Trajectory under harmonic potential and magnetic force

Eric Guiot

Trajectory under harmonic potential and magnetic force

Eric Guiot
Lyon, France
guiot.eric_1@yahoo.fr

Abstract: We study the trajectories of a point particle which is subjected to the simultaneous effect of an attractive restoring force and a force perpendicular to its speed, which can thus be a magnetic force. Solving the equations of motion, we present the mathematical expression of the trajectories we have obtained, which are in general cases, a part of the curves known as "Centered Trochoïd" curves. Several of them are detailed with geometrical properties such the radius of curvature and some characteristics of the motion. We also present the limiting cases and we discuss our results with references to works previously published, especially about the normal Zeeman effect.
.

Keywords: Central force; magnetic field; harmonic oscillator;

Introduction

The study of the trajectories of particles which move under the influence of restoring forces is an important topic in Physics, considering the many physical applications in almost all the areas. This is particularly true when these forces are attractive, i.e. when the mechanical system is referred to a harmonic motion, and when the particle is simultaneously submitted to the influence of a magnetic field.

This paper is devoted to this topic, and present results which can represent a significant progress. Indeed, naturally, this kind of motion has already been studied in the history of classical Physics, in particular when an atom was modelled as a simple harmonic oscillator, in order to explain the Zeeman effect ([1], [2]). However, it appears we have obtained others and original solutions, which contain the solutions already known as limiting cases.

These trajectories we have obtained are given with the two parametrized plane curves

$$\begin{Bmatrix} X = a\cos wt\cos\alpha t - b\sin wt\sin\alpha t \\ Y = a\cos wt\sin\alpha t + b\sin wt\cos\alpha t \end{Bmatrix} \quad (1.1)$$

And

$$\begin{Bmatrix} X = a\cos wt\cos\alpha t + b\sin wt\sin\alpha t \\ Y = a\cos wt\sin\alpha t - b\sin wt\cos\alpha t \end{Bmatrix} \quad (1.2)$$

Where : cartesian coordinate system is

$$(O;\ \vec{X};\ \vec{Y};\ \vec{Z})$$

a and b are two constant lengths
w and α are two constant pulsations; α is the frequency of Larmor,

t is the time.
Pulsations are linked to the spring constant k with

$$k = w^2 - \alpha^2$$

It appears that curves definite with relations (1) are, as we will detail it, already known under the name of “Centered Trochoïd” [3]. Among them we can distinguish two famous limiting cases, the ellipse of Lissajous – Bowditch, when $\alpha = 0$ (magnetic field is null, bi dimensional harmonic oscillator) and the rose curves, when $b = 0$.

In the first part of the paper we present the mathematical demonstration of our results, solving the equations of motion. In the second part the reader will find the descriptions of several characteristics of motions, such as energies and angular momentum. To conclude, we discuss the results and present possible applications, in particular about the about the Normal Zeeman and Pashen-Back effect.

Note this work is only constructed using the laws of classical Physics and consequently, we should remind the reader that we have not taken into consideration, at this step, relativistic or quantum effects.

1 – Dynamics

Note : In all the paper the mass of the point particle is considered equal to 1, such forces and accelerations are confused.

In this part we consider the motion of a charged point particle which is simultaneously submitted to an attractive restoring force and a magnetic force due to a constant magnetic field B perpendicular to the plane on which a charged particle moves. The force at distance we propose to study can be written

$$\vec{F} = -k\vec{r} + q\vec{V} * \vec{B} \qquad (2)$$

Where
k is the spring constant
q the charge of the particle
V its speed.

A limiting case is obtained when the magnetic field is null: this case is corresponding on the well-known harmonic oscillator and the point particle describes an ellipse of Lissajous – Bowditch.

