Traveling waves for the porous medium equation in the incompressible limit: asymptotic behavior and nonlinear stability

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Abstract

In this study, we analyze the behavior of monotone traveling waves of a one-dimensional porous medium equation modeling mechanical properties of living tissues. We are interested in the asymptotics where the pressure, which governs the diffusion process and limits the creation of new cells, becomes very stiff, and the porous medium equation degenerates towards a free boundary problem of Hele-Shaw type. This is the so-called *incompressible limit.* The solutions of the limit Hele-Shaw problem then couple "free dynamics" with zero pressure, and "incompressible dynamics" with positive pressure and constant density. In the first part of the work, we provide a refined description of the traveling waves for the porous medium equation in the vicinity of the transition between the free domain and the incompressible domain. The second part of the study is devoted to the analysis of the stability of the traveling waves. We prove that the linearized system enjoys a spectral gap property in suitable weighted L^2 spaces, and we give quantitative estimates on the rate of decay of solutions. The nonlinear terms are treated perturbatively, using an L^∞ control stemming from the maximum principle. As a consequence, we prove that traveling waves are stable under small perturbations. This constitutes the first nonlinear asymptotic stability result concerning smooth fronts of degenerate diffusion equations with a Fisher-KPP reaction term.

Keywords: porous medium equation, traveling waves, incompressible limit, mesa limit, stability, Hele-Shaw equations.

MSC: 35C07, 35K57, 35B40, 35B35.

1 Introduction

This paper is devoted to the asymptotic analysis and the stability of *traveling waves* (TWs) for the *porous medium equation* (PME). More precisely, let us consider the following nonlinear parabolic equation

$$\partial_t n - \partial_x\big(n\partial_x p(n)\big) = n\Phi(p(n))), \tag{1}$$

endowed with the boundary conditions

$$\lim_{x\to\pm\infty} n(t,x) = n_\pm,$$

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where $n_\pm$ are constant stationary states of the equation. This equation has been introduced in the literature to model tissue growth and, particularly, in the propagation of tumors (see, for instance, [5, 15, 22]). The left-hand side corresponds to the PME: the density of cells, n, is transported by a velocity given by the Darcy law $v = -\partial_x p$ where $p = p(n)$ denotes the mechanical pressure. The right-hand side models the cell proliferation in the medium, proliferation which is limited by the pressure. Hence, the function Φ is usually taken as a decreasing function of the pressure and is such that $\Phi(p_M) = 0$ for some $p_M > 0$ called the *homeostatic pressure*. In this study, we shall assume for simplicity that

$$p(n) = p_\gamma(n) = n^\gamma \text{ with } \gamma > 1, \qquad \Phi(p) = 1 - p. \tag{2}$$

In other words, the function Φ becomes negative above the threshold pressure $p_M = p_\gamma(n_M) = 1$, which means that cells are destroyed above the *maximal packing density* $n_M = 1$. We will also pick $n_- = n_M = 1$, and $n_+ = 0$.

This study aims to analyze the behavior of traveling wave solutions of (1) when the parameter γ appearing in the equation of state (2) tends to $+\infty$. For $\Phi(p) = 0$, *i.e.* without the reaction term in the equation, this limit $\gamma \to +\infty$ is referred as the *mesa limit* and has been studied for instance by Caffarelli and Friedman [2]. In this paper, the authors consider an initial datum larger than 1 on a nontrivial set and show that this upper part exceeding 1 collapses at $t = 0^+$ to $\{n = 1\}$. This phenomenon is due to the blow up of the diffusivity $np'_\gamma(n) = \gamma n^\gamma \to +\infty$ when $n > 1$. The singular limit $\gamma \to +\infty$ for solutions of the PME is then called the "mesa" limit in reference to the shape of the target density $n_\infty \in [0, 1]$ which is similar to flat-topped mountains. In the presence of a growth source term Φ, the limit $\gamma \to +\infty$ has been first tackled by Perthame *et al.* in [23]. As in the previous case, the blow-up of the pressure as $\gamma \to +\infty$ when $n > 1$ forces the limit density to lie in $[0, 1]$. The sequence $(n_\gamma)_{\gamma>1}$ of weak solutions to (1) is then shown to converge (for a suitable topology) towards a weak solution of the following Hele-Shaw system

$$\begin{cases} \partial_t n - \partial_x(n\partial_x p) = n\Phi(p), & \text{(3a)} \\ 0 \le n \le 1, \quad (1-n)p = 0, \quad p \ge 0, & \text{(3b)} \\ p\,\left(\partial_x^2 p + \Phi(p)\right) = 0. & \text{(3c)} \end{cases}$$

The transition between equation (1) and system (3) is usually called the *incompressible limit* in reference to the fact that, when the solution n of (3) reaches 1, it is blocked to this maximal value (the combination of the mass equation (3a) with the complementary relation (3c) yields formally $\partial_t n = 0$ in $\{n = 1\}$) and the medium cannot be further compressed. Beyond the physical and biological relevancy of system (1) seen as an approximation of (3), Mellet *et al.* [19] have shown that the incompressible limit can provide crucial qualitative information on the solutions of the Hele-Shaw system (3), like the regularity of the free boundary $\partial\{n = 1\}$. In [4], David *et al.* estimate the rate of convergence in a negative Sobolev norm in terms of γ when $\Phi = \Phi(t, x)$ is given.

To finish with the incompressible limit, let us mention that this type of singular limit has been studied in other frameworks: for other singular equations of state [12], in the case of coupling with the dynamics of nutrients [5], in the case of more than one type of cancerous cell as seen in [1, 7, 8], when the Darcy law is replaced by the Brinkman equation [22] or the Navier-Stokes equations [26].

To the best of our knowledge, the issue of TWs solutions to (3) remains rare in the literature (see [21] when nutrients are considered), even when the topic was intensively studied for nonlinear reaction-diffusion equations like (1). Indeed, TWs as a class of special solutions have been shown to provide valuable information on general solutions of these reaction-diffusion equations (see the books [11] and [28]). Most of the results concern the long-term behavior (convergence to TWs, asymptotic rate of propagation of disturbances) or the behavior close to interfaces of general solutions.

Regarding the issue of interfaces, Gilding and Kersner study in [10] the existence of *sharp* (or

finite) TWs whose support is bounded on one side in case of nonlinear degenerate diffusion, and deduce a result about the existence of an interface $\partial\{n = 0\}$ for general solutions. In [9], TWs are used to study the regularity of the general solutions near the free boundary $\partial\{n = 0\}$, as well as for the derivation of the interface motion. The essential tools of the analysis are then: the continuity of the flux across the interface and a comparison principle bracketing a general solution between two TWs.
Concerning the long-time behavior of solutions to reaction-diffusion scalar equations like (1), let us mention two types of results related to the nature of the wave-front. We shall indeed distinguish the case of *sharp fronts*, that is TWs with support bounded from above (or below) propagating at the critical speed c^*, from the case of *smooth fronts* traveling at speed $c > c^*$ which do not vanish on $\mathbb{R}$. The first result of stability of sharp fronts seems to be given by Hosono in [13] in the case of a Nagumo reaction term (it would correspond to a bistable situation where Φ vanishes at $\alpha \in (0, 1)$ and 1). He proves that, if the initial data has a support in a neighborhood of the support of the sharp front and is close to the profile in L^∞, then the difference between the perturbation and the sharp front remains small for all times. Later on, Kamin and Rosenau prove in [14] that solutions associated to initial data decaying sufficiently fast at infinity converge (in a specific sense) towards a sharp TW. The techniques they employ are inspired by L^1-stability theory of shock waves for viscous conservation laws (see for instance [24]): use of comparison principle (already mentioned above), derivation of L^1 conservation, and contraction principles with an exponential weight. It is worth pointing out that this latter result cannot be extended to *smooth fronts*. Indeed the weight used in [14] is specific to the critical speed c^* at which the sharp fronts travel (see Theorem 2.3 below) and is not suited for the smooth fronts propagating at speed $c > c^*$. Actually, as explained by Sherratt in [25], much less was known until very recently on the smooth fronts. To our knowledge, the only stability result dealing with smooth fronts is a spectral stability result obtained recently by Leyva and Plaza in [17] (see also [16] for the case of a Nagumo reaction term). In their work, the difficulties associated with the degeneracy of the diffusion term are overcome with the derivation of a kind relative entropy estimate with a well-suited exponential weight.

In this paper, the study of smooth TWs of (1) as $\gamma \to +\infty$ can be seen as a first step in the analysis of the free boundary $\partial\{n = 1\}$ for the limit Hele-Shaw system (3). Our contributions are twofold: we first give a qualitative and quantitative description (in terms of γ) of smooth TWs of (1) and show the convergence towards TWs of (3) that are discontinuous at the interface $\partial\{n = 1\}$; we also study the nonlinear asymptotic stability of the smooth TWs for small (quantified in terms of γ) general perturbations of these wave-fronts.
As in [9], our analysis relies strongly on the control of the flux around the interface (passage to the limit as $\gamma \to +\infty$, determination of the transmission conditions across the interface on the limit system); and the comparison principle (quantitative behavior of TWs as $\gamma \to +\infty$, control of general solutions lying between two TWs). Compared to the stability analysis of Leyva and Plaza [17], we have to deal with additional nonlinear contributions that we treat in a perturbative manner and control thanks to a Poincaré-type inequality. This latter also allows us to get a decay rate of the perturbation as $t \to +\infty$.

Statement of main results

In this paper, we focus on TW solutions of (1)-(2), *i.e.* solutions n_γ such that $n_\gamma(t,x) = N_\gamma(x-ct)$ where N_γ is the wave profile, $\xi = x - ct$ is the wave coordinate and c is the speed of propagation of the wave. The profile N_γ is then solution to the differential equation:

$$-cN_\gamma' - \gamma(N_\gamma^\gamma N_\gamma')' = N_\gamma(1 - N_\gamma^\gamma). \tag{4}$$

The above equation admits two equilibrium states: $N \equiv 0$ (unstable) and $N \equiv 1$ (stable), and, therefore, we seek wavefronts N_γ connecting these two states:

$$\lim_{\xi\to-\infty} N_\gamma(\xi) = 1, \qquad \lim_{\xi\to+\infty} N_\gamma(\xi) = 0. \tag{5}$$

The existence and uniqueness (up to a shift) of a monotone (decreasing) solution to (4)-(5), as well as the asymptotic behavior of N_γ close to $\pm\infty$, were previously investigated by Gilding and Kersner [10] for c larger than a threshold velocity $c^*_\gamma = \sqrt{\gamma/(\gamma+1)} > 0$ (see below Theorem 2.3 for a precise statement). In the present study, we intend to analyze further the behavior of N_γ and $P_\gamma(\xi) = (N_\gamma(\xi))^\gamma$, the associated pressure profile, with respect to the parameter γ. Our first main result concerns the qualitative and quantitative behaviors as $\gamma \to +\infty$.

Theorem 1.1. *Let $\gamma > 1$ sufficiently large, $c > 1$ be fixed, independent of γ, and let N_γ be the solution of (4)-(5) such that $P_\gamma(0) = \frac{1}{\gamma}$. Then the following properties hold true.*

- *There exist $\xi^-_\gamma, \xi^+_\gamma$ with $\xi^-_\gamma = O\left(\frac{1}{\sqrt{\gamma}}\right) < 0 < \xi^+_\gamma = O\left(\frac{1}{\gamma}\right)$, such that:*
  - *in the* congested zone *$\xi < \xi^-_\gamma$, the density N_γ converges uniformly to 1: there exists a constant $C > 0$ depending only on c such that*

$$\left(\frac{C}{\sqrt{\gamma}}\right)^{\frac{1}{\gamma}} \le N_\gamma(\xi) \le 1 \qquad \forall\, \xi \le \xi^-_\gamma, \tag{6}$$

    and there exist constants $C' \ge C > 0$ independent of γ such that

$$1 - \left(1 - \frac{C'}{\sqrt{\gamma}}\right) e^{(1-C\gamma^{-1/2})\xi} \le P_\gamma(\xi) \le 1 - \left(1 - \frac{C}{\sqrt{\gamma}}\right) e^{\xi} \qquad \forall\, \xi \le \xi^-_\gamma; \tag{7}$$

  - *in the* intermediate region *$\xi \in [\xi^-_\gamma, \xi^+_\gamma]$, N'_γ takes exponentially large values with respect to γ:*

$$\|N'_\gamma\|_{L^\infty(\xi^-_\gamma, \xi^+_\gamma)} = O\left(\left(1 - \frac{2}{c}\right)^{-\gamma}\right), \tag{8}$$

    while the pressure P_γ converges uniformly to 0 as $\gamma \to +\infty$: there exists $\delta \in (0, 1-c^{-1})$, independent of γ such that

$$\left(1 - \frac{1}{c} - \delta\right)^\gamma \le P_\gamma(\xi) \le \frac{C}{\sqrt{\gamma}} \qquad \forall\, \xi \in [\xi^-_\gamma, \xi^+_\gamma]; \tag{9}$$

  - *in the* free zone *$\xi > \xi^+_\gamma$, the pressure P_γ takes exponentially small values (w.r.t. γ): $P_\gamma(\xi) \le \left(1 - \frac{1}{2c}\right)^\gamma$ and N_γ decreases exponentially to 0 as $\xi \to +\infty$: there exists $\delta > 0$ independent of γ, such that for γ large enough*

$$\left(1 - \frac{1}{c} - \delta\right) \exp\left(-\left(\frac{1}{c} + \delta\right)\xi\right) \le N_\gamma(\xi) \le \left(1 - \frac{1}{c} + \delta\right) \exp\left(-\frac{1}{2c}\xi\right) \quad \forall\, \xi > \xi^+_\gamma; \tag{10}$$

- *As $\gamma \to +\infty$, there exists $(N_{HS}, P_{HS}) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ such that $N_\gamma \to N_{HS}$ in $L^p_{loc}(\mathbb{R})$ and $P_\gamma \to P_{HS}$ in $W^{1,p}_{loc}(\mathbb{R})$ for any $p \in [1, \infty[$, and (N_{HS}, P_{HS}) is a wave-front profile of the Hele-Shaw equations (3) such that $P_{HS}(\xi) = (1 - e^\xi)\mathbf{1}_{\xi \le 0}$, $\lim_{\xi \to 0^+} N_{HS} = 1 - \frac{1}{c}$.*

Remark 1.2. Concerning the convergence of (N_γ, P_γ) towards (N_{HS}, P_{HS}), a key ingredient of our proof is the uniform control of the flux $J_\gamma = cN_\gamma + N_\gamma P'_\gamma$ which is such $J'_\gamma = -N_\gamma(1-P_\gamma) \in [-1, 0]$. The control of J_γ implies in particular the control of P'_γ and thus yields the uniform convergence of $(P_\gamma)_\gamma$. It is important to note that this uniform convergence of (P_γ) is uncorrelated to the convergence of $(N_\gamma)_\gamma$. Indeed, we have $P'_\gamma = \gamma N_\gamma^{\gamma-1} N'_\gamma$ but the pre-factor $\gamma N_\gamma^{\gamma-1}$ which tends to 0 on a half-space, prevents us to get a uniform bound on N'_γ. Actually this derivative blows up as it can be observed on (8). The uniform convergence of the flux J_γ is also crucial to determine the value of N_{HS} on the right side of the interface $\xi = 0$. Since $J_{HS}(0) = c + \lim\limits_{\xi \to 0^-} P'_{HS}(\xi) = c - 1$, we deduce that $\lim_{\xi \to 0^+} N_{HS}(\xi) = c^{-1} J_{HS}(0) = 1 - \frac{1}{c}$.

Remark 1.3 (Passing to the limit in the pressure equation). One can also wonder whether it is possible to pass to the limit in the sense of distributions in products of the type $P_\gamma'' N_\gamma$, or more generally $P_\gamma'' f(N_\gamma)$, where f is a continuous function. Since the weak limit of P_γ'' involves a Dirac mass at the point where the limit of N_γ is discontinuous, the limit is not obvious, and is not an immediate consequence of the above theorem. However, looking at the equation on the pressure and using the control on the flux, it can be proved that (see Corollary 2.7)

$$N_\gamma P_\gamma'' \rightharpoonup -\mathbf{1}_{\xi<0} e^{\xi} - (c-1)\ln\left(1-\frac{1}{c}\right)\delta(\xi) \quad \text{in } \mathcal{D}'(\mathbb{R}),$$

where δ is the Dirac mass at $\xi = 0$, and more generally

$$P_\gamma'' f(N_\gamma) \rightharpoonup -\mathbf{1}_{\xi<0} f(1) e^{\xi} - (c-1) F\left(1-\frac{1}{c}\right)\delta(\xi),$$

for any function $f \in \mathcal{C}(\mathbb{R})$, where $F(z) = -\int_z^1 f(t)/t^2\, dt$ for $z > 0$.

This highlights the intricate relationship between N_γ and P_γ in the transition zone.

Remark 1.4. A legitimate question is the possible extension of the previous result to more general pressure laws (as for instance the singular potentials considered in [12] or [3]) and reaction terms Φ. Our analysis actually starts with the results obtained by Gilding and Kersner [10]. In particular in [10], the determination of the critical speed $c^* = c^*_\gamma$ is specific to the pressure law $p_\gamma(n) = n^\gamma$. To our knowledge, the explicit characterization of c^* has not been tackled in the literature. More precisely we would need an upper bound on c^* independent of the parameter characterizing the incompressible limit. The extension of [10] to the case of more general pressures and reaction terms is therefore out of the scope of the present paper but there is a reasonable hope for a generalization of the previous theorem once the existence of a profile N_γ for a fixed speed $c > c^*$ (independent of parameter γ) is ensured.

We believe that several steps of our strategy could be extended to other pressure laws (analysis of the phase portrait of the traveling wave and consequences, design of appropriate weights for the coercivity of the linearized operator, etc.) However, in several instances some quantitative arguments rely heavily on fine properties of N_γ (e.g. the description of the transition zone). It is unavoidable that such properties will depend on the exact nature of the pressure law, and that a case-by-case analysis needs to be performed.

Our second result is dedicated to the analysis of stability of the wavefront N_γ in weighted Sobolev spaces. To that end, we introduce the weight

$$W(\xi) := N_\gamma(\xi)^\gamma \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N_\gamma^\gamma}\right).$$

Note that W has a double exponential growth as $\xi \to +\infty$, and a (slow) exponential decay as $\xi \to -\infty$. Therefore, W will provide a very good control in the free zone $x - ct > 0$.

Before stating our stability result, let us recall some previous works regarding the Cauchy problem for equation (1). Since (1) is a nonlinear degenerate parabolic equation, its well-posedness is far from obvious. In particular, the solutions need to be understood in a weak sense. The existence and uniqueness of a bounded generalized solution of the porous medium equation (PME) was proved by Oleinik *et al.* in [20]. We refer the interested reader to the book by Vázquez [27] for a detailed study of the PME; in particular, Chapter 12 of [27] is dedicated to the analysis of the Cauchy problem of the PME for initial data satisfying appropriate growth assumptions at infinity. It can be checked as a Corollary of Theorems 12.8, 12.9 and 12.10 that for initial data in L^∞, the Cauchy problem is globally well-posed. We also refer to the work of de Pablo and Vázquez [6], which extends these results to the case of equation (1). In the rest of this paper, we will consider initial data n_γ^0 which belong to L^∞. The associated solution n_γ is the unique global generalized solution of (1).

