perp$ writes

$$\int_0^L \left\{(1+|\partial_s\eta|^2)^{1/2}\left[\mathbb{T}(v,p)n\right](t,s,1+\eta(t,s))\cdot e_2\right\}\ ds = 0 \tag{1.13}$$

and it determines the constant for the pressure: for system (1.3), the pressure is not determined up to a constant (as for the Navier-Stokes system without structure) (see [3] for more details). In what follows, we do not consider (1.13) and work only with (1.10), so that for our solutions, the pressure is determined up to a constant.

Note that the operator A_1 defined by (1.11) satisfies for $\theta \geqslant 0$,

$$\mathcal{D}(A_1^\theta) = H^{2\theta}_\#(\mathcal{I}). \tag{1.14}$$

In all what follows, we consider the following notations: L^α, H^k stand for the classical Lebesgue and Sobolev spaces and we write C^0 for the space of continuous maps and C^0_b for the space of continuous and bounded maps. We use the bold notation for the spaces of vector fields: $\mathbf{L}^\alpha = (L^\alpha)^2$, $\mathbf{H}^k = (H^k)^2$ etc. For $\eta : \mathcal{I} \to (-1, \infty)$, we also use the spaces

$$L^2_\#(\mathcal{F}_\eta) = \left\{ h \in L^2(\mathcal{F}_\eta) \mid \int_{\mathcal{F}_\eta} h dy = 0 \right\}, \quad H^1_\#(\mathcal{F}_\eta) = H^1(\mathcal{F}_\eta) \cap L^2_\#(\mathcal{F}_\eta),$$

and

$$H^{-1}_\#(\mathcal{F}_\eta) = \left\{ h \in [H^1(\mathcal{F}_\eta)]' \mid \langle h, 1 \rangle_{[H^1(\mathcal{F}_\eta)]', H^1(\mathcal{F}_\eta)} = 0 \right\}.$$

Observe that $H^{-1}_\#(\mathcal{F}_\eta)$ is the dual space of $H^1_\#(\mathcal{F}_\eta)$ with respect to the pivot space $L^2_\#(\mathcal{F}_\eta)$. To study (1.3), we need spaces associated with a moving fluid domain and we introduce spaces of the form $H^1(0, T; L^q(\mathcal{F}_\eta))$, $L^2(0, T; H^k(\mathcal{F}_\eta))$, etc. with $T \leqslant \infty$. If $\eta(t, \cdot) > -1$ $(t \in (0, T))$, then

$$v \in H^1(0, T; L^q(\mathcal{F}_\eta)) \quad \text{if} \quad y \mapsto v(t, y_1, y_2(1 + \eta(t, y_1)) \in H^1(0, T; L^q(\mathcal{F}_0))$$

and similarly, for the other spaces. Finally, in the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another.

In order to study the system (1.3), one can linearize it and we obtain the following linear system:

$$\begin{cases} \partial_t w - \operatorname{div} \mathbb{T}(w, q) = F, \quad t > 0, \ y \in \mathcal{F}_{\eta^*}, \\ \operatorname{div} w = G, \quad t > 0, \ y \in \mathcal{F}_{\eta^*}, \\ w(t, s, 1 + \eta^*(s)) = (\partial_t \eta)(t, s) e_2, \quad t > 0, \ s \in \mathcal{I}, \\ w = 0, \quad t > 0, \ y \in \Gamma_{-1}, \\ \partial_{tt} \eta + A_1 \eta = -\mathbb{H}_{\eta^*}(w, q) + H, \quad t > 0, \ s \in \mathcal{I}, \end{cases} \tag{1.15}$$

with the initial conditions

$$w(0, \cdot) = w^0 \ \text{ in } \ \mathcal{F}_{\eta^*}, \quad \eta(0, \cdot) = \zeta^0_1, \quad \partial_t \eta(0, \cdot) = \zeta^0_2, \tag{1.16}$$

and where $\eta^* : \mathcal{I} \to (-1, \infty)$ is a given function independent of time, so that $\mathcal{F}_{\eta^*}$ is a fixed spatial domain. The aim of this article is to show that the linear system (1.15)-(1.16) satisfies a maximal regularity property. For $T \in (0, +\infty]$ let us define the following Banach spaces

$$\mathfrak{F}_T(\mathcal{F}_{\eta^*}) \stackrel{\text{def}}{=} L^2(0, T; \mathbf{L}^2(\mathcal{F}_{\eta^*})) \times \left[L^2(0, T; H^1_\#(\mathcal{F}_{\eta^*})) \cap H^1(0, T; H^{-1}_\#(\mathcal{F}_{\eta^*})) \right] \times L^2(0, T; H^{1/2}_\#(\mathcal{I})), \tag{1.17}$$

and

$$\begin{aligned} \mathfrak{W}_T(\mathcal{F}_{\eta^*}) \stackrel{\text{def}}{=} & \left[L^2(0, T; \mathbf{H}^2(\mathcal{F}_{\eta^*})) \cap H^1(0, T; \mathbf{L}^2(\mathcal{F}_{\eta^*})) \right] \times L^2(0, T; H^1(\mathcal{F}_{\eta^*})/\mathbb{R}) \\ & \times \left[L^2(0, T; H^{5/2}_\#(\mathcal{I})) \cap H^1(0, T; H^{3/2}_\#(\mathcal{I})) \cap H^2(0, T; H^{1/2}_\#(\mathcal{I})) \right] \end{aligned} \tag{1.18}$$

endowed with the norms

$$\|(F, G, H)\|_{\mathfrak{F}_T(\mathcal{F}_{\eta^*})} \stackrel{\text{def}}{=} \|F\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}_{\eta^*}))} + \|G\|_{L^2(0,T;H^1_\#(\mathcal{F}_{\eta^*}))} + \|G\|_{H^1(0,T;H^1_\#(\mathcal{F}_{\eta^*})')} + \|H\|_{L^2(0,T;H^{1/2}_\#(\mathcal{I}))}$$

and

$$\begin{aligned} \|(w, q, \eta)\|_{\mathfrak{W}_T(\mathcal{F}_{\eta^*})} \stackrel{\text{def}}{=} & \|w\|_{L^2(0,T;\mathbf{H}^2(\mathcal{F}_{\eta^*}))} + \|w\|_{H^1(0,T;\mathbf{L}^2(\mathcal{F}_{\eta^*}))} + \|q\|_{L^2(0,T;H^1(\mathcal{F}_{\eta^*})/\mathbb{R})} \\ & + \|\eta\|_{L^2(0,T;H^{5/2}_\#(\mathcal{I}))} + \|\eta\|_{H^1(0,T;H^{3/2}_\#(\mathcal{I}))} + \|\eta\|_{H^2(0,T;H^{1/2}_\#(\mathcal{I}))}. \end{aligned}$$

Note that from Proposition 4.3, p.159 in [6], we have that the following map is well-defined and onto:

$$G \in L^2(0, \infty; H^1_\#(\mathcal{F}_{\eta^*})) \cap H^1(0, \infty; H^{-1}_\#(\mathcal{F}_{\eta^*})) \longmapsto G(0, \cdot) \in L^2_\#(\mathcal{F}_{\eta^*}). \tag{1.19}$$

Theorem 1.1. *Assume*

$$\eta^* \in H^2_\#(\mathcal{I}), \quad \eta^* > -1 \quad \textit{in } \mathcal{I}, \tag{1.20}$$

$$(F, G, H) \in \mathfrak{F}_\infty(\mathcal{F}_{\eta^*}), \tag{1.21}$$

$$\zeta_1^0 \in H^2_\#(\mathcal{I}), \quad \zeta_2^0 \in H^1_\#(\mathcal{I}), \quad w^0 \in \mathbf{H}^1(\mathcal{F}_{\eta^*}), \tag{1.22}$$

$$\operatorname{div} w^0 = G(0,\cdot) \textit{ in } \mathcal{F}_{\eta^*}, \quad w^0(s, 1+\eta^*(s)) = \zeta_2^0(s) e_2 \quad s \in \mathcal{I}, \quad w^0 = 0 \quad \textit{on } \Gamma_{-1}. \tag{1.23}$$

Then (1.15)-(1.16) *admits a unique solution*

$$(w, q, \eta) \in \mathfrak{W}_\infty(\mathcal{F}_{\eta^*}) \tag{1.24}$$

Moreover, there exists $C_L > 0$*, depending on* $\|\eta^*\|_{H^2(\mathcal{I})}$ *and on* $\inf_{\mathcal{I}} \eta^*$ *such that*

$$\|(w,q,\eta)\|_{\mathfrak{W}_\infty(\mathcal{F}_{\eta^*})} \leqslant C_L \left(\|w^0\|_{\mathbf{H}^1(\mathcal{F}_{\eta^*})} + \|\zeta_1^0\|_{H^2_\#(\mathcal{I})} + \|\zeta_2^0\|_{H^1_\#(\mathcal{I})} + \|(F,G,H)\|_{\mathfrak{F}_\infty(\mathcal{F}_{\eta^*})} \right). \tag{1.25}$$

For η^* satisfying (1.20), let denote by $\widehat{C}_L(\eta^*)$ the infimum of the constants C_L in the above theorem, that is the norm of the bounded map

$$(F,G,H,w^0,\zeta_1^0,\zeta_2^0) \in \mathfrak{F}_\infty(\mathcal{F}_{\eta^*}) \times \mathbf{H}^1(\mathcal{F}_{\eta^*}) \times H^2_\#(\mathcal{I}) \times H^1_\#(\mathcal{I}) \longmapsto (w,q,\eta) \in \mathfrak{W}_\infty(\mathcal{F}_{\eta^*})$$

where $(F,G,H,w^0,\zeta_1^0,\zeta_2^0)$ satisfies (1.23) and (w,q,η) is the corresponding solution of (1.15)-(1.16) given by Theorem 1.1. In what follows, we denote for $R > 0$

$$\mathcal{B}_R = \left\{ \eta^* \in H^2_\#(\mathcal{I}) \ ; \ \|\eta^*\|_{H^2_\#(\mathcal{I})} \leqslant R \quad \text{and} \quad \inf_{\mathcal{I}} \eta^* + 1 \geqslant 1/R \right\}. \tag{1.26}$$

Then we can show the following result:

Proposition 1.2. *For any* $R > 0$*, we have*

$$\sup_{\eta^* \in \mathcal{B}_R} \widehat{C}_L(\eta^*) < +\infty.$$

Using some standard extension results, we deduce from Proposition 1.2 the following result

Corollary 1.3. *Assume* $R > 0$*. Then there exists a constant* $C = C(R)$ *such that for any*

$$T \in (0,\infty], \quad \eta^* \in \mathcal{B}_R, \quad (F,G,H) \in \mathfrak{F}_T(\mathcal{F}_{\eta^*}), \quad (w^0,\zeta_1^0,\zeta_2^0) \in \times \mathbf{H}^1(\mathcal{F}_{\eta^*}) \times H^2_\#(\mathcal{I}) \times H^1_\#(\mathcal{I})$$

satisfying (1.23) *there exists a unique solution* $(w,q,\eta) \in \mathfrak{W}_T(\mathcal{F}_{\eta^*})$ *of* (1.15)-(1.16) *and we have the estimate*

$$\|(w,q,\eta)\|_{\mathfrak{W}_T(\mathcal{F}_{\eta^*})} \leqslant C \left(\|w^0\|_{\mathbf{H}^1(\mathcal{F}_{\eta^*})} + \|\zeta_1^0\|_{H^2_\#(\mathcal{I})} + \|\zeta_2^0\|_{H^1_\#(\mathcal{I})} + \|(F,G,H)\|_{\mathfrak{F}_T(\mathcal{F}_{\eta^*})} \right). \tag{1.27}$$

This result will be used for the study of the nonlinear system.

Let us now give some remarks on Theorem 1.1.

Remark 1.4. *The maximal regularity property stated in Theorem 1.1 is obtained by showing that the corresponding semigroup is analytic. More precisely, we define the corresponding operator* A_0 *in* (2.30), (2.31), *and* (2.32), *and we show in Theorem 2.5 that it is the infinitesimal generator of an analytic and exponentially stable semigroup. This result is suggested by [13] and in particular by Proposition 3.2 in this reference. However their proof is based on a regularization argument and on a priori estimates and we propose here a direct proof following the strategy already used in [3] and [4] consisting in showing a resolvent estimate, that is* (2.34)*. In particular, this work focuses on the resolvent equation*

$$\left\{ \begin{array}{c} \lambda v - \operatorname{div} \mathbb{T}(v,p) = f \quad \textit{in } \mathcal{F}_{\eta^*}, \\ \operatorname{div} v = 0 \quad \textit{in } \mathcal{F}_{\eta^*}, \\ v(s, 1+\eta^*(s)) = \eta_2(s) e_2, \quad s \in \mathcal{I}, \\ v = 0, \quad y \in \Gamma_{-1}, \\ \lambda \eta_1 - \eta_2 = g \\ \lambda \eta_2 + A_1 \eta_1 = -\mathbb{H}_{\eta^*}(v,p) + h. \end{array} \right. \tag{1.28}$$

Note that (1.15) *is coupling the Stokes system which is parabolic and the wave equation which is an hyperbolic system. It is thus not clear that the system* (1.15) *is a parabolic system. In particular, in [3] and [4], we consider the same problem with the beam equation instead of the wave equation and the corresponding semigroup is not analytic but only of Gevrey class. One of the key points relies on the analysis of the effect* *of the viscosity of the fluid on the structure equation. More precisely, if one take $f = 0$ and $g = 0$ in* (1.28)*, the resolvent equation reduces to*

$$V_\lambda \eta_1 = h, \quad \textit{where } V_\lambda = \lambda^2 I + \lambda L_\lambda + A_1$$

where L_λ corresponds to the influence of the fluid on the structure (see (3.18) *and* (3.7) *for the precise definition of V_λ and L_λ). Then the idea is to show that the operator L_λ plays the role of a structural damping (see [9]) and leads to a parabolic system.*

Theorem 1.1 is the main ingredient to prove the existence and uniqueness of strong solutions for the nonlinear system (1.3). First let us recall the definition of this notion:

Definition 1.5. *For $T \in (0, \infty)$ we say that (v, p, η) is a strong solution of* (1.3)-(1.4) *on $[0, T]$ if*

$$\eta(t, \cdot) > -1 \quad t \in [0, T], \tag{1.29}$$

$$v \in L^2(0, T; \mathbf{H}^2(\mathcal{F}_\eta)) \cap H^1(0, T; \mathbf{L}^2(\mathcal{F}_\eta)), \quad p \in L^2(0, T; H^1(\mathcal{F}_\eta)), \tag{1.30}$$

$$\begin{gathered} \eta \in L^2(0, T; H^{5/2}_\#(\mathcal{I})) \cap H^1(0, T; H^{3/2}_\#(\mathcal{I})), \\ \partial_t \eta \in L^2(0, T; H^{3/2}_\#(\mathcal{I})) \cap H^1(0, T; (H^{1/2}_\#(\mathcal{I}))), \end{gathered} \tag{1.31}$$

the first four equations of (1.3) *are satisfied almost everywhere or in the trace sense, the last equation in* (1.3) *holds in $L^2(0, T; H^{1/2}(\mathcal{I}))$ and* (1.4) *holds true.*

For $T \in (0, \infty]$, we say that (v, p, η) is a strong solution of (1.3)-(1.4) *on $[0, T)$ if for all $T' \in (0, T)$, (v, p, η) is a strong solution of* (1.3)-(1.4) *on $[0, T']$.*

We say that (v, p, η) is a stable strong solution of (1.3)-(1.4) *on $(0, \infty)$ if (v, p, η) is a strong solution of* (1.3)-(1.4) *on $[0, \infty)$ and (v, p, η) satisfies* (1.30)- (1.31) *with $T = \infty$.*

Remark 1.6. *Classical interpolation results imply that a strong solution (v, p, η) on $[0, T]$, for $T \in (0, \infty)$, satisfies*

$$\eta \in C^0([0, T]; H^2_\#(\mathcal{I})), \ \partial_t \eta \in C^0([0, T]; H^1_\#(\mathcal{I})), \ v \in C^0([0, T]; \mathbf{H}^1(\mathcal{F}_\eta)),$$

and that a stable strong solution (η, v, p) on $(0, \infty)$ satisfies

$$\eta \in C^0_b([0, \infty); H^2_\#(\mathcal{I})), \ \partial_t \eta \in C^0_b([0, \infty); H^1_\#(\mathcal{I})), \ v \in C^0_b([0, \infty); \mathbf{H}^1(\mathcal{F}_\eta)).$$

We assume that the initial conditions satisfy:

$$\eta_1^0 \in H^2_\#(\mathcal{I}), \quad \eta_2^0 \in H^1_\#(\mathcal{I}), \quad \eta_1^0 > -1 \quad \text{in } \mathcal{I}, \quad v^0 \in \mathbf{H}^1(\mathcal{F}_{\eta_1^0}), \tag{1.32}$$

$$\operatorname{div} v^0 = 0 \text{ in } \mathcal{F}_{\eta_1^0}, \quad v^0(s, 1 + \eta_1^0(s)) = \eta_2^0(s) e_2 \quad s \in \mathcal{I}, \quad v^0 = 0 \quad \text{on } \Gamma_{-1}. \tag{1.33}$$

First, we can obtain the existence and uniqueness of strong solutions of (1.3) for small times:

Theorem 1.7. *For any $(v^0, \eta_1^0, \eta_2^0)$ satisfying* (1.32)-(1.33)*, there exists a unique maximal strong solution (v, p, η) of* (1.3)-(1.4) *on $[0, T_{\max})$ with $T_{\max} \in (0, \infty]$ and with the following alternatives:*

- $T_{\max} = \infty$,
- $T_{\max} < \infty$ *and*

$$\limsup_{t \to T_{\max}} \|(v(t, \cdot), \eta(t, \cdot), \partial_t \eta(t, \cdot))\|_{\mathbf{H}^1(\mathcal{F}_\eta) \times H^2(\mathcal{I}) \times H^1(\mathcal{I})} + \left\| \frac{1}{1 + \eta(t, \cdot)} \right\|_{L^\infty(\mathcal{I})} = \infty. \tag{1.34}$$

This result is already known and proved in [13]. We can also deduce from Theorem 1.1 the existence and uniqueness of global strong solutions of (1.3) for small data:

Theorem 1.8. *There exists $c_1 > 0$ such that for any $(v^0, \eta_1^0, \eta_2^0)$ satisfying* (1.32)–(1.33)*, and*

$$\|\eta_1^0\|_{H^2(\mathcal{I})} + \|\eta_2^0\|_{H^1(\mathcal{I})} + \|v^0\|_{\mathbf{H}^1(\mathcal{F}_{\eta_1^0})} \leqslant c_1 \tag{1.35}$$

there exists a unique stable strong solution (v, p, η) of (1.3)-(1.4) *on $[0, \infty)$.*

In order to prove the above results, we first transform the equations of (1.3) by using a change of variables so that the spatial domain of the fluid becomes $\mathcal{F}_{\eta^*}$ where

$$\eta^* = \eta_1^0 \quad \text{(for Theorem 1.7)} \quad \eta^* = 0 \quad \text{(for Theorem 1.8)} \tag{1.36}$$

and then we use Theorem 1.1 and a fixed point argument.

Remark 1.9. *One can consider a more general structure equation*

$$\partial_{tt}\eta + \alpha_1 \partial_{ssss}\eta - \alpha_2 \partial_{ss}\eta - \delta \partial_{tss}\eta = -\widetilde{\mathbb{H}}_\eta(v,p). \tag{1.37}$$

A large part of the literature on this subject is done for the case of a beam equation $\alpha_1 > 0$: [8] (existence of weak solutions), [5], [15] and [12] (existence of strong solutions), [17] (stabilization of strong solutions), [2] (stabilization of weak solutions around a stationary state). In that case, at the opposite to the case of a wave equation, the viscosity of the fluid is not enough to modify the nature of the beam equation and one has to add the damping term $-\delta\partial_{tss}\eta$ with $\delta > 0$ to obtain a parabolic system.

Without this damping term (that is for $\delta = 0$ and $\alpha_1 > 0$), the existence of strong solutions is more difficult to achieve. It is obtained with an additional term of inertia of rotation ($-\partial_{ttss}\eta$ in (1.37)) in [13]. Without any additional terms, the corresponding fluid-structure system is studied in [3], where we proved the existence and uniqueness of strong solutions by an approach similar to the one done here: we also work with resolvent estimates but the corresponding semigroup is not analytic, only of Gevrey class. In particular the results for the linear and for the nonlinear system are weaker than the results presented here and we have a lost of regularity at initial time. Note that in [3], we focus in the case of small initial deformations and we only manage to remove this assumption in [4] by estimating the commutators between the operator L_λ introduced above (and defined in (3.7)) and the beam operator.

Here, we do not need in the analysis to consider these commutators and we obtain our result without assuming smallness of the initial deformations. This is mainly due to the fact that the wave operator is the square root of the beam operator so that the operator L_λ coming from the fluid has a stronger influence on the structure equation.

To finish this remark, we want to point out that in the framework of weak solutions, there have been some works studying the existence of weak solutions for a beam equation without dissipation (or a similar structure equation): [11], [16], [19], etc.

Remark 1.10. *The result of Theorem 1.1, that is the analyticity of the semigroup corresponding to the linear system is an important property and can be used for instance to show stabilization results. In the case of parabolic systems, there is a systematic method to show stabilization results with a control of finite dimension provided that a Fattorini-Hautus criterion is satisfied, see [1].*

Using the analyticity of the semigroup, we can also show that the solutions of (1.3) can be extended to holomorphic functions. For $\theta > 0$ and $T \in (0, +\infty]$ we introduce:

$$\mathcal{S}_{T,\theta} \stackrel{\text{def}}{=} \{z \in \mathbb{C}; 0 < |z| < T,\ |\arg(z)| < \theta\}.$$

Corollary 1.11. *Let us consider the strong solutions (v,p,η) of (1.3)-(1.4) obtained in Theorem 1.7 and in Theorem 1.8. Then they are analytic in time with values in $\mathbf{H}^2(\mathcal{F}_\eta) \times H^1_\#(\mathcal{F}_\eta) \times H^{5/2}_\#(\mathcal{I})$. More precisely,*

1. *If (v,p,η) is a strong solutions of (1.3)-(1.4) on $[0,T_0]$, $T_0 \in (0,\infty)$, there exists $\theta_0 > 0$ such that (v,p,η) admits an holomorphic extension in $\mathcal{S}_{T_0,\theta_0}$ with values in $\mathbf{H}^2(\mathcal{F}_\eta) \times H^1_\#(\mathcal{F}_\eta) \times H^{5/2}_\#(\mathcal{I})$.*
2. *There exist $c_2 \in (0, c_1]$ and $\theta_0 > 0$, where c_1 is the constant in Theorem 1.8, such that for any $(v^0, \eta_1^0, \eta_2^0)$ satisfying (1.32)–(1.33), and*

$$\|\eta_1^0\|_{H^2(\mathcal{I})} + \|\eta_2^0\|_{H^1(\mathcal{I})} + \|v^0\|_{\mathbf{H}^1(\mathcal{F}_{\eta_1^0})} \leqslant c_2 \tag{1.38}$$

then the strong solution (v,p,η) of (1.3) obtained in Theorem 1.8 admits an holomorphic extension in $\mathcal{S}_{\infty,\theta_0}$ with values in $\mathbf{H}^2(\mathcal{F}_\eta) \times H^1_\#(\mathcal{F}_\eta) \times H^{5/2}_\#(\mathcal{I})$.