1.1 solving

We begin introducing the cartesian system of coordinate $\left(O;\ \vec{X};\ \vec{Y};\ \vec{Z}\right)$ where O is the center of the restoring force. We can rewrite (2) as

$$\vec{F} = -k(X\vec{x} + Y\vec{y}) + +qB(\dot{X}\vec{y} - \dot{Y}\vec{x})$$

For convenience for the calculation we introduce the pulsation of the initial harmonic oscillator and the Larmor frequency (per mass unity) [4]

$$\begin{Bmatrix} w_0 = \sqrt{k} \\ \alpha = \frac{qB}{2} \end{Bmatrix}$$

System becomes using the Newton law of motion

$$\begin{Bmatrix} \ddot{X} = -{w_0}^2 X - 2\alpha\dot{Y} \\ \ddot{Y} = -{w_0}^2 Y + 2\alpha\dot{X} \end{Bmatrix}$$

A classical method to solve this kind of system is to introduce the complex number (solving can be found for example in references [1], [5])

$$u = X + jY$$

We obtain the differential equation

$$\frac{d^2u}{dt^2} + {w_0}^2 u - 2j\alpha\frac{du}{dt} = 0$$

To solve it we consider a limiting case whose we know an exact solution : it is naturally the ellipse of Lissajous-Bowditch we have evoked, corresponding on $\alpha = 0$. We name a and b respectivelly the semi major and minor axis of this conic. By choosing initial conditions such

$$\begin{Bmatrix} t = 0 \\ X = a \\ Y = 0 \end{Bmatrix}$$

The solution is given by the relations

$$\begin{Bmatrix} X = a\cos wt \\ Y = \pm b\sin wt \end{Bmatrix}$$

(Dependent on the direction of rotation). In this case the complex number u is

$$u = a\cos wt \pm jb\sin wt$$

Or,

$$\begin{Bmatrix} u = \frac{a+b}{2}e^{jwt} + \frac{a-b}{2}e^{-jwt} \\ u = \frac{a-b}{2}e^{jwt} + \frac{a+b}{2}e^{-jwt} \end{Bmatrix}$$

We add now a magnetic fields perpendicular to the plane of this ellipse. To begin the study we choose to investigate solutions of the form

$$u = a_0 e^{j(w+\alpha)t}$$

Introducing it in the differential equation we obtain

$$-(w+\alpha)^2 + {w_0}^2 + 2\alpha(w+\alpha) = 0$$

Consequently

$${w_0}^2 = w^2 - \alpha^2$$

Rewriting now our equation

$$\frac{d^2u}{dt^2} + (w^2 - \alpha^2)u - 2i\alpha\frac{du}{dt} = 0$$

We notice that a second solution exists

$$u = b_0 e^{j(-w+\alpha)t}$$

Where b_0 is our second constant length. Whereas these results we investigate thus a general solution such

$$u = a_0 e^{j(w+\alpha)t} + b_0 e^{j(-w+\alpha)t}$$

To determine the constants we study the initial conditions : considering this ellipse, i.e. for $\alpha = 0$, and considering the origin of the time we obtain

for $wt = 0$

$$a = a_0 + b_0$$

And for $wt = \frac{\pi}{2}$, we obtain two possibilities depending on the direction of rotation

$$\begin{Bmatrix} b = a_0 - b_0 \\ -b = a_0 - b_0 \end{Bmatrix}$$

Where a and b are successively the semi major and minor axis of the conic. We deduce that

$$\begin{Bmatrix} a_0 = \dfrac{a+b}{2} \\ b_0 = \dfrac{a-b}{2} \end{Bmatrix}$$

Or

$$\begin{Bmatrix} a_0 = \dfrac{a-b}{2} \\ b_0 = \dfrac{a+b}{2} \end{Bmatrix}$$

We obtain thus two general solutions

$$\begin{cases} u = \frac{a+b}{2} e^{j(+w+\alpha)t} + \frac{a-b}{2} e^{j(-w+\alpha)t} \\ u = \frac{a-b}{2} e^{j(+w+\alpha)t} + \frac{a+b}{2} e^{j(-w+\alpha)t} \end{cases}$$