Our result is the following:

Theorem 1.5. *There exist constants $\eta_1, \eta_2 \in]0,1[$, depending only on $c > 1$, such that the following result holds.*

Let $\gamma > 1$ be fixed, sufficiently large. We make the following assumptions on the initial data n_γ^0:

(H1) n_γ^0 lies between two shifts of N_γ, i.e. there exists $h > 0$ such that $n_\gamma^0(x) \in [N_\gamma(x+h), N_\gamma(x-h)]$ for all $x \in \mathbb{R}$;

(H2) The difference $n_\gamma^0 - N_\gamma$ is sufficiently decaying, namely

$$\int_{\mathbb{R}} \left(n_\gamma^0(x) - N_\gamma(x)\right)^2 W(x)dx < \infty.$$

Let n_γ be the solution of (1) associated with n_γ^0. Then, if $|h| \le \eta_2^\gamma$, the following inequality holds:

$$\int_{\mathbb{R}} (n_\gamma(t,x) - N_\gamma(x-ct))^2 W(x-ct)\ dx \le e^{-\eta_1^\gamma t} \int_{\mathbb{R}} \left(n_\gamma^0(x) - N_\gamma(x)\right)^2 W(x)dx, \qquad \forall t \ge 0.$$

Moreover, setting $u_\gamma(t,x) := (n_\gamma(t,x) - N_\gamma(x-ct))/N_\gamma'(x-ct)$, we have the additional dissipation of energy:

$$\gamma \int_0^\infty \int_{\mathbb{R}} (\partial_x u_\gamma(t,x))^2 (N_\gamma^\gamma (N_\gamma')^2 W)(x-ct)\ dx\, dt \le \int_{\mathbb{R}} \left(n_\gamma^0(x) - N_\gamma(x)\right)^2 W(x)dx.$$

Let us give a short sketch of proof of the above result. An important feature of equation (1) lies in the fact that its linearization around $N_\gamma(x-ct)$ is spectrally stable in suitable weighted Sobolev spaces. This property has been identified recently by Leyva and Plaza [17], using Sobolev spaces with an exponential weight. Here, we work with different weights, which we believe follow more closely the structure of the equation, see Lemma 3.3 and subsection 4.2, and which give a better control in the transition zone. One crucial point of our analysis lies in the derivation of a new weighted Poincaré inequality associated with this weight, see Proposition 3.5. This allows us to prove a spectral gap property, leading to the exponential decay announced in the above theorem. Once the dissipation properties of the linearized equation have been identified and quantified, we perform the nonlinear estimates by treating the quadratic terms as perturbations. In this regard, assumption (H1) allows us to have a uniform L^∞ control on $n_\gamma(t,x) - N_\gamma(x-ct)$, thanks to the parabolic nature of the equation.

Note that the rate of decay η_1^γ of the energy is exponentially small. This is linked to the exponential blow-up of N_γ' in the transition zone, see Theorem 1.1. This blow-up also imposes a strong limitation on the admissible size of the perturbation in L^∞, and thereby on the size of h. It is not clear whether this assumption could be substantially lowered, taking, for instance, initial perturbations that would be algebraically - but not exponentially - small. Indeed, it is possible that the strong variations of N_γ in the transition zone destabilize the flow.

Our study is organized as follows. In Section 2, we describe traveling fronts for both systems (1) and (3) and give a refined behavior of the profile N_γ in the transition zone between the congested region and the free region. Next, we prove in Section 3 the asymptotic stability of the profile N_γ (γ being fixed) for some L^2-weighted norm. Finally, we have postponed in Section 4 the proofs of some technical lemmas used in Section 3, and a list of important abscissas for the description of the profile N_γ.

2 Traveling waves for the Hele-Shaw system and the porous media equation

This section is devoted to studying the existence and properties of traveling fronts of both systems: Hele-Shaw and the mechanical model of tumor growth with "stiff pressure law" depending on the parameter γ. For the latter, an asymptotic expansion of traveling waves will be computed.

2.1 TW for the limit Hele-Shaw system

We look for traveling wave-type solutions of (3) of the form $(n,p) = (N,P)(x-ct)$, where $c > 0$ is a constant representing the TW speed and N, P are real nonnegative functions. We may assume that $c > 0$, since for $c = 0$ we find again the stationary solutions, and the case $c < 0$ can be reduced to $c > 0$ by reflection.

Lemma 2.1. *Let $c > 1$ be arbitrary, and let ξ denote the traveling wave variable $\xi = x - ct$.*

1. *Define the profile $(N_{HS}, P_{HS}) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ by*

$$N_{HS}(\xi) = \begin{cases} \left(1 - \dfrac{1}{c}\right) e^{-\frac{\xi}{c}} & \text{if } \xi > 0, \\ 1 & \text{if } \xi < 0, \end{cases} \qquad P_{HS}(\xi) = \begin{cases} 0 & \text{if } \xi > 0, \\ 1 - e^{\xi} & \text{if } \xi < 0. \end{cases} \tag{11}$$

Then $(N_{HS}, P_{HS})(x-ct)$ is a traveling wave moving at speed c solution of the Hele-Shaw system

$$cN' + (NP')' + N\Phi(P) = 0, \tag{12}$$

$$0 \le N \le 1, \quad (1-N)P = 0, \quad P \ge 0, \tag{13}$$

$$P(P'' + \Phi(P)) = 0. \tag{14}$$

2. *Let $(N,P) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ be a traveling wave profile moving at speed c of the Hele-Shaw system (3) satisfying the following conditions at $\pm\infty$*

$$\lim_{\xi \to -\infty} N(\xi) = \lim_{\xi \to -\infty} P(\xi) = 1, \quad \lim_{\xi \to +\infty} N(\xi) = \lim_{\xi \to +\infty} P(\xi) = 0.$$

Assume furthermore that $0 \le P \le 1$. Then there exists $\xi_0 \in \mathbb{R}$ such that $(N,P) = (N_{HS}, P_{HS})(\cdot - \xi_0)$.

Remark 2.2.
- The Lipschitz regularity assumption on P ensures that the term $P'N$ is well-defined, as a product of two L^∞ functions.
- An important feature of the analysis is the continuity of the flux $(c+P')N$ on $\mathbb{R}$ (in particular, at the transition point ξ_0). This property will determine the value of $N(\xi_0^+)$.

Proof. It is easily checked that (N_{HS}, P_{HS}) is a solution of (12)-(14). Hence the difficulty is to prove that all solutions are equal to (N_{HS}, P_{HS}) (up to a translation). As announced in Remark 2.2, we introduce the flux $J = cN + NP'$, which satisfies $J' = -N\Phi(P) \in [-1, 0]$. Hence J is Lipschitz continuous and decreasing. Using the values of N, P at $\pm\infty$, we find that $J(-\infty) = c$, $J(+\infty) = 0$, and therefore $0 \le J \le c$ a.e.

Since P is Lipschitz continuous, the set $\{P > 0\}$ is a countable union of disjoint open intervals, say $\bigcup_{j \in \mathcal{J}} (a_j, b_j)$. On any such interval (a_j, b_j), we have $N = 1$ and

$$-P'' = \Phi(P) = 1 - P, \qquad \forall \xi \in (a_j, b_j).$$

Hence, there exist $C_j^\pm$ such that

$$P(\xi) = 1 + C_j^+ e^{\xi} + C_j^- e^{-\xi}, \qquad \forall \xi \in (a_j, b_j).$$

Note that the case $b_j = +\infty$ is excluded, since $N(+\infty) = 0$, and that $C_j^- = 0$ if $a_j = -\infty$. Furthermore, on any interval (a_j, b_j), we have $J = c + P' \in [0, c]$, and $J' = P'' \le 0$. Hence P is non-increasing and concave on (a_j, b_j). If $a_j, b_j \in \mathbb{R}$, we have additionally $P(a_j) = P(b_j) = 0$, since $a_j, b_j \in \partial\{P > 0\}$. This entails that $P(\xi) = 0$ for all $\xi \in (a_j, b_j)$, which is absurd. Hence $\mathcal{J}$ is a singleton and there exists $\xi_0 \in \mathbb{R}$ such that $\{P > 0\} = (-\infty, \xi_0)$. Furthermore, since $P(\xi_0) = 0$, we find that

$$P(\xi) = 1 - e^{\xi - \xi_0}, \quad \forall \xi < \xi_0. \tag{15}$$

Let us now consider the free phase, i.e. the set $\{P=0\}=[\xi_0,+\infty[$. In (the interior of) this interval, the equation becomes

$$cN' = -N, \quad \forall \xi > \xi_0.$$

The solution of the above linear equation is of the form

$$N(\xi) = C\exp\left(-\frac{\xi-\xi_0}{c}\right).$$

We infer that in $(\xi_0,+\infty)$, $J = cC\exp\left(-\frac{\xi-\xi_0}{c}\right)$. By continuity of J at $\xi=\xi_0$, we obtain

$$c-1 = J(\xi_0^-) = J(\xi_0^+) = cC.$$

Thus $C=(c-1)/c$, and we find that $(N,P)=(N_{HS},P_{HS})(\cdot-\xi_0)$.

□

2.2 Qualitative properties of traveling waves for the porous medium equation (1)

Let us now consider traveling waves for the porous medium equation (1). We are interested in the behavior of such profiles in the limit $\gamma\to+\infty$, with a fixed velocity $c>0$. In the following two subsections, we aim to derive qualitative and quantitative information on the profiles when $\gamma \gg 1$.

The existence of a profile N_γ solution to (4)-(5) is ensured by a former study of Gilding and Kersner [10]. More precisely, as a particular case of [10], one has the following result.

Theorem 2.3 (Gilding & Kersner [10]). *Let* $c_\gamma^* = \sqrt{\frac{\gamma}{\gamma+1}}$.

1. *System* (4)-(5) *has a unique solution* N_γ *(up to a shift) for every* $c \geq c_\gamma^*$ *and no solution for* $c < c_\gamma^*$.
2. *When* $c=c_\gamma^*$, N_γ *is a sharp front, i.e. the support of* N_γ *is bounded from above, and, modulo translation,*
$$N_\gamma(\xi) = \begin{cases} (1-\exp(c\xi))^{1/\gamma} & \text{for} \quad \xi<0,\\ 0 & \text{for} \quad \xi \geq 0.\end{cases}$$
3. *When* $c > c_\gamma^*$, N_γ *is positive, strictly monotonic and satisfies*
$$(\ln(1-N_\gamma))'(\xi) \to \sqrt{1+\frac{c^2}{4\gamma^2}} - \frac{c}{2\gamma} = \frac{1}{\sqrt{1+\frac{c^2}{4\gamma^2}}+\frac{c}{2\gamma}} \quad \text{as} \quad \xi\to-\infty, \tag{16}$$
and
$$(\ln(N_\gamma))'(\xi) \to -\frac{1}{c}, \quad \text{as} \quad \xi \to +\infty.$$

The above theorem guarantees the existence (and the uniqueness up to a shift) of a TW N_γ for all $c \geq c_\gamma^* = \sqrt{\dfrac{\gamma}{\gamma+1}}$; the smoothness of N_γ when $c > c_\gamma^*$; the monotonically decreasing behavior of N_γ and its boundedness on $\mathbb{R}$. Notice that the sharp front with minimal speed $c=c_\gamma^*$ is only Hölder continuous with exponent $1/\gamma$ at $\xi=0$. The fact of $N_\gamma^{\gamma+1}$ being continuously differentiable in the whole domain means this traveling wave is a weak solution in the usual sense, while from the physics perspective, it indicates the presence of continuous flux.

Since Theorem 2.3 is adapted from Theorem 1 in [10], we refer to this work for a detailed proof.

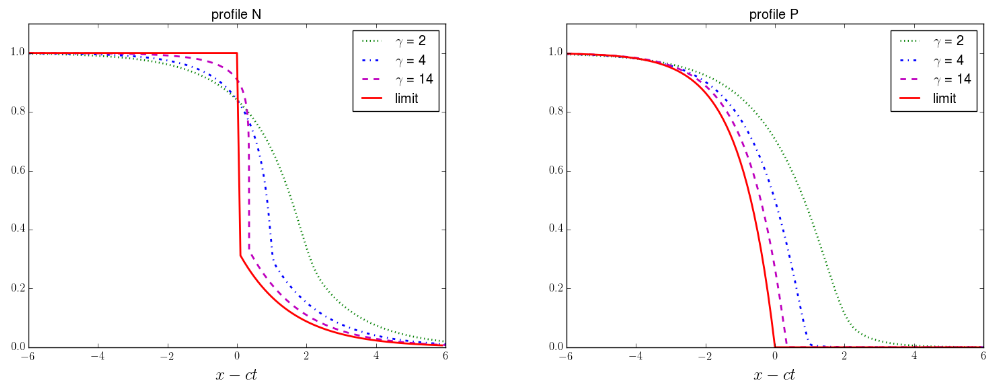

Figure 1: Density and pressure profiles for finite values of γ and limit profiles, $c = 1.5$.

From now on, we pick a velocity $c > 1$ independent of γ, so that $c > c_\gamma^*$[1]. We also fix the shift in N_γ by imposing

$$N_\gamma(0) = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma}}. \tag{17}$$

The goal of this subsection is to prove the following result:

Proposition 2.4. *Let $c > 1$ and let (N_γ, P_γ), $P_\gamma := p_\gamma(N_\gamma) = N_\gamma^\gamma$, be the unique bounded weak solution to (4) satisfying (17). Let $(N_{HS}, P_{HS}) \in L^\infty(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ be the reference traveling wave solution moving with speed c of the Hele-Shaw system, see (11).*

1. *The following convergence properties hold:*
   - *Weak-star convergence:*
   $$N_\gamma \rightharpoonup N_{HS} \quad \text{in } w^* - L^\infty, \quad P_\gamma \rightharpoonup P_{HS} \quad \text{in } w^* - W^{1,\infty};$$
   - *for any compact set $K \subset \mathbb{R}$*
   $$P_\gamma \to P_{HS} \quad \text{in} \quad \mathcal{C}(K);$$
   - *$N_\gamma \to 1$ uniformly on $\mathbb{R}_-$ and $P'_\gamma \to P'_{HS}$ uniformly in $\mathcal{C}(]-\infty, 0])$.*
2. *Pointwise bounds for P_γ on $\mathbb{R}_-$: setting $\lambda = (-c + \sqrt{c^2+4})/2$, we have,*

$$1 - \left(1 - \frac{1}{\gamma}\right) e^{\lambda\xi} \le P_\gamma(\xi) \le 1 - \left(1 - \frac{1}{\gamma}\right) e^{\xi}, \quad \forall\, \xi \le 0. \tag{18}$$

The rest of this subsection is devoted to the proof of Proposition 2.4.
L^∞ **bounds.** From Theorem 2.3, we know that $N'_\gamma \le 0$ with $\lim_{\xi\to-\infty} N_\gamma = 1$, $\lim_{\xi\to+\infty} N_\gamma = 0$, so that

$$0 \le N_\gamma(\xi),\ P_\gamma(\xi) \le 1, \qquad \forall\, \xi \in \mathbb{R}.$$

Therefore, there exist $(N, P) \in L^\infty \times L^\infty(\mathbb{R})$ such that up to the extraction of a subsequence, $N_\gamma \rightharpoonup N$, $P_\gamma \rightharpoonup P$ in $w^* - L^\infty(\mathbb{R})$. Furthermore, N, P are non-increasing. The choice of shift (17)

[1] All the results of this paper remain true with little or no modification if the velocity $c_\gamma > c_\gamma^*$ depends on γ in such a way that $c_\gamma \to \bar{c}$ with $\bar{c} > 1$. However for the sake of readability, we have chosen $c_\gamma \equiv c > 1$.

implies that $N_\gamma(0) \to 1, P_\gamma(0) \to 0$. Hence, since N_γ is non-increasing, N_γ converges uniformly towards 1 on $]-\infty, 0]$, and P_γ converges uniformly towards zero on $[0, +\infty[$. It follows that $N(\xi) = 1$ for $\xi < 0$ and $P(\xi) = 0$ for $\xi > 0$.

Strong convergence of P_γ and J_γ. Define the flux

$$J_\gamma := cN_\gamma + \gamma N_\gamma' N_\gamma^\gamma = cN_\gamma + N_\gamma P_\gamma'. \tag{19}$$

We observe that equation (4) can be written as

$$J_\gamma' = -N_\gamma(1 - N_\gamma^\gamma),$$

so that J_γ is decreasing on $\mathbb{R}$. Combining the latter with the L^∞ bounds on N_γ yields

$$\begin{gathered} -1 \le J_\gamma' \le 0, \\ 0 = J_\gamma(+\infty) \le J_\gamma \le J_\gamma(-\infty) = c. \end{gathered} \tag{20}$$

In particular, $cN_\gamma + N_\gamma P_\gamma' \ge 0$. Since we already know that P_γ is non-increasing, it follows that

$$-c \le P_\gamma' \le 0, \quad 0 \le P_\gamma \le 1. \tag{21}$$

From inequality (20) (resp. (21)) and Ascoli's theorem, J_γ (resp. P_γ) converges strongly, up to a subsequence, in $\mathcal{C}(K)$ for any compact set $K \subset \mathbb{R}$. Note also that $P_\gamma' \overset{*}{\rightharpoonup} P'$ in $L^\infty(\mathbb{R})$; since N_γ converges uniformly towards 1 on $\mathbb{R}_-$, we find that $J = cN + NP' = c + P'$ on $]-\infty, 0[$.

The exact same cannot be done with N_γ. Indeed, from (4) and (20), we can deduce the following bounds for N_γ

$$-c\frac{N_\gamma}{N_\gamma^\gamma \gamma} \le N_\gamma' \le c\frac{1 - N_\gamma}{N_\gamma^\gamma \gamma}. \tag{22}$$

Note that obtaining an L^∞ bound implies controlling $N_\gamma^{1-\gamma}\gamma^{-1}$ in L^∞ over any compact on $\mathbb{R}$ when $\gamma \to +\infty$. This is impossible as $N_\gamma \in (0, 1)$. In fact, we show in what follows that N is discontinuous at $\xi = 0$.

Passing to the limit in equation (4). We can write the diffusion term as

$$(N_\gamma P_\gamma')' = (\gamma N_\gamma' N_\gamma^\gamma)' = \frac{\gamma}{\gamma+1}(N_\gamma^{\gamma+1})'' = \frac{\gamma}{\gamma+1}(P_\gamma N_\gamma)''.$$

Since P_γ converges strongly in $\mathcal{C}(K)$ for all compact set $K \subset \mathbb{R}$, while N_γ converges weakly-* in $L^\infty(\mathbb{R})$, we can pass to the (weak) limit in equation (4).

We obtain that (N, P) satisfies the following equation in the sense of distributions

$$-cN' - (NP)'' = N(1 - P). \tag{23}$$

The same argument also shows that $J = cN + (NP)'$ on $\mathbb{R}$.

Limit in the free phase ($\xi > 0$). We recall that $P = 0$ in $\mathbb{R}_+$. Hence, in $(0, +\infty)$, equation (23) becomes

$$-cN' = N.$$

We recognize the ODE satisfied by N_{HS} in the free phase in the Hele-Shaw system. It follows that

$$N(\xi) = C\exp\left(-\frac{\xi}{c}\right) \quad \forall \xi > 0,$$

for some $C > 0$.

Limit in the congested phase ($\xi < 0$). We recall that $N = 1$ on $]-\infty, 0[$. Inserting this information into (23), the following elliptic equation (*complementarity equation*) is obtained

$$P'' + (1 - P) = 0 \quad \text{in} \quad \mathcal{D}'((-\infty, 0)). \tag{24}$$

From $P(0)=0$ (recall that P is continuous), it follows that $P(\xi)=1-e^{\xi}$ for $\xi\in\mathbb{R}_-$.

(N,P) satisfies (13)**.** We know that $P=0$ on $[0,+\infty)$ and $N=1$ on $\mathbb{R}_-$; hence, $P(1-N)=0$ on $\mathbb{R}$ as in (13).

Jump relation at $\xi=0$**.** We recall that the flux $J=cN+(NP)'$ is continuous on $\mathbb{R}$, and in particular at $\xi=0$. Thus,

$$\lim_{\xi\to 0^+} N(\xi)=1-\frac{1}{c}. \tag{25}$$

Gathering all the information, we find that $(N,P)=(N_{HS},P_{HS})$. Furthermore, since the limit is uniquely identified, we deduce that the whole sequence (N_γ,P_γ) converges (in the sense given above).