Remark 1.12. *Using the above corollary, one can obtain several properties on the solutions of (1.3). For instance, considering the change of variables X used to prove Theorem 1.7 or Theorem 1.8 and applying the Cauchy formula, one can show that the n-th time derivatives of $(v \circ X, \eta, \partial_t\eta)$ satisfy*

$$\left\| \left(\frac{d^{(n)}}{dt^{(n)}} v(t, X(t,\cdot)), \frac{d^{(n)}}{dt^{(n)}} \eta(t,\cdot), \frac{d^{(n+1)}}{dt^{(n+1)}} \eta(t,\cdot) \right) \right\|_{\mathbf{H}^1(\mathcal{F}_\eta) \times H^3_\#(\mathcal{I}) \times H^2_\#(\mathcal{I})} \leqslant \frac{C_n}{t^n},$$

for some constant $C_n > 0$ depending on n and the initial conditions.

The outline of the article is as follows: first in Section 2, we use a standard change of variables to write the system (1.3) in a fixed spatial domain and see how the corresponding linearization leads to the linear system (1.15). The analyticity of the semigroup for the linear system is stated in Theorem 2.5 with the corresponding resolvent estimate. Section 3.1 is devoted to the introduction of several useful operators together with their properties and in particular the operator L_λ corresponding to the force of the fluid acting on the structure. Before estimating the inverse of the operator V_λ introduced above (see (3.18) and (3.7) for its definition), we start by estimating the inverse of an approximation of V_λ in Section 3.2. In Section 3.3, we use these estimates to deduce the same estimates for V_λ and deduce the resolvent estimates that lead to the analyticity of the semigroup. Finally, in Section 4 we recall the idea of the proof of Theorem 1.7 and of Theorem 1.8 based on Theorem 1.1, by using a fixed point argument. Since this part of the analysis is now classical and has already been done for instance in [13], we only give the idea and postpone some technical details in the appendix. The time analyticity of the solutions, stated in Corollary 1.11, is proved in Section 5.

2 Change of variables and linearization

2.1 The system written in a fixed domain

In order to transform system (1.3) into a system with a fixed spatial fluid domain, we construct a change of variables. This change of variables is different from our previous articles [3, 4] and is similar to the one already considered in [13]. With our previous change of variables, we would be able to obtain the local in time existence, but we would not manage to obtain the criterion for the global existence stated in Theorem 1.7.

There exists a linear map $\mathcal{R}$ such that for any $\alpha > 0$,

$$\mathcal{R} : H^\alpha(\mathcal{I}) \to \left\{ w \in H^{\alpha+1/2}(\mathcal{F}_0) \ ; \ w = 0 \quad \text{on } \Gamma_{-1} \right\}, \quad \eta \mapsto \mathcal{R}_\eta \tag{2.1}$$

is continuous and satisfies $(\mathcal{R}_\eta)_{|\Gamma_0} = \eta$. Note that, in this article, we will use the above mapping for $\alpha = 1$, $\alpha = 3/2$, $\alpha = 5/4$, and $\alpha = 7/4$.

We consider the change of variables

$$X_{\eta^*,\eta} : \mathcal{F}_{\eta^*} \to \mathcal{F}_\eta, \quad (y_1, y_2) \mapsto \left(y_1, y_2 + \mathcal{R}_{\eta-\eta^*} \left(y_1, \frac{y_2}{1+\eta^*(y_1)} \right) \right). \tag{2.2}$$

Using the continuous embedding $H^{9/4}(\mathcal{F}_0) \hookrightarrow W^{1,\infty}(\mathcal{F}_0)$, and (2.1) for $\alpha = 7/4$, one can check that if $\eta^* \in H^2(\mathcal{I})$, if $\eta^* > -1$ in $\mathcal{I}$ and if

$$\|\eta^*\|_{H^2(\mathcal{I})} + \left\| \frac{1}{1+\eta^*} \right\|_{L^\infty(\mathcal{I})} \leqslant C_*$$

then there exists $C = C(C_*)$ such that

$$\|\nabla X_{\eta^*,\eta} - I_2\|_{L^\infty(\mathcal{F}_{\eta^*})} \leqslant C \|\eta - \eta^*\|_{H^{7/4}(\mathcal{I})}.$$

In particular, there exists $\kappa = \kappa(C_*) > 0$ such that if

$$\|\eta - \eta^*\|_{H^{7/4}(\mathcal{I})} \leqslant \kappa \tag{2.3}$$

then $X_{\eta^*,\eta}$ is a C^1-diffeomorphism.

In what follows, we assume that our deformation η depends on t, that $\eta(t)$ satisfies (2.3) for all t with η^* given by (1.36), and we use the simplified notation:

$$\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}_{\eta^*}. \tag{2.4}$$

If no confusion can arise, we write

$$X(t, \cdot) \stackrel{\text{def}}{=} X_{\eta^*,\eta(t)}, \quad Y(t, \cdot) \stackrel{\text{def}}{=} X(t, \cdot)^{-1} \tag{2.5}$$

so that $X(t,\cdot)$ transforms $\mathcal{F}$ onto $\mathcal{F}_{\eta(t)}$. Then, we write

$$w(t,y) \overset{\text{def}}{=} v(t,X(t,y)) \quad \text{and} \quad q(t,y) \overset{\text{def}}{=} p(t,X(t,y)), \tag{2.6}$$

so that

$$v(t,x) = w(t,Y(t,x)) \quad \text{and} \quad p(t,x) = q(t,Y(t,x)). \tag{2.7}$$

After some calculation (see Appendix A), system (1.3), (1.4) rewrites,

$$\begin{cases} \partial_t w - \operatorname{div} \mathbb{T}(w,q) = \widehat{F}(\eta,w,q) \quad \text{in } (0,\infty)\times\mathcal{F}, \\ \operatorname{div} w = \widehat{G}(\eta,w) \quad \text{in } (0,\infty)\times\mathcal{F}, \\ w(t,s,1+\eta^*(s)) = (\partial_t\eta)(t,s)e_2, \quad t>0,\ s\in\mathcal{I}, \\ w=0, \quad t>0,\ y\in\Gamma_{-1}, \\ \partial_{tt}\eta + A_1\eta = -\mathbb{H}_{\eta^*}(w,q) + \widehat{H}(\eta,w), \quad t>0, \end{cases} \tag{2.8}$$

with the initial conditions

$$\eta(0,\cdot)=\eta_1^0, \quad \partial_t\eta(0,\cdot)=\eta_2^0 \quad \text{and} \quad w(0,\cdot)=w^0 \overset{\text{def}}{=} v^0(X(0,\cdot)) \ \text{ in } \mathcal{F}, \tag{2.9}$$

where

$$\widehat{F}(\eta,w,q) = \widehat{F}_1(\eta,w) + \widehat{F}_2(\eta,w,q), \tag{2.10}$$

$$\begin{aligned} \widehat{F}_1(\eta,w)_i &\overset{\text{def}}{=} -\sum_j \frac{\partial w_i}{\partial y_j}\frac{\partial Y_j}{\partial t}(X) - \sum_{j,k} w_j \frac{\partial w_i}{\partial y_k}\frac{\partial Y_k}{\partial x_j}(X), \\ \widehat{F}_2(\eta,w,q)_i &\overset{\text{def}}{=} \nu\sum_{k,j}\frac{\partial^2 w_j}{\partial y_k\partial y_i}\left(\frac{\partial Y_k}{\partial x_j}(X)-\delta_{k,j}\right) + \nu\sum_{k,j,\ell}\frac{\partial w_j}{\partial y_k}\frac{\partial^2 Y_k}{\partial x_j\partial x_\ell}(X)\frac{\partial X_\ell}{\partial y_i} \\ &\quad + \nu\sum_{j,k,\ell}\frac{\partial^2 w_i}{\partial y_k\partial y_\ell}\left(\frac{\partial Y_k}{\partial x_j}(X)\frac{\partial Y_\ell}{\partial x_j}(X)-\delta_{k,j}\delta_{\ell,j}\right) + \nu\sum_{j,k}\frac{\partial w_i}{\partial y_k}\frac{\partial^2 Y_k}{\partial x_j^2}(X) \\ &\quad - \sum_k \frac{\partial q}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_i}(X)-\delta_{k,i}\right), \end{aligned} \tag{2.11}$$

$$\widehat{G}(\eta,w) \overset{\text{def}}{=} \operatorname{div}((I_2-\operatorname{Cof}(\nabla X)^*)w) = \nabla w : (I_2 - \operatorname{Cof}(\nabla X)), \tag{2.12}$$

and

$$\begin{aligned} \widehat{H}(\eta,w)(t,s) = M\Bigg[&\nu(\partial_s\eta^*-\partial_s\eta)\left(\frac{\partial w_1}{\partial y_2}+\frac{\partial w_2}{\partial y_1}\right)(t,s,1+\eta^*(s)) \\ &- \nu(\partial_s\eta)\sum_k\left(\frac{\partial w_1}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_2}(X)-\delta_{k,2}\right)+\frac{\partial w_2}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_1}(X)-\delta_{k,1}\right)\right)(t,s,1+\eta^*(s)) \\ &+ 2\nu\sum_k\left(\frac{\partial w_2}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_2}(X)-\delta_{k,2}\right)\right)(t,s,1+\eta^*(s))\Bigg]. \end{aligned} \tag{2.13}$$

In the above statements $\operatorname{Cof}(\nabla X)$ is the matrix of the cofactors of ∇X. Using the above change of variables, we can rewrite Theorem 1.7 and Theorem 1.8. The definitions of strong solutions are deduced from Definition 1.5:

Definition 2.1. *For $T\in(0,\infty)$ we say that (w,q,η) is a strong solution of* (2.8)-(2.13) *on* $[0,T]$ *if* (1.29) *holds, if $(w,q,\eta)\in\mathfrak{W}_T(\mathcal{F})$, if the first four equations of* (2.8) *are satisfied almost everywhere or in the trace sense, the last equation in* (2.8) *holds in $L^2(0,T;H^{1/2}(\mathcal{I}))$ and* (2.9) *holds true.*

For $T\in(0,\infty]$, we say that (w,q,η) is a strong solution of (2.8)-(2.13) *on* $[0,T)$ *if for all $T'\in(0,T)$, (w,q,η) is a strong solution of* (2.8)-(2.13) *on* $[0,T']$.

We say that (w,q,η) is a stable strong solution of (2.8)-(2.13) *on* $(0,\infty)$ *if (w,q,η) is a strong solution of* (2.8)-(2.13) *on* $[0,\infty)$ *and $(w,q,\eta)\in\mathfrak{W}_\infty(\mathcal{F})$.*

The hypotheses (1.32)-(1.33) on the initial conditions are transformed into

$$\eta_1^0 \in H^2_\#(\mathcal{I}), \quad \eta_2^0 \in H^1_\#(\mathcal{I}), \quad \eta_1^0 > -1 \quad \text{in } \mathcal{I}, \quad w^0 \in \mathbf{H}^1(\mathcal{F}), \tag{2.14}$$

$$\operatorname{div}\left(\operatorname{Cof}(\nabla X(0,\cdot))^* w^0\right) = 0 \text{ in } \mathcal{F}, \quad w^0(s, 1+\eta^*(s)) = \eta_2^0(s) e_2 \quad s \in \mathcal{I}, \quad w^0 = 0 \quad \text{on } \Gamma_{-1}. \tag{2.15}$$

With the above notations and definitions, the statements of Theorem 1.7 and Theorem 1.8 are transformed into the following theorems:

Theorem 2.2. *Let* $(w^0, \eta_1^0, \eta_2^0)$ *satisfying* (2.14)–(2.15)*, there exists a unique maximal strong solution* (w, q, η) *of* (2.8)-(2.13) *on* $[0, T_{\max})$ *with* $T_{\max} \in (0, \infty]$ *and with the following alternatives:*

- $T_{\max} = \infty$,
- $T_{\max} < \infty$ *and*

$$\limsup_{t \to T_{\max}} \|(w(t,\cdot), \eta(t,\cdot), \partial_t \eta(t,\cdot))\|_{\mathbf{H}^1(\mathcal{F}) \times H^2(\mathcal{I}) \times H^1(\mathcal{I})} + \left\| \frac{1}{1+\eta(t,\cdot)} \right\|_{L^\infty(\mathcal{I})} = \infty. \tag{2.16}$$

Theorem 2.3. *There exists* $c_1 > 0$ *such that for any* $(w^0, \eta_1^0, \eta_2^0)$ *satisfying* (2.14)–(2.15)*, and*

$$\|\eta_1^0\|_{H^2(\mathcal{I})} + \|\eta_2^0\|_{H^1(\mathcal{I})} + \|w^0\|_{\mathbf{H}^1(\mathcal{F})} \leqslant c_1 \tag{2.17}$$

there exists a unique stable strong solution (w, q, η) *of* (2.8)-(2.13) *on* $[0, \infty)$.

2.2 The linear system and the operator A_0

From the previous section, and in particular from system (2.8)-(2.9), we are led to consider the linear system (1.15)-(1.16) written in the fixed domain $\mathcal{F}$ (defined by (2.4)). We introduce the notation

$$\mathbb{C}^+ \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \; ; \; \operatorname{Re}(\lambda) \geqslant 0\}, \tag{2.18}$$

$$\mathbb{C}^+_\alpha \stackrel{\text{def}}{=} \left\{\lambda \in \mathbb{C}^+ \; ; \; |\lambda| > \alpha\right\}. \tag{2.19}$$

Let us consider the following functional spaces

$$\mathbf{V}^0_n(\mathcal{F}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{L}^2(\mathcal{F}) \; ; \; \operatorname{div} f = 0 \text{ in } \mathcal{F}, \quad f \cdot n = 0 \text{ on } \partial\mathcal{F} \right\},$$

$$\mathbf{V}^1_0(\mathcal{F}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{H}^1(\mathcal{F}) \; ; \; \operatorname{div} f = 0 \text{ in } \mathcal{F}, \quad f = 0 \text{ on } \partial\mathcal{F} \right\},$$

$$\mathbf{V}^\theta(\partial\mathcal{F}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{H}^\theta(\partial\mathcal{F}) \; ; \; \int_{\partial\mathcal{F}} f \cdot n \, d\gamma = 0 \right\} \quad (\theta \geqslant 0). \tag{2.20}$$

We introduce the operator $\Lambda : L^2(\mathcal{I}) \to \mathbf{L}^2(\partial\mathcal{F})$ defined by

$$\begin{aligned} &(\Lambda\eta)(y) = (M\eta(s))\, e_2 \quad \text{if} \quad y = (s, 1+\eta^*(s)) \in \Gamma_{\eta^*}, \\ &(\Lambda\eta)(y) = 0 \qquad\qquad \text{if} \quad y \in \Gamma_{-1}. \end{aligned} \tag{2.21}$$

The adjoint $\Lambda^* : \mathbf{L}^2(\partial\mathcal{F}) \to L^2(\mathcal{I})$ of Λ is given by

$$(\Lambda^* v)(s) = M\left((1 + |\partial_s \eta^*(s)|^2)^{1/2} v(s, 1+\eta^*(s)) \cdot e_2 \right). \tag{2.22}$$

We observe that $\Lambda(L^2(\mathcal{I})) \subset \mathbf{V}^0(\partial\mathcal{F})$, and since $\eta^* \in H^2(\mathcal{I})$, then for any $\theta \in [0,1]$,

$$\Lambda(H^\theta(\mathcal{I})) \subset \mathbf{V}^\theta(\partial\mathcal{F}), \tag{2.23}$$

$$\Lambda^*(\mathbf{H}^\theta(\partial\mathcal{F})) \subset \mathcal{D}(A_1^{\theta/2}) \tag{2.24}$$

and

$$\|\Lambda\eta\|_{\mathbf{H}^\theta(\partial\mathcal{F})} \geqslant c(\theta) \|A_1^{\theta/2}\eta\|_{\mathcal{H}_\mathcal{S}} \quad (\eta \in \mathcal{D}(A_1^{\theta/2})). \tag{2.25}$$

Using (2.23) for $\theta = 1/2$ and recalling (1.14), we deduce $\Lambda(\mathcal{D}(A_1^{1/4})) \subset \mathbf{H}^{1/2}(\partial\mathcal{F})$ so that

$$\Lambda^*(\mathbf{H}^{-1/2}(\partial\mathcal{F})) \subset \mathcal{D}(A_1^{1/4})'. \tag{2.26}$$

We consider the space $\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4})$ equipped with the scalar product:

$$\left\langle \left[w^{(1)}, \eta_1^{(1)}, \eta_2^{(1)}\right], \left[w^{(2)}, \eta_1^{(2)}, \eta_2^{(2)}\right] \right\rangle = \int_{\mathcal{F}} w^{(1)} \cdot w^{(2)} \, dy + \left(A_1^{3/4}\eta_1^{(1)}, A_1^{3/4}\eta_1^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}} + \left(A_1^{1/4}\eta_2^{(1)}, A_1^{1/4}\eta_2^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}},$$

(where $\mathcal{H}_{\mathcal{S}} \stackrel{\text{def}}{=} L^2_{\#}(\mathcal{I})$, see (1.11)) and we introduce the following spaces:

$$\mathcal{H} \stackrel{\text{def}}{=} \left\{[w, \eta_1, \eta_2] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}) \ ; \ w \cdot n = (\Lambda\eta_2) \cdot n \text{ on } \partial\mathcal{F}, \ \operatorname{div} w = 0 \text{ in } \mathcal{F}\right\}. \tag{2.27}$$

We also consider the space $\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$ equipped with the scalar product:

$$\left\langle \left[w^{(1)}, \eta_1^{(1)}, \eta_2^{(1)}\right], \left[w^{(2)}, \eta_1^{(2)}, \eta_2^{(2)}\right] \right\rangle_0 = \int_{\mathcal{F}} w^{(1)} \cdot w^{(2)} \, dy + \left(A_1^{1/2}\eta_1^{(1)}, A_1^{1/2}\eta_1^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}} + \left(\eta_2^{(1)}, \eta_2^{(2)}\right)_{\mathcal{H}_{\mathcal{S}}},$$

and we introduce the following space:

$$\mathcal{H}_0 \stackrel{\text{def}}{=} \left\{[w, \eta_1, \eta_2] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}} \ ; \ w \cdot n = (\Lambda\eta_2) \cdot n \text{ on } \partial\mathcal{F}, \ \operatorname{div} w = 0 \text{ in } \mathcal{F}\right\}. \tag{2.28}$$

Lemma 2.4. *The orthogonal projection P_0 from $\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$ onto $\mathcal{H}_0$ satisfies*

$$P_0 \in \mathcal{L}(\mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}), \mathcal{H}). \tag{2.29}$$

Proof. We have proven in [2, Proposition 3.1 and Proposition 3.2] that for any $[w, \eta_1, \eta_2] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \times \mathcal{H}_{\mathcal{S}}$,

$$P_0 \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} w - \nabla p \\ \eta_1 \\ \eta_2 + \Lambda^*(pn) \end{bmatrix}$$

where $p \in H^1(\mathcal{F})$. Hence, from the trace theorem and (2.24) with $\theta = 1/2$ we have

$$\begin{bmatrix} -\nabla p \\ 0 \\ \Lambda^*(pn) \end{bmatrix} \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4})$$

and we deduce the result. □

We now define the linear operator $A_0 : \mathcal{D}(A_0) \subset \mathcal{H} \to \mathcal{H}$:

$$\mathcal{D}(A_0) \stackrel{\text{def}}{=} \left\{[w, \eta_1, \eta_2] \in \mathbf{H}^2(\mathcal{F}) \times \mathcal{D}(A_1^{5/4}) \times \mathcal{D}(A_1^{3/4}) \ ; \ w = \Lambda\eta_2 \text{ on } \partial\mathcal{F}, \ \operatorname{div} w = 0 \text{ in } \mathcal{F}\right\}, \tag{2.30}$$

and for $[w, \eta_1, \eta_2] \in \mathcal{D}(A_0)$, we set

$$\widetilde{A}_0 \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \Delta w \\ \eta_2 \\ -A_1\eta_1 - \Lambda^*(2D(w)n) \end{bmatrix} \tag{2.31}$$

and

$$A_0 \stackrel{\text{def}}{=} P_0\widetilde{A}_0. \tag{2.32}$$

Note that A_0 is well-defined due to Lemma 2.4.

By using the above operators, we can rewrite the linear system (1.15) for $G = 0$, as follows

$$\frac{d}{dt} \begin{bmatrix} w \\ \eta \\ \partial_t\eta \end{bmatrix} = A_0 \begin{bmatrix} w \\ \eta \\ \partial_t\eta \end{bmatrix} + P_0 \begin{bmatrix} F \\ 0 \\ H \end{bmatrix}, \quad \begin{bmatrix} w \\ \eta \\ \partial_t\eta \end{bmatrix}(0) = \begin{bmatrix} w^0 \\ \zeta_1^0 \\ \zeta_2^0 \end{bmatrix}. \tag{2.33}$$

One of the main goals of this article is to show the following result:

Theorem 2.5. *The operator A_0 defined by* (2.30)–(2.32) *has compact resolvents, it is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}$. In particular, there exists $C_0 > 0$ such that for all $\lambda \in \mathbb{C}^+$,*

$$|\lambda| \left\|(\lambda I - A_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_0. \tag{2.34}$$

The proof of Theorem 2.5 is a consequence of Lemma 2.6 and Proposition 2.7. Indeed these results imply that $\mathbb{C}^+ \subset \rho(A_0)$ and that (2.34) holds for all $\lambda \in \mathbb{C}^+$. It is standard that (2.34) for all $\lambda \in \mathbb{C}^+$ and $0 \in \rho(A_0)$ imply that the semigroup generated by A_0 is analytic and exponentially stable. Let us recall how this can be done: proceeding as in the proof of Lemma 3.10 in [2], we write

$$(\lambda I - A_0) = (\imath \operatorname{Im} \lambda\, I - A_0)(I + \operatorname{Re} \lambda(\imath \operatorname{Im} \lambda\, I - A_0)^{-1})$$

and we deduce that there exists c_0 such that

$$\{\lambda \in \mathbb{C} \ ; \ |\operatorname{Re} \lambda| \leqslant c_0 |\operatorname{Im} \lambda|\} \subset \rho(A_0) \tag{2.35}$$

and that $|\lambda| \left\|(\lambda I - A_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H})}$ is bounded in this set. Then using [6, Theorem 2.10, p.109], we deduce that A_0 is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$.

Finally, using that $0 \in \rho(A_0)$ (and that $\rho(A_0)$ is an open set) and (2.35), we deduce that

$$\{\lambda \in \mathbb{C} \ ; \ \operatorname{Re} \lambda > -\varepsilon\} \subset \rho(A_0)$$

for some $\varepsilon > 0$. According to [6, Proposition 2.9, p.120] the semigroup $(e^{A_0 t})_{t \geqslant 0}$ is exponentially stable.