We recognize the expressions of the centered Trichoïd curves, given with [3]

$$u = r_1 e^{jw_1 t} + r_2 e^{jw_2 t}$$

And, by identifying the real and imaginary parts

$$\begin{cases} X = \frac{a+b}{2} \cos(w+\alpha)t + \frac{a-b}{2} \cos(w-\alpha)t \\ Y = \frac{a+b}{2} \sin(w+\alpha)t - \frac{a-b}{2} \sin(w-\alpha)t \end{cases} \quad (3.1)$$

Or

$$\begin{cases} X = \frac{a-b}{2} \cos(w+\alpha)t + \frac{a+b}{2} \cos(w-\alpha)t \\ Y = \frac{a-b}{2} \sin(w+\alpha)t - \frac{a+b}{2} \sin(w-\alpha)\, t \end{cases} \quad (3.2)$$

Finally

$$\begin{cases} X = a \cos wt \cos \alpha t - b \sin wt \sin \alpha t \\ Y = a \cos wt \sin \alpha t + b \sin wt \cos \alpha t \end{cases}$$

And

$$\begin{cases} X = a \cos wt \cos \alpha t + b \sin wt \sin \alpha t \\ Y = a \cos wt \sin \alpha t - b \sin wt \cos \alpha t \end{cases}$$

i.e. two possible parametrized curves dependent on the initial state of the harmonic oscillator . Note these trajectories are solutions to the problem if

$${w_0}^2 = k = w^2 - \alpha^2 > 0$$

Or, introducing the ratio of pulsations

$$n = \frac{\alpha}{w}$$

If

$$-1 < n < 1$$

This indicate that they are a part, corresponding to this condition, of the Centered Trichoïd curves.

1.2 – Verification

At this step we verify our solutions: considering for example the first of them and the successive derivatives with respect to the time

$$\begin{Bmatrix} \dot{X} = -a(w\sin wt\cos\alpha t + \alpha\cos wt\sin\alpha t) - b(w\cos wt\sin\alpha t + \alpha\sin wt\cos\alpha t) \\ \dot{Y} = -a(w\sin wt\sin\alpha t - \alpha\cos wt\cos\alpha t) + b(w\cos wt\cos\alpha t - \alpha\sin wt\sin\alpha t) \end{Bmatrix}$$

And

$$\begin{Bmatrix} \ddot{X} = a(-w^2\cos wt\cos\alpha t + 2w\alpha\sin wt\sin\alpha t - \alpha^2\cos wt\cos\alpha t) \\ +b(w^2\sin wt\sin\alpha t - 2w\alpha\cos wt\cos\alpha t + \alpha^2\sin wt\sin\alpha t) \\ \ddot{Y} = -a(w^2\cos wt\sin\alpha t + 2w\alpha\sin wt\cos\alpha t + \alpha^2\cos wt\sin\alpha t) \\ -b(w^2\sin wt\cos\alpha t + 2bw\alpha\cos wt\sin\alpha t + \alpha^2\sin wt\cos\alpha t) \end{Bmatrix}$$

A check by direct proof leads well to

$$\begin{Bmatrix} \ddot{X} = -(w^2-\alpha^2)X - 2\alpha\dot{Y} \\ \ddot{Y} = -(w^2-\alpha^2)Y + 2\alpha\dot{X} \end{Bmatrix}$$

Where

$$k = w^2 - \alpha^2$$

2. Presentation of the curves

2.1 several possible trajectories

In this second part we present the plane parametrized we obtained, choosing the first of them. Using the ratio n they are given by

$$\begin{Bmatrix} X = a\cos wt\cos nwt - b\sin wt\sin nwt \\ Y = a\cos wt\sin nwt + b\sin wt\cos nwt \end{Bmatrix}$$

Firstly note these equations are corresponding on a precession of the initial ellipse of Lissajous_Bowdith with an angle of precession equal to α . This precession is visible when n is small (see for example Figure 1). (curves corresponding on figures 1, 2, 3, 4, 5) are drawed for values $a = 1$, $e = 0.8$ in our system of coordinates) (e is the eccentricity of the conic).