Sub- and super-solution for P_γ **on** $\mathbb{R}_-$**.** Using (21), it follows that

$$-P_\gamma''N_\gamma=N_\gamma(1-P_\gamma)+(c+P_\gamma')N_\gamma'\le N_\gamma(1-P_\gamma), \tag{26}$$

whence

$$-P_\gamma''\le 1-P_\gamma \quad \text{on} \quad \mathbb{R}.$$

Now, let $\xi_1\in\mathbb{R}$ be arbitrary, and let $P_1:=P_\gamma(\xi_1)$. We have for $P_+:=1-(1-P_1)e^{\xi-\xi_1}$ that $-P_+''=1-P_+$. Thus

$$\begin{aligned} -(P_\gamma-P_+)''\le -(P_\gamma-P_+) \quad &\text{on} \quad (-\infty,\xi_1),\\ \text{and} \quad P_\gamma(\xi_1)=P_+(\xi_1), \qquad &P_\gamma(-\infty)=P_+(-\infty). \end{aligned}$$

It follows from the maximum principle that $P_\gamma\le P_+$ for $\xi\le\xi_1$. In particular, taking $\xi_1=0$ and $P_1=1/\gamma$,

$$0\le P_\gamma(\xi)\le -\left(1-\frac{1}{\gamma}\right)e^{\xi}, \qquad \forall\xi\le 0. \tag{27}$$

In a similar fashion, recalling that $\gamma P_\gamma\ge 1$ on $\mathbb{R}_-$ and $P_\gamma'\le 0$, we have

$$-P_\gamma''=1-P_\gamma+\frac{cP_\gamma'}{\gamma P_\gamma}+\frac{(P_\gamma')^2}{\gamma P_\gamma}\ge 1-P_\gamma+cP_\gamma'.$$

Arguing as before, we define $P_-(\xi)=1-\left(1-\frac{1}{\gamma}\right)e^{\lambda\xi}$, where λ is the positive root of $\lambda^2+c\lambda-1=0$ (i.e. $\lambda=(-c+\sqrt{c^2+4})/2$). By definition of λ, P_- satisfies

$$-P_-''=1-P_-+cP_-', \quad \lim_{\xi\to-\infty}P_-(\xi)=1, \quad P_-(0)=\frac{1}{\gamma}.$$

We infer from the maximum principle that

$$1-\left(1-\frac{1}{\gamma}\right)e^{\lambda\xi}\le P_\gamma(\xi), \qquad \forall\ \xi\le 0. \tag{28}$$

Uniform convergence of the flux and of P_γ' **on** $\mathbb{R}_-$**.**

We recall that $J_\gamma'=-N_\gamma(1-P_\gamma)$. The pointwise bounds on P_γ imply that

$$|J_\gamma'|\le e^{\lambda\xi}, \qquad \forall\xi\le 0,\ \forall\gamma>0.$$

It follows immediately that J_γ converges towards J_{HS} uniformly in $\mathcal{C}(\mathbb{R}_-)$. Since

$$P_\gamma'=\frac{J_\gamma}{N_\gamma}-c,$$

we infer that P_γ' also converges uniformly towards P_{HS}' in $\mathcal{C}(\mathbb{R}_-)$.

This concludes the proof of Proposition 2.4. □

2.3 Phase portrait of N_γ and further consequences

In this subsection, we derive other properties of the family $(N_\gamma)_{\gamma>0}$, which will be useful in our stability analysis. This will involve a thorough description of the behaviour of N_γ in the transition zone, i.e. in the vicinity of the point $\xi = 0$. To that end, we will introduce several remarkable abscissas, corresponding to points where the behavior of N_γ changes. For the reader's convenience, we included a list of these abscissas in Appendix A, together with an indication of where they are defined and their size.

The next lemma states that N_γ admits a unique inflection point.

Lemma 2.5. *There exists a unique $\xi_\gamma^0 \in \mathbb{R}$ such that N_γ^0 is concave on $(-\infty, \xi_\gamma^0)$ and convex on $(\xi_\gamma^0, +\infty)$.*

Proof. The proof relies crucially on the analysis of the phase portrait of N_γ. In order to plot the phase portrait of N_γ, we use the results of [18], together with the following remark: using equation (4), we have

$$\begin{aligned}\frac{dN'_\gamma}{dN_\gamma} &= \frac{dN'_\gamma}{d\xi}\frac{d\xi}{dN_\gamma}\\ &= -\frac{1}{\gamma N_\gamma^\gamma N'_\gamma}\left[cN'_\gamma + \gamma^2(N'_\gamma)^2N_\gamma^{\gamma-1} + N_\gamma(1-N_\gamma^\gamma)\right].\end{aligned}$$

The term in brackets in the right-hand side is a polynomial of degree two in N'_γ, with coefficients depending on N_γ. Hence dN'_γ/dN_γ vanishes if and only if $N_\gamma^\gamma(1-N_\gamma^\gamma) \le c^2/(4\gamma^2)$ and $N'_\gamma \in \{Q_-(N_\gamma), Q_+(N_\gamma)\}$, where

$$Q_\pm(N) = \frac{1}{2\gamma^2N^{\gamma-1}}\left(-c \pm \sqrt{c^2 - 4\gamma^2N^\gamma(1-N^\gamma)}\right). \tag{29}$$

Note that the curves $\Gamma_\pm = \{(N, Q_\pm(N)),\ N \in (0,1)\}$ each consist of two branches, for $N \in (0, N_1)$ and $N \in (N_2, 1)$. The points N_i are the roots of the discriminant, i.e. $N_i^\gamma(1-N_i^\gamma) = c^2/(4\gamma^2)$. The curves Γ_+ and Γ_- intersect at $N = N_1$ and at $N = N_2$. Note that N_1 and N_2 depend on γ, but we omit this dependency in order to lighten the notation. A straightforward analysis shows that

$$N_1 = 1 - \frac{2\ln\gamma}{\gamma} + o\left(\frac{\ln\gamma}{\gamma}\right), \quad N_2 = 1 - \frac{c^2}{4\gamma^3} + o\left(\frac{1}{\gamma^3}\right), \tag{30}$$

with

$$Q_\pm(N_1) \sim -\frac{2}{c}, \quad Q_\pm(N_2) \sim -\frac{c}{2\gamma^2}.$$

Furthermore, $Q_+(N) \sim -\frac{N}{c}$ for $N \ll 1$, while $Q_-(N) \to -\infty$ as $N \to 0$; and, $Q_+(N) \sim -\gamma(1-N)/c$ for $1 - N \ll 1$, while $Q_-(1) = -c/\gamma^2$.

Note also that with the normalisation of the previous section, i.e. $N_\gamma(0) = \gamma^{-1/\gamma}$, we have $N_\gamma(0) \in [N_1, N_2]$.

Now, let us denote by $\mathcal{T}$ (resp. $\mathcal{S}$) the interior region between the curves Γ_- and Γ_+ for $0 < N < N_1$ (resp. $N_2 < N < 1$). We also denote by Γ the curve (N_γ, N'_γ), which we orientate in the direction of growing N_γ. We make the following observations:

(i) for all $N_\gamma \in (N_1, N_2)$, $dN'_\gamma/dN_\gamma \ge 0$;

(ii) for all $N_\gamma \in (0, N_1)$ (resp. $N_\gamma \in (N_2, 1)$), $dN'_\gamma/dN_\gamma < 0$ iff $(N_\gamma, N'_\gamma) \in \mathcal{T}$ (resp. $(N_\gamma, N'_\gamma) \in \mathcal{S}$);

(iii) if Γ crosses one of the curves $\Gamma_\pm$, then $dN'_\gamma/dN_\gamma = 0$ at the crossing point and therefore the tangent to Γ at the crossing point is horizontal;

(iv) $\dfrac{dQ_\pm}{dN} \gtrless 0$ for all $N \in (N_2, 1)$;

(v) $\dfrac{dQ_\pm}{dN} \lesseqgtr 0$ for all $N \in (0, N_1)$;

(vi) when $\xi \to -\infty$, we have $N_\gamma(\xi) \to 1$, and $N_\gamma'(\xi) \sim -\left(\sqrt{1+\dfrac{c^2}{4\gamma^2}} - \dfrac{c}{2\gamma}\right)(1 - N_\gamma(\xi))$.

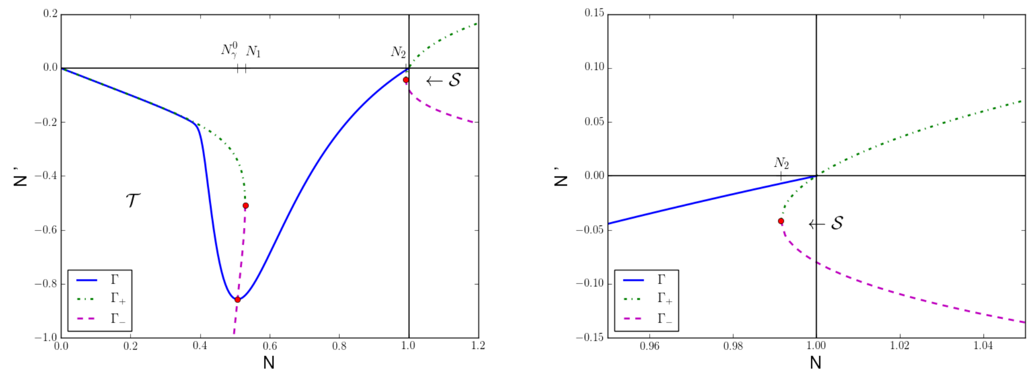

Figure 2: Trajectory Γ in the phase plane (N, N'), $c = 2$, $\gamma = 5$. On the right, enlargement around the point $(1, 0)$ and the region $\mathcal{S}$.

The proof of all items is easy and left to the reader, except for (v), which we prove below. Note that (vi) is a consequence of (16). It follows from (vi) that for N_γ in a neighborhood of 1 (the size of which depends on γ), the curve Γ is above Γ_+. Furthermore, (ii), (iii) and (iv) imply that if the curve Γ intersects the region $\mathcal{S}$, then it cannot exit $\mathcal{S}$. It follows that Γ lies strictly above Γ_+ for all $N \in (N_2, 1)$ (see Figure 2 on the right). *Consequently, for all* $N_\gamma \in (N_1, 1)$, $dN_\gamma'/dN_\gamma \geq 0$.

Let us now prove that $dQ_+/dN \leq 0$ for all $N \in (0, N_1)$ (the inequality for Q_- is easier and left to the reader). We have, setting $P = N^\gamma$,

$$
\begin{aligned}
\frac{dQ_+}{dN} &= \frac{d}{dN}\left[-\frac{c}{2\gamma^2 N^{\gamma-1}}\left(1 - \sqrt{1 - \frac{4\gamma^2}{c^2}N^\gamma(1-N^\gamma)}\right)\right] \\
&= -\frac{c}{2\gamma^2 N^{\gamma-1}}\left[-\frac{\gamma-1}{N}\left(1 - \sqrt{1 - \frac{4\gamma^2}{c^2}N^\gamma(1-N^\gamma)}\right) + \frac{2\gamma^3}{c^2}\frac{N^{\gamma-1}(1-2N^\gamma)}{\sqrt{1 - \frac{4\gamma^2}{c^2}N^\gamma(1-N^\gamma)}}\right] \\
&= -\frac{c}{2\gamma^2 N^{\gamma}}\left[-\frac{4\gamma^2(\gamma-1)P(1-P)}{c^2(1+\sqrt{1-\frac{4\gamma^2}{c^2}P(1-P)})} + \frac{2\gamma^3 P(1-2P)}{c^2\sqrt{1-\frac{4\gamma^2}{c^2}P(1-P)}}\right] \\
&= -\frac{\gamma(1-2P) - (\gamma - 2 + 2P)\sqrt{1-\frac{4\gamma^2}{c^2}P(1-P)}}{c(1+\sqrt{1-\frac{4\gamma^2}{c^2}P(1-P)})\sqrt{1-\frac{4\gamma^2}{c^2}P(1-P)}}.
\end{aligned}
$$

Note that for $0 < N < N_1$, $P = O(1/\gamma^2)$. In this regime, it can be easily checked that the numerator of the right-hand side is positive, and therefore $dQ_+/dN < 0$ for all $N \in (0, N_1)$. This completes the proof of (v).

We deduce that for $N \in (0, N_1)$, the curve Γ can cross Γ_+ at most once, as Γ exits the region $\mathcal{T}$. Now, let us argue by contradiction and assume that there exists $N_\gamma^1 \in (0, N_1)$ such that $(N_\gamma^1, (N_\gamma^1)') \in \Gamma$ lies above Γ_+. Then there are two possibilities:

- either (N_γ, N_γ') is above Γ_+ for all $N_\gamma \in (0, N_1)$. In that case, $dN_\gamma'/dN_\gamma \geq 0$ for all $N_\gamma \in (0, N_1)$. Since $(0,0) \in \Gamma$ and $N_\gamma' \leq 0$, it follows that $N_\gamma' = 0$ for all $N_\gamma \in (0, N_1)$, which contradicts Theorem 2.3;

- or there exists $N_\gamma^2 \in (0, N_\gamma^1)$ such that $(N_\gamma^2, (N_\gamma^2)') \in \Gamma \cap \mathcal{T}$. In that case, since Γ and Γ_+ intersect at most once, there exists $N_3 \in (0, N_1)$ such that for all $N_\gamma \in (0, N_3)$, $(N_\gamma, N_\gamma') \in \Gamma \cap \mathcal{T}$ and for $N_\gamma \in (N_3, N_1)$, (N_γ, N_γ') is above Γ_+. Since $dN_\gamma'/dN_\gamma > 0$ for $N \in (N_1, 1)$, N_γ' reaches a minimum for $N_\gamma = N_3$, and the value of this minimum is $Q_+(N_3) \geq Q_+(N_1) \sim -2/c$. Thus N_γ' is bounded in L^∞. Using Ascoli's theorem, we infer that N_γ converges uniformly on $\mathcal{C}(K)$ for any compact set $K \subset \mathbb{R}$ as $\gamma \to +\infty$. Since N_{HS} is discontinuous at $\xi = 0$, we have reached a contradiction.

We conclude that (N_γ, N_γ') remains below Γ_+ for all $N_\gamma \in (0, N_1)$, and therefore Γ does not cross Γ_+. Using once again the fact that $\min N_\gamma'$ must blow up as $\gamma \to +\infty$, we infer that Γ and Γ_- intersect exactly once, at a point where $N_\gamma = N_\gamma^0 \in (0, N_1)$, and N_γ^0 is such that $Q_-(N_\gamma^0) \to -\infty$ as $\gamma \to +\infty$. For all $N_\gamma \in (0, N_\gamma^0)$, $dN_\gamma'/dN_\gamma \leq 0$, and for $N_\gamma \in (N_\gamma^0, 1)$, $dN_\gamma'/dN_\gamma \geq 0$. Thus we obtain the phase portrait drawn in Figure 2.

Let us now go back to the analysis of $\xi \in \mathbb{R} \mapsto N_\gamma(\xi)$. There exists a unique $\xi_\gamma^0 \in \mathbb{R}$ such that $N_\gamma(\xi_\gamma^0) = N_\gamma^0$. Note that dN_γ'/dN_γ and N_γ'' have opposite signs. Hence, N_γ is concave on $(-\infty, \xi_\gamma^0)$ and convex on $(\xi_\gamma^0, +\infty)$. □

The following lemma summarizes properties on ξ_γ^0 and N_γ^0:

Lemma 2.6. *We normalize the function N_γ so that $N_\gamma(0) = \gamma^{-1/\gamma}$. We have the following properties:*

- $\xi_\gamma^0 > 0$ *and* $\lim_{\gamma \to +\infty} \xi_\gamma^0 = 0$*;*
- $\sup_{\gamma > 0} \sup_{\xi < 0} |N_\gamma'(\xi)| < +\infty$ *and* $\|N_\gamma'\|_{L^\infty(\mathbb{R})} = -Q_-(N_\gamma^0) \to +\infty$ *as* $\gamma \to +\infty$*;*
- $\lim_{\gamma \to +\infty} N_\gamma^0 = 1 - c^{-1}$*;*
- *For γ large enough, for all $\xi \geq \xi_\gamma^0$,*
$$0 \leq N_\gamma(\xi) \leq N_\gamma^0 \exp\left(-\frac{1}{2c}(\xi - \xi_\gamma^0)\right);$$
- $P_\gamma' \to P_{HS}'$ *and* $N_\gamma \to N_{HS}$ *in* $L^p_{loc}(\mathbb{R})$ *for all* $p \in [1, +\infty[$*;*
- *Let $\xi_\gamma^* > \xi_\gamma^0$ such that $N_\gamma'(\xi_\gamma^*) = -\frac{1}{c}\left(1 - \frac{1}{c}\right)$. Then, $\xi_\gamma^* \to 0$ and $N_\gamma(\xi_\gamma^*) \to 1 - c^{-1}$ as $\gamma \to +\infty$.*

Proof. • *First step: Upper bound on N_γ^0 and on ξ_γ^0.*

The analysis of the phase portrait entails immediately that $\|N_\gamma'\|_{L^\infty(\mathbb{R})} = -Q_-(N_\gamma^0)$. As recalled above, if $Q_-(N_\gamma^0)$ remains bounded, then N_γ converges strongly in $\mathcal{C}(K)$ for any compact set K, which is absurd since N_{HS} is discontinuous. Hence, $Q_-(N_\gamma^0)$ must blow up. Since

$$-\frac{c}{\gamma^2 (N_\gamma^0)^{\gamma-1}} \leq Q_-(N_\gamma^0) \leq -\frac{c}{2\gamma^2 (N_\gamma^0)^{\gamma-1}},$$

we deduce that $(N_\gamma^0)^\gamma = o(\gamma^{-2})) \ll \gamma^{-1} = N_\gamma(0)^\gamma$. Thus $\xi_\gamma^0 > 0$.

Since the flux J_γ is decreasing on $\mathbb{R}$, it follows that $J_\gamma(\xi_\gamma^0) \leq J_\gamma(0)$. Now

$$J_\gamma(\xi_\gamma^0) = cN_\gamma^0 + \gamma (N_\gamma^0)^\gamma Q_-(N_\gamma^0) = \left(c + O\left(\frac{1}{\gamma}\right)\right) N_\gamma^0, \tag{31}$$

and

$$J_\gamma(0) = \gamma^{-1/\gamma}\left(c + P_\gamma'(0)\right).$$

Thanks to the sub- and super-solutions for P_γ on $\mathbb{R}_-$ from Proposition 2.4, we know that for all $\xi < 0$,

$$\left(1 - \frac{1}{\gamma}\right)(1 - e^{\lambda\xi}) \leq P_\gamma - P_\gamma(0) \leq \left(1 - \frac{1}{\gamma}\right)(1 - e^{\xi}),$$

where $\lambda = (\sqrt{c^2+4}-c)/2$. Hence $P'_\gamma(0) \in [-(1-\gamma^{-1}), -(1-\gamma^{-1})\lambda]$. We deduce that $J_\gamma(0) \le c-(1-\gamma^{-1})\lambda$, and therefore

$$N_\gamma^0 \le \left(c + O\left(\frac{1}{\gamma}\right)\right)^{-1}\left(c - \left(1-\frac{1}{\gamma}\right)\lambda\right) \le 1 - \frac{\lambda}{c} + O\left(\frac{1}{\gamma}\right). \tag{32}$$

Hence $\limsup_{\gamma\to+\infty} N_\gamma^0 \le (c-\lambda)/c < 1$.