Lemma 2.6. *The operator A_0 defined by* (2.30)–(2.32) *satisfies $\mathbb{C}^+ \subset \rho(A_0)$.*

Proof. For that, assume $F = [f, g, h] \in \mathcal{H}$ and let us prove the existence and uniqueness of $[v, \eta_1, \eta_2] \in \mathcal{D}(A_0)$ solution of $(\lambda I - A_0)[v, \eta_1, \eta_2] = F$, that rewrites

$$\left\{\begin{array}{c} \lambda v - \operatorname{div} \mathbb{T}(v, p) = f \quad \text{in } \mathcal{F}, \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}, \\ v = \Lambda \eta_2 \quad \text{on } \partial\mathcal{F}, \\ \lambda \eta_1 - \eta_2 = g, \\ \lambda \eta_2 + A_1 \eta_1 = -\Lambda^* \left\{\mathbb{T}(v, p) n_{|\partial\mathcal{F}}\right\} + h. \end{array}\right. \tag{2.36}$$

Let us first assume $\lambda \in \mathbb{C}^+ \backslash \{0\}$ and consider a variational formulation associated with (2.36): find

$$[v, \eta_2] \in \mathbb{V} \stackrel{\text{def}}{=} \left\{[v, \eta_2] \in \mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1^{1/2}) \ ; \ \operatorname{div} v = 0 \ \text{ in } \mathcal{F}, \ v = \Lambda \eta_2 \ \text{ on } \partial\mathcal{F}\right\}, \tag{2.37}$$

such that for any $[\varphi, \zeta_2] \in \mathbb{V}$,

$$\begin{aligned} \lambda \left(\int_{\mathcal{F}} v \cdot \varphi dy + \left(\eta_2, \zeta_2\right)_{\mathcal{H}_{\mathcal{S}}} \right) + 2\nu \int_{\mathcal{F}} \mathbb{D} v : \mathbb{D} \varphi \ dy + \frac{1}{\lambda} \left(A_1^{1/2} \eta_2, A_1^{1/2} \zeta_2\right)_{\mathcal{H}_{\mathcal{S}}} \\ = \int_{\mathcal{F}} f \cdot \varphi \ dy + \left(h, \zeta_2\right)_{\mathcal{H}_{\mathcal{S}}} - \frac{1}{\lambda} \left(A_1^{1/2} g, A_1^{1/2} \zeta_2\right)_{\mathcal{H}_{\mathcal{S}}}. \end{aligned} \tag{2.38}$$

Using the Korn and the Poincaré inequalities, a trace inequality, (2.25) with $\theta = 1/2$ and the fact that $\operatorname{Re}(\lambda) \geqslant 0$, we can apply the Lax-Milgram theorem and we deduce the existence and uniqueness of $[v, \eta_2] \in \mathbb{V}$ satisfying (2.38) for any $[\varphi, \zeta_2] \in \mathbb{V}$. Taking $\zeta_2 = 0$ in (2.38) and using the De Rham theorem, we obtain the existence of q such that (w, q) is the weak solution of the Stokes system (the three first equations of (2.36)). From $f \in \mathbf{L}^2(\mathcal{F})$, we deduce $\operatorname{div} \mathbb{T}(v, p) \in \mathbf{L}^2(\mathcal{F})$. Hence, we have $\mathbb{T}(v, p) n_{|\partial\mathcal{F}} \in \mathbf{H}^{-1/2}(\partial\mathcal{F})$ and from (2.26) we deduce $\Lambda^* \left\{\mathbb{T}(v, p) n_{|\partial\mathcal{F}}\right\} \in \mathcal{D}(A_1^{1/4})'$.

Writing $\eta_1 = \lambda^{-1}(\eta_2 + g)$ and using that (w, q) satisfies the three first equations of (2.36), we obtain from (2.38) the last equation of (2.36):

$$\lambda \eta_2 + A_1 \eta_1 = -\Lambda^* \left\{\mathbb{T}(v, p) n_{|\partial\mathcal{F}}\right\} + h \quad \text{in } \mathcal{D}(A_1^{1/4})'. \tag{2.39}$$

We deduce from the above system that $\eta_1 \in \mathcal{D}(A_1^{3/4})$ and thus $\eta_2 = \lambda \eta_1 - g \in \mathcal{D}(A_1^{3/4})$. Since $f \in \mathbf{L}^2(\mathcal{F})$, from regularity results for the Stokes system with H^2-boundary (see [12, Lemma 1]) we obtain $v \in \mathbf{H}^2(\mathcal{F})$

and $p \in H^1(\mathcal{F})$. In particular, from trace results and (2.24), we obtain $\Lambda^* \{\mathbb{T}(v,p)n\} \in \mathcal{D}(A_1^{1/4})$. Then from (2.39) we get $\eta_1 \in \mathcal{D}(A_1^{5/4})$. Finally, we have proved that $[w, \eta_1, \eta_2] \in \mathcal{D}(A_0)$, and thus, that $\mathbb{C}^+ \backslash \{0\} \subset \rho(A_0)$.

Consider now the case $\lambda = 0$. The system (2.36) rewrites

$$\left\{ \begin{array}{c} -\operatorname{div} \mathbb{T}(v,p) = f \quad \text{in } \mathcal{F}, \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}, \\ v = -\Lambda g \quad \text{on } \partial\mathcal{F}, \\ \eta_2 = -g \\ A_1 \eta_1 = -\Lambda^* \left\{ \mathbb{T}(v,p) n_{|\partial\mathcal{F}} \right\} + h. \end{array} \right. \tag{2.40}$$

We observe that in that case, the Stokes system can be solved independently from the structure equation. Hence, from regularity results for the Stokes system with H^2-boundary (see [12, Lemma 1]), we obtain the existence and uniqueness of $w \in \mathbf{H}^2(\mathcal{F})$ and $q \in H^1(\mathcal{F})$ satisfying the first three equality in (2.40). Next, from $\mathbb{T}(w,q)n \in \mathbf{H}^{1/2}(\partial\mathcal{F})$ and (2.24) we get $\Lambda^* (\mathbb{T}(w,q)n) \in \mathcal{D}(A_1^{1/4})$ and the last equality in (2.40) admits a unique solution $\eta_1 \in \mathcal{D}(A_1^{5/4})$. Finally, $\eta_2 = -g \in \mathcal{D}(A_1^{3/4})$ and we have prove the existence and uniqueness of $[w, \eta_1, \eta_2] \in \mathcal{D}(A_0)$ satisfying (2.40).

□

In particular, we deduce from Lemma 2.6 and from the properties of the resolvent, that for any $\alpha > 0$ we have

$$\sup_{\lambda \in \mathbb{C}^+, |\lambda| \leqslant \alpha} |\lambda| \|(\lambda - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.41}$$

Proposition 2.7. *There exists $\alpha > 0$ such that*

$$\sup_{\lambda \in \mathbb{C}_\alpha^+} |\lambda| \|(\lambda - A_0)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.42}$$

The proof of this result is the core of the article and is done in Section 3.3. The proof of the following proposition is standard but for sake of completeness we give it in the next section after the introduction of some notation.

Proposition 2.8. *The operator A_0 defined by* (2.30)-(2.32) *satisfies*

$$[\mathcal{H}, \mathcal{D}(A_0)]_{1/2} = \left\{ [w, \eta_1, \eta_2] \in \mathbf{H}^1(\mathcal{F}) \times \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2}) \ ; \ w = \Lambda \eta_2 \text{ on } \partial\mathcal{F}, \ \operatorname{div} w = 0 \text{ in } \mathcal{F} \right\}. \tag{2.43}$$

2.3 Proofs of Theorem 1.1

We are now in a position to prove Theorem 1.1:

Proof of Theorem 1.1. First, we introduce the lifting operator $\mathcal{R}$ defined by $\mathcal{R}(g) = z$ where z is the solution of

$$\left\{ \begin{array}{c} -\Delta z + \nabla \chi = 0 \quad \text{in } \mathcal{F}, \\ \operatorname{div} z = g \quad \text{in } \mathcal{F}, \\ z = 0 \quad \text{on } \partial\mathcal{F}. \end{array} \right.$$

According to [18, Lemma 8.1 and Lemma 8.2] we have

$$\mathcal{R} \in \mathcal{L}(H_{\#}^{-1}(\mathcal{F}), \mathbf{L}^2(\mathcal{F})) \cap \mathcal{L}(H_{\#}^1(\mathcal{F}), \mathbf{H}^2(\mathcal{F})). \tag{2.44}$$

Then we consider $\widetilde{w} = w - \mathcal{R}(G)$ so that (1.15), (1.16) becomes

$$\left\{ \begin{array}{c} \partial_t \widetilde{w} - \operatorname{div} \mathbb{T}(\widetilde{w}, q) = \widetilde{F}, \quad t > 0, \ y \in \mathcal{F}, \\ \operatorname{div} \widetilde{w} = 0, \quad t > 0, \ y \in \mathcal{F}, \\ \widetilde{w}(t, s, 1 + \eta^*(s)) = (\partial_t \eta)(t, s) e_2, \quad t > 0, \ s \in \mathcal{I}, \\ \widetilde{w} = 0, \quad t > 0, \ y \in \Gamma_{-1}, \\ \partial_{tt} \eta + A_1 \eta = -\Lambda^* \left\{ \mathbb{T}(\widetilde{w}, q) n_{|\partial\mathcal{F}} \right\} + \widetilde{H}, \quad t > 0, \ s \in \mathcal{I}, \end{array} \right. \tag{2.45}$$

with the initial conditions

$$\widetilde{w}(0, \cdot) = \widetilde{w}^0, \quad \eta(0, \cdot) = \zeta_1^0, \quad \partial_t \eta(0, \cdot) = \zeta_2^0 \tag{2.46}$$

and where

$$\widetilde{F} \stackrel{\text{def}}{=} F - \mathcal{R}(\partial_t G) + 2\nu \operatorname{div} \mathbb{D}(\mathcal{R}(G)),$$

$$\widetilde{H} \stackrel{\text{def}}{=} H - M\left\{(1+|\partial_s\eta^*|^2)^{1/2}\left[2\nu\mathbb{D}(\mathcal{R}(G))n\right](t,s,1+\eta^*(s))\cdot e_2\right\},$$
$$\widetilde{w}^0 = w^0 - \mathcal{R}(G(0,\cdot)).$$

From (2.44), we have

$$\mathcal{R}(G) \in L^2(0,\infty;\mathbf{H}^2(\mathcal{F})) \cap H^1(0,\infty;\mathbf{L}^2(\mathcal{F})) \hookrightarrow C^0([0,\infty);\mathbf{H}^1(\mathcal{F})),$$

so that from (1.21),

$$\widetilde{F} \in L^2(0,\infty;\mathbf{L}^2(\mathcal{F})), \quad \widetilde{H} \in L^2(0,\infty;\mathcal{D}(A_1^{1/4})),$$

and from (1.22), (1.23), Proposition 2.8 and Lemma 2.4,

$$\begin{bmatrix}\widetilde{w}^0\\ \zeta_1^0\\ \zeta_2^0\end{bmatrix} \in [\mathcal{H},\mathcal{D}(A_0)]_{1/2}, \quad P_0\begin{bmatrix}\widetilde{F}\\ 0\\ \widetilde{H}\end{bmatrix} \in L^2(0,\infty;\mathcal{H}).$$

The linear system (2.45), (2.46) rewrites as (2.33), and maximal regularity results for analytic semigroups (see e.g. [6, Theorem 3.1, p. 143]) ensure that

$$\begin{bmatrix}\widetilde{w}\\ \eta\\ \partial_t\eta\end{bmatrix} \in L^2(0,\infty;\mathcal{D}(A_0)) \cap H^1(0,\infty;\mathcal{H}), \tag{2.47}$$

with

$$\left\|\begin{bmatrix}\widetilde{w}\\ \eta\\ \partial_t\eta\end{bmatrix}\right\|_{L^2(0,\infty;\mathcal{D}(A_0))\cap H^1(0,\infty;\mathcal{H})} \leqslant C\left(\left\|\begin{bmatrix}\widetilde{w}^0\\ \zeta_1^0\\ \zeta_2^0\end{bmatrix}\right\|_{[\mathcal{H},\mathcal{D}(A_0)]_{1/2}} + \left\|P_0\begin{bmatrix}\widetilde{F}\\ 0\\ \widetilde{H}\end{bmatrix}\right\|_{L^2(0,\infty;\mathcal{H})}\right), \tag{2.48}$$

and using the above formula on $\widetilde{w}$, $\widetilde{F}$ and $\widetilde{H}$, we deduce (1.25). This concludes the proof of Theorem 1.1. □

2.4 Proof of Proposition 1.2 and Corollary 1.3

Proof of Proposition 1.2. Assume $R > 0$ and $\eta^* \in \mathcal{B}_R$. We show the existence of $\kappa^* > 0$ and $C > 0$, depending only on R such that for any $\widetilde{\eta} \in \mathcal{B}_R$

$$\|\widetilde{\eta} - \eta^*\|_{H^{7/4}(\mathcal{I})} < \kappa^* \implies \widehat{C}_L(\widetilde{\eta}) \leqslant C\widehat{C}_L(\eta^*). \tag{2.49}$$

If the above implication holds true, one can end the proof of Proposition 1.2: assume $(\widetilde{\eta}_n)$ is a sequence of $\mathcal{B}_R$ such that

$$\sup_n \widehat{C}_L(\widetilde{\eta}_n) = \infty. \tag{2.50}$$

Using standard compactness results and the embedding $H^{7/4}_\#(\mathcal{I}) \hookrightarrow L^\infty(\mathcal{I})$, there exists $\eta^* \in \mathcal{B}_R$ such that, up to a subsequence,

$$\widetilde{\eta}_n \rightharpoonup \eta^* \quad \text{weakly in } H^2_\#(\mathcal{I}), \quad \widetilde{\eta}_n \to \eta^* \quad \text{strongly in } H^{7/4}_\#(\mathcal{I}).$$

Then relation (2.49) contradicts (2.50).

It thus remains to show (2.49) to conclude the proof of Proposition 1.2. We thus assume that $\widetilde{\eta} \in \mathcal{B}_R$ satisfies

$$\|\widetilde{\eta} - \eta^*\|_{H^{7/4}(\mathcal{I})} < \kappa^*. \tag{2.51}$$

Taking $\kappa^* \leqslant \kappa$ (see (2.3)), we can consider the change of variable $X_{\eta^*,\widetilde{\eta}} : \mathcal{F}_{\eta^*} \to \mathcal{F}_{\widetilde{\eta}}$ defined by (2.2). For sake of simplicity, in what follows we use the notations

$$\mathcal{F} \stackrel{\text{def}}{=} \mathcal{F}_{\eta^*}, \quad X \stackrel{\text{def}}{=} X_{\eta^*,\widetilde{\eta}}, \quad Y \stackrel{\text{def}}{=} X^{-1}. \tag{2.52}$$

Let us consider

$$(\widetilde{F},\widetilde{G},H) \in \mathfrak{F}_\infty(\mathcal{F}_{\widetilde{\eta}}), \tag{2.53}$$

and

$$\zeta_1^0 \in H^2_{\#}(\mathcal{I}), \quad \zeta_2^0 \in H^1_{\#}(\mathcal{I}), \quad \widetilde{w}^0 \in \mathbf{H}^1(\mathcal{F}_{\widetilde{\eta}}), \tag{2.54}$$

such that

$$\operatorname{div} \widetilde{w}^0 = \widetilde{G}(0,\cdot) \text{ in } \mathcal{F}_{\widetilde{\eta}}, \quad \widetilde{w}^0(s, 1+\widetilde{\eta}(s)) = \zeta_2^0(s) e_2 \quad s \in \mathcal{I}, \quad \widetilde{w}^0 = 0 \quad \text{on } \Gamma_{-1}. \tag{2.55}$$

Let us consider the solution

$$(\widetilde{w}, \widetilde{q}, \eta) \in \mathfrak{W}_\infty(\mathcal{F}_{\widetilde{\eta}}) \tag{2.56}$$

of (1.15)-(1.16) given by Theorem 1.1 (with η^* replaced by $\widetilde{\eta}$).

Thus, we set

$$w(t,y) \stackrel{\text{def}}{=} \widetilde{w}(t, X(y)), \quad q(t,y) \stackrel{\text{def}}{=} \widetilde{q}(t, X(y)),$$

$$F(t,y) \stackrel{\text{def}}{=} \widetilde{F}(t, X(y)), \quad G(t,y) \stackrel{\text{def}}{=} \det(\nabla X(y)) \widetilde{G}(t, X(y)), \quad w^0(y) \stackrel{\text{def}}{=} \widetilde{w}^0(X(y)), \tag{2.57}$$

and we verify that (w, q, η) is solution to the system

$$\left\{ \begin{array}{c} \partial_t w - \operatorname{div} \mathbb{T}(w,q) = \widehat{F}_2(\widetilde{\eta}, w, q) + F \quad \text{in } (0,\infty) \times \mathcal{F}, \\ \operatorname{div} w = \widehat{G}(\widetilde{\eta}, w) + G \quad \text{in } (0,\infty) \times \mathcal{F}, \\ w(t, s, 1+\eta^*(s)) = (\partial_t \eta)(t,s) e_2, \quad t > 0, \ s \in \mathcal{I}, \\ w = 0, \quad t > 0, \ y \in \Gamma_{-1}, \\ \partial_{tt}\eta + A_1 \eta = -\mathbb{H}_{\eta^*}(w,q) + \widehat{H}(\widetilde{\eta}, w) + H, \quad t > 0, \end{array} \right. \tag{2.58}$$

with the initial conditions

$$\eta(0,\cdot) = \zeta_1^0, \quad \partial_t \eta(0,\cdot) = \zeta_2^0 \quad \text{and} \quad w(0,\cdot) = w^0 \quad \text{in } \mathcal{F}, \tag{2.59}$$

where the mappings $\widehat{F}_2(\widetilde{\eta}, w, q)$, $\widehat{G}(\widetilde{\eta}, w)$ and $\widehat{H}(\widetilde{\eta}, w)$ are given by (2.11), (2.12) and (2.13). In Appendix B.3 we prove that

$$\left\| \left(\widehat{F}_2(\widetilde{\eta}, w, q), \widehat{G}(\widetilde{\eta}, w), \widehat{H}(\widetilde{\eta}, w) \right) \right\|_{\mathfrak{F}_\infty(\mathcal{F})} \leqslant C \|\widetilde{\eta} - \eta^*\|_{H^{7/4}(\mathcal{I})} \|(w,q,\eta)\|_{\mathfrak{W}_\infty(\mathcal{F})}, \tag{2.60}$$

where $C = C(R) > 0$. The above estimate and the definition of $\widehat{C}_L(\eta^*)$ yield

$$\begin{aligned} \|(w,q,\eta)\|_{\mathfrak{W}_\infty(\mathcal{F})} \leqslant \widehat{C}_L(\eta^*) \left[\|(w^0, \zeta_1^0, \zeta_2^0)\|_{\mathbf{H}^1(\mathcal{F}) \times H^2_{\#}(\mathcal{I}) \times H^1_{\#}(\mathcal{I})} + \|(F,G,H)\|_{\mathfrak{F}_\infty(\mathcal{F})} \right] \\ + C \widehat{C}_L(\eta^*) \|\widetilde{\eta} - \eta^*\|_{H^{7/4}_{\#}(\mathcal{I})} \|(w,q,\eta)\|_{\mathfrak{W}_\infty(\mathcal{F})}, \end{aligned}$$

and for $\kappa^* = \kappa^*(R)$ small enough in (2.51) we obtain

$$\|(w,q,\eta)\|_{\mathfrak{W}_\infty(\mathcal{F})} \leqslant 2\widehat{C}_L(\eta^*) \left(\|(w^0, \zeta_1^0, \zeta_2^0)\|_{\mathbf{H}^1(\mathcal{F}) \times H^2_{\#}(\mathcal{I}) \times H^1_{\#}(\mathcal{I})} + \|(F,G,H)\|_{\mathfrak{F}_\infty(\mathcal{F})} \right).$$

Moreover, we prove in Appendix B.3 that there exists a constant $C = C(R)$ such that

$$\|(\widetilde{w}, \widetilde{q}, \eta)\|_{\mathfrak{W}_\infty(\mathcal{F}_{\widetilde{\eta}})} \leqslant C \|(w,q,\eta)\|_{\mathfrak{W}_\infty(\mathcal{F})}, \quad \|w^0\|_{\mathbf{H}^1(\mathcal{F})} \leqslant C \|\widetilde{w}^0\|_{\mathbf{H}^1(\mathcal{F}_{\widetilde{\eta}})} \tag{2.61}$$

and

$$\|(F,G,H)\|_{\mathfrak{F}_\infty(\mathcal{F})} \leqslant C \|(\widetilde{F}, \widetilde{G}, H)\|_{\mathfrak{F}_\infty(\mathcal{F}_{\widetilde{\eta}})}. \tag{2.62}$$

Thus we deduce that

$$\|(\widetilde{w}, \widetilde{q}, \eta)\|_{\mathfrak{W}_\infty(\mathcal{F}_{\widetilde{\eta}})} \leqslant C \widehat{C}_L(\eta^*) \left(\|(\widetilde{w}^0, \zeta_1^0, \zeta_2^0)\|_{\mathbf{H}^1(\mathcal{F}_{\widetilde{\eta}}) \times H^2_{\#}(\mathcal{I}) \times H^1_{\#}(\mathcal{I})} + \|(\widetilde{F}, \widetilde{G}, H)\|_{\mathfrak{F}_\infty(\mathcal{F}_{\widetilde{\eta}})} \right),$$

for some constant $C = C(R)$. Then (2.49) follows. □

Proof of Corollary 1.3. Using Theorem 1.1 and Proposition 1.2, we only need to extend $(F,G,H) \in \mathfrak{F}_T(\mathcal{F}_{\eta^*})$ in functions defined on $(0,\infty)$ with a control on the norm of the extensions. For F and H we simply extend them by 0 in (T,∞). For G we first use the surjectivity of the map defined by (1.19). Using the open mapping theorem, there exists

$$G^{(1)} \in L^2(0,\infty; H^1_{\#}(\mathcal{F}_{\eta^*})) \cap H^1(0,\infty; H^{-1}_{\#}(\mathcal{F}_{\eta^*}))$$

such that

$$G^{(1)}(0,\cdot) = G(0,\cdot), \quad \|G^{(1)}\|_{L^2(0,\infty;H^1_\#(\mathcal{F}_{\eta^*}))\cap H^1(0,\infty;H^{-1}_\#(\mathcal{F}_{\eta^*}))} \leqslant C\|G(0,\cdot)\|_{L^2_\#(\mathcal{F}_{\eta^*})} \leqslant C\|w^0\|_{\mathbf{H}^1(\mathcal{F}_{\eta^*})}.$$

Then we set

$$G^{(2)} \overset{\text{def}}{=} G - G^{(1)} \in L^2(0,T;H^1_\#(\mathcal{F}_{\eta^*})) \cap H^1(0,T;H^{-1}_\#(\mathcal{F}_{\eta^*}))$$

and we have $G^{(2)}(0,\cdot) = 0$. We extend $G^{(2)}$ in $(T,2T)$ by setting $G^{(2)}(t) = G^{(2)}(2T-t)$ so that

$$G^{(2)} \in L^2(0,2T;H^1_\#(\mathcal{F}_{\eta^*})) \cap H^1_0(0,2T;H^{-1}_\#(\mathcal{F}_{\eta^*}))$$

and then we extend $G^{(2)}$ by 0 in $(2T,\infty)$. This gives us an extension of (F,G,H) such that $(F,G,H) \in \mathfrak{F}_\infty(\mathcal{F}_{\eta^*})$ and

$$\|(F,G,H)\|_{\mathfrak{F}_\infty(\mathcal{F}_{\eta^*})} \leqslant C\|(F,G,H)\|_{\mathfrak{F}_T(\mathcal{F}_{\eta^*})}$$

for some constant C independent of (F,G,H) and of T. □

3 Resolvent estimates of A_0

This section is devoted to the proof of Proposition 2.7 that allows us to obtain Theorem 2.5, that is the analyticity and the exponential stability of the semigroup generated by A_0.

3.1 Definition and properties of some operators

We define the Stokes operator

$$\mathcal{D}(\mathbb{A}) \overset{\text{def}}{=} \mathbf{V}^1_0(\mathcal{F}) \cap \mathbf{H}^2(\mathcal{F}), \quad \mathbb{A} \overset{\text{def}}{=} \nu\mathbb{P}\Delta : \mathcal{D}(\mathbb{A}) \to \mathbf{V}^0_n(\mathcal{F}), \tag{3.1}$$

where $\mathbb{P} : \mathbf{L}^2(\mathcal{F}) \to \mathbf{V}^0_n(\mathcal{F})$ is the Leray projection operator.