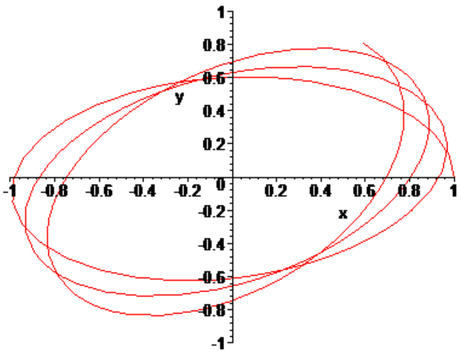

Figure 1 : $n = 0.05$

For a revolution angle of precession is simply given by

$$\hat{p} = 2\pi n$$

Trajectories are closed if n i a rational number. When this ratio varies we obtain a large variety of curves, See for example the following centered Trochoïd curves

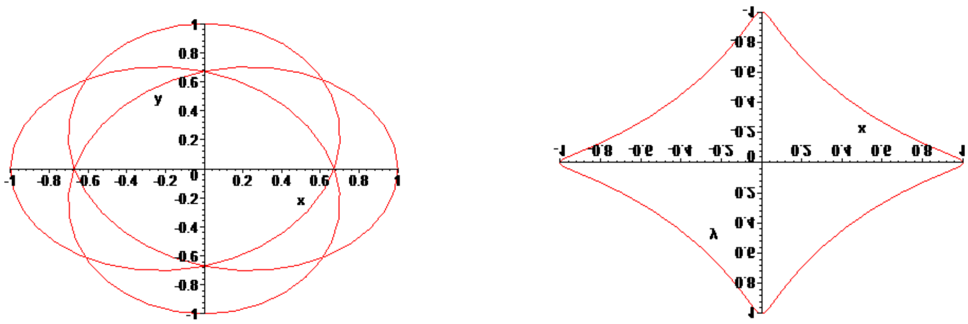

Figure 2 : $n = \pm 0.5$

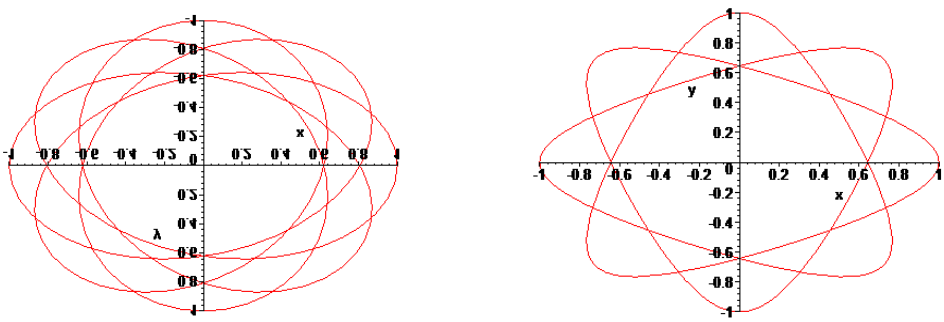

Figure 3 : $n = \pm 0.25$

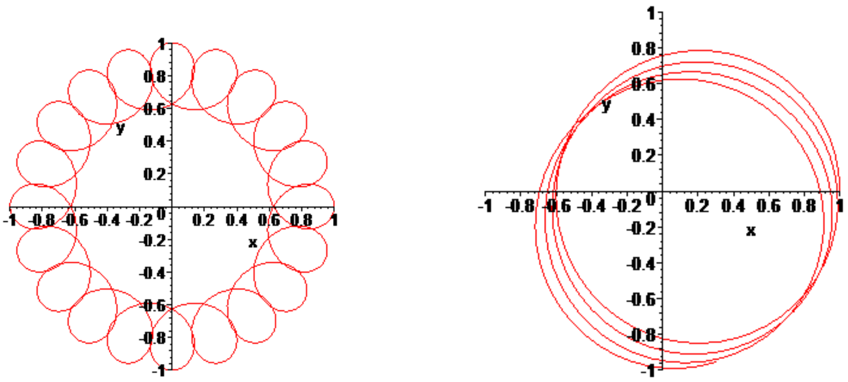

Figure 4 - $n = -0.9$ and $n = 0.9$

2.3 – Energy and angular momentum

For the reason that magnetic force doesn't work Mechanical Energy E (sum of the kinetic energy $T$and the harmonic potential V) is conserved on all the trajectories

$$E = T + V = \frac{1}{2}\left(\dot{X}^2 + \dot{Y}^2 + k(X^2 + Y^2)\right) = \frac{1}{2}w(wb^2 + 2ab\alpha + a^2 w)$$