The bound on $P'_\gamma(0)$ also implies the boundedness of N'_γ on $\mathbb{R}_-$. Indeed, since $\xi_\gamma^0 > 0$, N'_γ is decreasing and negative on $\mathbb{R}_-$, and

$$\sup_{\xi<0} |N'_\gamma(\xi)| = |N'_\gamma(0)| = -\frac{P'_\gamma(0)}{\gamma N_\gamma(0)^{\gamma-1}} = O(1).$$

Let us now address the upper bound on ξ_γ^0. We recall that N_γ is concave on $(-\infty, \xi_\gamma^0)$. Consequently, for all $\xi \in (0, \xi_\gamma^0)$,

$$0 \le N_\gamma(\xi) \le N_\gamma(0) + N'_\gamma(0)\xi.$$

In particular, taking $\xi = \xi_\gamma^0$, we deduce that

$$\xi_\gamma^0 \le -\frac{N_\gamma(0)}{N'_\gamma(0)} = -\frac{\gamma P_\gamma(0)}{P'_\gamma(0)} = -\frac{1}{P'_\gamma(0)},$$

since $P_\gamma(0) = \gamma^{-1}$ by choice of our normalization. We deduce in particular that

$$0 \le \xi_\gamma^0 \le \frac{1}{\lambda(1-\gamma^{-1})}.$$

• *Second step: Super-solution for N_γ on $(\xi_\gamma^0, +\infty)$.*

We recall that N_γ is convex on $(\xi_\gamma^0, +\infty)$. As a consequence, using the equation on N_γ, we have

$$-cN'_\gamma = N_\gamma(1-N_\gamma^\gamma) + \gamma N''_\gamma N_\gamma^\gamma + \gamma^2 (N'_\gamma)^2 N_\gamma^{\gamma-1} \ge N_\gamma(1-(N_\gamma^0)^\gamma) \quad \forall \xi \in (\xi_\gamma^0, +\infty). \tag{33}$$

The Grönwall Lemma then implies that

$$N_\gamma(\xi) \le N_\gamma^0 \exp\left(-\frac{1-(N_\gamma^0)^\gamma}{c}(\xi - \xi_\gamma^0)\right), \qquad \forall \xi \ge \xi_\gamma^0. \tag{34}$$

Recalling (32), we deduce that for γ large enough, for all $\xi \in (\xi_\gamma^0, +\infty)$,

$$N_\gamma(\xi) \le N_\gamma^0 \exp\left(-\frac{1}{2c}(\xi - \xi_\gamma^0)\right). \tag{35}$$

• *Third step: Strong convergence of P'_γ and N_γ.*

We start with an L^2 bound for P'_γ. From (4), P_γ is solution to

$$-cP'_\gamma - \gamma P_\gamma P''_\gamma - (P'_\gamma)^2 = \gamma P_\gamma(1-P_\gamma). \tag{36}$$

Integrating equation (36) over $\mathbb{R}$ gives

$$(\gamma-1)\int_{\mathbb{R}} |P'_\gamma(\xi)|^2 d\xi = -c + \gamma \int_{\mathbb{R}} P_\gamma(1-P_\gamma)\, d\xi.$$

Hence, we get the following inequality

$$\|P'_\gamma\|^2_{L^2(\mathbb{R})} \le \frac{\gamma}{\gamma-1}\left(\|P_\gamma\|_{L^1(\mathbb{R}_+)} + \|1-P_\gamma\|_{L^1(\mathbb{R}_-)}\right).$$

The right-hand side is uniformly bounded with respect to γ thanks to sub-solution for P_γ on $\mathbb{R}_-$ (see Proposition 2.4) and to the upper bound for N_γ on $(\xi_\gamma^0, +\infty)$ (see (35)). On the interval $(0, \xi_\gamma^0)$, we simply use the fact that ξ_γ^0 is bounded and $P_\gamma \leq P_\gamma(0)$. Hence, $(P'_\gamma)_{\gamma>1}$ is bounded in $L^2(\mathbb{R})$.

We now show an additional strong convergence of $(P'_\gamma)_\gamma$ in L^2. Going back to Equation (36) and taking into account that $(P'_\gamma)_\gamma$ is uniformly bounded in $L^2(\mathbb{R})$, we have for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} \psi P_\gamma \left[P''_\gamma + (1-P_\gamma)\right] \, d\xi = -\frac{1}{\gamma} \int_{\mathbb{R}} \psi \left[cP'_\gamma + (P'_\gamma)^2\right] \, d\xi \to 0 \quad \text{as } \gamma \to +\infty.$$

Hence, by integration by parts in the left-hand side:

$$-\int_{\mathbb{R}} \psi (P'_\gamma)^2 \, d\xi - \int_{\mathbb{R}} \psi' P_\gamma P'_\gamma \, d\xi + \int_{\mathbb{R}} \psi P_\gamma (1-P_\gamma) \, d\xi \to 0 \quad \text{as } \gamma \to +\infty.$$

From the previous bounds, it is clear that

$$-\int_{\mathbb{R}} \psi' P_\gamma P'_\gamma \, d\xi + \int_{\mathbb{R}} \psi P_\gamma (1-P_\gamma) \underset{\gamma\to+\infty}{\to} -\int_{\mathbb{R}} \psi' P_{HS} \, P'_{HS} \, d\xi + \int_{\mathbb{R}} \psi P_{HS}(1-P_{HS}) \, d\xi = \int_{\mathbb{R}} \psi (P'_{HS})^2,$$

using the complementarity equation (24). Finally,

$$\int_{\mathbb{R}} \psi (P'_\gamma)^2 \, d\xi \to \int_{\mathbb{R}} \psi (P'_{HS})^2 \, d\xi \quad \text{as } \gamma \to +\infty,$$

which means that $(P'_\gamma)_\gamma$ converges strongly in $L^2_{\text{loc}}(\mathbb{R})$ to P'_{HS}.

We then recall that $J_\gamma = N_\gamma(c + P'_\gamma)$. Since $(P'_\gamma)_\gamma$ converges in L^2_{loc}, there exists a subsequence (which we still denote by P'_γ) which also converges almost everywhere. Recall that $(J_\gamma)_\gamma$ converges in $\mathcal{C}(K)$ for any compact set $K \subset \mathbb{R}$. Therefore $(N_\gamma)_\gamma$ converges almost everywhere - up to a subsequence - on any set of the form $\cap_{\gamma>0}\{c + P'_\gamma \geq \delta\}$, with $\delta > 0$.

Let K be a compact set in $\mathbb{R}$, and let $M = \sup_K J_{HS} < c$, $m = \inf_K J_{HS} > 0$. There exists $\gamma_K > 0$ such that for $\gamma \geq \gamma_K$, $J_\gamma(K) \subset [m/2, (c+M)/2]$. Since $J_\gamma \leq c + P'_\gamma$, we deduce that $c + P'_\gamma \geq m/2$ on K for $\gamma \geq \gamma_K$. Whence $(N_\gamma)_\gamma$ converges almost everywhere on K, up to a subsequence. Since N_γ is bounded in L^∞, Lebesgue's dominated convergence theorem implies that $(N_\gamma)_\gamma$ converges towards N_{HS} in $L^p(K)$ for any $p \in [1, +\infty[$. Note that the limit is uniquely identified. Hence the whole sequence $(N_\gamma)_\gamma$ converges in L^p_{loc}.

• *Fourth step: Convergence of* ξ_γ^0 *and* N_γ^0.

We argue by contradiction and assume that $\limsup_{\gamma\to+\infty} \xi_\gamma^0 > 0$. Then there exists $\bar{\xi} > 0$ such that $\bar{\xi} \leq \xi_\gamma^0$ for a subsequence. We recall that $N''_\gamma \leq 0$ on $(0, \bar{\xi})$. Passing to the limit in the sense of distributions along this subsequence, we obtain that $N''_{HS} \leq 0$ in $\mathcal{D}'((0, \bar{\xi}))$, which is absurd. Thus $\xi_\gamma^0 \to 0$ as $\gamma \to +\infty$.

Let us now go back to (31). We recall that J_γ converges uniformly towards J_{HS}, and that $J_{HS}(0) = c - 1$. It follows that $\lim_{\gamma\to+\infty} N_\gamma^0 = 1 - c^{-1}$.

• *Fifth step: Asymptotic behavior of* ξ_γ^* *and* $N_\gamma(\xi_\gamma^*)$.

First, notice that ξ_γ^* is well-defined since N'_γ is increasing on $(\xi_\gamma^0, +\infty)$, with $N'_\gamma(+\infty) = 0$ and $\lim_{\gamma\to+\infty} N'_\gamma(\xi_\gamma^0) = -\infty$. Furthermore, since $N_\gamma(\xi_\gamma^*) \leq N_\gamma^0$, $\limsup_{\gamma\to+\infty} N_\gamma(\xi_\gamma^*) \leq 1 - c^{-1}$.

In order to prove that $\lim_{\gamma\to+\infty} \xi_\gamma^* = 0$, we argue once again by contradiction and we assume that $\limsup_{\gamma\to+\infty} \xi_\gamma^* > 0$. Thus there exists $\delta > 0$ such that $\xi_\gamma^* \geq \delta$ along a subsequence. By monotony of N'_γ, we know that $N'_\gamma(\xi) \leq N'_\gamma(\xi_\gamma^*) = -c^{-1}(1-c^{-1})$ for all $\xi \in (\xi_\gamma^0, \delta) \subset (\xi_\gamma^0, \xi_\gamma^*)$. Thus, passing to the weak limit, we find that there exists a non-empty open interval included in $(0, +\infty)$ on which $N'_{HS} \leq -c^{-1}(1-c^{-1})$. This contradicts the explicit expression of N'_{HS} on $\mathbb{R}_+$, namely $N'_{HS}(\xi) = -c^{-1}(1-c^{-1})e^{-\xi/c}$ for $\xi > 0$; and therefore $\lim_{\gamma\to+\infty} \xi_\gamma^* = 0$.

The convergence of $N_\gamma(\xi_\gamma^*)$ towards $1 - c^{-1}$ follows from the same arguments as the one of N_γ^0: we note that

$$J_\gamma(\xi_\gamma^*) = cN_\gamma(\xi_\gamma^*) - \gamma \frac{1}{c}\left(1 - \frac{1}{c}\right) N_\gamma(\xi_\gamma^*)^\gamma.$$

Since $\limsup_{\gamma\to+\infty} N_\gamma(\xi_\gamma^*) < 1$, the second term in the right-hand side converges towards zero exponentially fast. We also recall that by uniform convergence of J_γ, $J_\gamma(\xi_\gamma^*) \to J_{HS}(0) = c-1$ as $\gamma \to +\infty$. Hence $\lim_{\gamma\to+\infty} N_\gamma(\xi_\gamma^*) = 1 - c^{-1}$.

□

Note that thanks to the above Lemma, we can pass to the limit in the pressure equation (26), i.e. take the limit of quantities such as $N_\gamma P_\gamma''$, even though P_{HS}'' has a Dirac mass at the point where N_{HS} has a discontinuity.

Corollary 2.7. *As $\gamma \to +\infty$,*

$$N_\gamma P_\gamma'' \rightharpoonup -\mathbf{1}_{\xi<0} e^{\xi} - (c-1)\ln\left(1 - \frac{1}{c}\right)\delta(\xi) \quad in \quad \mathcal{D}'(\mathbb{R})$$

where δ is the Dirac mass at $\xi = 0$.

Proof. Let us rewrite the pressure equation (26) as

$$-N_\gamma P_\gamma'' = N_\gamma(1 - P_\gamma) + J_\gamma(\ln N_\gamma)'.$$

According to Proposition 2.4 and Lemma 2.6,

$$\begin{aligned}
N_\gamma(1 - P_\gamma) &\to \mathbf{1}_{\xi<0} e^{\xi} + \mathbf{1}_{\xi>0}\left(1 - \frac{1}{c}\right) e^{-\xi/c} \quad \text{in } L^p_{\text{loc}}(\mathbb{R}) \\
J_\gamma \to J_{HS} &= \mathbf{1}_{\xi\leq 0}(c - e^{\xi}) + \mathbf{1}_{\xi>0}(c-1)e^{-\xi/c} \quad \text{in } \mathcal{C}(K), \\
\ln(N_\gamma) &\to \mathbf{1}_{\xi>0}\left(\ln\left(1 - \frac{1}{c}\right) - \frac{\xi}{c}\right) \quad \text{in } L^p_{\text{loc}}(\mathbb{R})
\end{aligned}$$

for all $1 \leq p < +\infty$ and for any compact set $K \subset \mathbb{R}$.
The result follows easily.

□

The same method also allows us to pass to the limit in products such as $P_\gamma'' f(N_\gamma)$, for any continuous function f. We find that

$$P_\gamma'' f(N_\gamma) \rightharpoonup -\mathbf{1}_{\xi<0} f(1) e^{\xi} - (c-1) F\left(1 - \frac{1}{c}\right)\delta(\xi),$$

where $F(z) = -\int_z^1 f(t)/t^2\,dt$ for $z > 0$.

2.4 Quantitative bounds for the profiles N_γ

In order to prove our quantitative stability result in Theorem 1.5, we will need some quantitative information on the asymptotic behavior of N_γ and its derivatives (e.g., the size of $\|N_\gamma'\|_{L^\infty}$). This subsection is devoted to the proof of such bounds.
More precisely, we prove the following result:

Lemma 2.8. *There exists a constant $C > 1$, depending only on c, such that the following properties hold, for any $\gamma > 0$:*

$$\sup_{0<|h|\leq 1} \sup_{x\in\mathbb{R}} \frac{1}{|h|} \frac{|N_\gamma(x+h) - N_\gamma(x)|}{N_\gamma(x)} + \left\|\frac{N_\gamma'}{N_\gamma}\right\|_\infty \leq C^\gamma,$$

$$\sup_{0<|h|\leq 1} \sup_{x\in\mathbb{R}} \frac{1}{|h|} \left|\frac{N_\gamma(x+h) - N_\gamma(x)}{N_\gamma'(x)}\right| \leq C^\gamma, \qquad \sup_{\xi<0}\left|\frac{1 - P_\gamma(\xi)}{P_\gamma'(\xi)}\right| \leq C.$$

Proof. **Bound on N_γ'/N_γ in the free zone $\xi > \xi_\gamma^*$.**

We set $L_\gamma := \frac{N_\gamma'}{N_\gamma} + c^{-1}$. Using the equation and the convexity of N_γ in $\xi > \xi_\gamma^*$, see (33), we have

$$-cN_\gamma'(\xi) \geq N_\gamma(\xi)(1-(N_\gamma(\xi))^\gamma) \implies L_\gamma(\xi) = \frac{N_\gamma'(\xi)}{N_\gamma(\xi)} + \frac{1}{c} \leq \frac{(N_\gamma(\xi))^\gamma}{c} \leq \frac{(N_\gamma(\xi_\gamma^*))^\gamma}{c}, \qquad \forall\, \xi \geq \xi_\gamma^*.$$

Furthermore, since $N_\gamma'(\xi_\gamma^*) = -c^{-1}(1-c^{-1})$ and $N_\gamma(\xi_\gamma^*) \to 1 - c^{-1}$, we immediately infer that $L_\gamma(\xi_\gamma^*)$ vanishes as $\gamma \to +\infty$. We now derive an equation for L_γ in order to obtain a lower bound on L_γ. We have, using the equation on N_γ,

$$\begin{aligned} L_\gamma' &= \frac{N_\gamma''}{N_\gamma} - \frac{(N_\gamma')^2}{N_\gamma^2} \\ &= -\frac{1}{\gamma N_\gamma^{\gamma+1}}\left(N_\gamma(1-N_\gamma^\gamma) + cN_\gamma' + \gamma^2 (N_\gamma')^2 N_\gamma^{\gamma-1}\right) - \frac{(N_\gamma')^2}{N_\gamma^2} \\ &= -\frac{cL_\gamma}{\gamma N_\gamma^\gamma} + \frac{1}{\gamma} - (\gamma+1)\left(L_\gamma - \frac{1}{c}\right)^2. \end{aligned}$$

Thus L_γ satisfies the differential equation

$$L_\gamma' + \left[(\gamma+1)L_\gamma + \frac{c}{\gamma N_\gamma^\gamma} - \frac{2(\gamma+1)}{c}\right] L_\gamma = \frac{1}{\gamma} - \frac{\gamma+1}{c^2}.$$

Note that the coefficient $\frac{c}{\gamma N_\gamma^\gamma} - \frac{2(\gamma+1)}{c}$ is exponentially large in the free zone, and drives a strong convergence of L_γ towards zero. Thus the whole idea is to prove that the quadratic term $(\gamma+1)L_\gamma^2$ does not perturb the linear behavior. This easily follows from a bootstrap argument. First, note that there exists a non-empty open interval on the right of ξ_γ^* on which $L_\gamma > -2|L_\gamma(\xi_\gamma^*)|$. Let us set

$$\tilde{\xi}_\gamma := \sup\{\xi > \xi_\gamma^*,\ L_\gamma > -2|L_\gamma(\xi_\gamma^*)| \text{ on } (\xi_\gamma^*, \xi)\}.$$

On the interval $(\xi_\gamma^*, \tilde{\xi}_\gamma)$, we have

$$-2|L_\gamma(\xi_\gamma^*)| \leq L_\gamma(\xi) \leq \frac{(N_\gamma(\xi_\gamma^*))^\gamma}{c},$$

and therefore L_γ converges uniformly towards zero on this interval. If $\tilde{\xi}_\gamma = +\infty$, we obtain a uniform bound on N_γ'/N_γ in the free zone. If $\tilde{\xi}_\gamma < +\infty$, then $L_\gamma(\tilde{\xi}_\gamma) = -2|L_\gamma(\xi_\gamma^*)| < 0$. Thus at $\xi = \tilde{\xi}_\gamma$, for γ large enough

$$(\gamma+1)L_\gamma + \frac{c}{\gamma N_\gamma^\gamma} - \frac{2(\gamma+1)}{c} > \frac{c}{2\gamma N_\gamma^\gamma}. \tag{37}$$

Thus by continuity, this property remains true on a non-empty open interval on the right of $\tilde{\xi}_\gamma$. We now define

$$\xi_{\max} := \sup\{\xi > \tilde{\xi}_\gamma,\ (37) \text{ holds and } L_\gamma < 0 \text{ on } (\tilde{\xi}_\gamma, \xi)\}.$$

Then $\xi_{\max} > \tilde{\xi}_\gamma > \xi_\gamma^*$, and on the interval $(\tilde{\xi}_\gamma, \xi_{\max})$, we have, since $L_\gamma(\xi) < 0$

$$L_\gamma' + \frac{c}{\gamma N_\gamma^\gamma} L_\gamma \leq \frac{1}{\gamma} - \frac{\gamma+1}{c^2} \leq -\frac{\gamma}{2c^2},$$

$$L_\gamma' + \frac{c}{2\gamma N_\gamma^\gamma} L_\gamma \geq \frac{1}{\gamma} - \frac{\gamma+1}{c^2} \geq -\frac{\gamma+1}{c^2}.$$

The Grönwall Lemma then implies that for all $\xi \in (\tilde{\xi}_\gamma, \xi_{\max})$,

$$L_\gamma(\xi) \leq -2|L_\gamma(\xi_\gamma^*)| \exp\left(-\int_{\tilde{\xi}_\gamma}^{\xi} \frac{c}{\gamma N_\gamma^\gamma}\right) - \frac{\gamma}{2c^2} \int_{\tilde{\xi}_\gamma}^{\xi} \exp\left(-\int_{\xi'}^{\xi} \frac{c}{\gamma N_\gamma^\gamma}\right) d\xi',$$

$$L_\gamma(\xi) \geq -2|L_\gamma(\xi_\gamma^*)| \exp\left(-\int_{\tilde{\xi}_\gamma}^{\xi} \frac{c}{2\gamma N_\gamma^\gamma}\right) - \frac{\gamma+1}{c^2} \int_{\tilde{\xi}_\gamma}^{\xi} \exp\left(-\int_{\xi'}^{\xi} \frac{c}{2\gamma N_\gamma^\gamma}\right) d\xi'.$$

Now, we recall that for $\xi > \xi_\gamma^*$, for γ sufficiently large,

$$N_\gamma(\xi) \leq N_\gamma(\xi_\gamma^*) \leq 1 - \frac{1}{2c}.$$

Thus for all $\xi \in (\tilde{\xi}_\gamma, \xi_{\max})$,

$$\begin{aligned} L_\gamma(\xi) \quad &\geq \quad -2|L_\gamma(\xi^*)| \exp\left(-(\xi - \tilde{\xi}_\gamma)\frac{c}{2\gamma}\left(1 - \frac{1}{2c}\right)^{-\gamma}\right) \\ &\quad -\frac{\gamma+1}{c^2} \int_{\tilde{\xi}_\gamma}^{\xi} \exp\left(-(\xi - \xi')\frac{c}{2\gamma}\left(1 - \frac{1}{2c}\right)^{-\gamma}\right) d\xi' \\ &\geq \quad -2|L_\gamma(\xi^*)| \exp\left(-(\xi - \tilde{\xi}_\gamma)\frac{c}{2\gamma}\left(1 - \frac{1}{2c}\right)^{-\gamma}\right) - \frac{2\gamma(\gamma+1)}{c^3}\left(1 - \frac{1}{2c}\right)^{\gamma}. \end{aligned}$$

Note that the right-hand side of the above inequality converges uniformly towards zero. In particular, for γ sufficiently large, $L_\gamma(\xi) \geq -1$ for all $\xi \in (\tilde{\xi}_\gamma, \xi_{\max})$. It follows that

$$(\gamma+1)L_\gamma + \frac{c}{\gamma N_\gamma^\gamma} - \frac{2(\gamma+1)}{c} \geq \frac{c}{\gamma N_\gamma^\gamma} - \frac{(c+2)(\gamma+1)}{c} \geq \frac{3c}{4\gamma N_\gamma^\gamma} \quad \forall \xi \in (\tilde{\xi}_\gamma, \xi_{\max}).$$

The upper bound on $L_\gamma(\xi)$ also shows that the threshold $L_\gamma(\xi) = 0$ is never reached for finite ξ. By a bootstrap argument, we deduce that $\xi_{\max} = +\infty$. The above inequalities imply, in particular, that

$$L_\gamma \to 0 \text{ uniformly on } (\xi_\gamma^*, +\infty).$$

Remark 2.9. The uniform convergence of L_γ towards zero yields the existence of sub-solutions of N_γ in the zone $\xi > \xi_\gamma^*$. Indeed, let $\delta > 0$ be arbitrary. Then for γ large enough, $L_\gamma \geq -\delta$, and therefore $\frac{N_\gamma'}{N_\gamma} \geq -(c^{-1} + \delta)$. By the Grönwall Lemma, we obtain

$$N_\gamma(\xi) \geq N_\gamma(\xi_\gamma^*) \exp\left(-\left(\frac{1}{c} + \delta\right)(\xi - \xi_\gamma^*)\right). \tag{38}$$

Bound on N_γ'/N_γ and on the first difference quotient in L^∞.