Let us consider $v_{\lambda,f} \overset{\text{def}}{=} (\lambda I - \mathbb{A})^{-1}\mathbb{P}f$, that is the solution of

$$\left\{\begin{array}{rl} \lambda\widehat{v}_{\lambda,f} - \operatorname{div}\mathbb{T}(\widehat{v}_{\lambda,f},\widehat{p}_{\lambda,f}) = f & \text{in } \mathcal{F}, \\ \operatorname{div}\widehat{v}_{\lambda,f} = 0 & \text{in } \mathcal{F}, \\ \widehat{v}_{\lambda,f} = 0 & \text{on } \partial\mathcal{F}. \end{array}\right. \tag{3.2}$$

We can define the following operator (see (2.22))

$$\mathcal{T}_\lambda \in \mathcal{L}(\mathbf{L}^2(\mathcal{F}), \mathcal{D}(A_1^{1/4})), \quad \mathcal{T}_\lambda f \overset{\text{def}}{=} -\Lambda^*\left\{\mathbb{T}(\widehat{v}_{\lambda,f},\widehat{p}_{\lambda,f})n_{|\partial\mathcal{F}}\right\}. \tag{3.3}$$

Using a trace theorem, regularity results for Stokes system with H^2-boundary (see [12, Lemma 1]) and resolvent estimates for the Stokes operator $\mathbb{A}$ (defined by (3.1)), we have that

$$\sup_{\lambda\in\mathbb{C}^+} \|\mathcal{T}_\lambda\|_{\mathcal{L}(\mathbf{L}^2(\mathcal{F}),\mathcal{D}(A_1^{1/4}))} < \infty. \tag{3.4}$$

Let us consider the following system for $\lambda \in \mathbb{C}^+$:

$$\left\{\begin{array}{rl} \lambda w_{\lambda,\eta} - \operatorname{div}\mathbb{T}(w_{\lambda,\eta},q_{\lambda,\eta}) = 0 & \text{in } \mathcal{F}, \\ \operatorname{div} w_{\lambda,\eta} = 0 & \text{in } \mathcal{F}, \\ w_{\lambda,\eta} = \Lambda\eta & \text{on } \partial\mathcal{F}, \end{array}\right. \tag{3.5}$$

where Λ is defined by (2.21). Since $\eta^* \in H^2(\mathcal{I})$ we can apply [12, Lemma 1]: for any $\eta \in \mathcal{D}(A_1^{3/4})$, the above system admits a unique solution $(w_{\lambda,\eta},q_{\lambda,\eta}) \in \mathbf{H}^2(\mathcal{F}) \times H^1_\#(\mathcal{F})$. We can thus define the operators

$$W_\lambda \in \mathcal{L}(\mathcal{D}(A_1^{3/4}), \mathbf{H}^2(\mathcal{F})), \quad Q_\lambda \in \mathcal{L}(\mathcal{D}(A_1^{3/4}), H^1_\#(\mathcal{F})), \quad L_\lambda \in \mathcal{L}(\mathcal{D}(A_1^{3/4}), \mathcal{D}(A_1^{1/4})) \tag{3.6}$$

as

$$W_\lambda\eta \overset{\text{def}}{=} w_{\lambda,\eta}, \quad Q_\lambda\eta \overset{\text{def}}{=} q_{\lambda,\eta}, \quad L_\lambda\eta \overset{\text{def}}{=} \Lambda^*\left\{\mathbb{T}(w_{\lambda,\eta},q_{\lambda,\eta})n_{|\partial\mathcal{F}}\right\}. \tag{3.7}$$

Note that we have

$$W_\lambda \in \mathcal{L}(\mathcal{D}(A_1^{1/4}), \mathbf{H}^1(\mathcal{F})) \cap \mathcal{L}(\mathcal{D}(A_1^{1/4})', \mathbf{L}^2(\mathcal{F})), \tag{3.8}$$

and the following estimate

$$\|W_\lambda \eta\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\|A_1^{-1/4}\eta\|_{\mathcal{H}_\mathcal{S}} \quad (\eta \in \mathcal{D}(A_1^{1/4})'), \tag{3.9}$$

where the constant C does not depend on λ (see [4]).

We can now prove Proposition 2.8:

Proof of Proposition 2.8. With the above notation, we see that

$$\mathcal{D}(A_0) = \left\{[w, \eta_1, \eta_2] \in \mathbf{H}^2(\mathcal{F}) \times \mathcal{D}(A_1^{5/4}) \times \mathcal{D}(A_1^{3/4}) \ ; \ w - W_0\eta_2 \in \mathcal{D}(\mathbb{A})\right\}$$

and

$$\mathcal{H} = \left\{[w, \eta_1, \eta_2] \in \mathbf{L}^2(\mathcal{F}) \times \mathcal{D}(A_1^{3/4}) \times \mathcal{D}(A_1^{1/4}) \ ; \ w - W_0\eta_2 \in \mathbf{V}_n^0(\mathcal{F})\right\}.$$

We deduce by interpolation that:

$$[\mathcal{D}(A_0), \mathcal{H}]_{1/2} = \left\{[w, \eta_1, \eta_2] \in \mathcal{H} \ ; \ (\eta_1, \eta_2) \in \mathcal{D}(A_1) \times \mathcal{D}(A_1^{1/2}), w - W_0\eta_2 \in [\mathcal{D}(\mathbb{A}), \mathbf{V}_n^0(\mathcal{F})]_{1/2}\right\}.$$

Then from [10, Theorem 1.1] (which remains true for nonsmooth boundary provided H^2-regularity for Stokes system holds) and [14] we deduce

$$[\mathcal{D}(\mathbb{A}), \mathbf{V}_n^0(\mathcal{F})]_{1/2} = [\mathbf{H}^2(\mathcal{F}) \cap \mathbf{H}_0^1(\mathcal{F}), \mathbf{L}^2(\mathcal{F})]_{1/2} \cap \mathbf{V}_n^0(\mathcal{F}) = \mathbf{V}_0^1(\mathcal{F}).$$

□

In system (3.5), we can write

$$(w_{\lambda,\eta}, q_{\lambda,\eta}) = (w_{0,\eta}, q_{0,\eta}) + \lambda(z_{\lambda,\eta}, \pi_{\lambda,\eta}) \tag{3.10}$$

where

$$\begin{cases} \lambda z_{\lambda,\eta} - \operatorname{div} \mathbb{T}(z_{\lambda,\eta}, \pi_{\lambda,\eta}) = -w_{0,\eta} & \text{in } \mathcal{F}, \\ \operatorname{div} z_{\lambda,\eta} = 0 & \text{in } \mathcal{F}, \\ z_{\lambda,\eta} = 0 & \text{on } \partial\mathcal{F}. \end{cases} \tag{3.11}$$

From (3.5) and (3.7), we have the following relations

$$\langle L_\lambda \eta, \zeta\rangle = \lambda \int_\mathcal{F} w_{\lambda,\eta} \cdot \overline{w_{\lambda,\zeta}} \ dy + 2\nu \int_\mathcal{F} \mathbb{D}(w_{\lambda,\eta}) : \mathbb{D}(\overline{w_{\lambda,\zeta}}) \ dy \tag{3.12}$$

and

$$2\nu \int_\mathcal{F} \mathbb{D}(w_{0,\eta}) : \mathbb{D}(\overline{z_{\lambda,\zeta}}) \ dy = 2\nu \int_\mathcal{F} \mathbb{D}(z_{\lambda,\eta}) : \mathbb{D}(\overline{w_{0,\zeta}}) \ dy = 0.$$

Combining the above relations, we deduce the following decomposition

$$L_\lambda = L_0 + \lambda K_\lambda^{(1)} + |\lambda|^2 K_\lambda^{(2)} = L_0 + \lambda K_\lambda, \quad K_\lambda \stackrel{\text{def}}{=} K_\lambda^{(1)} + \overline{\lambda} K_\lambda^{(2)}, \tag{3.13}$$

where

$$\langle K_\lambda^{(1)} \eta, \zeta\rangle \stackrel{\text{def}}{=} \int_\mathcal{F} w_{\lambda,\eta} \cdot \overline{w_{\lambda,\zeta}} \ dy \quad \text{and} \quad \langle K_\lambda^{(2)} \eta, \zeta\rangle \stackrel{\text{def}}{=} 2\nu \int_\mathcal{F} \mathbb{D}(z_{\lambda,\eta}) : \mathbb{D}(\overline{z_{\lambda,\zeta}}) \ dy. \tag{3.14}$$

We have the following properties on these operators:

Proposition 3.1. *The operator L_0 can be extended as a self-adjoint positive operator*

$$L_0 \in \mathcal{L}(\mathcal{D}(A_1^{1/4}), \mathcal{D}(A_1^{1/4})')$$

and satisfies

$$\rho_1 \|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}}^2 \leqslant \langle L_0 \eta, \eta\rangle_{\mathcal{D}(A_1^{1/4})', \mathcal{D}(A_1^{1/4})} \leqslant \rho_2 \|A_1^{1/4}\eta\|_{\mathcal{H}_\mathcal{S}}^2 \quad (\eta \in \mathcal{D}(A_1^{1/4})), \tag{3.15}$$

for some constants $\rho_1, \rho_2 > 0$.

The operators

$$K_\lambda^{(1)} \in \mathcal{L}(\mathcal{D}(A_1^{1/4})', \mathcal{D}(A_1^{1/4})) \quad \textit{and} \quad K_\lambda^{(2)} \in \mathcal{L}(\mathcal{D}(A_1^{1/4})', \mathcal{D}(A_1^{1/4}))$$

are positive and self-adjoint and there exists $\rho_3 > 0$ such that for any λ such that $\operatorname{Re} \lambda > 0$, we have

$$0 \leqslant \langle K_\lambda^{(1)} \eta, \eta\rangle_{\mathcal{H}_S} \leqslant \rho_3 \|A_1^{-1/4}\eta\|_{\mathcal{H}_\mathcal{S}}^2 \quad (\eta \in (\mathcal{D}(A_1^{1/4}))'), \tag{3.16}$$

$$0 \leqslant \langle K_\lambda^{(2)} \eta, \eta\rangle_{\mathcal{H}_S} \leqslant \frac{\rho_3}{|\lambda|} \|A_1^{-1/4}\eta\|_{\mathcal{H}_\mathcal{S}}^2 \quad (\eta \in (\mathcal{D}(A_1^{1/4}))'). \tag{3.17}$$

Proof. The first part of the proposition comes from (3.12), from (3.8) and from Korn's inequality with a trace theorem. The properties of $K_\lambda^{(1)}$ are a consequence of (3.9). The properties of $K_\lambda^{(2)}$ are a consequence of (3.9) with the estimate (see [4, Proposition 3.2, (3.15)]):

$$\|z_{\lambda,\eta}\|_{\mathbf{H}^1(\mathcal{F})} \leqslant C|\lambda|^{-1/2}\|w_{0,\eta}\|_{\mathbf{L}^2(\mathcal{F})}.$$

□

Next, we define the operator

$$V_\lambda = \lambda^2 I + \lambda L_\lambda + A_1 = \lambda^2(I + K_\lambda) + \lambda L_0 + A_1, \tag{3.18}$$

and an "approximation" of V_λ:

$$\widetilde{V}_\lambda \stackrel{\text{def}}{=} \lambda^2(I + K_\lambda) + 2\rho\lambda A_1^{1/2} + A_1, \tag{3.19}$$

where $\rho > 0$ is a constant to be fixed later.

The operator V_λ is crucial is the forthcoming analysis to prove Theorem 2.5. It appears naturally when we consider the equation $(\lambda I - A_0)[v, \eta_1, \eta_2] = [f, g, h]$. Indeed, if $f = 0$ and $g = 0$, this equation is transformed into

$$v = W_\lambda \eta_2, \quad \eta_2 = \lambda\eta_1, \quad V_\lambda \eta_1 = h. \tag{3.20}$$

We thus need to study the inverse of V_λ. In particular from the second step of Theorem 2.5 that we already did, we know that $V_\lambda : \mathcal{D}(A_1^{5/4}) \to \mathcal{D}(A_1^{1/4})$ is invertible.

We have the following properties on K_λ:

Lemma 3.2. *The operator K_λ defined by (3.13) satisfies for any $\lambda \in \mathbb{C}_0^+$:*

$$\|(I + K_\lambda)\eta\|_{\mathcal{H}_S} \geqslant \|\eta\|_{\mathcal{H}_S} \quad (\eta \in \mathcal{H}_S), \tag{3.21}$$

$$\|A_1^{1/4} K_\lambda \eta\|_{\mathcal{H}_S} \leqslant C\|A_1^{-1/4}\eta\|_{\mathcal{H}_S} \quad (\eta \in (\mathcal{D}(A_1^{1/4}))'), \tag{3.22}$$

$$\|A_1^{1/4}\eta\|_{\mathcal{H}_S} \leqslant C\|A_1^{1/4}(I + K_\lambda)\eta\|_{\mathcal{H}_S} \quad (\eta \in (\mathcal{D}(A_1^{1/4}))), \tag{3.23}$$

where the constant C does not depend on η or on λ.

Proof. First, from (3.16)-(3.17) we deduce that $\operatorname{Re}\langle K_\lambda \eta, \eta\rangle_{\mathcal{H}_S} \geqslant 0$ if $\operatorname{Re}(\lambda) \geqslant 0$, which yields (3.21).

Second, multiplying the first equation of (3.11) by $w_{\lambda,\zeta}$ and integrating by parts, we find that

$$K_\lambda \eta = \Lambda^* \left\{ \mathbb{T}(z_{\lambda,\eta}, \pi_{\lambda,\eta}) n_{|\partial\mathcal{F}} \right\}. \tag{3.24}$$

Thus, using the trace inequality, classical resolvent estimates for the Stokes operator and (3.9), we obtain

$$\begin{aligned}\left\|A_1^{1/4} K_\lambda \eta\right\|_{\mathcal{H}_S} &\leqslant C \left\|\mathbb{T}(z_{\lambda,\eta}, \pi_{\lambda,\eta}) n\right\|_{\mathbf{H}^{1/2}(\partial\mathcal{F})} \leqslant C \left\|(z_{\lambda,\eta}, \pi_{\lambda,\eta})\right\|_{\mathbf{H}^2(\mathcal{F}) \times H^1_\#(\mathcal{F})} \\ &\leqslant C\|w_{0,\lambda}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\|A_1^{-1/4}\eta\|_{\mathcal{H}_S}.\end{aligned}$$

Finally, for the last relation, we use (3.21) and (3.22):

$$\begin{aligned}\left\|A_1^{1/4}\eta\right\|_{\mathcal{H}_S} &\leqslant \left\|A_1^{1/4}(I + K_\lambda)\eta\right\|_{\mathcal{H}_S} + \left\|A_1^{1/4} K_\lambda \eta\right\|_{\mathcal{H}_S} \leqslant \left\|A_1^{1/4}(I + K_\lambda)\eta\right\|_{\mathcal{H}_S} + C\left\|\eta\right\|_{\mathcal{H}_S} \\ &\leqslant \left\|A_1^{1/4}(I + K_\lambda)\eta\right\|_{\mathcal{H}_S} + C\left\|(I + K_\lambda)\eta\right\|_{\mathcal{H}_S} \leqslant C\left\|A_1^{1/4}(I + K_\lambda)\eta\right\|_{\mathcal{H}_S}.\end{aligned}$$

□

We have the same properties for the adjoint of K_λ:

Lemma 3.3. *The operator K_λ defined by (3.13) satisfies for any $\lambda \in \mathbb{C}_0^+$:*

$$\|(I + K_\lambda^*)\eta\|_{\mathcal{H}_S} \geqslant \|\eta\|_{\mathcal{H}_S} \quad (\eta \in \mathcal{H}_S), \tag{3.25}$$

$$\|A_1^{1/4} K_\lambda^* \eta\|_{\mathcal{H}_S} \leqslant C\|A_1^{-1/4}\eta\|_{\mathcal{H}_S} \quad (\eta \in (\mathcal{D}(A_1^{1/4}))'), \tag{3.26}$$

$$\|A_1^{1/4}\eta\|_{\mathcal{H}_S} \leqslant C\|A_1^{1/4}(I + K_\lambda^*)\eta\|_{\mathcal{H}_S} \quad (\eta \in (\mathcal{D}(A_1^{1/4}))), \tag{3.27}$$

where the constant C does not depend on η or on λ.

Proof. We note that (3.13) yields

$$K_\lambda^* = K_\lambda^{(1)} + \lambda K_\lambda^{(2)},$$

and thus $\operatorname{Re}\langle K_\lambda^* \eta, \eta\rangle_{\mathcal{H}_S} \geqslant 0$ if $\operatorname{Re}(\lambda) \geqslant 0$, which yields (3.25). Relation (3.26) is a direct consequence of (3.22) Finally, the proof of (3.27) is the same as the proof of (3.23). □

3.2 Estimation of $\widetilde{V}_\lambda^{-1}$

The aim of this section is to estimate the inverse of $\widetilde{V}_\lambda$ defined by (3.19). We recall that the notation $\mathbb{C}_\alpha^+$ is introduced in (2.19). First we show the following result on an "approximation" of $\widetilde{V}_\lambda$:

Lemma 3.4. *There exists a constant C_1 such that for all $\lambda \in \mathbb{C}^+$,*

$$\left\|(\lambda^2 I + 2\rho\lambda A_1^{1/2} + A_1)\eta\right\|_{\mathcal{H}_S} \geqslant C_1 \left(|\lambda|^2 \|\eta\|_{\mathcal{H}_S} + \|A_1\eta\|_{\mathcal{H}_S}\right) \quad (\eta \in \mathcal{D}(A_1)). \tag{3.28}$$

Proof. We write

$$\begin{aligned}\left\|(\lambda^2 I + 2\rho\lambda A_1^{1/2} + A_1)\eta\right\|_{\mathcal{H}_S}^2 &= |\lambda|^4 \|\eta\|_{\mathcal{H}_S}^2 + 4\rho^2|\lambda|^2 \left\|A_1^{1/2}\eta\right\|_{\mathcal{H}_S}^2 + \|A_1\eta\|_{\mathcal{H}_S}^2 \\ &\quad + 4\rho \operatorname{Re}\lambda \left(\left\|\lambda A_1^{1/4}\eta\right\|_{\mathcal{H}_S}^2 + \left\|A_1^{3/4}\eta\right\|_{\mathcal{H}_S}^2\right) + 2\operatorname{Re}\lambda^2 \left\|A_1^{1/2}\eta\right\|_{\mathcal{H}_S}^2 \\ &\geqslant |\lambda|^4 \|\eta\|_{\mathcal{H}_S}^2 + 2(2\rho^2-1)|\lambda|^2 \left\|A_1^{1/2}\eta\right\|_{\mathcal{H}_S}^2 + \|A_1\eta\|_{\mathcal{H}_S}^2 .\end{aligned} \tag{3.29}$$

If $2\rho^2 - 1 \geqslant 0$, then we deduce

$$\left\|(\lambda^2 I + 2\rho\lambda A_1^{1/2} + A_1)\eta\right\|_{\mathcal{H}_S}^2 \geqslant |\lambda|^4 \|\eta\|_{\mathcal{H}_S}^2 + \|A_1\eta\|_{\mathcal{H}_S}^2 .$$

Else, we have $2\rho^2 - 1 < 0$ and we deduce from (3.29) that

$$\begin{aligned}\left\|(\lambda^2 I + 2\rho\lambda A_1^{1/2} + A_1)\eta\right\|_{\mathcal{H}_S}^2 &\geqslant |\lambda|^4 \|\eta\|_{\mathcal{H}_S}^2 + 2(2\rho^2-1)|\lambda|^2 \|\eta\|_{\mathcal{H}_S} \|A_1\eta\|_{\mathcal{H}_S} + \|A_1\eta\|_{\mathcal{H}_S}^2 \\ &= (1-2\rho^2)(|\lambda|^2 \|\eta\|_{\mathcal{H}_S} - \|A_1\eta\|_{\mathcal{H}_S})^2 + 2\rho^2 \left(|\lambda|^4 \|\eta\|_{\mathcal{H}_S}^2 + \|A_1\eta\|_{\mathcal{H}_S}^2\right).\end{aligned} \tag{3.30}$$

□

Theorem 3.5. *There exists $\alpha > 0$ such that for all $\lambda \in \mathbb{C}_\alpha^+$ the operator $\widetilde{V}_\lambda : \mathcal{D}(A_1^{5/4}) \to \mathcal{D}(A_1^{1/4})$ is an isomorphism and for $(\theta, \beta) \in [-1/4, 5/4]^2$ such that $0 \leqslant \theta + \beta \leqslant 1$, the following estimate holds*

$$\sup_{\lambda \in \mathbb{C}_\alpha^+} |\lambda|^{2-2\theta-2\beta} \|A_1^\theta \widetilde{V}_\lambda^{-1} A_1^\beta\|_{\mathcal{L}(\mathcal{H}_S)} < +\infty, \tag{3.31}$$

$$\sup_{\lambda \in \mathbb{C}_\alpha^+} |\lambda|^{2-2\theta-2\beta} \|A_1^\theta (\widetilde{V}_\lambda^*)^{-1} A_1^\beta\|_{\mathcal{L}(\mathcal{H}_S)} < +\infty. \tag{3.32}$$

Proof. We combine (3.19) with Lemma 3.2

$$\begin{aligned}\left\|A_1^{1/4}\widetilde{V}_\lambda\eta\right\|_{\mathcal{H}_S} &\geqslant \left\|A_1^{1/4}(I+K_\lambda)\left[\lambda^2\eta + 2\rho\lambda A_1^{1/2}\eta + A_1\eta\right]\right\|_{\mathcal{H}_S} - \left\|A_1^{1/4}K_\lambda\left[2\rho\lambda A_1^{1/2}\eta + A_1\eta\right]\right\|_{\mathcal{H}_S} \\ &\geqslant \left\|\lambda^2 A_1^{1/4}\eta + 2\rho\lambda A_1^{3/4}\eta + A_1^{5/4}\eta\right\|_{\mathcal{H}_S} - C\left(|\lambda|\left\|A_1^{1/4}\eta\right\|_{\mathcal{H}_S} + \left\|A_1^{3/4}\eta\right\|_{\mathcal{H}_S}\right).\end{aligned} \tag{3.33}$$

Applying Lemma 3.4, we deduce

$$\left\|A_1^{1/4}\widetilde{V}_\lambda\eta\right\|_{\mathcal{H}_S} \geqslant C_1\left(|\lambda|^2 \|A_1^{1/4}\eta\|_{\mathcal{H}_S} + \|A_1^{5/4}\eta\|_{\mathcal{H}_S}\right) - C\left(|\lambda|\left\|A_1^{1/4}\eta\right\|_{\mathcal{H}_S} + \left\|A_1^{3/4}\eta\right\|_{\mathcal{H}_S}\right). \tag{3.34}$$

On the other hand, from an interpolation inequality and the Young inequality we have

$$\left\|A_1^{3/4}\eta\right\|_{\mathcal{H}_S} \leqslant C|\lambda|^{-1}|\lambda| \left\|A_1^{1/4}\eta\right\|_{\mathcal{H}_S}^{1/2} \left\|A_1^{5/4}\eta\right\|_{\mathcal{H}_S}^{1/2} \leqslant C|\lambda|^{-1}\left(|\lambda|^2 \|A_1^{1/4}\eta\|_{\mathcal{H}_S} + \|A_1^{5/4}\eta\|_{\mathcal{H}_S}\right). \tag{3.35}$$

Combining the above inequality with (3.34), we deduce that for α large enough, and for $\lambda \in \mathbb{C}_\alpha^+$,

$$\left\|A_1^{1/4}\widetilde{V}_\lambda\eta\right\|_{\mathcal{H}_S} \geqslant C\left(|\lambda|^2 \|A_1^{1/4}\eta\|_{\mathcal{H}_S} + \|A_1^{5/4}\eta\|_{\mathcal{H}_S}\right). \tag{3.36}$$

Since

$$\widetilde{V}_\lambda^* = \overline{\lambda}^2(I + K_\lambda^*) + 2\rho\overline{\lambda}A_1^{1/2} + A_1,$$

and since K_λ^* satisfies the same properties as K_λ (see Lemma 3.3), we also deduce that for α large enough, and for $\lambda \in \mathbb{C}_\alpha^+$,

$$\left\| A_1^{1/4}\widetilde{V}_\lambda^*\eta \right\|_{\mathcal{H}_S} \geqslant C\left(|\lambda|^2 \|A_1^{1/4}\eta\|_{\mathcal{H}_S} + \|A_1^{5/4}\eta\|_{\mathcal{H}_S} \right). \tag{3.37}$$

From (3.36) and (3.37), we deduce that $\widetilde{V}_\lambda : \mathcal{D}(A_1^{5/4}) \to \mathcal{D}(A_1^{1/4})$ is a closed operator (since A_1 is a closed operator) and has a closed range. Applying [7, Corollary II.17, p.28], we deduce that $\widetilde{V}_\lambda$ is invertible. Moreover (3.36) and (3.37) yield (3.31) and (3.32) for $(\theta, \beta) = (1/4, -1/4)$ and $(\theta, \beta) = (5/4, -1/4)$. By a duality argument, this implies (3.31) and (3.32) for $(\theta, \beta) = (-1/4, 1/4)$ and $(\theta, \beta) = (-1/4, 5/4)$. We deduce the result by an interpolation argument. □

3.3 Estimation of V_λ^{-1}

In order to show the resolvent estimate for the operator A_0 defined by (2.31)–(2.32), assume $\lambda \in \mathbb{C}_\alpha^+$ for $\alpha > 0$ given in Theorem 3.5 and $[f, g, h] \in \mathcal{H}$. We have that $[v, \eta_1, \eta_2] \stackrel{\text{def}}{=} (\lambda I - A_0)^{-1}[f, g, h]$ satisfies

$$\begin{cases} \lambda v - \operatorname{div} \mathbb{T}(v, p) = f \quad \text{in } \mathcal{F}, \\ \operatorname{div} v = 0 \quad \text{in } \mathcal{F}, \\ v = \Lambda\eta_2 \quad \text{on } \partial\mathcal{F}, \\ \lambda\eta_1 - \eta_2 = g \\ \lambda\eta_2 + A_1\eta_1 = -\Lambda^*\left\{\mathbb{T}(v, p)n_{|\partial\mathcal{F}}\right\} + h, \end{cases} \tag{3.38}$$

for some pressure function p.