Angular momentum is given at O by the relation

$$\vec{p} = \vec{r} * \vec{V}$$

And can be written in our system of coordinate

$$\vec{p} = (X\vec{x} + Y\vec{y}) * \left(\dot{X}\vec{x} + \dot{Y}\vec{y}\right) = \left(X\dot{Y} - Y\dot{X}\right)\vec{z}$$

Using previously relations we obtain after simplifications

$$\vec{p} = (abw + \alpha a^2[1 - e^2 sin^2 wt])\vec{z}$$

2.4 Radius of curvature

Radius of curvature is naturally an important property of the curves. This one, for plane parametrized curves depending on the time is given by

$$R(t) = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

Where the time derivatives are written by

$$\begin{Bmatrix} \dot{x} = \dfrac{dx}{dt} \\ \ddot{x} = \dfrac{d^2x}{dt^2} \end{Bmatrix}$$

After calculations we have obtained

$$R(t) = \frac{A^{3/2}}{abw^3 + \alpha[A + w(b^2 w + ab\alpha + wa^2)]}$$

where

$$A = (aw + b\alpha)^2 - (w^2 - \alpha^2)(a^2 - b^2)cos^2 wt$$

It extends along an axis (IM) where M is a point of the curve

$$M(X, Y, 0)$$

And I is point located on the X axis such

$$I(I_x, 0,0)$$

To obtain it we write the condition

$$\overrightarrow{IM}.\vec{V} = 0$$

Where $\vec{V}$ is the speed (naturally tangential to the curve)

$$\vec{V}(\dot{X}; \dot{Y}; 0)$$

Thus

$$(I_x - X).\dot{X} - Y.\dot{Y} = 0$$

And point I is defined by

$$I_x = \frac{w(a^2 - b^2)(\cos wt \sin wt)}{(aw + b\alpha)(\sin wt \cos \alpha t) + (a\alpha + bw)(\cos wt \sin \alpha t)}$$

Two interesting limiting cases can be noted : indeed, when $\alpha = \pm w$, radius of curvature R are constant. It is also the case of the speed V (tab. 1)

$n = 1$	$R = \frac{1}{2}(a + b)$	$I_x = \frac{1}{2}(a - b)$	$V = \alpha(a + b)$
$n = -1$	$R = \frac{1}{2}(a - b)$	$I_x = \frac{1}{2}(a + b)$	$V = \alpha(a - b)$

Tab 1 : particular cases : $\alpha = \pm w$

We obtain thus two circles

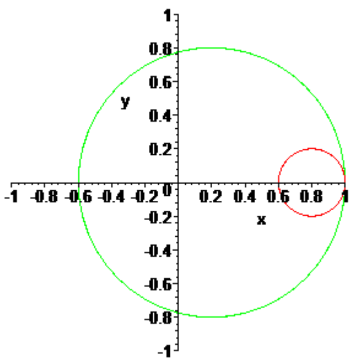

Figure 5 : $n = \pm 1$

In fact it corresponds to a well-known situation. Indeed, in these cases the coefficient k given by

$$k = (w^2 - \alpha^2)$$

Becomes equal to zero : This indicates that the harmonic force is null and, consequently, point particle is only submitted to the constant magnetic field. Trajectories are thus circular and we re-obtain for each cases the classical relation

$$V = qBR$$

2.5 – Polar equation

It is often difficult to write a system of parametric equations in a polar system of coordinate $(O, \overrightarrow{e_r}, \overrightarrow{e_\theta})$. In our case we managed to do it only for certain limiting cases, writing the classical equations