We distinguish between $\xi < \xi_\gamma^*$ and $\xi > \xi_\gamma^*$ and we write, for γ sufficiently large,

$$\begin{aligned} \left\|\frac{N_\gamma'}{N_\gamma}\right\|_\infty \quad &= \quad \max\left(\sup_{\xi<\xi_\gamma^*} \frac{|N_\gamma'|}{N_\gamma}, \sup_{\xi>\xi_\gamma^*} \left|L_\gamma - \frac{1}{c}\right|\right) \\ &\leq \quad \max\left(\frac{1}{N_\gamma(\xi_\gamma^*)}\|N_\gamma'\|_\infty, \frac{1}{c} + 1\right) \\ &\leq \quad C|Q_-(N_\gamma^0)| \leq \left(1 - \frac{1}{2c}\right)^{-\gamma}. \end{aligned}$$

Let us now consider the difference quotient

$$\frac{1}{|h|}\frac{|N_\gamma(x+h) - N_\gamma(x)|}{N_\gamma(x)}.$$

We will need to distinguish several cases:

- Case $x < \xi^*_\gamma$: in that case, $N_\gamma(x) \geq N_\gamma(\xi^*_\gamma) \to 1 - c^{-1}$, and therefore the difference quotient is bounded by $C\|N'_\gamma\|_\infty$.
- Case $x > \xi^*_\gamma$:
  - Sub-case $h > 0$: we write $N_\gamma(x+h) - N_\gamma(x) = \int_0^h N'_\gamma(x+y)\,dy$, and we recall that since L_γ is uniformly bounded, $|N'_\gamma| \leq CN_\gamma$ for some constant C in $(\xi^*_\gamma, +\infty)$. Using the monotony of N_γ, we deduce that the difference quotient is bounded.
  - Sub-case $h < 0$ and $x + h > \xi^*_\gamma$: an argument similar to the sub-case $h > 0$ applies. In that case, we obtain, using a variant of Remark 2.9,
$$\frac{1}{|h|}\frac{|N_\gamma(x+h) - N_\gamma(x)|}{N_\gamma(x)} \leq C\frac{N_\gamma(x+h)}{N_\gamma(x)} \leq C.$$
  - Sub-case $x + h \leq \xi^*_\gamma$: in that case, note that $x = x + h - h \leq \xi^*_\gamma + 1$ since $|h| \leq 1$. Hence, $N_\gamma(x) \geq N_\gamma(\xi^*_\gamma + 1)$, which is uniformly bounded from below thanks to (38). Thus the difference quotient is bounded by $C\|N'_\gamma\|_\infty$.

Gathering these results, we obtain the bounds announced in the Lemma.

Bound on $(1 - P_\gamma)/P'_\gamma$ on $\mathbb{R}_-$.

Let $M_\gamma := (1 - P_\gamma)/P'_\gamma$. According to Proposition 2.4, $M_\gamma \to (1 - P_{HS})/P'_{HS} = -1$ locally uniformly on $\mathbb{R}_-$. So, for γ sufficiently large, $M_\gamma(\xi) \in [-3/2, -1/2]$ for all $\xi \in [-1, 0]$. Furthermore we know that
$$\frac{N'_\gamma}{P'_\gamma}(\xi) = \frac{1}{\gamma(N_\gamma(\xi))^{\gamma-1}} \underset{\xi\to-\infty}{\to} \frac{1}{\gamma}, \quad \lim_{\xi\to-\infty}\frac{1 - P_\gamma(\xi)}{1 - N_\gamma(\xi)} = \lim_{N\to1^-}\frac{1 - N^\gamma}{1 - N} = \gamma,$$
so that, thanks to Theorem 2.3,
$$M_\gamma(\xi) = \frac{N'_\gamma(\xi)}{P'_\gamma(\xi)}\frac{1 - N_\gamma(\xi)}{N'_\gamma(\xi)}\frac{1 - P_\gamma(\xi)}{1 - N_\gamma(\xi)} \to -\left(\sqrt{1 + \frac{c^2}{4\gamma^2}} - \frac{c}{2\gamma}\right)^{-1}, \quad \text{as } \xi \to -\infty.$$

Now, let us consider the interval $(-\infty, -1]$. There are two possibilities:

- either $M_\gamma(\xi)$ is between $M_\gamma(-1)$ and $M_\gamma(-\infty)$ for all $\xi \in (-\infty, -1]$. In that case, for γ sufficiently large, $M_\gamma(\xi) \in [-3/2, -1/2]$ for all $\xi \in (-\infty, -1]$;
- or M_γ takes values outside the interval $[M_\gamma(-1), M_\gamma(-\infty)]$. In that case M_γ reaches a local extremum at some $\xi_M \in (-\infty, -1)$, and therefore $M'_\gamma(\xi_M) = 0$.

  Let us compute M'_γ. Using the equation satisfied by P_γ (26), we have
$$\begin{aligned} M'_\gamma &= -1 - \frac{P''_\gamma(1 - P_\gamma)}{(P'_\gamma)^2} \\ &= -1 + \frac{1 - P_\gamma}{(P'_\gamma)^2}\left(1 - P_\gamma + \frac{cP'_\gamma}{\gamma P_\gamma} + \frac{(P'_\gamma)^2}{\gamma P_\gamma}\right) \\ &= -1 + M_\gamma^2 + c\frac{M_\gamma}{\gamma P_\gamma} + \frac{1 - P_\gamma}{\gamma P_\gamma}. \end{aligned}$$
  At $\xi = \xi_M$, the right-hand side vanishes, and therefore
$$M_\gamma(\xi_M) = \frac{1}{2}\left(-\frac{c}{\gamma P_\gamma(\xi_M)} \pm \sqrt{4 + \frac{c^2}{\gamma^2 P_\gamma(\xi_M)^2} - 4\frac{1 - P_\gamma(\xi_M)}{\gamma P_\gamma(\xi_M)}}\right).$$
  Note that, thanks to (18), $P_\gamma(\xi_M) \geq P_\gamma(-1) \geq 1 - e^{-\lambda} > 0$. Hence, $M_\gamma(\xi_M) = \pm 1 + O(\gamma^{-1})$. Recalling that $M_\gamma < 0$ on $\mathbb{R}_-$, we deduce that $M_\gamma(\xi_M) = -1 + O(\gamma^{-1})$.

  Once again, for γ sufficiently large, we find that $M_\gamma(\xi) \in [-3/2, -1/2]$ for all $\xi \in (-\infty, -1]$.

Hence, we deduce in all cases that for γ sufficiently large,

$$-\frac{3}{2} \leq \frac{1-P_\gamma}{P'_\gamma} \leq -\frac{1}{2}, \qquad \forall \xi \in \mathbb{R}_-. \tag{39}$$

Note that these bounds (which are stronger than what is announced in the statement of the lemma) imply in particular the following inequalities, which are easy consequences of the Grönwall Lemma: for all $\xi \leq \xi' \leq 0$, for γ large enough,

$$(1-P_\gamma(\xi))\exp\left(-2(\xi'-\xi)\right) \leq 1-P_\gamma(\xi') \leq (1-P_\gamma(\xi))\exp\left(-\frac{2}{3}(\xi'-\xi)\right). \tag{40}$$

Bound on the second difference quotient.

We now address the bound on

$$\sup_{0<|h|\leq 1}\sup_{x\in\mathbb{R}} \frac{1}{|h|}\left|\frac{N_\gamma(x+h)-N_\gamma(x)}{N'_\gamma(x)}\right|.$$

Once again, we will need to distinguish between several zones. First, note that

$$\frac{1}{|h|}\left|\frac{N_\gamma(x+h)-N_\gamma(x)}{N'_\gamma(x)}\right| = \frac{1}{|h|}\left|\frac{N_\gamma(x+h)-N_\gamma(x)}{N_\gamma(x)}\right| \left|\frac{N_\gamma(x)}{N'_\gamma(x)}\right|.$$

Hence, for $x>-2$, this difference quotient is bounded by

$$\sup_{x\in\mathbb{R}}\sup_{0<|h|\leq 1}\frac{1}{|h|}\left|\frac{N_\gamma(x+h)-N_\gamma(x)}{N_\gamma(x)}\right| \sup_{x>-2}\frac{N_\gamma(x)}{|N'_\gamma(x)|}.$$

For $x>\xi^*_\gamma$, $N_\gamma/N'_\gamma = (L_\gamma - c^{-1})^{-1}$, and we recall that L_γ converges uniformly towards zero on $(\xi^*_\gamma,+\infty)$. Hence, N_γ/N'_γ is uniformly bounded on $(\xi^*_\gamma,+\infty)$. And looking at the variations of N'_γ, we infer that

$$\sup_{x\in(-2,\xi^*_\gamma)}\frac{N_\gamma(x)}{|N'_\gamma(x)|} \leq \max\left(\frac{1}{|N'_\gamma(-2)|}, \frac{1}{|N'_\gamma(\xi^*_\gamma)|}\right) \leq C\gamma.$$

Thus

$$\sup_{0<|h|\leq 1}\sup_{x\in(-2,+\infty)}\frac{1}{|h|}\left|\frac{N_\gamma(x+h)-N_\gamma(x)}{N'_\gamma(x)}\right| \leq \gamma C^\gamma \leq C_1^\gamma,$$

for some constant $C_1>C$.

We now consider the interval $(-\infty,-2)$. Since $|h|\leq 1$, we have $x+h\leq -1$. Hence, x and $x+h$ are in the congested zone. We write

$$\frac{1}{h}\frac{N_\gamma(x+h)-N_\gamma(x)}{N'_\gamma(x)} = \int_0^1 \frac{N'_\gamma(x+\tau h)}{N'_\gamma(x)}\,d\tau.$$

Recall that $N'_\gamma = \gamma^{-1}P'_\gamma N_\gamma^{-(\gamma-1)}$. Hence,

$$\frac{N'_\gamma(x+\tau h)}{N'_\gamma(x)} = \frac{P'_\gamma(x+\tau h)}{P'_\gamma(x)}\,\frac{N_\gamma(x)^{\gamma-1}}{N_\gamma(x+\tau h)^{\gamma-1}}.$$

Note that $N_\gamma^{\gamma-1} = P_\gamma/N_\gamma$ is uniformly bounded from above and from below on $(-\infty,-1)$. Thus we focus on the quotient $P'_\gamma(x+\tau h)/P'_\gamma(x)$, which we further decompose as

$$\frac{P'_\gamma(x+\tau h)}{P'_\gamma(x)} = \frac{P'_\gamma(x+\tau h)}{1-P_\gamma(x+\tau h)}\,\frac{1-P_\gamma(x+\tau h)}{1-P_\gamma(x)}\,\frac{1-P_\gamma(x)}{P'_\gamma(x)} = \frac{M_\gamma(x)}{M_\gamma(x+\tau h)}\,\frac{1-P_\gamma(x+\tau h)}{1-P_\gamma(x)}.$$

Using (40) and (39), we deduce that

$$\left|\frac{P_\gamma'(x+\tau h)}{P_\gamma'(x)}\right| \leq Ce^{2|h|}.$$

Hence,

$$\sup_{0<|h|\leq 1}\sup_{x\leq -2}\frac{1}{|h|}\left|\frac{N_\gamma(x+h)-N_\gamma(x)}{N_\gamma'(x)}\right| \leq C.$$

□

Our nonlinear stability result will hold in weighted Sobolev spaces. The weights will depend on the function N_γ and its derivative, and therefore will have abrupt changes in the transition zone $(0,\xi_\gamma^*)$. In order to monitor precisely these changes, we introduce two additional abscissas ξ_γ^- and ξ_γ^+, which we define as follows:

Definition 2.10 (Definition of ξ_γ^- and ξ_γ^+)**.**

- The abscissa $\xi_\gamma^-\in\mathbb{R}$ is the unique point where

$$P_\gamma(\xi_\gamma^-)=\left(\frac{c^3}{(c-1)(\gamma+1)}\right)^{1/2}. \tag{41}$$

- The abscissa $\xi_\gamma^+\in\mathbb{R}$ is the unique point such that $N_\gamma(\xi_\gamma^+)\in(0,N_\gamma^0)$ and

$$N_\gamma'(\xi_\gamma^+)=-\frac{c-1}{4\gamma^2 N_\gamma(\xi_\gamma^+)^{\gamma-1}}.$$

Remark 2.11.
- Note that ξ_γ^- is well-defined by monotony of P_γ, and $\xi_\gamma^-<0$ since $P_\gamma(\xi_\gamma^-)>P_\gamma(0)$;
- The definition of ξ_γ^+ is a little more intricate. We recall that for all $N_\gamma\in(0,N_\gamma^0)$, $Q_-(N)<N_\gamma'<0$, where Q_- is defined in (29) and $dN_\gamma'/dN_\gamma\leq 0$ for all $N_\gamma\in(0,N_\gamma^0)$; we refer to the analysis of the phase portrait in the previous subsection (see also Fig. 2).

  Now, define $\tilde{Q}(N)$ by

$$\tilde{Q}(N):=-\frac{c-1}{4\gamma^2N^{\gamma-1}}.$$

  It is clear from the definition of $\tilde{Q}$ and Q_- that $Q_-<\tilde{Q}$ for all $N\in(0,N_\gamma^0)$, and $\tilde{Q}$ is monotone increasing on that interval. Consequently, the curve (N_γ,N_γ') intersects the curve $(N,\tilde{Q}(N))$ exactly once on the interval $(0,N_\gamma^0)$ (see Figure 3). We denote the abscissa of the intersection point as N_γ^+, and ξ_γ^+ is defined implicitly as $N_\gamma(\xi_\gamma^+)=N_\gamma^+$.

Let us now give some properties of ξ_γ^+ and ξ_γ^-, which will be used in the next section:

Lemma 2.12 (Properties of ξ_γ^+ and ξ_γ^-)**.** *For γ large enough, the following properties hold:*

- $\xi_\gamma^-<0<\xi_\gamma^0<\xi_\gamma^+<\xi_\gamma^*$. *As a consequence,* $\lim_{\gamma\to+\infty}N_\gamma(\xi_\gamma^+)=1-c^{-1}$*;*
- $\xi_\gamma^-=O(\gamma^{-1/2})$*, and* $\xi_\gamma^+=O(\gamma^{-1})$*;*
- $P_\gamma'\leq-\gamma^{-1}(c-1)/4$ *for all* $\xi\in(\xi_\gamma^-,\xi_\gamma^+)$.

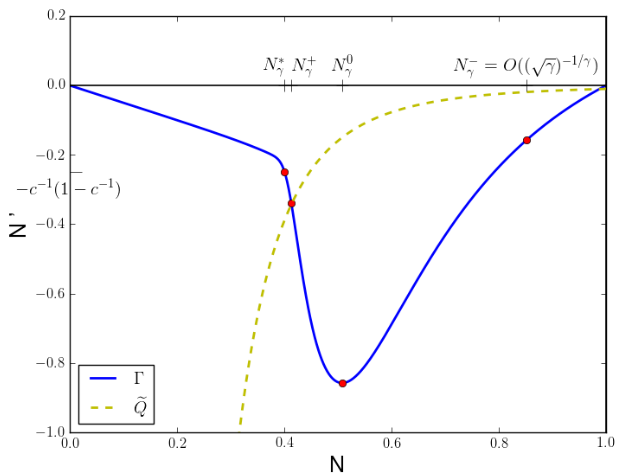

Figure 3: Definition of the point N_γ^+ in the phase plane (N, N'), $c = 2$, $\gamma = 5$.

Proof. **Relative positions of $\xi_\gamma^0, \xi_\gamma^+, \xi_\gamma^*$.**

By definition of ξ_γ^+, $N_\gamma(\xi_\gamma^+) < N_\gamma^0$, and thus $\xi_\gamma^0 < \xi_\gamma^+$. Furthermore, we recall that N_γ' is monotone increasing on $(\xi_\gamma^0, +\infty)$, and

$$N_\gamma'(\xi_\gamma^+) = -\frac{c-1}{4\gamma^2 N_\gamma(\xi_\gamma^+)^{\gamma-1}} \leq -\frac{c-1}{4\gamma^2 (N_\gamma^0)^{\gamma-1}} \to -\infty.$$

Whence $N_\gamma'(\xi_\gamma^+) < N_\gamma'(\xi_\gamma^*)$, and therefore $\xi_\gamma^+ < \xi_\gamma^*$. The limit of $N_\gamma(\xi_\gamma^+)$ follows from the monotony of N_γ and the fact that $\lim_{\gamma\to+\infty} N_\gamma(\xi_\gamma^0) = \lim_{\gamma\to+\infty} N_\gamma(\xi_\gamma^*) = 1 - c^{-1}$ (see Lemma 2.6).

Size of ξ_γ^-.

First, considering the sub-solution for P_γ defined in (18), we see that $\xi_\gamma^- > -1$. Using (39), we recall that P_γ' is bounded away from zero on $(-1, 0)$, for γ large enough. Hence,

$$\frac{|P_\gamma(\xi_\gamma^-) - P_\gamma(0)|}{\sup_{(\xi_\gamma^-,0)} |P_\gamma'|} \leq |\xi_\gamma^-| \leq \frac{|P_\gamma(\xi_\gamma^-) - P_\gamma(0)|}{\inf_{(\xi_\gamma^-,0)} |P_\gamma'|},$$

and thus

$$\frac{C^{-1}}{\sqrt{\gamma}} \leq |\xi_\gamma^-| \leq \frac{C}{\sqrt{\gamma}}.$$

Lower bound for $|P_\gamma'|$ on $(\xi_\gamma^-, \xi_\gamma^+)$ and size of ξ_γ^+.

Let us introduce yet another intermediate point ξ_γ^{int} such that

$$N_\gamma(\xi_\gamma^{\text{int}}) = 1 - (2c)^{-1}. \tag{42}$$

We recall that $N_\gamma(\xi_\gamma^0) \to 1 - c^{-1}$, and therefore $\xi_\gamma^{\text{int}} \in (0, \xi_\gamma^0)$ for γ large enough. Now, for $\xi \in (\xi_\gamma^-, \xi_\gamma^{\text{int}})$, we have $N_\gamma(\xi) \in [1 - (2c)^{-1}, 1]$, and

$$P_\gamma' = \frac{J_\gamma - cN_\gamma}{N_\gamma} \leq \frac{J_\gamma}{1 - \frac{1}{2c}} - c.$$

We recall that $J_\gamma(\xi) \to c - 1$ uniformly on that interval. Thus $P_\gamma' \leq -C < 0$ on $(\xi_\gamma^-, \xi_\gamma^{\text{int}})$ for γ sufficiently large, for some uniform constant C.