First we decompose the fluid velocity of the above system by using W_λ and $\mathbb{A}$ introduced in (3.7) and (3.1):

$$v = W_\lambda\eta_2 + (\lambda I - \mathbb{A})^{-1}\mathbb{P}f.$$

This allows us to rewrite the system (3.38) as

$$\begin{cases} \lambda\eta_1 - \eta_2 = g \\ \lambda\eta_2 + A_1\eta_1 + L_\lambda\eta_2 = \mathcal{T}_\lambda f + h, \end{cases} \tag{3.39}$$

where $L_\lambda \in \mathcal{L}(\mathcal{D}(A_1^{3/4}), \mathcal{D}(A_1^{1/4}))$ and $\mathcal{T}_\lambda \in \mathcal{L}(\mathbf{L}^2(\mathcal{F}), \mathcal{D}(A_1^{1/4}))$ are defined by (3.7) and (3.3). Then, we define

$$\mathcal{A}_\lambda \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -I \\ A_1 & L_\lambda \end{bmatrix}, \tag{3.40}$$

and we can write (3.39) as follows:

$$(\lambda I + \mathcal{A}_\lambda)\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} g \\ \mathcal{T}_\lambda f + h \end{bmatrix}. \tag{3.41}$$

We recall that since $\mathbb{C}^+ \subset \rho(A_0)$, we know that V_λ defined in (3.18) is invertible and some calculation yields the following formulas for the inverse of $\lambda I + \mathcal{A}_\lambda$ and of $\lambda I - A_0$:

$$(\lambda I + \mathcal{A}_\lambda)^{-1} = \begin{bmatrix} \dfrac{I - V_\lambda^{-1}A_1}{\lambda} & V_\lambda^{-1} \\ -V_\lambda^{-1}A_1 & \lambda V_\lambda^{-1} \end{bmatrix}, \tag{3.42}$$

and

$$(\lambda I - A_0)^{-1} = \begin{bmatrix} (\lambda I - \mathbb{A})^{-1}\mathbb{P} + \lambda W_\lambda V_\lambda^{-1}\mathcal{T}_\lambda & -W_\lambda V_\lambda^{-1}A_1 & \lambda W_\lambda V_\lambda^{-1} \\ V_\lambda^{-1}\mathcal{T}_\lambda & \dfrac{I - V_\lambda^{-1}A_1}{\lambda} & V_\lambda^{-1} \\ \lambda V_\lambda^{-1}\mathcal{T}_\lambda & -V_\lambda^{-1}A_1 & \lambda V_\lambda^{-1} \end{bmatrix}. \tag{3.43}$$

Here, we estimate the inverse of the operator V_λ defined in (3.18) for $\lambda \in \mathbb{C}_\alpha^+$ and $\alpha > 0$ given in Theorem 3.5. From now on, we fix $\rho < \rho_1/4$ where ρ_1 is defined in Proposition 3.1. The main result of this section is the following:

Theorem 3.6. *There exists $\alpha > 0$ such that, for $(\theta, \beta) \in [-1/4, 3/4]^2$, such that $0 \leqslant \theta + \beta \leqslant 1$, the following estimate holds*

$$\sup_{\lambda \in \mathbb{C}^+_\alpha} |\lambda|^{2-2\theta-2\beta} \|A_1^\theta V_\lambda^{-1} A_1^\beta\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})} < +\infty. \tag{3.44}$$

Proof. Comparing (3.18) and (3.19), we see that

$$V_\lambda - \widetilde{V}_\lambda = \lambda S, \quad S \stackrel{\text{def}}{=} L_0 - 2\rho A_1^{1/2}$$

and thus

$$\widetilde{V}_\lambda^{-1} = [I + \lambda \widetilde{V}_\lambda^{-1} S] V_\lambda^{-1}. \tag{3.45}$$

In the above relation, note that from (3.6), $S : \mathcal{D}(A_1^{3/4}) \to \mathcal{D}(A_1^{1/4})$ so that $\widetilde{V}_\lambda^{-1} S V_\lambda^{-1} : \mathcal{D}(A_1^{1/4}) \to \mathcal{D}(A_1^{5/4})$. Moreover, from Proposition 3.1 and in particular (3.15), we have with our choice of ρ that $S : \mathcal{D}(A_1^{1/4}) \to \mathcal{D}(A_1^{1/4})'$ is a positive self-adjoint operator satisfying

$$\forall \eta \in \mathcal{H}_\mathcal{S}, \quad \|S^{1/2} A_1^{-1/4} \eta\|_{\mathcal{H}_\mathcal{S}} \leqslant C \|\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{3.46}$$

Moreover, (3.46) with a duality argument yields

$$\forall \eta \in \mathcal{H}_\mathcal{S}, \quad \|A_1^{-1/4} S^{1/2} \eta\|_{\mathcal{H}_\mathcal{S}} \leqslant C \|\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{3.47}$$

Combining the above inequalities with (3.31) we obtain for $\beta \in [-1/4, 3/4]$,

$$\forall \eta \in \mathcal{H}_\mathcal{S}, \quad \|S^{1/2} \widetilde{V}_\lambda^{-1} A_1^\beta \eta\|_{\mathcal{H}_\mathcal{S}} \leqslant C|\lambda|^{-3/2+2\beta} \|\eta\|_{\mathcal{H}_\mathcal{S}}, \tag{3.48}$$

and for $\theta \in [-1/4, 3/4]$,

$$\forall \eta \in \mathcal{H}_\mathcal{S}, \quad \|A_1^\theta \widetilde{V}_\lambda^{-1} S^{1/2} \eta\|_{\mathcal{H}_\mathcal{S}} \leqslant C|\lambda|^{-3/2+2\theta} \|\eta\|_{\mathcal{H}_\mathcal{S}}. \tag{3.49}$$

Let us multiply (3.45) by $S V_\lambda^{-1} \eta$ with $\eta \in \mathcal{H}_\mathcal{S}$:

$$\left\langle S^{1/2} \widetilde{V}_\lambda^{-1} \eta, S^{1/2} V_\lambda^{-1} \eta \right\rangle_{\mathcal{H}_\mathcal{S}} = \left\| S^{1/2} V_\lambda^{-1} \eta \right\|^2_{\mathcal{H}_\mathcal{S}} + \left\langle \lambda \widetilde{V}_\lambda^{-1} S V_\lambda^{-1} \eta, S V_\lambda^{-1} \eta \right\rangle_{\mathcal{H}_\mathcal{S}}. \tag{3.50}$$

On the other hand, from (3.19), (3.13), (3.16) and (3.17), we deduce for any $\lambda \in \mathbb{C}^+$ and for any $\zeta \in \mathcal{D}(A_1)$,

$$\begin{aligned} \operatorname{Re}\langle \lambda\zeta, \widetilde{V}_\lambda \zeta \rangle_{\mathcal{H}_\mathcal{S}} = \operatorname{Re}(\lambda) \|\lambda\zeta\|^2_{\mathcal{H}_\mathcal{S}} + |\lambda|^2 \operatorname{Re}(\lambda) \langle K_\lambda^{(1)} \zeta, \zeta \rangle_{\mathcal{H}_\mathcal{S}} + |\lambda|^4 \langle K_\lambda^{(2)} \zeta, \zeta \rangle_{\mathcal{H}_\mathcal{S}} \\ + 2\rho |\lambda|^2 \|A_1^{1/4} \zeta\|^2_{\mathcal{H}_\mathcal{S}} + \operatorname{Re}(\lambda) \|A_1^{1/2} \zeta\|^2_{\mathcal{H}_\mathcal{S}} \geqslant 0. \end{aligned} \tag{3.51}$$

Using $\zeta = \widetilde{V}_\lambda^{-1} S V_\lambda^{-1} \eta$ in the above relation and combining it with (3.50), we deduce

$$\forall \eta \in \mathcal{H}_\mathcal{S}, \quad \|S^{1/2} V_\lambda^{-1} \eta\|_{\mathcal{H}_\mathcal{S}} \leqslant \|S^{1/2} \widetilde{V}_\lambda^{-1} \eta\|_{\mathcal{H}_\mathcal{S}}.$$

Thus, coming back to equality (3.45) we deduce that for $\eta \in \mathcal{H}_\mathcal{S}$, and for $(\theta, \beta) \in [-1/4, 3/4]^2$ such that $0 \leqslant \theta + \beta \leqslant 1$,

$$\begin{aligned} \|A_1^\theta V_\lambda^{-1} A_1^\beta \eta\|_{\mathcal{H}_\mathcal{S}} &\leqslant \|A_1^\theta \widetilde{V}_\lambda^{-1} A_1^\beta \eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda| \|A_1^\theta \widetilde{V}_\lambda^{-1} S V_\lambda^{-1} A_1^\beta \eta\|_{\mathcal{H}_\mathcal{S}} \\ &\leqslant \|A_1^\theta \widetilde{V}_\lambda^{-1} A_1^\beta \eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda| \|A_1^\theta \widetilde{V}_\lambda^{-1} S^{1/2}\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})} \|S^{1/2} V_\lambda^{-1} A_1^\beta \eta\|_{\mathcal{H}_\mathcal{S}} \\ &\leqslant \|A_1^\theta \widetilde{V}_\lambda^{-1} A_1^\beta \eta\|_{\mathcal{H}_\mathcal{S}} + |\lambda| \|A_1^\theta \widetilde{V}_\lambda^{-1} S^{1/2}\|_{\mathcal{L}(\mathcal{H}_\mathcal{S})} \|S^{1/2} \widetilde{V}_\lambda^{-1} A_1^\beta \eta\|_{\mathcal{H}_\mathcal{S}}. \end{aligned}$$

Then using estimates (3.31), (3.48), (3.49) and a density argument yields (3.44). □

3.4 Proof of Proposition 2.7

We are now in position to prove Proposition 2.7 and thus completes the proof of Theorem 2.5.

Proof of Proposition 2.7. Let us consider $\alpha > 0$ given by Theorem 3.6 and let us assume $\lambda \in \mathbb{C}_\alpha^+$ (see (2.19)). From (3.43) and (2.27), we have

$$\left\| \lambda(\lambda I - A_0)^{-1} \begin{bmatrix} f \\ g \\ h \end{bmatrix} \right\|_{\mathcal{H}}^2 = \left\| \lambda(\lambda I - \mathbb{A})^{-1}\mathbb{P}f + \lambda^2 W_\lambda V_\lambda^{-1}\mathcal{T}_\lambda f - \lambda W_\lambda V_\lambda^{-1} A_1 g + \lambda^2 W_\lambda V_\lambda^{-1} h \right\|_{\mathbf{L}^2(\mathcal{F})}^2$$
$$+ \left\| \lambda A_1^{3/4} V_\lambda^{-1}\mathcal{T}_\lambda f + A_1^{3/4}\left(I - V_\lambda^{-1}A_1\right) g + \lambda A_1^{3/4} V_\lambda^{-1} h \right\|_{\mathcal{H}_\mathcal{S}}^2$$
$$+ \left\| \lambda^2 A_1^{1/4} V_\lambda^{-1}\mathcal{T}_\lambda f - \lambda A_1^{1/4} V_\lambda^{-1} A_1 g + \lambda^2 A_1^{1/4} V_\lambda^{-1} h \right\|_{\mathcal{H}_\mathcal{S}}^2 \tag{3.52}$$

Using that $\mathbb{A}$ is the generator of an analytic semigroup, we have

$$|\lambda| \|(\lambda - \mathbb{A})^{-1}\mathbb{P}f\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\|f\|_{\mathbf{L}^2(\mathcal{F})}. \tag{3.53}$$

From (3.9), (3.44) with $(\theta, \beta) = (0, 0)$ and (3.4),

$$|\lambda|^2 \|W_\lambda V_\lambda^{-1}\mathcal{T}_\lambda f\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\|\mathcal{T}_\lambda f\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|f\|_{\mathbf{L}^2(\mathcal{F})}. \tag{3.54}$$

From (3.44) with $(\theta, \beta) = (3/4, -1/4)$ and (3.4),

$$|\lambda| \|A_1^{3/4} V_\lambda^{-1}\mathcal{T}_\lambda f\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|f\|_{\mathbf{L}^2(\mathcal{F})}. \tag{3.55}$$

From (3.44) with $(\theta, \beta) = (1/4, -1/4)$ and (3.4),

$$|\lambda|^2 \|A_1^{1/4} V_\lambda^{-1}\mathcal{T}_\lambda f\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|f\|_{\mathbf{L}^2(\mathcal{F})}. \tag{3.56}$$

From (3.9) and (3.44) with $(\theta, \beta) = (1/4, 1/4)$

$$|\lambda| \|W_\lambda V_\lambda^{-1} A_1 g\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\|A_1^{3/4} g\|_{\mathcal{H}_\mathcal{S}}. \tag{3.57}$$

From (3.44) with $(\theta, \beta) = (3/4, 1/4)$

$$\left\| A_1^{3/4}(I - V_\lambda^{-1}A_1) g \right\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|A_1^{3/4} g\|_{\mathcal{H}_\mathcal{S}}. \tag{3.58}$$

From (3.44) with $(\theta, \beta) = (1/4, 1/4)$

$$|\lambda| \left\| A_1^{1/4} V_\lambda^{-1} A_1 g \right\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|A_1^{3/4} g\|_{\mathcal{H}_\mathcal{S}}. \tag{3.59}$$

From (3.9) and (3.44) with $(\theta, \beta) = (0, 0)$

$$|\lambda|^2 \|W_\lambda V_\lambda^{-1} h\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\|A_1^{1/4} h\|_{\mathcal{H}_\mathcal{S}}. \tag{3.60}$$

From (3.44) with $(\theta, \beta) = (3/4, -1/4)$

$$|\lambda| \left\| A_1^{3/4} V_\lambda^{-1} h \right\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|A_1^{1/4} h\|_{\mathcal{H}_\mathcal{S}}. \tag{3.61}$$

From (3.44) with $(\theta, \beta) = (1/4, -1/4)$

$$|\lambda|^2 \left\| A_1^{1/4} V_\lambda^{-1} h \right\|_{\mathcal{H}_\mathcal{S}} \leqslant C\|A_1^{1/4} h\|_{\mathcal{H}_\mathcal{S}}. \tag{3.62}$$

Combining the above estimates, we deduce (2.42). □

4 Strong solutions for (1.3)-(1.4)

In this section, we recall the main steps to obtain the existence and uniqueness results for the nonlinear system (1.3)-(1.4), or equivalently for the system (2.8)-(2.13).

4.1 Proof of Theorem 1.7

This section is devoted to the proof of Theorem 1.7 (or equivalently Theorem 2.2).

Local in time existence. First, for $(w^0, \eta_1^0, \eta_2^0)$ satisfying (2.14)–(2.15) we show the existence of a local in time solution for (2.8)-(2.13) by a fixed point argument. We recall that in that case $\eta^* = \eta_1^0$. In particular, $\mathrm{Cof}(\nabla X(0,\cdot)) = I_2$ and the divergence condition in (2.15) rewrites $\operatorname{div} w^0 = 0$ in $\mathcal{F}$.

For $R > 0$ and $T > 0$ we consider the closed subset of $\mathfrak{F}_T(\mathcal{F})$ (see (1.17)):

$$\mathfrak{B}_{R,T} \stackrel{\text{def}}{=} \left\{(F, G, H) \in \mathfrak{F}_T(\mathcal{F}) \ ; \ G(0, \cdot) = 0 \text{ in } \mathcal{F}, \quad \|(F, G, H)\|_{\mathfrak{F}_T(\mathcal{F})} \leqslant R\right\}. \tag{4.1}$$

To simplify, we assume here that $T \leqslant 1$. For any $(F, G, H) \in \mathfrak{B}_{R,T}$, we consider the solution (w, η, q) of (1.15)-(1.16) given by Corollary 1.3 with $(\zeta_1^0, \zeta_2^0) = (\eta_1^0, \eta_2^0)$. In particular, using (2.14), we can take R large enough such that

$$R \geqslant 1 + \|[w^0, \eta_1^0, \eta_2^0]\|_{\mathbf{H}^1(\mathcal{F})\times H^2(\mathcal{I})\times H^1(\mathcal{I})}, \quad \inf_{\mathcal{I}} \eta_1^0 + 1 \geqslant 1/R, \tag{4.2}$$

and then there exists a constant $C = C(R)$ such that

$$\|(w, q, \eta)\|_{\mathfrak{W}_T(\mathcal{F})} \leqslant C. \tag{4.3}$$

First we notice that by interpolation, there exists a constant $C = C(R)$ such that

$$\|\eta\|_{H^{3/4}(0,T;H^{7/4}(\mathcal{I}))} + \|\partial_t \eta\|_{L^4(0,T;H^{5/4}(\mathcal{I}))} + \|w\|_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))} \leqslant C. \tag{4.4}$$

Using Proposition A.1 in [3] we deduce

$$\|\eta - \eta_1^0\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))} \leqslant CT^{1/6}\|\eta - \eta_1^0\|_{H^{3/4}(0,T;H^{7/4}(\mathcal{I}))} \leqslant CT^{1/6} \tag{4.5}$$

for some constants $C = C(R)$.

From the above estimate, we see that for T small enough, (2.3) holds and we can construct the change of variables X and Y defined by (2.5) (with $\eta^* = \eta_1^0$). In the appendix, we express X and Y, and their derivatives (see (A.12)–(A.17)). We can then consider the mapping

$$\mathcal{Z} : (F, G, H) \mapsto (\widehat{F}(\eta, w, q), \widehat{G}(\eta, w), \widehat{H}(\eta, w)) \tag{4.6}$$

where the maps $\widehat{F}$, $\widehat{G}$ and $\widehat{H}$ are defined by (2.10), (2.12) and (2.13), and (w, q, η) defined as above. The estimations of $\widehat{F}$, $\widehat{G}$ and $\widehat{H}$ follow from standard arguments and we postpone them in Appendix B. Using (B.27), (B.34) and (B.36), we deduce that for $T \leqslant 1$, $\mathcal{Z}$ is well-defined from $\mathfrak{B}_{R,T}$ into $\mathfrak{F}_T(\mathcal{F})$ and satisfies

$$\|\mathcal{Z}(F, G, H)\|_{\mathfrak{F}_T(\mathcal{F})} \leqslant CT^{1/6}, \tag{4.7}$$

for some constant $C = C(R)$. From (4.7), for all $T > 0$ small enough, we deduce

$$\mathcal{Z}(F, G, H) \in \mathfrak{B}_{R,T}.$$

With estimates similar to (B.27), (B.34) and (B.36), one can also show that, taking T possibly smaller, $\mathcal{Z}$ is a strict contraction on $\mathfrak{B}_{R,T}$ and using the Banach fixed point theorem, we deduce the existence and uniqueness of $(F, G, H) \in \mathfrak{B}_{R,T}$ such that

$$\mathcal{Z}((F, G, H)) = (F, G, H).$$

The corresponding solution (w, q, η) of system (1.15)-(1.16) is a solution of (2.8)-(2.13).

Uniqueness. Assume that $(v^{(i)}, p^{(i)}, \eta^{(i)})$, $i = 1, 2$ are strong solutions of (1.3)-(1.4) on $[0, T^{(i)})$, $T^{(i)} > 0$, associated with $(v^0, \eta_1^0, \eta_2^0)$ (satisfying (1.32)-(1.33)).

Assume that $t_0 \in [0, \min(T^{(1)}, T^{(2)}))$ is such that

$$(\eta^{(1)}(t_0, \cdot), \partial_t \eta^{(1)}(t_0, \cdot), v^{(1)}(t_0, \cdot)) = (\eta^{(2)}(t_0, \cdot), \partial_t \eta^{(2)}(t_0, \cdot), v^{(2)}(t_0, \cdot)).$$

Then we show below that there exists $T > 0$ such that,

$$(v^{(1)}, p^{(1)}, \eta^{(1)}) = (v^{(2)}, p^{(2)}, \eta^{(2)}) \quad \text{in } [t_0, t_0 + T]. \tag{4.8}$$

To prove the above implication, we can assume $t_0 = 0$, the general case follows by changing t to $t - t_0$. First using an estimate of the type (4.5), there exists $\widehat{T} > 0$ depending on $\eta^{(1)}$ and on $\eta^{(2)}$ such that

$$\left\| \eta^i(t, \cdot) - \eta^i(0, \cdot) \right\|_{H^{7/4}(\mathcal{I})} \leqslant \kappa \quad (t \in [0, \widehat{T}], \quad i = 1, 2).$$

In particular, (see (2.3)), we can construct the change of variables X and Y defined by (2.5) with $\eta^* = \eta^{(1)}(0, \cdot)$ and transform $(v^{(i)}, p^{(i)})$ into $(w^{(i)}, q^{(i)})$ so that $(w^{(i)}, q^{(i)}, \eta^{(i)})$ is a strong solution of (2.8)–(2.13) on $[0, T]$, for $T \in (0, \widehat{T}]$. Let us write, for $i = 1, 2$,

$$(F^{(i)}, G^{(i)}, H^{(i)}) \stackrel{\text{def}}{=} (\widehat{F}(\eta^{(i)}, w^{(i)}, q^{(i)}), \widehat{G}(\eta^{(i)}, w^{(i)}), \widehat{H}(\eta^{(i)}, w^{(i)}))$$

where the maps $\widehat{F}$, $\widehat{G}$ and $\widehat{H}$ are defined by (2.10), (2.12) and (2.13). There exists R large enough such that for any $T \in (0, \widehat{T}]$,

$$(F^{(i)}, G^{(i)}, H^{(i)}) \in \mathfrak{B}_{R,T}, \quad (i = 1, 2),$$

where $\mathfrak{B}_{R,T}$ is defined by (4.1) and such that the analogue of (4.2) holds true. They are both fixed points of $\mathcal{Z}$ defined by (4.6), and from the above proof, $\mathcal{Z}$ admits a unique fixed point on $\mathfrak{B}_{R,T}$ for T small enough. We thus deduce (4.8).