$$\begin{Bmatrix} X = R\cos\theta \\ Y = R\sin\theta \end{Bmatrix}$$

Where R is given by

$$R = \sqrt{X^2 + Y^2} = \sqrt{a^2cos^2wt + b^2sin^2wt}$$

Indeed we can obtain the angular speed using the relation

$$\dot{\theta} = -\frac{\left(\frac{\dot{X}}{R}\right)}{\frac{Y}{R}}$$

Which leads to

$$\dot{\theta} = \frac{(a^2cos^2wt + b^2sin^2wt)\alpha + wab)}{a^2cos^2wt + b^2sin^2wt}$$

We have noted note that the angular speed is constant and given by $\dot{\theta} = \alpha$ if $w = 0$, $a = 0$ or $b = 0$

Which allows to obtain in these cases the polar angle

$$\theta = \alpha t$$

Considering for example the case where $b = 0$. The curves becomes

$$\begin{Bmatrix} X = a\cos wt \cos nwt \\ Y = a\cos wt \sin nwt \end{Bmatrix}$$

And, according to our previously results, can be written in the polar system of coordinate

$$\begin{cases} X = a\cos\dfrac{\theta}{n}\cos\theta \\ Y = a\cos\dfrac{\theta}{n}\sin\theta \end{cases}$$

Which is corresponding on a well known familly of curves, called in the history of mathematics as « Rose Curves » since they have been studied by the mathematician Guido Grandi ([6], [7]). To illustrate the paper we draw the case $n = 1/3$ (« regular trifolium ») (Figure 6)

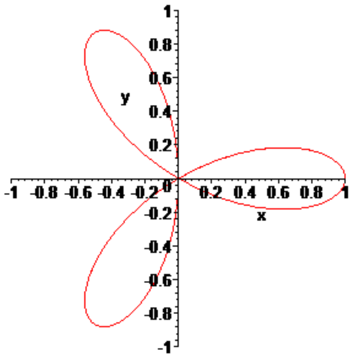

Figure 6 : $n = 1/3$

This indicates that when the initial oscillator is moved in a straight (i.e. when it is a one-dimensional oscillator) the point particle (after addition of the magnetic field) describes one of these curves, where the angular speed is constant and simply given by $\dot{\theta} = \alpha$. It is interesting to note that these properties seem not have been noticed until today. The speed of the point particle along these lines are thus

$$V = \frac{a\alpha}{n}\sqrt{1 + (n^2 - 1)cos^2\frac{\theta}{n}}$$

We list the extrema in the following table:

$\theta = 2kn\pi$	$V_{min} = a\alpha$
$\theta = n\pi\left(\dfrac{2k+1}{2}\right)$	$V_{max} = \dfrac{a\alpha}{n}$

Tab 2. Speed on the rose curves

Speed is well at the most when the point particle reaches the origin of the restoring force.

3. Discussion

As we introduced it we can compare our work with important results of the Classical Physics: we think in particular to the Normal Zeeman Effect (if the magnetic field is weak) and the Paschen-Back effect (if it is strong) [8]. Indeed, these effects describes the behavior of an atom (modeled as a harmonic oscillator) submitted to a constant magnetic field. The theoretical explanation of the Normal Zeeman Effect is historically due to Lorentz, solving equation (1), as it can be found in reference [1].

Our own results can thus be comparable with these works when the spring coefficient given by

$$k = {w_0}^2 = w^2 - \alpha^2$$

Is positive, thus when

$$-1 < n < 1$$

We can list several differences between our results and theses demonstrations. Indeed, it appears these solutions have been obtained considering that the angle between the direction of the constant magnetic field and the plane of the initial trajectory is random, which is not the case of our work. They have also generally been obtained considering approximations, for example assuming that $\alpha \ll w_0$ [1] or that the point particle describes initially a circular orbit around the center of harmonic force [5].