In particular, since $P_\gamma(\xi_\gamma^{\text{int}}) = (1-(2c)^{-1})^\gamma$ is exponentially small, it follows that

$$\xi_\gamma^{\text{int}} \le \frac{|P_\gamma(\xi_\gamma^{\text{int}}) - P_\gamma(0)|}{\inf_{[0,\xi_\gamma^{\text{int}}]}|P'_\gamma|} \le \frac{C}{\gamma}.$$

Let us now consider the intervals $(\xi_\gamma^{\text{int}}, \xi_\gamma^0)$ and $(\xi_\gamma^0, \xi_\gamma^+)$. Using the notations introduced in Lemma 2.5 and (30), it is easily checked that $N_\gamma(\xi_\gamma^{\text{int}}) \le N_1$. As a result, using the phase portrait of N_γ (see Figure 2), $N'_\gamma \le Q_-(N_\gamma)$ for all $\xi \in (\xi_\gamma^{\text{int}}, \xi_\gamma^0)$. Consequently,

$$\begin{aligned} P'_\gamma &= \gamma N'_\gamma(\xi)\big(N_\gamma(\xi)\big)^{\gamma-1} \\ &\le \gamma\big(N_\gamma(\xi)\big)^{\gamma-1} \times \frac{1}{2\gamma^2\big(N_\gamma(\xi)\big)^{\gamma-1}}\Big(-c-\sqrt{c^2-4\gamma^2\big(N_\gamma(\xi)\big)^\gamma\big(1-\big(N_\gamma(\xi)\big)^\gamma\big)}\Big) \\ &\le -\frac{c}{2\gamma}, \qquad \forall \xi \in (\xi_\gamma^{\text{int}}, \xi_\gamma^0). \end{aligned}$$

For $\xi \in (\xi_\gamma^0, \xi_\gamma^+)$, the argument is similar. On this interval, $N'_\gamma \ge Q_-(N_\gamma)$, but $N'_\gamma \le \tilde{Q}(N_\gamma)$ by definition of ξ_γ^+ (see Figure 3). Thus

$$P'_\gamma(\xi) \le \gamma\big(N_\gamma(\xi)\big)^{\gamma-1} \times \left(-\frac{c-1}{4\gamma^2\big(N_\gamma(\xi)\big)^{\gamma-1}}\right) \le -\frac{c-1}{4\gamma}, \qquad \forall \xi \in (\xi_\gamma^0, \xi_\gamma^+).$$

We obtain the desired lower bound on $|P'_\gamma|$ on $(\xi_\gamma^{\text{int}}, \xi_\gamma^+)$. It follows that

$$\xi_\gamma^+ - \xi_\gamma^{\text{int}} \le \frac{|P_\gamma(\xi_\gamma^+) - P_\gamma(\xi_\gamma^{\text{int}})|}{\inf_{(\xi_\gamma^{\text{int}},\xi_\gamma^+)}|P'_\gamma|} \le C\gamma\left(1-\frac{1}{2c}\right)^\gamma = o(\gamma^{-1}).$$

Hence, ξ_γ^+ and ξ_γ^{int} are exponentially close. The estimate on ξ_γ^+ follows.

□

As an immediate consequence of the previous lemma, we can compute the size of an integral which will play an important role in the next section:

Lemma 2.13. *There exists a constant $C > 0$, such that as $\gamma \to +\infty$,*

$$\int_{\xi_\gamma^-}^{\xi_\gamma^+} \frac{1}{\gamma N_\gamma^\gamma(z)}dz \le C\gamma.$$

Proof. Using Lemma 2.12, we recall that $|P'_\gamma| = \gamma|N'_\gamma|N_\gamma^{\gamma-1} \ge (c-1)\gamma^{-1}/4$ on $(\xi_\gamma^-, \xi_\gamma^+)$. Hence,

$$\begin{aligned} \int_{\xi_\gamma^-}^{\xi_\gamma^+} \frac{1}{\gamma N_\gamma^\gamma} &= \int_{\xi_\gamma^-}^{\xi_\gamma^+} \frac{|N'_\gamma|}{\gamma|N'_\gamma|N^\gamma} \\ &\le \frac{4}{c-1}\gamma\int_{\xi_\gamma^-}^{\xi_\gamma^+}\frac{|N'_\gamma|}{N_\gamma} \\ &\le \frac{4}{c-1}\gamma\ln\left(\frac{N_\gamma(\xi_\gamma^-)}{N_\gamma(\xi_\gamma^+)}\right) \le C\gamma. \end{aligned}$$

□

Let us conclude this section by saying a few words about the proof of Theorem 1.1. The sizes and signs of ξ_γ^- and ξ_γ^+ are given in Lemma 2.12. Inequality (6) follows from the monotony of N_γ and from the definition of ξ_γ^-. Let us briefly discuss the inequality claimed in (7). Actually, the reader may check that the derivation of sub- and super-solutions on $\mathbb{R}_-$ made in (27)-(28) can be easily adapted to the interval $(-\infty, \xi_\gamma^-]$, using the fact that $P_\gamma(\xi_\gamma^-) = O(\gamma^{-1/2})$ and $\gamma P_\gamma \geq C\sqrt{\gamma}$ on $(-\infty, \xi_\gamma^-]$. It follows that

$$1 - \left(1 - P_\gamma(\xi_\gamma^-)\right) e^{\mu_\gamma \xi} \leq P_\gamma(\xi) \leq 1 - \left(1 - P_\gamma(\xi_\gamma^-)\right) e^{\xi},$$

where μ_γ is the positive root of $\mu^2 + \frac{c}{C\sqrt{\gamma}}\mu - 1 = 0$. It is straightforward that $\mu_\gamma = 1 - O(\gamma^{-1/2})$, which leads to inequality (7).

The size of $\|N_\gamma'\|_\infty$ in the intermediate region $(\xi_\gamma^-, \xi_\gamma^+)$ is an easy consequence of Lemma 2.6, and the bounds on the pressure in that zone follow from the monotony of P_γ, the definition of ξ_γ^- and the asymptotic behavior of $N_\gamma(\xi_\gamma^+)$ (see Lemma 2.12).

Eventually, the lower and bounds on N_γ in the free zone follow from (38) and (35) respectively.

The convergence properties for N_γ, P_γ at the end of Theorem 1.1 are a consequence of Lemma 2.6 and Proposition 2.4.

3 Stability of the profiles N_γ

The goal of this section is to prove that the solution of the equation

$$\partial_t n_\gamma - \gamma \partial_x \left(n_\gamma^\gamma \partial_x n_\gamma\right) = n_\gamma \left(1 - n_\gamma^\gamma\right) \tag{43}$$

associated to an initial datum that lies between two shifts of the profile N_γ, converges (in a sense specified below) towards N_γ as $t \to +\infty$. Let us recall that according to the work of de Pablo and Vázquez [6] (see also [20, 27]), equation (43) associated with such an initial datum has a unique global generalized solution. Furthermore, it will follow from the comparison principle that this solution remains bounded from below by a shift of N_γ. In particular, n_γ remains strictly positive everywhere, and the solution is in fact a classical solution.

After a presentation of the general strategy, we discuss in depth the two main steps of the proof: the analysis of the linearized system and, next, the control of the nonlinear contributions. To keep the presentation as seamless as possible, we have postponed the proof of some technical lemmas to the next section.

This section contains rather technical ingredients. Therefore, in order to alleviate the notation as much as possible, *we will systematically drop the dependency with respect to* γ in the computations and proofs: N_γ will be denoted by N, n_γ will be denoted by n, etc. We only keep track of this dependency in the statement of our main result.

In the whole section, for all weights and coefficients $f(t, x)$ that only depend on $\xi = x - ct$, we denote $f(t, x) = \bar{f}(x - ct)$.

3.1 Overall strategy

We define here our notion of stability and convergence towards the profile N_γ. We introduce a weight

$$\overline{w}_0(\xi) = K N_\gamma^\gamma(\xi) (N_\gamma'(\xi))^2 \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N_\gamma(z)} dz\right), \tag{44}$$

with a normalization constant K chosen so that $\overline{w}_0(\xi_\gamma^-) = 1$. The definition of $\overline{w}_0$ is dictated by a certain ODE (a kind of dual problem) that the weight should satisfy, see (51) and Section 4.1, which leads to formula (44).

We will prove that for sufficiently small and decaying initial data,

$$\int_{\mathbb{R}} \left| \frac{n_\gamma(t, x) - N_\gamma(x - ct)}{N_\gamma'(x - ct)} \right|^2 \overline{w}_0(x - ct)\, dx \to 0 \qquad \text{as} \quad t \to +\infty. \tag{45}$$

The result is summarized in the following theorem.

Theorem 3.1. *There exists $\eta_1, \eta_2 \in]0,1[$ such that the following result holds. Let $\gamma > 1$ be fixed, sufficiently large. Let us assume that n_γ^0 lies between two shifts of N_γ, i.e. there exists $h > 0$ such that $n_\gamma^0(x) \in [N_\gamma(x+h), N_\gamma(x-h)]$ for all $x \in \mathbb{R}$. Let n_γ be the solution of (1) associated with n_γ^0 and*

$$u_\gamma(t,x) := \frac{n_\gamma(t,x) - N_\gamma(x-ct)}{N_\gamma'(x-ct)}.$$

Assume that

$$\int_{\mathbb{R}} |u_\gamma(0,x)|^2 \overline{w}_0(x)\, dx < +\infty.$$

If $h \leq \eta_2^\gamma$, the following inequalities hold

$$\begin{aligned}\int_{\mathbb{R}} |u_\gamma(t,x)|^2 \overline{w}_0(x-ct)\ dx &\leq e^{-\eta_1^\gamma t} \int_{\mathbb{R}} |u_\gamma(0,x)|^2 \overline{w}_0(x)\ dx \qquad \forall t \geq 0,\\ \gamma \int_0^\infty \int_{\mathbb{R}} |\partial_x u_\gamma(t,x)|^2 N_\gamma^\gamma(x-ct) \overline{w}_0(x-ct)\, dx\, dt &\leq \int_{\mathbb{R}} |u_\gamma(0,x)|^2 \overline{w}_0(x)\ dx.\end{aligned} \tag{46}$$

Note that this statement is merely a rephrasing of Theorem 1.5 in terms of the unknown u_γ. We emphasize that u_γ is a natural variable when linearizing equation (43) around $N_\gamma(x-ct)$. Indeed, since equation (43) has constant coefficients and $N_\gamma(x-ct)$ is a particular solution of the equation, it is classical that $\partial_x N_\gamma(x-ct)$ is a solution of the linearized equation around $N_\gamma(x-ct)$ (and we also recall that $\partial_x N_\gamma$ does not vanish on $\mathbb{R}$). Moreover, $n_\gamma(t,x) - N_\gamma(x-ct)$ is also a solution of the linearized equation, up to a quadratic remainder which we will treat perturbatively. Therefore working with energies depending on u_γ is similar to deriving relative entropies for the system.

The result relies on two main estimates: a L^∞ control on $n - N$ (almost immediate, see below Lemma 3.9) and a more complicated L^2 weighted estimate on the variable u. Indeed, an easy computation (see subsection 4.1) shows that u satisfies the equation

$$\partial_t u + b\partial_x u - a\partial_x^2 u = \frac{\gamma}{\gamma+1} \frac{\partial_x^2 G(u)}{N'(x-ct)} - \frac{G(u)}{N'(x-ct)}, \tag{47}$$

with $a(t,x) = \bar{a}(x-ct)$, $b(t,x) = \bar{b}(x-ct)$ and

$$\bar{a} := \gamma N^\gamma, \quad \bar{b} := -2\gamma \frac{\left(N^\gamma N'\right)'}{N'} = -2\gamma^2 N^{\gamma-1} N' - 2\gamma N^\gamma \frac{N''}{N'},$$

and

$$\begin{aligned}G(u) &:= n^{\gamma+1} - N^{\gamma+1}(x-ct) - (\gamma+1) N^\gamma(x-ct)(n - N(x-ct))\\ &= \left(N(x-ct) + uN'(x-ct)\right)^{\gamma+1} - N^{\gamma+1}(x-ct) - (\gamma+1)N^\gamma(x-ct) uN'(x-ct).\end{aligned} \tag{48}$$

Let us make a few remarks before exposing the main ingredients of the proof. First, we emphasize that all unknowns and coefficients depend on γ (i.e. b, a, u, G, N). As mentioned above, we chose not to make this dependency explicit in our notation. Second, equation (47) has a structure of the type

$$\partial_t u + \mathcal{L} u = \mathcal{G}[u],$$

where $\mathcal{L}$ is a linear operator, corresponding to the linearization of equation (43) around $n = N$, and $\mathcal{G}[u]$ is a quadratic operator in the sense of (50) below.

Quite classically, the core of our proof relies on the two following observations:

- **The linear operator $\mathcal{L}$ is coercive in some weighted H^1 space.** More precisely, there exists a weight $\overline{w}$ and a constant $\delta_\gamma > 0$ with the following property: for any $v \in \mathcal{C}_c^2(\mathbb{R})$,

$$\int_{\mathbb{R}} (\bar{b}\partial_\xi v - \bar{a}\partial_\xi^2 v) v \overline{w} \geq \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a} \overline{w} + \frac{\delta_\gamma}{2} \int_{\mathbb{R}} |v|^2 e^{\sqrt{\gamma}\xi} - \frac{c}{2} \int_{\mathbb{R}} |v|^2 \partial_\xi \overline{w}. \tag{49}$$

Note that the last term will enter the time derivative of the energy $\int |u|^2 w$ when we perform energy estimates.

This type of coercivity property had been identified by Leyva and Plaza in [17], without the L^2 term $\int_{\mathbb{R}} |v|^2 e^{\sqrt{\gamma}\xi}$, which will play a crucial role in the energy estimates.

- **The nonlinear term $\mathcal{G}[u]$ is quadratic.** More precisely, for all $u \in H^1_{\rm loc}(\mathbb{R})$,

$$|\mathcal{G}[u]| \leq C_\gamma |u|(|u| + |\partial_x u|). \tag{50}$$

Hence, if $\|u\|_{L^\infty}$ is small enough, we can hope to absorb this term in the energy dissipation provided by the coercivity of $\mathcal{L}$.

The remainder of the section is devoted to a more rigorous statement and to the proofs of the above heuristic arguments. Concerning the smallness of the L^∞ bound, a possible strategy could be to differentiate equation (47) with respect to x and to derive uniform, high regularity bounds on u. This strategy is likely to succeed. However, it will probably come at a high technical cost. Consequently, to simplify the proof and the presentation, we chose here to take advantage of the parabolic structure of this scalar equation and use the comparison principle (or maximum principle), which immediately implies an L^∞ bound on n and u.

Remark 3.2. Let us mention by anticipation that the constant δ_γ in (49) will be small, while the constant C_γ in (50) will be very large. Whence we will need $\|u\|_{L^\infty}$ to be very small (in fact, exponentially small) to treat the quadratic term as a perturbation. This is related to the strong singularities in N'_γ which were highlighted in the previous section (recall that $\|N'_\gamma\|_{L^\infty}$ blows up exponentially, see Lemmas 2.6 and 2.8).

Let us now present the main ideas of the proof.

Structure of the linearized system - weighted L^2 estimate

We start from a reference weight $\overline{w}_0$, which is defined as the solution of the differential equation

$$\begin{cases} (\bar{a}\overline{w}_0)'(\xi) + \big(\bar{b}(\xi) - c\big)\overline{w}_0 = 0 \quad \text{for } \xi \in \mathbb{R}, \\ \overline{w}_0(\xi_0) = 1 \quad \text{for some } \xi_0 \in \mathbb{R}. \end{cases} \tag{51}$$

Below, we will take $\xi_0 = \xi_\gamma^-$, where ξ_γ^- is defined in Definition 2.10. This weight is identical to that of Leyva and Plaza in [17, Section 3.1], although our derivation differs from theirs, see subsections 4.1 and 4.3. For this weight $\overline{w}_0$, we have the following

Lemma 3.3 (Stability estimates for the linearized system)**.** *Let u be a smooth solution to*

$$\partial_t u + b\partial_x u - a\partial_x^2 u = S, \tag{52}$$

where S is a general source term. The following equality holds, with $w_0(t,x) = \overline{w}_0(x - ct)$

$$\int_{\mathbb{R}} |u(t,x)|^2 w_0(t,x)\ dx + 2\int_0^t \int_{\mathbb{R}} a(s,x)(\partial_x u(s,x))^2 w_0(s,x)\ dx\ ds \\ = \int_{\mathbb{R}} |u^0(x)|^2 \overline{w}_0(x)\ dx + 2\int_0^t \int_{\mathbb{R}} S(s,x)u(s,x)w_0(s,x)\ dxds. \tag{53}$$

Furthermore the weight $\overline{w}_0$ fulfills the following properties:

Lemma 3.4 (Asymptotic behaviors of $\overline{w}_0$)**.** *The solution of (51) with $\xi_0 = \xi_\gamma^-$ is given by*

$$\overline{w}_0(\xi) = KN^\gamma(\xi)(N'(\xi))^2 \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\bar{a}(z)} dz\right), \tag{54}$$

where the normalization constant K is chosen so that $\overline{w}_0(\xi_\gamma^-) = 1$. We find that $K \propto \gamma^{3/2}$. Consequently $\overline{w}_0$ has the following asymptotic behaviors:

- *as $\xi \to +\infty$, $\overline{w}_0$ has a double exponential growth: there exist $C_1, C_2, C > 0$ independent of γ such that for all $\xi \geq C$,*

$$\exp\left(\exp\left(C_1\gamma\xi\right)\right) \leq \overline{w}_0(\xi) \leq \exp\left(\exp\left(C_2\gamma\xi\right)\right); \tag{55}$$

- *as $\xi \to -\infty$, $\overline{w}_0$ decreases exponentially to 0: there exists $C > 0$ independent of γ such that for all $\xi \leq -C$,*

$$C^{-1}\frac{K}{\gamma^2}\exp\left(2\left(1+\frac{C}{\sqrt{\gamma}}\right)\xi\right) \leq \overline{w}_0(\xi) \leq C\frac{K}{\gamma^2}\exp\left(2\left(1-\frac{C}{\sqrt{\gamma}}\right)\xi\right). \tag{56}$$

Lemmas 3.3 and 3.4 will be proved in subsections 4.2 and 4.3 respectively.

Spectral gap and Poincaré inequality

However, the sole weight w_0 is not entirely sufficient to have an exponential decay in time of the energy $\int_{\mathbb{R}} |u|^2 w_0$. Indeed, in order to prove such an exponential decay, we need a Poincaré inequality of the type

$$\int_{\mathbb{R}} |v|^2 \overline{w}_0 \leq C_\gamma \int_{\mathbb{R}} (\partial_x v)^2 \bar{a}\overline{w}_0, \qquad \forall\, v \in \mathcal{C}^1_c(\mathbb{R}).$$

In other words, we need to prove a spectral gap inequality. To the best of our knowledge, such an inequality does not hold for the weight $\overline{w}_0$. However, we are able to prove a variant of such an inequality, with an additional L^2 term in the right-hand side:

Proposition 3.5 (Weighted Poincaré-type inequality)**.** *There exists a constant $\bar{C}$ independent of γ and a constant $C_\gamma \leq C^\gamma$ such that, for any $v \in \mathcal{C}^1_c(\mathbb{R})$,*

$$\int_{-\infty}^{\xi_\gamma^-} v^2 \gamma N_\gamma^\gamma \overline{w}_0 d\xi + \int_{\xi_\gamma^+}^{+\infty} v^2 \frac{1}{\gamma N_\gamma^\gamma}\overline{w}_0 d\xi \leq \bar{C}\int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a}\overline{w}_0\, d\xi + C_\gamma \int_{\mathbb{R}} v^2 e^{\sqrt{\gamma}\xi}\, d\xi. \tag{57}$$

In particular, there exists $\eta_1 \in\,]0,1[$ independent of γ, such that

$$\eta_1^\gamma \int_{\mathbb{R}} v^2 \overline{w}_0 d\xi \leq \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a}\overline{w}_0\, d\xi + \int_{\mathbb{R}} |v|^2 \exp(\sqrt{\gamma}\xi) d\xi.$$

Proposition 3.5 is proved in Section 4.4.

Remark 3.6.