Now, let us consider

$$T_0 = \sup \Big\{ T \in [0, \min(T^{(1)}, T^{(2)})) \ ; \\ (\eta^{(1)}(t, \cdot), \partial_t \eta^{(1)}(t, \cdot), v^{(1)}(t, \cdot)) = (\eta^{(2)}(t, \cdot), \partial_t \eta^{(2)}(t, \cdot), v^{(2)}(t, \cdot)) \quad \forall t \in [0, T] \Big\}. \tag{4.9}$$

Using the above property for $t_0 = 0$, the above supremum is well-defined and by continuity of the solutions, if $T_0 < \min(T^{(1)}, T^{(2)})$ then, we have

$$(\eta^{(1)}(T_0, \cdot), \partial_t \eta^{(1)}(T_0, \cdot), v^{(1)}(T_0, \cdot)) = (\eta^{(2)}(T_0, \cdot), \partial_t \eta^{(2)}(T_0, \cdot), v^{(2)}(T_0, \cdot))$$

so that the above argument with $t_0 = T_0$ contradicts that T_0 satisfies (4.9). We thus deduce the uniqueness.

Criterion for global existence. Assume that (v, p, η) is the maximal strong solution on $[0, T_{\max})$ of (1.3)-(1.4), associated with $(v^0, \eta_1^0, \eta_2^0)$ (satisfying (1.32)-(1.33)). Let us assume that

$$T_{\max} < \infty, \qquad \sup_{t \in (0, T_{\max})} \|(v(t, \cdot), \eta(t, \cdot), \partial_t \eta(t, \cdot))\|_{\mathbf{H}^1(\mathcal{F}_\eta) \times H^2(\mathcal{I}) \times H^1(\mathcal{I})} + \left\| \frac{1}{1 + \eta(t, \cdot)} \right\|_{L^\infty(\mathcal{I})} < \infty. \tag{4.10}$$

Then, there exists $R > 0$ such that for any $t \in [0, T_{\max})$

$$R \geqslant 1 + \|[v(t, \cdot), \eta(t, \cdot), \partial_t \eta(t, \cdot)]\|_{\mathbf{H}^1(\mathcal{F}_\eta) \times H^2(\mathcal{I}) \times H^1(\mathcal{I})}, \quad \inf_{\mathcal{I}} \eta(t, \cdot) + 1 \geqslant 1/R. \tag{4.11}$$

In particular, from the first part of the proof (local in time existence), there exists $T > 0$ depending only on R such that we can construct a strong solution of (1.3) on $[t, t+T]$, with initial conditions $(v(t, \cdot), \eta(t, \cdot), \partial_t \eta(t, \cdot))$. This shows that we can go beyond $T_{\max}$ and leads to a contradiction with the fact that (v, p, η) is a maximal solution.

4.2 Proof of Theorem 1.8

The proof of Theorem 1.8 (or equivalently Theorem 2.3) is similar to the proof of Theorem 1.7. We use again a fixed point argument to show the global in time existence of strong solutions for (2.8)-(2.13). We recall that in that case $\eta^* = 0$.

For $R > 0$, we consider the closed subset of $\mathfrak{F}_\infty(\mathcal{F})$ (see (1.17)):

$$\mathfrak{B}_R \stackrel{\text{def}}{=} \left\{(F, G, H) \in \mathfrak{F}_\infty(\mathcal{F}) \ ; \ G(0, \cdot) = \operatorname{div} w^0, \quad \|(F, G, H)\|_{\mathfrak{F}_\infty(\mathcal{F})} \leqslant R\right\}.$$

Note that using (2.14)–(2.15), we can check that $\operatorname{div} w^0 \in L^2_\#(\mathcal{F}) = \left[H^1_\#(\mathcal{F}), H^{-1}_\#(\mathcal{F})\right]_{1/2}$. Thus using the trace theorems, there exists a constant c such that if

$$R \geqslant c\|[w^0, \eta_1^0, \eta_2^0]\|_{\mathbf{H}^1(\mathcal{F})\times H^2(\mathcal{I})\times H^1(\mathcal{I})}, \tag{4.12}$$

then $\mathfrak{B}_R$ is nonempty.

For any $(F, G, H) \in \mathfrak{B}_R$, we consider the solution $(w, q, \eta) \in \mathfrak{W}_\infty(\mathcal{F})$ of (1.15)-(1.16) given by Theorem 1.1. In particular (1.25) and (4.12) imply

$$\|(w, q, \eta)\|_{\mathfrak{W}_\infty(\mathcal{F})} \leqslant CR \tag{4.13}$$

for some constant C independent of R.

In particular, for R small enough (that is, with (4.12), for initial conditions small enough) the condition (2.3) holds with $\eta^* = 0$. Then we can consider our change of variables X and Y defined by (2.5) with $\eta^* = 0$, and the mapping $\mathcal{Z}$ introduced in (4.6) is well-defined.

Moreover, we notice that by interpolation, (4.13) yields

$$\|w\|_{L^8(0,\infty;\mathbf{H}^{5/4}(\mathcal{F}))} + \|w\|_{L^{8/3}(0,\infty;\mathbf{H}^{7/4}(\mathcal{F}))} \leqslant CR, \tag{4.14}$$

and using the estimates (B.44), (B.45), (B.46) and (B.47), we deduce that for R small enough, $\mathcal{Z}$ is well-defined from $\mathfrak{B}_R$ into $\mathfrak{F}_\infty(\mathcal{F})$ and satisfies

$$\|\mathcal{Z}(F, G, H)\|_{\mathfrak{F}_\infty(\mathcal{F})} \leqslant CR^2. \tag{4.15}$$

From (4.15), for all $R > 0$ small enough, we deduce

$$\mathcal{Z}(F, G, H) \in \mathfrak{B}_R.$$

Similarly, taking R possibly smaller, we can also show that $\mathcal{Z}$ is a strict contraction on $\mathfrak{B}_R$ and using the Banach fixed point theorem, we deduce the existence and uniqueness of $(F, G, H) \in \mathfrak{B}_R$ such that

$$\mathcal{Z}((F, G, H)) = (F, G, H).$$

The corresponding solution (w, q, η) of system (1.15)-(1.16) is a solution of (2.8)-(2.13).

5 Time analyticity of the solutions of (1.3)

In this section we prove Corollary 1.11. We first start with some general results for the time analyticity of the solutions of parabolic systems.

5.1 General results

Let us consider $\mathcal{H}$ and $\mathcal{V}$ two Banach spaces such that $\mathcal{V} \subset \mathcal{H}$, and for any $T \in (0, +\infty]$ and $\theta \in (0, \pi/2)$ let us consider the sector

$$\mathcal{S}_{T,\theta} \stackrel{\text{def}}{=} \{z \in \mathbb{C} \ ; \ 0 < |z| < T, \ |\arg(z)| < \theta\}.$$

We denote by $\mathbf{Hol}(\mathcal{S}_{T,\theta}; \mathcal{H})$ the space of functions holomorphic in $\mathcal{S}_{T,\theta}$ with values in $\mathcal{H}$. Let us consider the following norms

$$\|f\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})} \stackrel{\text{def}}{=} \sup_{|\phi|<\theta} \left(\int_0^T \|f(e^{\imath\phi}t)\|^2_{\mathcal{H}} dt\right)^{1/2}, \quad \|f\|_{\mathcal{W}^2(\mathcal{S}_{T,\theta};\mathcal{V},\mathcal{H})} \stackrel{\text{def}}{=} \|f\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{V})} + \|f'\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})}$$

and the following subspaces of $\mathbf{Hol}(\mathcal{S}_{T,\theta};\mathcal{H})$:

$$\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Hol}(\mathcal{S}_{T,\theta};\mathcal{H}) \ ; \ \|f\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})} < +\infty \right\},$$

$$\mathcal{W}^2(\mathcal{S}_{T,\theta};\mathcal{V},\mathcal{H}) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Hol}(\mathcal{S}_{T,\theta};\mathcal{V}) \ ; \ \|f\|_{\mathcal{W}^2(\mathcal{S}_{T,\theta};\mathcal{V},\mathcal{H})} < +\infty \right\}.$$

Theorem 5.1. *The spaces $\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})$ and $\mathcal{W}^2(\mathcal{S}_{T,\theta};\mathcal{V},\mathcal{H})$ are Banach spaces.*

Proof. We only give the proof for $\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})$, the proof is similar for the other space. Suppose that (f_n) is a Cauchy sequence of $\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})$. Then the sequence $(\tilde{f}_n)$ defined by $\tilde{f}_n(s,\phi) = f_n(se^{\imath\phi})$ is a Cauchy sequence of the Banach space $L^2(-\theta,\theta;L^2(0,T))$, and thus converges to some $\tilde{f} \in L^2(-\theta,\theta;L^2(0,T))$. Then the function $f : \mathcal{S}_{T,\theta} \to \mathcal{H}$ defined by $f(se^{\imath\phi}) = \tilde{f}(s,\phi)$ satisfies $\|f\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})} < \infty$ and

$$\lim_{n\to+\infty} \|f_n - f\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})} = 0.$$

It remains to show that $f \in \mathbf{Hol}(\mathcal{S}_{T,\theta};\mathcal{H})$. For that we define the analytic functions:

$$\forall z \in \mathcal{S}_{T,\theta}, \quad F_n(z) = \int_{[0,z]} f_n(\xi) d\xi.$$

By the Cauchy-Schwarz inequality, for any $T_0 < T \leqslant \infty$,

$$\|F_n - F_m\|_{L^\infty(\mathcal{S}_{T_0,\theta};\mathcal{H})} \leqslant T_0^{1/2} \|f_n - f_m\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})},$$

from which we deduce that (F_n) converges to some $F \in L^\infty(\mathcal{S}_{T_0,\theta};X)$, with F analytic in $\mathcal{S}_{T_0,\theta}$. Thus $f = F'$ and we obtain that f is analytic in $\mathcal{S}_{T_0,\theta}$ for any $T_0 < T$ and thus analytic in $\mathcal{S}_{T,\theta}$. □

Theorem 5.2. *Assume that $A : \mathcal{D}(A) \to \mathcal{H}$ is the infinitesimal generator of an exponentially stable semigroup $(e^{tA})_{t\geqslant 0}$ on $\mathcal{H}$ that is analytic in the sector $\mathcal{S}_{\infty,\theta'}$. Assume $T \in (0,+\infty]$ and $\theta \in (0,\theta')$. Then for any $y_0 \in [\mathcal{D}(A),\mathcal{H}]_{1/2}$ and $f \in \mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})$, there exists a unique solution $y \in \mathcal{W}^2(\mathcal{S}_{T,\theta};\mathcal{D}(A),\mathcal{H})$ of*

$$y' = Ay + f, \quad y(0) = y_0. \tag{5.1}$$

Moreover, there exists $C > 0$ independent of T such that

$$\|y\|_{\mathcal{W}^2(\mathcal{S}_{T,\theta};\mathcal{D}(A),\mathcal{H})} \leqslant C \left(\|y_0\|_{[\mathcal{D}(A),\mathcal{H}]_{1/2}} + \|f\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathcal{H})} \right). \tag{5.2}$$

Proof. First, we note that since $z \mapsto e^{zA}$ and $t \mapsto Ae^{zA}$ are analytic in $\mathcal{S}_{\infty,\theta'}$, then

$$\frac{d}{dz} e^{zA} = Ae^{zA} \quad (z \in \mathcal{S}_{\infty,\theta}).$$

From the Duhamel formula,

$$y(z) = e^{zA} y_0 + \int_{[0,z]} e^{(z-s)A} f(s)\ ds \quad (z \in \mathcal{S}_{T,\theta}).$$

Let us consider a closed disc $D(z_0,r) \subset \mathcal{S}_{T,\theta}$ with $r > 0$. Since $\mathcal{S}_{T,\theta}$ is an open set, there exists $\varepsilon \in (0,T)$ (small enough) such that $D(z_0,r) - \varepsilon \subset \mathcal{S}_{T,\theta}$. In particular, for any $z \in D(z_0,r)$, $z - \varepsilon \in \mathcal{S}_{T,\theta}$. Let us denote by $\mathcal{K}$ the compact set defined by

$$\mathcal{K} \stackrel{\text{def}}{=} \{[\varepsilon, z] \ ; \ z \in D(z_0,r)\} \subset \mathcal{S}_{T,\theta}.$$

Since $z \mapsto e^{zA}$ and f are analytic in $\mathcal{S}_{T,\theta}$, there exists $C_1 > 0$,

$$\forall z \in \mathcal{K},\ \forall k \geqslant 0,\ \left\| \frac{d^k e^{zA}}{dz^k} \right\|_{\mathcal{L}(\mathcal{H})} + \|f^{(k)}(z)\|_{\mathcal{H}} \leqslant C_1^k (k!). \tag{5.3}$$

We can write

$$y(z) = e^{(z-\varepsilon)A} y(\varepsilon) + \int_{[\varepsilon,z]} e^{(z-s)A} f(s)\ ds \quad (z \in D(z_0,r)).$$

We already know that $z \mapsto e^{(z-\varepsilon)A} y(\varepsilon) \in \mathbf{Hol}(D(z_0,r);\mathcal{H})$. We show that y_1 defined by

$$y_1(z) = \int_{[\varepsilon,z]} e^{(z-s)A} f(s)\ ds = \int_{[0,z-\varepsilon]} e^{sA} f(z-s)\ ds$$

satisfies $y_1 \in \mathbf{Hol}(D(z_0,r);\mathcal{H})$. We have

$$\frac{d^n}{dz^n} y_1(z) = \sum_{k=0}^{n-1} \frac{d^{n-1-k}}{dz^{n-1-k}} e^{(z-\varepsilon)A} \frac{d^k}{dt^k} f(\varepsilon) + \int_{[0,z-\varepsilon]} e^{sA} \frac{d^n}{dz^n} f(z-s)\ ds.$$

Thus, from (5.3) we deduce that for all $z \in D(z_0,r)$,

$$\left\| \frac{d^n}{dt^n} y_1(z) \right\|_{\mathcal{H}} \leqslant C C_1^n (n!).$$

This proves that $y \in \mathbf{Hol}(\mathcal{S}_{T,\theta};\mathcal{H})$ and using (5.1), we deduce that $y \in \mathbf{Hol}(\mathcal{S}_{T,\theta};\mathcal{D}(A))$.

Now, assume $|\phi| < \theta$ and let us write

$$y_\phi(t) \stackrel{\text{def}}{=} y(e^{\imath\phi} t), \quad f_\phi(t) \stackrel{\text{def}}{=} f(e^{\imath\phi} t) \quad (t \in (0,T)).$$

We have

$$y_\phi' = e^{\imath\phi} A y_\phi + e^{\imath\phi} f_\phi, \quad y_\phi(0) = y_0. \tag{5.4}$$

We extend f_ϕ by 0 on (T,∞) and we consider the solution of (5.4) on $(0,+\infty)$ which is an extension of y_ϕ on (T,∞).

We first consider the case $y_0 = 0$: we extend y_ϕ, f_ϕ by 0 on $(-\infty,0)$ and we take the Fourier transform of (5.4):

$$\widehat{y}_\phi(\xi) = (\imath e^{-\imath\phi}\xi - A)^{-1} \widehat{f}_\phi(\xi) \quad (\xi \in \mathbb{R}).$$

By adapting, for instance the proof of $(iii) \Rightarrow (i)$ of [6, Paragraph II.1, Theorem 2.11, p.112], we deduce that for $\theta \in (0,\theta')$,

$$\sup_{\lambda \in \mathcal{S}_{\infty,\frac{\pi}{2}+\theta}} \left\{ \|A(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} + |\lambda| \|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \right\} < +\infty.$$

We thus deduce

$$\|A\widehat{y}_\phi\|_{L^2(\mathbb{R};\mathcal{H})} + \left\| \widehat{y_\phi'} \right\|_{L^2(\mathbb{R};\mathcal{H})} \leqslant C \|\widehat{f_\phi}\|_{L^2(\mathbb{R};\mathcal{H})},$$

where the constant C is independent on ϕ. Hence, (5.2) follows in the case $y_0 = 0$.

Now we consider the case $f = 0$: since $[\mathcal{D}(A),\mathcal{H}]_{1/2}$ is the trace space of $L^2(0,\infty;\mathcal{D}(A)) \cap H^1(0,\infty;\mathcal{H})$, there exists $u \in L^2(0,\infty;\mathcal{D}(A)) \cap H^1(0,\infty;\mathcal{H})$ such that $u(0) = y_0$ and

$$\|u\|_{L^2(0,\infty;\mathcal{D}(A))} + \|u'\|_{L^2(0,\infty;\mathcal{H})} \leqslant C \|y_0\|_{[\mathcal{D}(A),\mathcal{H}]_{1/2}}.$$

Hence, $\widetilde{y}_\phi \stackrel{\text{def}}{=} y_\phi - u$ satisfies

$$\widetilde{y}_\phi' = e^{\imath\phi} A \widetilde{y}_\phi + e^{\imath\phi} A u - u', \quad \widetilde{y}_\phi(0) = 0,$$

and we are reduced to the first case. □

5.2 Proof of Corollary 1.11

We are now in position to prove Corollary 1.11. First we use Theorem 2.5 to obtain an angle $\theta > 0$ such that (e^{zA}) on $\mathcal{H}$ is analytic in the sector $\mathcal{S}_{\infty,\theta}$.

For $T \in (0,+\infty]$ let us define the following spaces (see (1.17) and (1.18))

$$\mathfrak{F}_T^\theta(\mathcal{F}_{\eta^*}) \stackrel{\text{def}}{=} \mathcal{L}^2(\mathcal{S}_{T,\theta};\mathbf{L}^2(\mathcal{F}_{\eta^*})) \times \mathcal{W}^2(\mathcal{S}_{T,\theta};H^1_\#(\mathcal{F}_{\eta^*}),H^{-1}_\#(\mathcal{F}_{\eta^*})) \times \mathcal{L}^2(\mathcal{S}_{T,\theta};H^{1/2}_\#(\mathcal{I})))$$

and

$$\mathfrak{W}_T^\theta(\mathcal{F}_{\eta^*}) \stackrel{\text{def}}{=} \Big\{ (w,q,\eta)\ ;\ w \in \mathcal{W}^2(\mathcal{S}_{T,\theta};\mathbf{H}^2(\mathcal{F}_{\eta^*}),\mathbf{L}^2(\mathcal{F}_{\eta^*})),\ q \in \mathcal{L}^2(\mathcal{S}_{T,\theta};H^1(\mathcal{F}_{\eta^*})/\mathbb{R}),$$
$$\eta \in \mathcal{W}^2(\mathcal{S}_{T,\theta};H^{5/2}_\#(\mathcal{I}),H^{3/2}_\#(\mathcal{I})),\ \eta' \in \mathcal{W}^2(\mathcal{S}_{T,\theta};H^{3/2}_\#(\mathcal{I}),H^{1/2}_\#(\mathcal{I})) \Big\}.$$

We endow these spaces with the norms

$$\|(F,G,H)\|_{\mathfrak{F}_T^\theta(\mathcal{F}_{\eta^*})} \overset{\text{def}}{=} \|F\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};\mathbf{L}^2(\mathcal{F}_{\eta^*}))} + \|G\|_{\mathcal{W}^2(\mathcal{S}_{T,\theta};H^1_{\#}(\mathcal{F}_{\eta^*}),H^{-1}_{\#}(\mathcal{F}_{\eta^*}))} + \|H\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};H^{1/2}_{\#}(\mathcal{I}))} \tag{5.5}$$

and

$$\|(w,q,\eta)\|_{\mathfrak{W}_T^\theta(\mathcal{F}_{\eta^*})} \overset{\text{def}}{=} \|w\|_{\mathcal{W}^2(\mathcal{S}_{T,\theta};\mathbf{H}^2(\mathcal{F}_{\eta^*}),\mathbf{L}^2(\mathcal{F}_{\eta^*}))} + \|q\|_{\mathcal{L}^2(\mathcal{S}_{T,\theta};H^1(\mathcal{F}_{\eta^*})/\mathbb{R})} + \|\eta\|_{\mathcal{W}^2(\mathcal{S}_{T,\theta};H^{5/2}_{\#}(\mathcal{I}),H^{3/2}_{\#}(\mathcal{I}))} + \|\eta'\|_{\mathcal{W}^2(\mathcal{S}_{T,\theta};H^{3/2}_{\#}(\mathcal{I}),H^{1/2}_{\#}(\mathcal{I}))}.$$

We deduce from Theorem 5.1 that the above spaces are Banach spaces.

Combining Theorem 2.5 and Theorem 5.2 we obtain the analogue of Theorem 1.1 that provides a unique solution (w,q,η) of (1.15)-(1.16) in $\mathcal{S}_{T,\theta}$ with

$$\|(w,q,\eta)\|_{\mathfrak{W}_T^\theta(\mathcal{F}_{\eta^*})} \leqslant C\left(\|w^0\|_{\mathbf{H}^1(\mathcal{F}_{\eta^*})} + \|\zeta_1^0\|_{H^2_{\#}(\mathcal{I})} + \|\zeta_2^0\|_{H^1_{\#}(\mathcal{I})} + \|(F,G,H)\|_{\mathfrak{F}_T^\theta(\mathcal{F}_{\eta^*})}\right). \tag{5.6}$$

Moreover, following the proof of Proposition 1.2 and Corollary 1.3, we can choose the constant C independent of T and so that its dependence on η^* comes only from R provided $\eta^* \in \mathcal{B}_R$ (see (1.26)).

Then the proof of Corollary 1.11 follows the proofs in Section 4.1 and in Section 4.2. For instance, taking $\eta^* = \eta_1^0$ and setting $\mathcal{F} = \mathcal{F}_{\eta^*}$, we can define the following closed subset of $\mathfrak{F}_T^\theta(\mathcal{F})$:

$$\mathfrak{B}_{R,T}^\theta \overset{\text{def}}{=} \left\{(F,G,H) \in \mathfrak{F}_T^\theta(\mathcal{F}) \ ; \ G(0,\cdot) = 0 \text{ in } \mathcal{F}, \quad \|(F,G,H)\|_{\mathfrak{F}_T^\theta(\mathcal{F})} \leqslant R\right\}.$$

Then taking R large enough such that (4.2) holds true, there exists a constant $C = C(R)$ such that for any $(F,G,H) \in \mathfrak{B}_{R,T}^\theta$ there exists a unique solution (w,q,η) of (1.15)-(1.16) with $(\zeta_1^0,\zeta_2^0) = (\eta_1^0,\eta_2^0)$ in $\mathcal{S}_{T,\theta}$ with

$$\|(w,q,\eta)\|_{\mathfrak{W}_T^\theta(\mathcal{F})} \leqslant C. \tag{5.7}$$

Thus using the same proof as in Section 4.1, we deduce the existence of $T = T(R)$ small enough such that we have a solution of (2.8)-(2.13) in $\mathcal{S}_{T,\theta}$. By uniqueness, this solution extends the solution that we obtained in Section 4.1.

To complete the first point of Corollary 1.11, assume that we have a strong solution on $[0,T_0]$. Then, by continuity of η, there exists $R > 0$ such that $\eta(t,\cdot) \in \mathcal{B}_R$ (see (1.26)) for $t \in [0,T_0]$. From the above proof, it implies that there exists a uniform T such that we can extend the solution of (2.8)-(2.13) in $t + \mathcal{S}_{T,\theta}$ for all $t \in [0,T_0]$. Then the domain of analyticity contains the union of $t + \mathcal{S}_{T,\theta}$, $t \in [0,T_0]$, and we deduce the result by choosing $\theta_0 \in [0,\theta)$ small enough.