The solutions presented in the paper seems different because we have chosen different conditions:

- Initial trajectory of the point particle (before the addition of the magnetic field) is a conic (and not necessary a circle): It is in this case, an ellipse of Hooke (or Lissajous-Bowditch).
- Angle between the magnetic field and the osculating plane of the trajectory is right.
- We didn't do approximation on the ratio α/w_0

Consequently, the family of curve is different but can be compared when initial trajectory is circular, i.e. when the two lengths a and b are equals, and when the orientation of the magnetic field is perpendicular. Indeed, our two possible trajectories are in this case given by

$$\begin{Bmatrix} X = a[\cos wt \cos \alpha t - \sin wt \sin \alpha t] \\ Y = a[\cos wt \sin \alpha t + \sin wt \cos \alpha t] \end{Bmatrix}$$

and

$$\begin{Bmatrix} X = a[\cos wt \cos \alpha t + \sin wt \sin \alpha t] \\ Y = a[\cos wt \sin \alpha t - \sin wt \cos \alpha t] \end{Bmatrix}$$

Or, simpler

$$\begin{Bmatrix} X = a\cos(w+\alpha)t \\ Y = a\sin(w+\alpha)t \end{Bmatrix}$$

and

$$\begin{Bmatrix} X = a\cos(w-\alpha)t \\ Y = a\sin(w-\alpha)t \end{Bmatrix}$$

Can be compared with the solutions presented in reference [2], for these conditions, which are respectively

$$\begin{Bmatrix} X = a\cos(w_0+\alpha)t \\ Y = a\sin(w_0+\alpha)t \end{Bmatrix}$$

And

$$\begin{Bmatrix} X = a\cos(w_0 - \alpha)t \\ Y = a\sin(w_0 - \alpha)t \end{Bmatrix}$$

We see that the two expressions are in correct agreement as much the magnetic field is weak. Indeed classical explaination of the effect provides that “The circular motion of the oscillating electron in the xy-plane at angular frequencies ω0 + ΩL and ω0 − ΩL produces radiation at these frequencies » [1] (ΩL is the frequency of Larmor). We note we obtain well the same gap between the two frequencies. However we also note a difference about the central position, which isn’t in our solution w_0 but w. This difference is thus

$$\Delta = w_0 - \sqrt{w_0{}^2 + \alpha^2}$$

Naturally this difference is small in the case of the Normal Zeeman effect, because the magnetic fields is weak. But, when it increases (case of the Paschen Back effect) the difference could also increase. This point can thus be considered as a prediction of the model, to the extent that, naturally, classical mechanics can describe this effect.

4. Conclusion

We have presented a family of plane parametrized curves which are obtained by performing a precession around the center of an initial ellipse. We have showed that a part of these curves can describe the trajectory of a point-particle simultaneously submitted to a harmonic force and a force perpendicular to its speed. Besides their mathematical and physical interest, these results could have applications in some areas such, for example, the classical electrodynamic, or others. We suggest a prediction about the normal Zeeman and the Paschen-Back effects. Research perspectives could be to verify this prediction and to extend the study, for example to a three-dimensional harmonic oscillator. Another major point should to consider the consequences of quantum effects.

References:

[1] Christopher Foot *Atomic Physics* Oxford University Press (2005) pp 13 – 16
[2] A.J. Kox. *The discovery of the electron : II. The Zeeman effect*. Eur.J.Phys.18. (1997) pp 139-144
[3] https://mathcurve.com/courbes2d/trochoid/trochoidacentre.shtml
[4] Hand L.N, Finch J.D., *Analytical Mechanics*, Cambridge University Press (1998) pp 220
[5] J.M. Malherbe, Cours, L’observatoire de Paris, MOOC (Astro)Physique I : Électromagnétisme, Oscillateur harmonique en présence de champ magnétique et effet Zeeman
MOOC-ELM-DOC-ELM-A-V-VII.pdf (fun-mooc.fr)
[6] https://mathcurve.com
[7] O’Connor John J., Robertson Edmund F., *Mac Tutor History of mathematics archive*
[8] Paschen F., Back E, *Liniengruppen magnetisch vervollständigt*, Physica (1921) pp 261 - 273