- We recall that we defined ξ_γ^- so that $P_\gamma(\xi_\gamma^-) = N_\gamma(\xi_\gamma^-)^\gamma = \left(\frac{c^3}{(c-1)(\gamma+1)}\right)^{1/2}$. Consequently, in the first integral of (57), the term γN_γ^γ is bounded from below by $C\sqrt{\gamma}$.
- In a similar way, for $\xi > \xi_\gamma^+$, we have $N_\gamma \leq 1 - (2c)^{-1}$, so that the term $\frac{1}{\gamma N_\gamma^\gamma}$ in the second integral in the left-hand side of (57) is exponentially large.
- We stated this result for $v \in \mathcal{C}^1_c(\mathbb{R})$, but the result can be extended to v in suitable weighted Sobolev spaces by a classical density argument.
- Let us give a few motivations for the weight $e^{\sqrt{\gamma}\xi}$ in the right-hand side of (57). We actually have some freedom in the choice of the coefficient of the exponential that we take equal to $\alpha_\gamma = \sqrt{\gamma}$. We could a priori take a larger coefficient α_γ with respect to γ. However α_γ must satisfy a number of conditions. First, an important feature is that the growth (resp. decay) of this weight as $\xi \to +\infty$ (resp. $\xi \to -\infty$) is lower (resp. stronger) than the one of $\overline{w}_0$. By doing so, the energy estimate (53) with weight w_0 will not be perturbed when w_0 is replaced by $w = w_0\phi$ with ϕ defined below in (58).
  Moreover, the energy dissipation provides a very good control of the energy in the two zones $\xi < \xi_\gamma^-$ and $\xi > \xi_\gamma^+$, as we can see in inequality (57). The additional term $\int_{\mathbb{R}} v^2 e^{\sqrt{\gamma}\xi}$ is

only needed in the transition zone $(\xi_\gamma^-, \xi_\gamma^+)$, as we shall see in the course of the proof. Our choice $\alpha_\gamma = \sqrt{\gamma}$ is actually motivated by the need to control, uniformly with respect to γ, the exponential $\exp(\alpha_\gamma \xi_\gamma^-)$ (see in particular (80)). Since $\xi_\gamma^- = O(\gamma^{-1/2})$, it leads us to set $\alpha_\gamma = \sqrt{\gamma}$.

- The proof of Proposition 3.5 relies on the quantitative estimations of Lemma 2.8, and will be performed in subsection 4.4.

As a consequence, if we are able to have an additional lower-order dissipation term in the energy estimate (the term $\int |v|^2 e^{\sqrt{\gamma}\xi}$ in the right-hand side), the exponential decay of the energy for the linearized system will follow. In order to get this extra dissipation, it is sufficient to modulate slightly the weight $\overline{w}_0$. More precisely, we define $\overline{w} = \overline{w}_0 \bar{\phi}$ where

$$\begin{aligned} \bar{\phi}(-\infty) &= 2, \\ \bar{\phi}'(\xi) &= -\frac{\delta_\gamma}{\sqrt{\gamma}} \exp\left(\sqrt{\gamma}\xi\right)\left(\bar{a}(\xi)\overline{w}_0(\xi)\right)^{-1}, \end{aligned} \tag{58}$$

and the constant $\delta_\gamma > 0$ is chosen such that $\bar{\phi}(+\infty) \geq 1$.

Lemma 3.4 ensures that $\bar{\phi}' \in L^1(\mathbb{R})$, and therefore $\bar{\phi}$ is well-defined and monotonous. Note that since $1 \leq \bar{\phi} \leq 2$ by construction, the weights $\overline{w}_0$ and $\overline{w}$ are equivalent. However, choosing $\overline{w}$ gives us the following additional control:

Lemma 3.7. *Under the same assumptions and notation as in Lemma 3.3, we have*

$$\begin{aligned} \int_{\mathbb{R}} |u(t,x)|^2 w(t,x)\ dx + 2\int_0^t \int_{\mathbb{R}} (\partial_x u)^2 a w\ dxds + \delta_\gamma \int_0^t \int_{\mathbb{R}} |u(s,x)|^2 \exp\left(\sqrt{\gamma}(x-cs)\right) dxds \\ = \int_{\mathbb{R}} |u^0(x)|^2 \overline{w}(x)\ dx + 2\int_0^t \int_{\mathbb{R}} S(s,x)u(s,x)w(s,x)\ dxds, \end{aligned} \tag{59}$$

with

$$\delta_\gamma \geq \eta_3^\gamma, \tag{60}$$

for some constant $\eta_3 \in]0,1[$ independent of γ.

Lemma 3.7 is proved in Section 4.2.

Definition 3.8. In the rest of the paper, we set

$$\mathcal{D}_\gamma(t) := 2\int_{\mathbb{R}} (\partial_x u(t,x))^2 a(t,x) w(t,x) dx + \delta_\gamma \int_{\mathbb{R}} |u(t,x)|^2 \exp(\sqrt{\gamma}(x-ct)) dx,$$

which is the total dissipation term.

Note that Proposition 3.5 allows us to control the energy by the total dissipation term $\mathcal{D}_\gamma$, up to an exponentially small multiplicative constant. Gathering Proposition 3.5 and Lemma 3.7, we see that any solution of the linearized equation (52) with $S = 0$, with an initial datum such that $\int_{\mathbb{R}} |u_0|^2 \overline{w}_0 < \infty$, decays exponentially (at a rate $(\eta_1\eta_3)^\gamma$, which we rename as η_1^γ) as $t \to +\infty$.

L^∞ estimate

In order to prove that the dynamics of the nonlinear equation (47) are driven by the linearized part of the equation, and that the nonlinear term in the right-hand side of (47) can be treated perturbatively, we will need a last ingredient, which is a direct consequence of the comparison principle:

Lemma 3.9 (L^∞ estimate)**.** *Let $h > 0$ be small enough and assume that n^0 lies between two shifts of the reference profile N:*

$$N(x+h) \leq n^0(x) \leq N(x-h).$$

Then, for all $t \geq 0$, for all $x \in \mathbb{R}$,

$$N(x+h-ct) \leq n(t,x) \leq N(x-h-ct).$$

From Lemma 2.8, we have

$$\|u\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} + \left\| \frac{n - N(x-ct)}{N(x-ct)} \right\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \leq C^\gamma h, \tag{61}$$

where C is a positive constant independent of γ.

Equipped with this estimate and the control in L^∞ from Lemma 3.9, we can control the nonlinear contributions and deduce an exponential decay of the L^2 weighted norm as $t \to +\infty$, as stated in Theorem 3.1. The next subsection is devoted to the control of the nonlinear terms. We then give a proof of Theorem 1.5 at the end of section 3.

3.2 Control of the nonlinear terms and long-time behavior

We now address the proof of Theorem 3.1 using the tools described above. Let u be a smooth solution to (47), we get by applying (59):

$$\begin{aligned}
&\int_{\mathbb{R}} |u|^2 w_0 \phi \; dx + 2\int_0^t \int_{\mathbb{R}} a(\partial_x u)^2 w_0 \phi \; dxds + \int_0^t \int_{\mathbb{R}} |u|^2 \delta_\gamma \exp(\sqrt{\gamma}(x-cs)) \; dxds \\
&= \int_{\mathbb{R}} |u_0| \overline{w} \; dx + \frac{2\gamma}{\gamma+1} \int_0^t \int_{\mathbb{R}} \frac{\partial_x^2 G(u)}{\partial_x N} u w_0 \phi \; dxds - 2\int_0^t \int_{\mathbb{R}} \frac{G(u)}{\partial_x N} u w_0 \phi \; dxds. \qquad (62)
\end{aligned}$$

Observe that the first term of the right-hand side comes from the nonlinear diffusion while the second comes from the reaction term. We also recall that

$$G(u) = (N + u\partial_x N)^{\gamma+1} - N^{\gamma+1} - (\gamma+1)N^\gamma u \partial_x N.$$

First, let us estimate $G(u)$.

Lemma 3.10. *Assume that*

$$\left\| u \frac{\partial_x N}{N} \right\|_\infty = \left\| \frac{n(t,x) - N(x-ct)}{N(x-ct)} \right\|_\infty \leq \frac{1}{\gamma},$$

where $N, \partial_x N$ are evaluated at $x - ct$.
Then

$$\begin{aligned}
|G(u)| &\leq C\gamma^2 (u\partial_x N)^2 N^{\gamma-1}, && (63)\\
|\partial_x G(u)| &\leq C\gamma^3 |\partial_x N| \; N^{\gamma-2} (u\partial_x N)^2 + C\gamma^2 N^{\gamma-1} |u\partial_x N| \; |\partial_x (u \partial_x N)| && (64)\\
&\leq C\gamma^3 |\partial_x N| \; N^{\gamma-2} (u\partial_x N)^2 + C\gamma^2 N^{\gamma-1} (\partial_x N)^2 |u| \; |\partial_x u| \\
&\quad + C\gamma u^2 |\partial_x N|^2 N^{-1} + C\gamma u^2 |\partial_x N|,
\end{aligned}$$

for some constant $C > 0$ independent of γ.

Proof. The first estimate can be easily proved by writing

$$G(u) = N^{\gamma+1} g\left(\frac{u\partial_x N}{N}\right),$$

where $g(X) = (1+X)^{\gamma+1} - 1 - (\gamma+1)X$. A Taylor expansion at order two close to $X = 0$ shows that if $|X| \leq \gamma^{-1}$,

$$|g(X)| \leq \frac{1}{2}\gamma(\gamma+1)\left(1 + \frac{1}{\gamma}\right)^{\gamma-1} |X|^2 \lesssim \gamma^2 |X|^2.$$

Estimate (63) follows. We also know by convexity that $G(u) \geq 0$.

For the second estimate, we differentiate G and get

$$\partial_x G(u) = (\gamma+1)\partial_x N\Big((N + u\partial_x N)^\gamma - N^\gamma - \gamma N^{\gamma-1} u\partial_x N\Big) \\ + (\gamma+1)\partial_x(u\partial_x N)\Big(\big(N + u\partial_x N\big)^\gamma - N^\gamma\Big).$$

Reasoning as before, we infer that

$$|\partial_x G(u)| \leq C(\gamma+1)\gamma^2|\partial_x N||u\partial_x N|^2 N^{\gamma-2} + C(\gamma+1)\gamma|\partial_x(u\partial_x N)||u\partial_x N|N^{\gamma-1}.$$

To obtain the last set of inequalities, we use Equation (4) on N, and we recall that

$$\gamma\partial_x^2 N N^\gamma = -c\partial_x N - \gamma^2(\partial_x N)^2 N^{\gamma-1} - N(1 - N^\gamma),$$

which concludes the proof of the lemma. □

Lemma 3.11 (Control of the nonlinear reaction term)**.** *There exists a constant* $\eta_2 \in]0,1[$ *such that if*

$$\left\|\frac{u\partial_x N}{N}\right\|_\infty \leq \eta_2^\gamma,$$

then the following inequality holds

$$\left|\int_{\mathbb{R}} \frac{G(u)}{\partial_x N} u w_0 \phi \; dx\right| \leq \frac{1}{4}\mathcal{D}_\gamma. \tag{65}$$

Proof. Using Lemma 3.10, we have

$$\left|\int_{\mathbb{R}} \frac{G(u)}{\partial_x N} u w_0 \phi \; dx\right| \leq C\gamma^2 \int_{\mathbb{R}} |u|^3 |\partial_x N| N^{\gamma-1} w_0 \phi \; dx,$$

that we want to absorb in the left-hand side of the equality (62) thanks to the diffusion and damping terms:

$$\mathcal{D}_\gamma = 2\int_{\mathbb{R}} a(\partial_x u)^2 w_0 \phi \; dx + \int_{\mathbb{R}} \delta_\gamma \exp(\sqrt{\gamma}(x - ct)) u^2 \; dx,$$

with $\delta_\gamma \geq \eta_3^\gamma$, $\eta_3 \in \,]0,1[$. Recalling Proposition 3.5, we observe that it suffices to have

$$C\gamma^2 \left\|\frac{u\partial_x N}{N}\right\|_\infty \leq \frac{(\eta_1\eta_3)^\gamma}{4},$$

which concludes the proof, choosing $\eta_2 < \eta_1\eta_3$.

□

Lemma 3.12 (Control of the nonlinear diffusion term)**.** *There exists a constant* $\eta_2 \in]0,1[$ *such that if*

$$\|u\|_\infty + \left\|\frac{u\partial_x N}{N}\right\|_\infty \leq \eta_2^\gamma,$$

then the following inequality holds

$$\left|\int_{\mathbb{R}} \frac{\partial_x^2(G(u))}{\partial_x N} u w_0 \phi \, dx\right| \leq \frac{1}{4}\mathcal{D}_\gamma. \tag{66}$$

Proof. Integrating by parts the nonlinear term stemming from the diffusion, we have

$$\int_{\mathbb{R}} \frac{\partial_x^2(G(u))}{\partial_x N} u w_0 \phi \, dx = -\int_{\mathbb{R}} \partial_x G(u) \partial_x u \left(\frac{w_0\phi}{\partial_x N}\right) - \int_{\mathbb{R}} \partial_x G(u) u \partial_x \left(\frac{w_0\phi}{\partial_x N}\right). \tag{67}$$

• We first address the first term in the right-hand side of (67), using the estimate on $\partial_x G(u)$ from Lemma 3.10. It follows that

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x G(u)\partial_x u \left(\frac{w_0\phi}{\partial_x N}\right)\right| \quad &\leq \quad C\int_{\mathbb{R}} \gamma^3 N^{\gamma-2}|u|^2|\partial_x u|(\partial_x N)^2 w_0\phi \\ &\quad + C\int_{\mathbb{R}} \gamma^2 |u||\partial_x u|^2 N^{\gamma-1}|\partial_x N| w_0\phi \\ &\quad + C\int_{\mathbb{R}} \gamma |u|^2|\partial_x u| \frac{|\partial_x N|}{N} w_0\phi \\ &\quad + C\int_{\mathbb{R}} \gamma |u|^2 |\partial_x u| w_0\phi \\ &= \quad \sum_{i=1}^4 I_i.\end{aligned}$$

We then address each term I_i separately. We start with the term I_2. Recalling that $\bar{a} = \gamma N^\gamma$, we simply write

$$I_2 \leq C\gamma \left\| u\frac{\partial_x N}{N}\right\|_{L^\infty} \int_{\mathbb{R}} (\partial_x u)^2 a w_0\phi,$$

which is smaller than $\mathcal{D}_\gamma/16$, provided $\|u\partial_x N/N\|_\infty \leq (16C\gamma)^{-1}$.

For all other terms, we first perform a Cauchy-Schwarz inequality. We have,

$$I_1 \leq C\left(\int_{\mathbb{R}} (\partial_x u)^2 a w\right)^{1/2} \left(\gamma^5 \int_{\mathbb{R}} |u|^4 N^{\gamma-4}(\partial_x N)^4 w_0\phi\right)^{1/2},$$

$$I_3 \leq C\left(\int_{\mathbb{R}} (\partial_x u)^2 a w\right)^{1/2} \left(\int_{\mathbb{R}} \gamma |u|^4 \frac{(\partial_x N)^2}{N^{\gamma+2}} w_0\phi\right)^{1/2},$$

$$I_4 \leq C\left(\int_{\mathbb{R}} (\partial_x u)^2 a w\right)^{1/2} \left(\int_{\mathbb{R}} \gamma |u|^4 N^{-\gamma} w_0\phi\right)^{1/2}.$$

We then bound each integral with $|u|^4$ in the right-hand side by using the Poincaré inequality from Proposition 3.5 and the L^∞ estimate on u. The simplest term is I_1, for which we have

$$\begin{aligned} I_1 &\leq C\gamma^{5/2}\|u\|_{L^\infty} \left\|\frac{\partial_x N}{N}\right\|_{L^\infty}^2 \left(\int_{\mathbb{R}} (\partial_x u)^2 a w\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 w_0\phi\right)^{1/2} \\ &\leq C\gamma^{5/2}\|u\|_{L^\infty} \left\|\frac{\partial_x N}{N}\right\|_{L^\infty}^2 (\eta_1\eta_3)^{-\gamma/2}\mathcal{D}_\gamma. \end{aligned}$$

Concerning the term I_3, we have

$$\int_{\mathbb{R}} \gamma |u|^4 \frac{(\partial_x N)^2}{N^{\gamma+2}} w_0\phi \leq \gamma \left\| u\frac{\partial_x N}{N}\right\|_{L^\infty}^2 \int_{\mathbb{R}} |u|^2 \frac{w_0\phi}{N^\gamma}.$$

Using Proposition 3.5, we have

$$\begin{aligned}\int_{\mathbb{R}} |u|^2 \frac{w_0\phi}{N^\gamma} \quad &= \quad \int_{\xi>\xi_\gamma^+} |u|^2 \frac{w_0\phi}{N^\gamma} + \int_{\xi<\xi_\gamma^+} |u|^2 \frac{w_0\phi}{N^\gamma} \\ &\leq \quad \bar{C}\gamma \int_{\mathbb{R}} (\partial_x u)^2 a w_0\phi + \gamma C_\gamma \int_{\mathbb{R}} u^2 e^{\sqrt{\gamma}(x-ct)} dx \\ &\quad + \frac{1}{N(\xi_\gamma^+)^\gamma} \int_{\xi<\xi_\gamma^+} |u|^2 w_0\phi \\ &\leq \quad C'_\gamma \mathcal{D}_\gamma,\end{aligned}$$

for some exponentially large constant C'_γ. The above inequality also allows us to bound I_4. Thus, provided

$$\begin{aligned}\|u\|_{L^\infty}\left\|\frac{\partial_x N}{N}\right\|^2_{L^\infty} &\leq \delta_0(\eta_1\eta_3)^{\gamma/2}\gamma^{-5/2},\\ \|u\|_{L^\infty} &\leq \delta_0(C'_\gamma)^{-1/2}\gamma^{-1/2},\\ \left\|u\frac{\partial_x N}{N}\right\|_{L^\infty} &\leq \delta_0 \inf\left((C'_\gamma)^{-1/2}\gamma^{-1/2}, \frac{1}{\gamma}\right),\end{aligned}$$

for some small constant δ_0 independent of γ, we infer that

$$\left|\int_{\mathbb{R}} \partial_x G(u)\partial_x u \left(\frac{w_0\phi}{\partial_x N}\right)\right| \leq \frac{1}{8}\mathcal{D}_\gamma.$$

• Let us now consider the second term in the right-hand side of (67). Computing the weight in the right-hand side and using the definitions of ϕ (58), w_0 (54) and the equation satisfied by N, we find

$$\begin{aligned}\partial_\xi\left(\frac{\overline{w}_0\bar\phi}{\partial_\xi N}\right) &= \partial_\xi\bar\phi\frac{\overline{w}_0}{\partial_\xi N} + \bar\phi\partial_\xi\left(\frac{\overline{w}_0}{\partial_\xi N}\right)\\ &= \partial_\xi\bar\phi\frac{\overline{w}_0}{\partial_\xi N} + K\bar\phi\partial_\xi(\partial_\xi N N^\gamma)\exp\left(\int_{\xi_\gamma^-}^{\xi}\frac{c}{\bar a}\right)\\ &\quad + K\bar\phi\partial_\xi N N^\gamma\frac{c}{\bar a}\exp\left(\int_{\xi_\gamma^-}^{\xi}\frac{c}{\bar a}\right)\\ &= -\frac{\delta_\gamma}{\sqrt{\gamma}}\frac{\exp(\sqrt{\gamma}\xi)}{\gamma N^\gamma\partial_\xi N} - \frac{K}{\gamma}\bar\phi N(1-N^\gamma)\exp\left(\int_{\xi_\gamma^-}^{\xi}\frac{c}{\bar a}\right)\\ &=: W_1 + W_2.\end{aligned}$$

We then use the estimate on $\partial_x G(u)$ from Lemma 3.10, treating W_1 and W_2 separately. We have, concerning the terms with W_1,

$$\begin{aligned}\left|\int_{\mathbb{R}}\partial_x G(u)uW_1\right| &\leq C\frac{\delta_\gamma}{\sqrt{\gamma}}\left[\int_{\mathbb{R}}\gamma^2|u|^3\frac{(\partial_x N)^2}{N^2}e^{\sqrt{\gamma}\xi} + \int_{\mathbb{R}}\gamma\frac{|\partial_x N|}{N}|u|^2|\partial_x u|e^{\sqrt{\gamma}\xi}\right]\\ &\quad + C\frac{\delta_\gamma}{\sqrt{\gamma}}\int_{\mathbb{R}}|u|^3N^{-\gamma}\left(1+\frac{|\partial_x N|}{N}\right)e^{\sqrt{\gamma}\xi}.\end{aligned}$$

Using a Cauchy-Schwarz inequality for the second integral, we get

$$\begin{aligned}\left|\int_{\mathbb{R}}\partial_x G(u)uW_1\right| &\leq C\delta_\gamma\gamma^{3/2}\left\|u\frac{\partial_x N}{N}\right\|_{L^\infty}\left\|\frac{\partial_x N}{N}\right\|_{L^\infty}\int_{\mathbb{R}}u^2e^{\sqrt{\gamma}\xi}\\ &\quad + C\delta_\gamma\left(\int_{\mathbb{R}}(\partial_x u)^2aw\right)^{1/2}\left\|u\frac{\partial_x N}{N}\right\|_{L^\infty}\left(\int_{\mathbb{R}}\frac{u^2}{N^\gamma w_0}e^{2\sqrt{\gamma}\xi}\right)^{1/2}\\ &\quad + C\frac{\delta_\gamma}{\sqrt{\gamma}}\left(\|u\|_{L^\infty} + \left\|u\frac{\partial_x N}{N}\right\|_{L^\infty}\right)\int_{\mathbb{R}}\frac{u^2}{N^\gamma}e^{\sqrt{\gamma}\xi}.\end{aligned}$$

Thanks to the growth and decay properties of the weight w_0 at $\pm\infty$, we claim that there exists a constant $C > 1$ such that

$$e^{\sqrt{\gamma}\xi} \leq C^\gamma\overline{w}(\xi), \qquad \forall\xi\in\mathbb{R}. \tag{68}$$

We postpone the proof of this inequality to the end of Section 4.3.