With similar arguments but adapting Section 4.2 instead of Section 4.1, we deduce the second point of Corollary 1.11.

A Formula for the change of variables

Here we recall standard formula that allow us to obtain the expressions (2.10)-(2.13) and similar ones (see Section 2.4).

Starting from

$$v(t,x) = w(t,Y(t,x)) \quad \text{and} \quad p(t,x) = q(t,Y(t,x)), \tag{A.1}$$

we have

$$\partial_t v_i(t,X(t,y)) = \partial_t w_i(t,y) + \sum_j \frac{\partial w_i}{\partial y_j}(t,y)\frac{\partial Y_j}{\partial t}(t,X(t,y)), \tag{A.2}$$

$$((v\cdot\nabla)v)_i(t,X(t,y)) = \sum_{j,k} w_j(t,y)\frac{\partial w_i}{\partial y_k}(t,y)\frac{\partial Y_k}{\partial x_j}(t,X(t,y)), \tag{A.3}$$

$$\frac{\partial v_i}{\partial x_j}(t,X(t,y)) = \sum_k \frac{\partial w_i}{\partial y_k}(t,y)\frac{\partial Y_k}{\partial x_j}(t,X(t,y)), \tag{A.4}$$

$$\frac{\partial^2 v_i}{\partial x_j \partial x_m}(t, X(t,y)) = \sum_{k,\ell} \frac{\partial^2 w_i}{\partial y_k \partial y_\ell}(t,y) \frac{\partial Y_k}{\partial x_j}(t, X(t,y)) \frac{\partial Y_\ell}{\partial x_m}(t, X(t,y)) \\ + \sum_k \frac{\partial w_i}{\partial y_k}(t,y) \frac{\partial^2 Y_k}{\partial x_j \partial x_m}(t, X(t,y)), \quad \text{(A.5)}$$

$$\frac{\partial p}{\partial x_i}(t, X(t,y)) = \sum_k \frac{\partial q}{\partial y_k}(t,y) \frac{\partial Y_k}{\partial x_i}(t, X(t,y)), \quad \text{(A.6)}$$

$$\operatorname{div} v(t, X(t,y)) = \sum_{k,j} \frac{\partial w_j}{\partial y_k}(t,y) \frac{\partial Y_k}{\partial x_j}(t, X(t,y)). \quad \text{(A.7)}$$

In particular

$$\det(\nabla X) \operatorname{div} v(X) = \nabla w : \operatorname{Cof}(\nabla X) = \operatorname{div}(\operatorname{Cof}(\nabla X)^* w). \quad \text{(A.8)}$$

We have used here that $\operatorname{div}(\operatorname{Cof}(\nabla X)) = 0$.

From (1.6), we deduce

$$\widetilde{\mathbb{H}}_\eta(v,p)(t,s) = -\nu(\partial_s \eta) \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right)(t, s, 1+\eta(t,s)) + 2\nu \frac{\partial v_2}{\partial x_2}(t, s, 1+\eta(t,s)) - p(t, s, 1+\eta(t,s)). \quad \text{(A.9)}$$

Thus

$$\widetilde{\mathbb{H}}_\eta(v,p)(t,s) = \widetilde{\mathbb{H}}_{\eta^*}(w,q)(t,s) + \nu(\partial_s \eta^* - \partial_s \eta) \left(\frac{\partial w_1}{\partial y_2} + \frac{\partial w_2}{\partial y_1} \right)(t, s, 1+\eta^*(s)) \\ - \nu(\partial_s \eta) \sum_k \left(\frac{\partial w_1}{\partial y_k} \left(\frac{\partial Y_k}{\partial x_2}(X) - \delta_{k,2} \right) + \frac{\partial w_2}{\partial y_k} \left(\frac{\partial Y_k}{\partial x_1}(X) - \delta_{k,1} \right) \right)(t, s, 1+\eta^*(s)) \\ + 2\nu \sum_k \left(\frac{\partial w_2}{\partial y_k} \left(\frac{\partial Y_k}{\partial x_2}(X) - \delta_{k,2} \right) \right)(t, s, 1+\eta^*(s)). \quad \text{(A.10)}$$

We can write our change of variables defined by (2.2) and (2.5) as

$$X(t, y_1, y_2) = (y_1, y_2 + \zeta(t, y_1, y_2)) \quad \text{and} \quad \zeta(t, y_1, y_2) \stackrel{\text{def}}{=} \mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)} \right). \quad \text{(A.11)}$$

Then we have the following formulas:

$$\nabla X = I_2 + \begin{bmatrix} 0 & 0 \\ \partial_{y_1}\zeta & \partial_{y_2}\zeta \end{bmatrix}, \quad \operatorname{Cof}(\nabla X) = I_2 + \begin{bmatrix} \partial_{y_2}\zeta & -\partial_{y_1}\zeta \\ 0 & 0 \end{bmatrix}, \quad \text{(A.12)}$$

$$\det \nabla X = 1 + \partial_{y_2}\zeta, \quad \nabla Y(X) = I_2 - \begin{bmatrix} 0 & 0 \\ \dfrac{\partial_{y_1}\zeta}{1+\partial_{y_2}\zeta} & \dfrac{\partial_{y_2}\zeta}{1+\partial_{y_2}\zeta} \end{bmatrix}, \quad \text{(A.13)}$$

$$\frac{\partial^2 Y_1}{\partial x_i \partial x_j}(X) = 0 \quad (i,j \in \{1,2\}), \quad \text{(A.14)}$$

$$\frac{\partial^2 Y_2}{\partial x_1^2}(X) = -\frac{\partial^2_{y_1 y_1}\zeta}{1+\partial_{y_2}\zeta} + \frac{2(\partial_{y_1}\zeta)(\partial^2_{y_1 y_2}\zeta)}{(1+\partial_{y_2}\zeta)^2} - \frac{(\partial_{y_1}\zeta)^2(\partial^2_{y_2 y_2}\zeta)}{(1+\partial_{y_2}\zeta)^3}, \quad \text{(A.15)}$$

$$\frac{\partial^2 Y_2}{\partial x_1 \partial x_2}(X) = -\frac{\partial^2_{y_1 y_2}\zeta}{(1+\partial_{y_2}\zeta)^2} + \frac{(\partial_{y_1}\zeta)(\partial^2_{y_2 y_2}\zeta)}{(1+\partial_{y_2}\zeta)^3}, \quad \frac{\partial^2 Y_2}{\partial x_2^2}(X) = -\frac{\partial^2_{y_2 y_2}\zeta}{(1+\partial_{y_2}\zeta)^3}, \quad \text{(A.16)}$$

$$\partial_t Y(X) = \begin{bmatrix} 0 \\ -\dfrac{\partial_t \zeta}{1+\partial_{y_2}\zeta} \end{bmatrix}, \quad \partial_t \operatorname{Cof}(\nabla X) = \begin{bmatrix} \partial_{t,y_2}\zeta & -\partial_{t,y_1}\zeta \\ 0 & 0 \end{bmatrix}, \quad \text{(A.17)}$$

$$\partial_{y_1}\zeta(t, y_1, y_2) = \partial_{x_1}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)} \right) - \frac{y_2 \partial_{y_1}\eta^*(y_1)}{(1+\eta^*(y_1))^2} \partial_{x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)} \right), \quad \text{(A.18)}$$

$$\partial_{y_2}\zeta(t, y_1, y_2) = \frac{1}{1+\eta^*(y_1)} \partial_{x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)} \right), \quad \text{(A.19)}$$

$$
\begin{aligned}
\partial_{y_1y_1}\zeta(t,y_1,y_2) &= \partial_{x_1x_1}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right) - \frac{2y_2\partial_{y_1}\eta^*(y_1)}{(1+\eta^*(y_1))^2}\partial_{x_1x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right)\\
&- \frac{y_2\partial_{y_1y_1}\eta^*(y_1)}{(1+\eta^*(y_1))^2}\partial_{x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right) + \frac{2y_2(\partial_{y_1}\eta^*(y_1))^2}{(1+\eta^*(y_1))^3}\partial_{x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right)\\
&+ \frac{y_2^2(\partial_{y_1}\eta^*(y_1))^2}{(1+\eta^*(y_1))^4}\partial_{x_2x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right),
\end{aligned} \tag{A.20}
$$

$$
\begin{aligned}
\partial_{y_1y_2}\zeta(t,y_1,y_2) &= \frac{1}{1+\eta^*(y_1)}\partial_{x_1x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right)\\
&- \frac{\partial_{y_1}\eta^*(y_1)}{(1+\eta^*(y_1))^2}\partial_{x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right)\\
&- \frac{y_2\partial_{y_1}\eta^*(y_1)}{(1+\eta^*(y_1))^3}\partial_{x_2x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right),
\end{aligned} \tag{A.21}
$$

$$
\partial_{y_2y_2}\zeta(t,y_1,y_2) = \frac{1}{(1+\eta^*(y_1))^2}\partial_{x_2x_2}\mathcal{R}_{\eta(t,\cdot)-\eta^*}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right), \tag{A.22}
$$

$$
\partial_t\zeta(t,y_1,y_2) = \mathcal{R}_{\partial_t\eta(t,\cdot)}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right), \tag{A.23}
$$

$$
\partial_{t,y_1}\zeta(t,y_1,y_2) = \partial_{x_1}\mathcal{R}_{\partial_t\eta(t,\cdot)}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right) - \frac{y_2\partial_{y_1}\eta^*(y_1)}{(1+\eta^*(y_1))^2}\partial_{x_2}\mathcal{R}_{\partial_t\eta(t,\cdot)}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right), \tag{A.24}
$$

$$
\partial_{t,y_2}\zeta(t,y_1,y_2) = \frac{1}{1+\eta^*(y_1)}\partial_{x_2}\mathcal{R}_{\partial_t\eta(t,\cdot)}\left(y_1, \frac{y_2}{1+\eta^*(y_1)}\right). \tag{A.25}
$$

B Estimates of the nonlinear terms

This section is devoted to the proof of (4.7), (4.15), and (2.60)-(2.62). We assume

$$
\|\eta^*\|_{H^2(\mathcal{I})} + \left\|\frac{1}{1+\eta^*}\right\|_{L^\infty(\mathcal{I})} \leqslant C_*, \tag{B.1}
$$

and all the constants below depend only on C_*. For instance, using Sobolev embeddings, the above estimate yields

$$
\|\eta^*\|_{W^{1,\infty}(\mathcal{I})} + \left\|\frac{1}{1+\eta^*}\right\|_{H^2(\mathcal{I})} \leqslant C. \tag{B.2}
$$

B.1 Preliminary results

In this section, we state and prove some preliminary results that allow us to deal with the fixed points of this article. In order to do prove these results, we need to introduce a change of variables different from (2.2) in order to transform $\mathcal{F}_0 = \mathcal{I}\times(0,1)$ into $\mathcal{F}_{\eta^*}$: for any η^1, η^2 satisfying (B.1), we write

$$
\widetilde{X}_{\eta^1,\eta^2} : \mathcal{F}_{\eta^1} \to \mathcal{F}_{\eta^2}, \quad (y_1,y_2) \mapsto \left(y_1, y_2\frac{1+\eta^2(y_1)}{1+\eta^1(y_1)}\right). \tag{B.3}
$$

One can check that $\widetilde{X}_{\eta^1,\eta^2}$ transforms $\mathcal{F}_{\eta^1}$ into $\mathcal{F}_{\eta^2}$ and its inverse is $\widetilde{X}_{\eta^2,\eta^1}$.

In this section, we thus use the change of variables $\widetilde{X}_{0,\eta^*}$ and for any $f : \mathcal{F}_{\eta^*} \to \mathbb{R}$, we set

$$
\widetilde{f} \overset{\text{def}}{=} f \circ \widetilde{X}_{0,\eta^*}. \tag{B.4}
$$

Then we have the following result.

Lemma B.1. *Assume* (B.1). *Assume $p \in [1,\infty]$ and $\alpha \in [0,1]$. Then there exist $C_1, C_2 > 0$ depending only on C_* and on p, α such that*

$$
C_1\|\widetilde{f}\|_{L^p(\mathcal{F}_0)} \leqslant \|f\|_{L^p(\mathcal{F}_{\eta^*})} \leqslant C_2\|\widetilde{f}\|_{L^p(\mathcal{F}_0)}, \tag{B.5}
$$

$$
C_1\|\widetilde{f}\|_{H^\alpha(\mathcal{F}_0)} \leqslant \|f\|_{H^\alpha(\mathcal{F}_{\eta^*})} \leqslant C_2\|\widetilde{f}\|_{H^\alpha(\mathcal{F}_0)}. \tag{B.6}
$$

Proof. The proof of (B.5) follows from a change of variables in the integral. For (B.6), we can prove it for $\alpha = 0$ and $\alpha = 1$ by standard calculation. For $\alpha \in (0, 1)$, we recall that

$$\begin{aligned}
\|\widetilde{f}\|^2_{H^\alpha(\mathcal{F}_0)} &= \|\widetilde{f}\|^2_{L^2(\mathcal{F}_0)} + \int_{\mathcal{F}_0}\int_{\mathcal{F}_0} \frac{\left|\widetilde{f}(y^{(1)}) - \widetilde{f}(y^{(2)})\right|^2}{|y^{(1)} - y^{(2)}|^{2\alpha+2}}\ dy^{(1)}dy^{(2)} \\
&= \|\widetilde{f}\|^2_{L^2(\mathcal{F}_0)} + \int_{\mathcal{F}_{\eta^*}}\int_{\mathcal{F}_{\eta^*}} \frac{\left|f(x^{(1)}) - f(x^{(2)})\right|^2}{\left|\widetilde{X}_{\eta^*,0}(x^{(1)}) - \widetilde{X}_{\eta^*,0}(x^{(2)})\right|^{2\alpha+2}} \frac{1}{1+\eta^*(x_1^{(1)})} \frac{1}{1+\eta^*(x_1^{(2)})}\ dx^{(1)}dx^{(2)}. \quad \text{(B.7)}
\end{aligned}$$

Using (B.2), we can check that

$$\left|\widetilde{X}_{\eta^*,0}(x^{(1)}) - \widetilde{X}_{\eta^*,0}(x^{(2)})\right| \geqslant C\left|x^{(1)} - x^{(2)}\right|$$

and thus we deduce that $\|\widetilde{f}\|_{H^\alpha(\mathcal{F}_0)} \leqslant C\|f\|_{H^\alpha(\mathcal{F}_{\eta^*})}$. The other estimate is obtained similarly. □

Lemma B.2. *Assume* (B.1).

- *If $f \in H^\alpha(\mathcal{F}_{\eta^*})$, $\alpha > 1/2$ and if $g \in L^2(\mathcal{I})$, then $fg \in L^2(\mathcal{F}_{\eta^*})$ and there exists a constant C depending on C_* and on α such that*
$$\|fg\|_{L^2(\mathcal{F}_{\eta^*})} \leqslant C\|g\|_{L^2(\mathcal{I})}\|f\|_{H^\alpha(\mathcal{F}_{\eta^*})}. \quad \text{(B.8)}$$
- *If $f \in H^\alpha(\mathcal{F}_{\eta^*})$, $\alpha \in [0, 1]$, and if $g \in H^1(\mathcal{I})$, then $fg \in H^\alpha(\mathcal{F}_{\eta^*})$ and there exists a constant C depending on C_* and on α such that*
$$\|fg\|_{H^\alpha(\mathcal{F}_{\eta^*})} \leqslant C\|g\|_{H^1(\mathcal{I})}\|f\|_{H^\alpha(\mathcal{F}_{\eta^*})}. \quad \text{(B.9)}$$

Proof. For both relations, we use (B.4) to work in $\mathcal{F}_0$. By interpolation we can check that

$$\|\widetilde{f}\|_{L^2(0,1;H^\alpha(\mathcal{I}))} \leqslant C\|\widetilde{f}\|_{H^\alpha(\mathcal{F}_0)}$$

and thus by the Sobolev embedding, we deduce that if $\alpha > 1/2$,

$$\|\widetilde{f}\|_{L^2(0,1;L^\infty(\mathcal{I}))} \leqslant C\|\widetilde{f}\|_{H^\alpha(\mathcal{F}_0)}.$$

From the above estimates with (B.5), (B.6) we deduce (B.8).

For the second point, it is sufficient to prove it for $\alpha = 0$ and $\alpha = 1$. For $\alpha = 0$, the relation comes from the fact that $g \in L^\infty(\mathcal{I})$. For $\alpha = 1$, we use the first point of this lemma with

$$\partial_{y_1}(\widetilde{f}g) = (\partial_{y_1}\widetilde{f})g + \widetilde{f}\partial_{y_1}g \in L^2(\mathcal{F}_0), \quad \partial_{y_2}(\widetilde{f}g) = (\partial_{y_2}\widetilde{f})g \in L^2(\mathcal{F}_0),$$

and we can conclude the proof of the second point of the lemma by interpolation. □

Assume now $\alpha \in [1, 2]$ and $f \in H^\alpha(\mathcal{F}_{\eta^*})$, and let us consider $\widetilde{f}$ defined by (B.4). Then using the above lemma, we deduce that for some constants C_1, C_2 depending only on C_* and on α,

$$C_1\|\nabla\widetilde{f}\|_{H^{\alpha-1}(\mathcal{F}_0)} \leqslant \|\nabla f\|_{H^{\alpha-1}(\mathcal{F}_{\eta^*})} \leqslant C_2\|\nabla\widetilde{f}\|_{H^{\alpha-1}(\mathcal{F}_0)}.$$

In particular, we deduce that (B.6) is valid for $\alpha \in [0, 2]$:

Corollary B.3. *Assume* (B.1) *and assume $\alpha \in [0, 2]$. Then there exist $C_1, C_2 > 0$ depending only on C_* and on α such that*

$$C_1\|\widetilde{f}\|_{H^\alpha(\mathcal{F}_0)} \leqslant \|f\|_{H^\alpha(\mathcal{F}_{\eta^*})} \leqslant C_2\|\widetilde{f}\|_{H^\alpha(\mathcal{F}_0)}. \quad \text{(B.10)}$$

B.2 Estimates of X and Y

In this section $T \in (0, \infty]$ and X, Y are defined by (2.2), and either (2.5) or (2.52). We write below the estimates in the case (2.5), to obtain them in the case (2.52), it is sufficient to replace η by $\widetilde{\eta}$ that does not depend on time (so that several estimates below are trivial in that case).

Using the continuous embedding $H^{9/4}(\mathcal{F}_0) \hookrightarrow W^{1,\infty}(\mathcal{F}_0)$ and (A.18), (A.19), (B.1), we deduce that there exists a constant C depending on C_* such that

$$\|\zeta\|_{L^\infty(0,T;C^1(\overline{\mathcal{F}}))} \leqslant C \|\mathcal{R}_{\eta-\eta^*}\|_{L^\infty(0,T;C^1(\overline{\mathcal{F}_0}))} \leqslant C \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}. \tag{B.11}$$

In particular, assuming that η satisfies (2.3) for all t with κ small enough, then there exists a constant C depending only C_* and κ such that

$$\left\| \frac{1}{1+\partial_{y_2}\zeta} \right\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} \leqslant C. \tag{B.12}$$

Combining (B.11), (B.12), (A.12), and (A.13), we deduce

$$\|\nabla X - I_2\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} + \|\operatorname{Cof}(\nabla X) - I_2\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} + \|\nabla Y(X) - I_2\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} \leqslant C \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))} \tag{B.13}$$

and

$$\|\det \nabla X\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} \leqslant 1 + C \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}. \tag{B.14}$$

From (A.23) and (2.1) for $\alpha = 1$, we also deduce

$$\|\partial_t\zeta\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} \leqslant C \|\mathcal{R}_{\partial_t\eta}\|_{L^\infty(0,T;H^{3/2}(\mathcal{F}_0))} \leqslant C \|\partial_t\eta\|_{L^\infty(0,T;H^1(\mathcal{I}))}. \tag{B.15}$$

The above estimate, (A.17) and (B.12) yield

$$\|\partial_t Y(X)\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} \leqslant C \|\partial_t\eta\|_{L^\infty(0,T;H^1(\mathcal{I}))}. \tag{B.16}$$

Combining (B.15) with (A.24), (A.25) and (B.5), we also deduce that

$$\|\partial_{t,y_i}\zeta\|_{L^\infty(0,T;L^2(\mathcal{F}))} \leqslant C \|\partial_t\eta\|_{L^\infty(0,T;H^1(\mathcal{I}))}. \tag{B.17}$$

The above estimate and (A.17) yield

$$\|\partial_t \operatorname{Cof}(\nabla X)\|_{L^\infty(0,T;L^2(\mathcal{F}))} \leqslant C \|\partial_t\eta\|_{L^\infty(0,T;H^1(\mathcal{I}))}. \tag{B.18}$$

From (A.20)–(A.22), (B.1) and Sobolev embeddings, we deduce that

$$\partial_{y_i y_j}\zeta = \partial_{y_1 y_1}\eta^* \Xi_1^{i,j} + \Xi_2^{i,j} \tag{B.19}$$

with

$$\|\Xi_1^{i,j}\|_{L^\infty(0,T;L^\infty(\mathcal{F}))} + \|\Xi_2^{i,j}\|_{L^\infty(0,T;L^{8/3}(\mathcal{F}))} \leqslant C \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}. \tag{B.20}$$

This decomposition, (B.11), (B.12) and (A.14)–(A.16) yield that

$$\frac{\partial^2 Y_i}{\partial x_j \partial x_k}(X) = \partial_{y_1 y_1}\eta^* \Xi_3^{i,j,k} + \Xi_4^{i,j,k} \tag{B.21}$$

with

$$\|\Xi_3^{i,j,k}\|_{L^\infty(0,T;L^\infty(\mathcal{F}))} + \|\Xi_4^{i,j,k}\|_{L^\infty(0,T;L^{8/3}(\mathcal{F}))} \leqslant C \left(1 + \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}\right)^2 \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}. \tag{B.22}$$

We can also use the decomposition (B.19), (B.20) and (A.12) to obtain

$$\frac{\partial}{\partial y_k} \operatorname{Cof}(\nabla X)_{i,j} = \partial_{y_1 y_1}\eta^* \Xi_5^{i,j,k} + \Xi_6^{i,j,k} \tag{B.23}$$

with

$$\|\Xi_5^{i,j,k}\|_{L^\infty(0,T;L^\infty(\mathcal{F}))} + \|\Xi_6^{i,j,k}\|_{L^\infty(0,T;L^{8/3}(\mathcal{F}))} \leqslant C \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}. \tag{B.24}$$

B.3 Estimates of $\widehat{F}$, $\widehat{G}$ and $\widehat{H}$

In this section, we estimate the nonlinearities $\widehat{F}$, $\widehat{G}$ and $\widehat{H}$ defined by (2.10), (2.12) and (2.13). These estimates are used in Section 4 for the proofs of Theorem 1.7 and of Theorem 1.8 (see (4.7) and (4.15)) and in Section 2.4 for the proof of Proposition 1.2 (see (2.60), (2.61), (2.62)).

Proof of (4.7)**.** We first prove (4.7) and in that case, we recall that we assume $T \leqslant 1$, that $\eta^* = \eta_1^0$ satisfies (B.1) with $C_* = 2R$ (see (4.2)). In all this part, the constants C may depend on R and we already have the estimates (4.3) and (4.4) on (w, q, η). Since $\eta^* = \eta_1^0$, we can use Proposition A.1 in [3] and with (4.4) we deduce

$$\|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))} \leqslant CT^{1/6} \|\eta - \eta^*\|_{H^{3/4}(0,T;H^{7/4}(\mathcal{I}))} \leqslant CT^{1/6}. \tag{B.25}$$

The above estimate and (B.13) yield

$$\|\nabla X - I_2\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} + \|\mathrm{Cof}(\nabla X) - I_2\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} + \|\nabla Y(X) - I_2\|_{L^\infty(0,T;C^0(\overline{\mathcal{F}}))} \leqslant CT^{1/6}. \tag{B.26}$$

We first estimate the nonlinearity $\widehat{F}$ given by (2.10). Combining (4.3) and (B.16), we deduce

$$\left\| \frac{\partial w_i}{\partial y_j} \frac{\partial Y_j}{\partial t}(X) \right\|_{L^2(0,T;L^2(\mathcal{F}))} \leqslant CT^{1/2} \|\partial_t \eta\|_{L^\infty(0,T;H^1(\mathcal{I}))} \|w\|_{L^\infty(0,T;\mathbf{H}^1(\mathcal{F}))} \leqslant CT^{1/2}.$$

Combining (4.4) and (B.13), we deduce

$$\left\| w_j \frac{\partial w_i}{\partial y_k} \frac{\partial Y_k}{\partial x_j}(X) \right\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leqslant CT^{1/4} (1 + \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}) \|w\|^2_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))} \leqslant CT^{1/4}.$$

Combining (4.3) and (B.26), we deduce

$$\left\| \frac{\partial^2 w_j}{\partial y_k \partial y_i} \left(\frac{\partial Y_k}{\partial x_j}(X) - \delta_{k,j} \right) \right\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \left\| \frac{\partial^2 w_i}{\partial y_k \partial y_\ell} \left(\frac{\partial Y_k}{\partial x_j}(X) \frac{\partial Y_\ell}{\partial x_j}(X) - \delta_{k,j}\delta_{\ell,j} \right) \right\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}$$
$$+ \left\| \frac{\partial q}{\partial y_k} \left(\frac{\partial Y_k}{\partial x_i}(X) - \delta_{k,i} \right) \right\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leqslant CT^{1/6}.$$

Combining (4.3), the embedding $H^1(\mathcal{F}) \subset L^8(\mathcal{F})$, (B.8) for $\alpha = 1$, (B.21), (B.22), (B.13) and (B.25), we deduce

$$\left\| \frac{\partial w_i}{\partial y_k} \frac{\partial^2 Y_k}{\partial x_j^2}(X) \right\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \left\| \frac{\partial w_j}{\partial y_k} \frac{\partial^2 Y_k}{\partial x_j \partial x_\ell}(X) \frac{\partial X_\ell}{\partial y_i} \right\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}$$
$$\leqslant C(1 + \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))})^3 \|\eta - \eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))} \|w\|_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))} \leqslant CT^{1/6},$$

The above estimates yields that

$$\left\| \widehat{F}(\eta, w, q) \right\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leqslant CT^{1/6}. \tag{B.27}$$

The estimate of $\widehat{G}(\eta, w)$ defined by (2.12) in $L^2(0, T; H^1_\#(\mathcal{F}))$ leads to estimate

$$\frac{\partial^2 w_i}{\partial y_k \partial y_j} [\delta_{i,j} - \mathrm{Cof}(\nabla X)_{i,j}] - \frac{\partial w_i}{\partial y_j} \frac{\partial}{\partial y_k} [\mathrm{Cof}(\nabla X)_{i,j}] \tag{B.28}$$

in $L^2(0, T; L^2(\mathcal{F}))$. Combining (4.3), the embedding $H^1(\mathcal{F}) \subset L^8(\mathcal{F})$, (B.8) for $\alpha = 1$, (B.23)–(B.24), (B.25), and (B.26), we deduce

$$\left\| \widehat{G}(\eta, w) \right\|_{L^2(0,T;H^1_\#(\mathcal{F}))} \leqslant CT^{1/6}. \tag{B.29}$$

Thus, from the boundary condition of (1.15) and (A.12) we deduce that $(I_2 - \mathrm{Cof}(\nabla X)^*)w = 0$ on $\partial\mathcal{F}$. Hence, from (2.12), if $\varphi \in H^1_\#(\mathcal{F})$, then

$$\langle \widehat{G}(\eta, w), \varphi \rangle_{H^{-1}_\#(\mathcal{F}), H^1_\#(\mathcal{F})} = -\int_{\mathcal{F}} \nabla \varphi \cdot ((I_2 - \mathrm{Cof}(\nabla X)^*)w) \, dy. \tag{B.30}$$

The above relation yield that

$$\left\|\widehat{G}(\eta,w)\right\|_{H^{-1}_{\#}(\mathcal{F})} \leqslant C\left\|(I_2-\operatorname{Cof}(\nabla X))w\right\|_{\mathbf{L}^2(\mathcal{F})} \tag{B.31}$$

and similarly,

$$\left\|\partial_t\widehat{G}(\eta,w)\right\|_{H^{-1}_{\#}(\mathcal{F})} \leqslant C\left(\|\partial_t \operatorname{Cof}(\nabla X)w\|_{\mathbf{L}^2(\mathcal{F})}+\|(\operatorname{Cof}(\nabla X)-I_2)\partial_t w\|_{\mathbf{L}^2(\mathcal{F})}\right). \tag{B.32}$$

The two above relations, combined with (4.3), (4.4), (B.18), (B.26) and (B.25), imply

$$\begin{aligned}\left\|\widehat{G}(\eta,w)\right\|_{H^1(0,T;H^{-1}_{\#}(\mathcal{F}))} &\leqslant CT^{3/8}\|\partial_t\eta\|_{L^\infty(0,T;H^1(\mathcal{I}))}\|w\|_{L^8(0,T;\mathbf{H}^{5/4}(\mathcal{F}))}\\ &\quad + CT^{1/6}\|w\|_{H^1(0,T;\mathbf{L}^2(\mathcal{F}))}\leqslant CT^{1/6}.\end{aligned} \tag{B.33}$$

Gathering the above estimate and (B.29), we obtain

$$\left\|\widehat{G}(\eta,w)\right\|_{L^2(0,T;H^1_{\#}(\mathcal{F}))}+\left\|\widehat{G}(\eta,w)\right\|_{H^1(0,T;H^{-1}_{\#}(\mathcal{F}))}\leqslant CT^{1/6}. \tag{B.34}$$

Finally, to estimate $\widehat{H}(\eta,w)$ given by (2.13) in $L^2(0,T;H^{1/2}_{\#}(\mathcal{I}))$, we can estimate

$$\begin{aligned}\nu\widetilde{\zeta}\left(\frac{\partial w_1}{\partial y_2}+\frac{\partial w_2}{\partial y_1}\right)-\nu\widehat{\zeta}\sum_k\left(\frac{\partial w_1}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_2}(X)-\delta_{k,2}\right)+\frac{\partial w_2}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_1}(X)-\delta_{k,1}\right)\right)\\ +2\nu\sum_k\left(\frac{\partial w_2}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_2}(X)-\delta_{k,2}\right)\right)\end{aligned} \tag{B.35}$$

in $L^2(0,T;H^1(\mathcal{F}))$. Here

$$\widetilde{\zeta}(t,y_1,y_2)\stackrel{\text{def}}{=}\mathcal{R}_{\partial_s\eta^*-\partial_s\eta(t,\cdot)}\left(y_1,\frac{y_2}{1+\eta^*(y_1)}\right),\quad \widehat{\zeta}(t,y_1,y_2)\stackrel{\text{def}}{=}\mathcal{R}_{\partial_s\eta(t,\cdot)}\left(y_1,\frac{y_2}{1+\eta^*(y_1)}\right),$$

where we recall that $\mathcal{R}$ is the lifting defined by (2.1). In particular, using (B.10),

$$\begin{aligned}\left\|\widetilde{\zeta}\right\|_{L^\infty(0,T;H^{5/4}(\mathcal{F}))}\leqslant C\|\eta-\eta^*\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}\leqslant CT^{1/6},\\ \left\|\widehat{\zeta}\right\|_{L^\infty(0,T;H^{5/4}(\mathcal{F}))}\leqslant C\|\eta\|_{L^\infty(0,T;H^{7/4}(\mathcal{I}))}\leqslant C.\end{aligned}$$

The estimate of (B.35) in $L^2(0,T;H^1(\mathcal{F}))$ leads to the same kind of estimates as above so that

$$\left\|\widehat{H}(\eta,w)\right\|_{L^2(0,T;H^{1/2}_{\#}(\mathcal{I}))}\leqslant CT^{1/6}. \tag{B.36}$$

Gathering (B.27), (B.34), and (B.36) we deduce (4.7).

Proof of (2.60)-(2.62)**.** The proofs of (2.60), (2.61) and (2.62) are quite similar. Here we recall that $T=\infty$, that $\eta^*,\widetilde{\eta}\in\mathcal{B}_R$ (see (1.26)), $\mathcal{F}=\mathcal{F}_{\eta^*}$ and that (2.51) holds. In particular, we use the estimates in Appendix B.2 where the constants C may depend on R (see (B.1)). We can also assume that $\kappa^*\leqslant 1$ for simplicity.

Let us estimate $\widehat{F}_2(\widetilde{\eta},w,q)$ given by (2.11). With (B.13), we first have

$$\begin{aligned}&\left\|\frac{\partial w_j}{\partial y_k\partial y_i}\left(\frac{\partial Y_k}{\partial x_j}(X)-\delta_{k,j}\right)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}+\left\|\frac{\partial^2 w_i}{\partial y_k\partial y_\ell}\left(\frac{\partial Y_k}{\partial x_j}(X)\frac{\partial Y_\ell}{\partial x_j}(X)-\delta_{k,j}\delta_{\ell,j}\right)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\\ &+\left\|\frac{\partial q}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_i}(X)-\delta_{k,i}\right)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\leqslant C\|\widetilde{\eta}-\eta^*\|_{H^{7/4}(\mathcal{I})}\left(\|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}+\|\nabla q\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\right).\end{aligned}$$

Combining (B.13), the embedding $H^1(\mathcal{F}) \subset L^8(\mathcal{F})$, (B.8) for $\alpha = 1$ and (B.21)–(B.22), we deduce

$$\left\| \frac{\partial w_j}{\partial y_k} \frac{\partial^2 Y_k}{\partial x_j \partial x_\ell}(X) \frac{\partial X_\ell}{\partial y_i} \right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \left\| \frac{\partial w_i}{\partial y_k} \frac{\partial^2 Y_k}{\partial x_j^2}(X) \right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \leqslant C \|\widetilde{\eta} - \eta^*\|_{H^{7/4}(\mathcal{I})} \|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}.$$

The above estimates yield that

$$\left\| \widehat{F}_2(\widetilde{\eta}, w, q) \right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \leqslant C \|\widetilde{\eta} - \eta^*\|_{H^{7/4}(\mathcal{I})} \left(\|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} + \|\nabla q\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \right). \tag{B.37}$$

The estimate of $\widehat{G}(\widetilde{\eta}, w)$ defined by (2.12) in $L^2(0,\infty; H^1_\#(\mathcal{F}))$ leads to estimate (B.28) in $L^2(0,\infty; L^2(\mathcal{F}))$. With the same argument leading to (B.29) we obtain

$$\left\| \widehat{G}(\widetilde{\eta}, w) \right\|_{L^2(0,\infty;H^1_\#(\mathcal{F}))} \leqslant C \|\widetilde{\eta} - \eta^*\|_{H^{7/4}(\mathcal{I})} \|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}. \tag{B.38}$$

Moreover, combining (B.31), (B.32), (B.13), (B.18) and (4.13) implies

$$\left\| \widehat{G}(\widetilde{\eta}, w) \right\|_{H^1(0,\infty;H^{-1}_\#(\mathcal{F}))} \leqslant C \|\widetilde{\eta} - \eta^*\|_{H^{7/4}(\mathcal{I})} \left(\|w\|_{H^1(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \|w\|_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))} \right). \tag{B.39}$$

Finally, to estimate $\widehat{H}(\eta, w)$ in $L^2(0,\infty; H^{1/2}_\#(\mathcal{I}))$, we estimate (B.35) in $L^2(0,\infty; H^1(\mathcal{F}))$. This leads to the same estimates as above so that

$$\left\| \widehat{H}(\widetilde{\eta}, w) \right\|_{L^2(0,\infty;H^{1/2}_\#(\mathcal{I}))} \leqslant C \|\widetilde{\eta} - \eta^*\|_{H^{7/4}(\mathcal{I})} \|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}. \tag{B.40}$$

Gathering (B.37), (B.38), (B.39), (B.40) yields (2.60).

The proof of (2.61)-(2.62) follows the same approach. For instance, one can use (A.5) and the above estimates for $\widehat{F}_2(\widetilde{\eta}, w, q)$ to deduce

$$\begin{aligned} \left\| \frac{\partial^2 \widetilde{w}_i}{\partial x_j \partial x_m} \right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}_{\widetilde{\eta}}))} \leqslant & \sum_{k,\ell} \left\| \frac{\partial^2 w_i}{\partial y_k \partial y_\ell} \frac{\partial Y_k}{\partial x_j}(X) \frac{\partial Y_\ell}{\partial x_m}(X) \sqrt{\det(\nabla X)} \right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \\ & + \sum_k \left\| \frac{\partial w_i}{\partial y_k} \frac{\partial^2 Y_k}{\partial x_j \partial x_m}(X) \sqrt{\det(\nabla X)} \right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \leqslant C \|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} \end{aligned} \tag{B.41}$$

and we obtain similarly $\|(\widetilde{w}, \widetilde{q}, \eta)\|_{\mathfrak{W}_\infty(\mathcal{F}_{\widetilde{\eta}})} \leqslant C \|(w, q, \eta)\|_{\mathfrak{W}_\infty(\mathcal{F})}$. For the other estimates of (2.61)-(2.62), we can give the ideas to prove

$$\|G\|_{L^2(0,\infty;H^1_\#(\mathcal{F})) \cap H^1(0,\infty;H^{-1}_\#(\mathcal{F}))} \leqslant C \|\widetilde{G}\|_{L^2(0,\infty;H^1_\#(\mathcal{F}_{\widetilde{\eta}})) \cap H^1(0,\infty;H^{-1}_\#(\mathcal{F}_{\widetilde{\eta}}))}$$

the other relations are obtained in the same way. First, from (2.57) and (A.13) we can write

$$\frac{\partial G}{\partial y_i} = (\partial_{y_i y_2} \zeta) \widetilde{G}(X) + (1 + \partial_{y_2} \zeta) \sum_k \frac{\partial \widetilde{G}}{\partial x_k}(X) \frac{\partial X_k}{\partial y_i}.$$

Combining (B.11), (B.13), the embedding $H^1(\mathcal{F}) \subset L^8(\mathcal{F})$, (B.8) for $\alpha = 1$ and (B.20)–(B.20), we deduce

$$\|G\|_{L^2(0,\infty;H^1_\#(\mathcal{F}))} \leqslant C \|\widetilde{G}\|_{L^2(0,\infty;H^1_\#(\mathcal{F}_{\widetilde{\eta}}))}.$$

Then differentiating the relation for G in (2.57) with respect to time, we obtain

$$\frac{\partial G}{\partial t} = \det(\nabla X) \frac{\partial \widetilde{G}}{\partial t}(X) \tag{B.42}$$

and thus

$$\left\langle \frac{\partial G}{\partial t}, \varphi \right\rangle_{H^{-1}_\#(\mathcal{F}), H^1_\#(\mathcal{F})} = \left\langle \frac{\partial \widetilde{G}}{\partial t}, \varphi(Y) \right\rangle_{H^{-1}(\mathcal{F}_{\widetilde{\eta}}), H^1(\mathcal{F}_{\widetilde{\eta}})}. \tag{B.43}$$

We notice that $\|\varphi(Y)\|_{H^1(\mathcal{F}_{\widetilde{\eta}})} \leqslant C \|\varphi\|_{H^1(\mathcal{F})}$. This leads us to the estimate

$$\|G\|_{H^1(0,\infty;H^{-1}_\#(\mathcal{F}))} \leqslant C \|\widetilde{G}\|_{H^1(0,\infty;H^{-1}_\#(\mathcal{F}_{\widetilde{\eta}}))}.$$

Proof of (4.15)**.** The proof is similar to the two previous ones. Here $T=\infty$, $\eta^*=0$ so that it satisfies (B.1) with $C_*=1$. At the contrary to the two previous proofs, the constants C do not depend on R, and we have assumed $R\leqslant 1$ to simplify. We use the estimates in Appendix B.2 where the constants C do not depend on R since here $\eta^*=0$. We have already the estimates (4.13) and (4.14) on (w,q,η).

First, let us estimate $\widehat{F}(\eta,w,q)$ defined by (2.10). With (B.13), we deduce

$$\left\|\frac{\partial w_j}{\partial y_k\partial y_i}\left(\frac{\partial Y_k}{\partial x_j}(X)-\delta_{k,j}\right)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}+\left\|\frac{\partial^2 w_i}{\partial y_k\partial y_\ell}\left(\frac{\partial Y_k}{\partial x_j}(X)\frac{\partial Y_\ell}{\partial x_j}(X)-\delta_{k,j}\delta_{\ell,j}\right)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}$$
$$+\left\|\frac{\partial q}{\partial y_k}\left(\frac{\partial Y_k}{\partial x_i}(X)-\delta_{k,i}\right)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}$$
$$\leqslant C(1+\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))})\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))}\left(\|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}+\|\nabla q\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\right)\leqslant CR^2.$$

Combining (B.13), the embedding $H^1(\mathcal{F})\subset L^8(\mathcal{F})$ and (B.21)–(B.22), we deduce

$$\left\|\frac{\partial w_j}{\partial y_k}\frac{\partial^2 Y_k}{\partial x_j\partial x_\ell}(X)\frac{\partial X_\ell}{\partial y_i}\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}+\left\|\frac{\partial w_i}{\partial y_k}\frac{\partial^2 Y_k}{\partial x_j^2}(X)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}$$
$$\leqslant C(1+\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))})^3\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))}\|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\leqslant CR^2.$$

Moreover, combining (4.13) and (B.16), we deduce

$$\left\|\frac{\partial w_i}{\partial y_j}\frac{\partial Y_j}{\partial t}(X)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\leqslant C\|\partial_t\eta\|_{L^\infty(0,\infty;H^1(\mathcal{I}))}\|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\leqslant CR^2,$$

and combining (4.14) and (B.13), we deduce

$$\left\|w_j\frac{\partial w_i}{\partial y_k}\frac{\partial Y_k}{\partial x_j}(X)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\leqslant C(1+\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))})\|w\|_{L^8(0,\infty;\mathbf{H}^{5/4}(\mathcal{F}))}\|w\|_{L^{8/3}(0,\infty;\mathbf{H}^{7/4}(\mathcal{F}))}\leqslant CR^2.$$

The above estimates yield

$$\left\|\widehat{F}(\eta,w,q)\right\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\leqslant CR^2. \tag{B.44}$$

The estimate of $\widehat{G}(\eta,w)$ defined by (2.12) in $L^2(0,\infty;H^1_\#(\mathcal{F}))$ leads to estimate (B.28) in $L^2(0,\infty;L^2(\mathcal{F}))$. With the same argument leading to (B.29) we obtain

$$\left\|\widehat{G}(\eta,w)\right\|_{L^2(0,\infty;H^1_\#(\mathcal{F}))}\leqslant C(1+\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))})^3\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))}\|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\leqslant CR^2. \tag{B.45}$$

Moreover, combining (B.31), (B.32), (B.13), (B.18) and (4.13) implies

$$\left\|\widehat{G}(\eta,w)\right\|_{H^1(0,\infty;H^{-1}_\#(\mathcal{F}))}\leqslant C\,\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))}\,\|w\|_{H^1(0,\infty;\mathbf{L}^2(\mathcal{F}))}$$
$$+C\|\partial_t\eta\|_{L^\infty(0,\infty;H^1(\mathcal{I}))}\|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\leqslant CR^2. \tag{B.46}$$

Finally, to estimate $\widehat{H}(\eta,w)$ in $L^2(0,\infty;H^{1/2}_\#(\mathcal{I}))$, we estimate (B.35) in $L^2(0,\infty;H^1(\mathcal{F}))$. This leads to the same estimates as above so that

$$\left\|\widehat{H}(\eta,w)\right\|_{L^2(0,\infty;H^{1/2}_\#(\mathcal{I}))}\leqslant C(1+\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))})^2\|\eta\|_{L^\infty(0,\infty;H^{7/4}(\mathcal{I}))}\|w\|_{L^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\leqslant CR^2. \tag{B.47}$$

Gathering (B.44), (B.45), (B.46) and (B.47), we deduce (4.15).

References

[1] Mehdi Badra and Takéo Takahashi. On the Fattorini criterion for approximate controllability and stabilizability of parabolic systems. *ESAIM Control Optim. Calc. Var.*, 20(3):924–956, 2014.

[2] Mehdi Badra and Takéo Takahashi. Feedback boundary stabilization of 2D fluid-structure interaction systems. *Discrete Contin. Dyn. Syst.*, 37(5):2315–2373, 2017.

[3] Mehdi Badra and Takéo Takahashi. Gevrey regularity for a system coupling the Navier-Stokes system with a beam equation. *SIAM J. Math. Anal.*, 51(6):4776–4814, 2019.

[4] Mehdi Badra and Takéo Takahashi. Gevrey regularity for a system coupling the navier-stokes system with a beam: the non-flat case. *Funkcialaj Ekvacioj*, to appear. https://hal.archives-ouvertes.fr/hal-02303258/.

[5] Hugo Beirão da Veiga. On the existence of strong solutions to a coupled fluid-structure evolution problem. *J. Math. Fluid Mech.*, 6(1):21–52, 2004.

[6] Alain Bensoussan, Giuseppe Da Prato, Michel C. Delfour, and Sanjoy K. Mitter. *Representation and control of infinite dimensional systems*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, second edition, 2007.

[7] Haïm Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[8] Antonin Chambolle, Benoît Desjardins, Maria J. Esteban, and Céline Grandmont. Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *J. Math. Fluid Mech.*, 7(3):368–404, 2005.

[9] Shu Ping Chen and Roberto Triggiani. Proof of extensions of two conjectures on structural damping for elastic systems. *Pacific J. Math.*, 136(1):15–55, 1989.

[10] Hiroshi Fujita and Hiroko Morimoto. On fractional powers of the Stokes operator. *Proc. Japan Acad.*, 46:1141–1143, 1970.

[11] Céline Grandmont. Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *SIAM J. Math. Anal.*, 40(2):716–737, 2008.

[12] Céline Grandmont and Matthieu Hillairet. Existence of global strong solutions to a beam-fluid interaction system. *Arch. Ration. Mech. Anal.*, 220(3):1283–1333, 2016.

[13] Céline Grandmont, Matthieu Hillairet, and Julien Lequeurre. Existence of local strong solutions to fluid-beam and fluid-rod interaction systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(4):1105–1149, 2019.

[14] Pierre Grisvard. Caractérisation de quelques espaces d'interpolation. *Arch. Rational Mech. Anal.*, 25:40–63, 1967.

[15] Julien Lequeurre. Existence of strong solutions to a fluid-structure system. *SIAM J. Math. Anal.*, 43(1):389–410, 2011.

[16] Boris Muha and Suncica Canić. Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls. *Arch. Ration. Mech. Anal.*, 207(3):919–968, 2013.

[17] Jean-Pierre Raymond. Feedback stabilization of a fluid-structure model. *SIAM J. Control Optim.*, 48(8):5398–5443, 2010.

[18] Jean-Pierre Raymond. Stokes and Navier-Stokes equations with a nonhomogeneous divergence condition. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1537–1564, 2010.

[19] Sunčica Čanić, Boris Muha, and Martina Bukač. Fluid-structure interaction in hemodynamics: modeling, analysis, and numerical simulation. In *Fluid-structure interaction and biomedical applications*, Adv. Math. Fluid Mech., pages 79–195. Birkhäuser/Springer, Basel, 2014.