It follows that

$$\int_{\mathbb{R}}u^2e^{\sqrt{\gamma}\xi}\,dx \leq C^\gamma\int_{\mathbb{R}}u^2w \leq C^\gamma(\eta_1\eta_3)^{-\gamma}\mathcal{D}_\gamma.$$

Concerning the integral $\int_{\mathbb{R}} \frac{u^2}{N^\gamma} e^{\sqrt{\gamma}\xi}$, using Proposition 3.5 together with (68) gives

$$\begin{aligned}\int_{\mathbb{R}} \frac{u^2}{N^\gamma} e^{\sqrt{\gamma}\xi} &\leq C^\gamma \mathcal{D}_\gamma + \int_{x-ct<\xi_\gamma^+} \frac{u^2}{N^\gamma} e^{\sqrt{\gamma}\xi} \\ &\leq C^\gamma \mathcal{D}_\gamma + C^\gamma \int_{x-ct<\xi_\gamma^+} u^2 e^{\sqrt{\gamma}\xi} \\ &\leq C^\gamma \mathcal{D}_\gamma,\end{aligned}$$

with a constant $C > 1$ which changes from line to line. The last term is treated in the same fashion, noticing that $e^{2\sqrt{\gamma}\xi}/\overline{w}_0 \leq C^\gamma e^{\sqrt{\gamma}\xi}$. We obtain

$$\left| \int_{\mathbb{R}} \partial_x G(u) u W_1 \right| \leq C^\gamma \left(\left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} + \|u\|_{L^\infty} \right) \mathcal{D}_\gamma \leq \frac{1}{8} \mathcal{D}_\gamma,$$

provided $\|u\|_{L^\infty}$ and $\|u\partial_x N/N\|_{L^\infty}$ are small enough (exponentially small with γ).

There remains to address the terms containing W_2. We have, using Lemma 3.10 and recalling the expression of the weight $\overline{w}_0$ from Lemma 3.4,

$$\begin{aligned}&\left| \int_{\mathbb{R}} \partial_x G(u) u W_2 \right| \\ \leq\ & C\gamma^2 \int_{\mathbb{R}} \frac{|\partial_x N|}{N} |u|^3 w + C\gamma \int_{\mathbb{R}} u^2 |\partial_x u| w + \int_{\mathbb{R}} |u|^3 \frac{w}{N^\gamma} + \int_{\mathbb{R}} \frac{|u|^3 (1 - N^\gamma)}{|\partial_x N| N^{\gamma-1}} w.\end{aligned}$$

Using the same arguments as before, we find that the first three terms are bounded by

$$\left(\gamma^2 \left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} (\eta_1 \eta_3)^{-\gamma} + \gamma \|u\|_{L^\infty} C_\gamma + \gamma \|u\|_{L^\infty} (\eta_1 \eta_3)^{-\gamma} C^\gamma \right) \mathcal{D}_\gamma,$$

and can be absorbed in $\mathcal{D}_\gamma$ under the assumptions of the lemma, provided η_2 is sufficiently small.

There remains the last term, which has an additional singularity in the congested zone because of the $\partial_x N$ factor in the denominator (note that in the free zone $\xi > \xi_\gamma^+$, $|\partial_x N| \gtrsim N$, so that this singularity can be treated thanks to the weighted Poincaré inequality from Proposition 3.5.) However this singularity is compensated by the factor $(1 - N^\gamma)$ in the numerator. More precisely, we have, for $\xi \leq \xi_\gamma^+$ and using Lemmas 2.8 and 2.12

$$\left| \frac{1 - N^\gamma}{\partial_x N\, N^{\gamma-1}} \right| = \gamma \left| \frac{1 - P}{P'} \right| \lesssim \gamma^2.$$

Hence,

$$\int_{x-ct\leq\xi_\gamma^+} \frac{|u|^3 (1 - N^\gamma)}{|\partial_x N| N^{\gamma-1}} w \leq C\gamma^2 \|u\|_{L^\infty} \int |u|^2 w \leq C\gamma^2 \|u\|_{L^\infty} (\eta_1 \eta_3)^{-\gamma} \mathcal{D}_\gamma.$$

Gathering all the terms, we obtain the inequality announced in the Lemma.

□

3.3 Proof of Theorem 1.5

Let us now complete the proof of Theorem 1.5. First, we choose h so that $h \leq \eta_2/C$, where $\eta_2 \in]0, 1[$ is the constant appearing in Lemmas 3.11 and 3.12, and C is the constant in (61). Then Lemma 3.9 entails that

$$\|u\|_\infty + \left\| u \frac{\partial_x N}{N} \right\|_\infty \leq \eta_2^\gamma.$$

It follows from Lemmas 3.11 and 3.12 that the sum of the two nonlinear terms in the right-hand side of (62) is bounded by $\int_0^t \mathcal{D}_\gamma/2$. Therefore, we obtain for all $t \geq 0$,

$$\int_{\mathbb{R}} |u|^2 w + \frac{1}{2} \int_0^t \mathcal{D}_\gamma \leq \int_{\mathbb{R}} |u_0|^2 w.$$

Letting $t \to +\infty$, we obtain the control of the diffusion announced in Theorem 1.5. Now, applying the Poincaré inequality from Proposition 3.5 and Lemma 3.7, we have for all $t \geq 0$,

$$\int_{\mathbb{R}} |u|^2 w + \frac{\eta_3^\gamma \eta_1^\gamma}{2} \int_0^t \int_{\mathbb{R}} |u|^2 w \leq \int_{\mathbb{R}} |u_0|^2 w.$$

The exponential decay with a rate $(\eta_1\eta_3)^\gamma/2$ (which we rename η_1^γ) follows easily from the Grönwall Lemma.

4 Proofs of some technical results

This section is devoted to the proof of several results used in section 3: we start with the derivation of the equation satisfied by $u(t,x) = (n(t,x) - N(x-ct))/\partial_x N(x-ct)$, for which we analyze the structure of the linearized system. We therefore justify the introduction of the weights w and w_0. We then prove the growth and decay estimates on $\overline{w}$ and $\overline{w}_0$ announced in Lemma 3.4. Eventually, we prove the Poincaré inequality announced in Proposition 3.5.

4.1 Derivation of the equation on u

In this subsection, we prove that u defined by

$$u(t,x) = \frac{n(t,x) - N(x-ct)}{\partial_x N(x-ct)},$$

is a solution of (62). As in the previous section, we omit the dependency in γ for simplicity.

First, we recall that n and $N(x-ct)$ are both solutions of (43), in which we rewrite the diffusion term as

$$\gamma \partial_x (n^\gamma \partial_x n) = \frac{\gamma}{\gamma+1} \partial_{xx} n^{\gamma+1}.$$

Recalling the definition of $G(u)$ from (48), where $u(t,x) := (n(t,x) - N(x-ct))/\partial_x N(x-ct)$, we write (omitting $x - ct$ in the argument of N)

$$n^{\gamma+1} - N^{\gamma+1} = (\gamma+1) N^\gamma (n - N) + G(u).$$

Introducing $\nu(t,x) = n(t,x) - N(x-ct)$, we find that ν is a solution of

$$\partial_t \nu - \gamma \partial_x^2 (N^\gamma \nu) - \nu \left(1 - (\gamma+1) N^\gamma\right) = \frac{\gamma}{\gamma+1} \partial_x^2 (G(u)) - G(u). \tag{69}$$

Observe that, from (4), $\partial_x N(x-ct)$ is a (negative) solution of the linearized equation

$$\partial_t \partial_x N(x-ct) - \gamma \partial_x^2 (N^\gamma \partial_x N(x-ct)) - \left(1 - (\gamma+1) N^\gamma\right) \partial_x N(x-ct) = 0.$$

Let us compute the equation satisfied by $u = \nu/\partial_x N$. Using the identity

$$\begin{aligned} \partial_x^2 u &= \partial_x \left(\frac{\partial_x \nu}{\partial_x N} - \frac{\nu \partial_x^2 N}{(\partial_x N)^2} \right) \\ &= \frac{\partial_x^2 \nu}{\partial_x N} - \frac{\nu}{(\partial_x N)^2} \partial_x^3 N - 2 \frac{\partial_x^2 N}{\partial_x N} \partial_x u, \end{aligned}$$

we infer that

$$\partial_t u + b \partial_x u - a \partial_x^2 u = \frac{\gamma}{\gamma+1} \frac{\partial_x^2 G(u)}{\partial_x N} - \frac{G(u)}{\partial_x N}, \tag{70}$$

where $a(t,x) = \bar{a}(x-ct)$, $b(t,x) = \bar{b}(x-ct)$ and

$$\bar{b} := -2\gamma \partial_x N^\gamma - 2\gamma N^\gamma \frac{\partial_x^2 N}{\partial_x N}, \qquad \bar{a} := \gamma N^\gamma. \tag{71}$$

4.2 Structure of the linearized system: Lemmas 3.3 and equality (59)

Proof of Lemma 3.3. Multiplying (47) by $2uw_0$ and integrating on $\mathbb{R}$, we obtain formally

$$\frac{d}{dt}\int_{\mathbb{R}}|u|^2 w_0 - \int_{\mathbb{R}}|u|^2\partial_t w_0 + \int_{\mathbb{R}} 2u\partial_x u(bw_0 + \partial_x(aw_0)) + 2\int_{\mathbb{R}} aw_0(\partial_x u)^2 = 2\int_{\mathbb{R}} Suw_0.$$

Integrating by parts the middle term gives

$$\int_{\mathbb{R}} 2u\partial_x u(bw_0 + \partial_x(aw_0)) = -\int_{\mathbb{R}}|u|^2\partial_x(bw_0 + \partial_x(aw_0)).$$

Gathering all the terms yields

$$\frac{d}{dt}\int_{\mathbb{R}}|u|^2 w_0 - \int_{\mathbb{R}}|u|^2\left[\partial_t w_0 + \partial_x(bw_0 + \partial_x(aw_0))\right] + \int_{\mathbb{R}} aw_0(\partial_x u)^2 = 2\int_{\mathbb{R}} Suw_0.$$

Now, let us look at the term between brackets. As $w_0 = \overline{w}_0(x-ct)$,

$$\begin{aligned}
\left[\partial_t w_0 + \partial_x(bw_0 + \partial_x(aw_0))\right] &= \left[-c\overline{w}_0' + (\bar{b}\overline{w}_0 + (\bar{a}\overline{w}_0)')'\right](x-ct),\\
&= \partial_x\left((\bar{a}\overline{w}_0)' + (\bar{b}-c)\overline{w}_0\right)(x-ct),\\
&= 0,
\end{aligned}$$

from the definition of $\overline{w}_0$ given in (51). This implies

$$\frac{d}{dt}\int_{\mathbb{R}}|u|^2 w_0 + 2\int_{\mathbb{R}} aw_0(\partial_x u)^2 = 2\int_{\mathbb{R}} Suw_0,$$

and therefore, integrating with respect to t,

$$\int_{\mathbb{R}}|u(t,x)|^2 w_0(t,x)dx + 2\int_0^t\int_{\mathbb{R}} a(s,x)w_0(s,x)(\partial_x u(s,x))^2 dxds$$
$$= \int_{\mathbb{R}}|u^0(x)|^2 w_0(0,x)dx + 2\int_0^t\int_{\mathbb{R}} Suw_0.$$

Note that $w_0(0,x) = \overline{w}_0(x)$. Hence, we obtain the identity announced in the Lemma. □

Proof of equality (59). This proof is very similar to that of Lemma 3.3. Observe first that for $w = w_0\phi$, one has

$$\begin{aligned}
-[\partial_t w + \partial_x(bw + \partial_x(aw))](t,x) &= \left(c\overline{w}_0\bar{\phi} - \bar{b}\overline{w}_0\bar{\phi} - (\bar{a}\overline{w}_0\bar{\phi})'\right)'(x-ct),\\
&= -\left(\bar{a}\overline{w}_0\bar{\phi}'\right)'(x-ct).
\end{aligned}$$

Note that from the definition (58) of ϕ, $\bar{a}\overline{w}_0\bar{\phi}' = -\frac{\delta_\gamma}{\sqrt{\gamma}}\exp(\sqrt{\gamma}\xi)$. Hence,

$$-[\partial_t w + \partial_x(bw + \partial_x(aw))](t,x) = \delta_\gamma\exp(\sqrt{\gamma}(x-ct)).$$

Proceeding exactly as in the proof of Lemma 3.3, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}}|u|^2 w - \int_{\mathbb{R}}|u|^2\left[\partial_t w + \partial_x(bw + \partial_x(aw))\right] + 2\int_{\mathbb{R}} aw(\partial_x u)^2 = 2\int_{\mathbb{R}} Suw,$$
$$\frac{d}{dt}\int_{\mathbb{R}}|u|^2 w + \delta_\gamma\int_{\mathbb{R}}|u(t,x)|^2\exp(\sqrt{\gamma}(x-ct))dx + 2\int_{\mathbb{R}} aw(\partial_x u)^2 = 2\int_{\mathbb{R}} Suw,$$

and therefore, integrating again with respect to t gives for all $t \geq 0$,

$$\int_{\mathbb{R}}|u(t,x)|^2 w(t,x)dx + \delta_\gamma\int_0^t\int_{\mathbb{R}}|u(s,x)|^2\exp(\sqrt{\gamma}(x-cs))dx\,ds$$
$$+2\int_0^t\int_{\mathbb{R}} a(s,x)w(s,x)(\partial_x u(s,x))^2 dx\,ds = \int_{\mathbb{R}}|u^0(x)|^2\overline{w}(x)dx + 2\int_0^t\int_{\mathbb{R}} Suw.$$

The Poincaré inequality stated in Lemma 3.7 is an easy consequence of Proposition 3.5 and of the equivalence between the weights $\overline{w}$ and $\overline{w}_0$. □

4.3 Properties of the weights $\overline{w}_0$ and $\overline{w}$: Lemma 3.4, estimate (60) and inequality (68)

Proof of Lemma 3.4. Let us rewrite equation (51) as

$$(\bar{a}\overline{w}_0)' + \frac{\bar{b}-c}{\bar{a}}(\bar{a}\overline{w}_0) = 0,$$

which yields, since $\overline{w}_0(\xi_\gamma^-) = 1$,

$$\bar{a}\overline{w}_0(\xi) = \bar{a}(\xi_\gamma^-)\exp\left(\int_{\xi_\gamma^-}^{\xi}\frac{c-\bar{b}}{\bar{a}}\right), \qquad \forall \xi \in \mathbb{R}.$$

We recall that

$$\bar{a} = \gamma N^\gamma, \qquad \bar{b} = -2\gamma^2 N^{\gamma-1}\partial_x N - 2\gamma N^\gamma \frac{\partial_x^2 N}{\partial_x N},$$

hence

$$\begin{aligned}
-\int_{\xi_\gamma^-}^{\xi}\frac{\bar{b}(z)}{\bar{a}(z)}dz &= 2\int_{\xi_\gamma^-}^{\xi}\left[\frac{(N^\gamma)'(z)}{N^\gamma(z)} + \frac{N''(z)}{N'(z)}\right]dz \\
&= 2\ln\left(\frac{(N(\xi))^\gamma}{(N(\xi_\gamma^-))^\gamma}\right) + 2\ln\left(\frac{|N'(\xi)|}{|N'(\xi_\gamma^-)|}\right),
\end{aligned}$$

and therefore

$$\begin{aligned}
\overline{w}_0(\xi) &= \frac{\bar{a}(\xi_\gamma^-)}{\bar{a}(\xi)}\exp\left(\int_{\xi_\gamma^-}^{\xi}\frac{c}{\bar{a}}\right)\exp\left(-\int_{\xi_\gamma^-}^{\xi}\frac{\bar{b}}{\bar{a}}\right) \qquad (72)\\
&= \frac{1}{(N(\xi_\gamma^-))^\gamma(N'(\xi_\gamma^-))^2}(N(\xi))^\gamma(N'(\xi))^2\exp\left(\int_{\xi_\gamma^-}^{\xi}\frac{c}{\bar{a}}dz\right). \qquad (73)
\end{aligned}$$

Therefore we find the expression announced in Lemma 3.4, with a normalization constant

$$K := \frac{1}{(N(\xi_\gamma^-))^\gamma(N'(\xi_\gamma^-))^2}.$$

Let us now estimate K. We recall that ξ_γ^- is defined in (41). Since $N' = \gamma^{-1}P'P^{\frac{1}{\gamma}-1}$, it follows that

$$N'(\xi_\gamma^-) = \frac{1}{\gamma}P'(\xi_\gamma^-)\left(\frac{(c-1)(\gamma+1)}{c^3}\right)^{\frac{1}{2}-\frac{1}{2\gamma}},$$

and thus

$$K = \left(\frac{c^3}{(c-1)(\gamma+1)}\right)^{\frac{1}{2}-\frac{1}{\gamma}}\frac{\gamma^2}{P'(\xi_\gamma^-)^2}.$$

The sub- and super-solutions for P (see Proposition 2.4) entail that $P'(\xi_\gamma^-)$ is bounded from above and below. Hence, K is of order $\gamma^{3/2}$.

For $\xi \geq \xi_\gamma^*$, we know from (10) that there exist $0 < A_1 < A_2 < 1$ (close to $1 - c^{-1}$) such that

$$A_1 e^{-\frac{2}{c}\xi} \leq N(\xi) \leq A_2 e^{-\frac{\xi}{2c}}. \qquad (74)$$

Moreover, remember that $L_\gamma = \frac{N'}{N} + \frac{1}{c}$ converges uniformly to 0 on $[\xi_\gamma^*, +\infty)$ as $\gamma \to +\infty$ (see the proof of Lemma 2.8). Hence, for any $\eta > 0$, there exists γ_0 such that for all $\gamma > \gamma_0$, $N'/N \in [-1/c - \eta, -1/c + \eta]$ for all $\xi \in [\xi_\gamma^*, +\infty)$. By (74), we deduce that

$$\tilde{A}_1 e^{-\frac{2}{c}\xi} \leq |N'(\xi)| \leq \tilde{A}_2 e^{-\frac{\xi}{2c}}, \qquad (75)$$

with $\tilde{A}_{1,2} \in (0,1)$. As a consequence $\overline{w}_0$ has a double exponential growth as $\xi \to +\infty$: for all $\xi > \xi_\gamma^*$,

$$
\begin{aligned}
&K A_1^\gamma \tilde{A}_1^2 \exp\left(-2\frac{\gamma+2}{c}\xi\right)\exp\left(\frac{2c^2}{\gamma^2}A_2^{-\gamma}\left(\exp\left(\frac{\gamma}{2c}\xi\right)-\exp\left(\frac{\gamma}{2c}\xi_\gamma^*\right)\right)\right)\\
&\quad \le \overline{w}_0(\xi) \qquad\qquad (76)\\
&\qquad \le K A_2^\gamma \tilde{A}_2^2 \exp\left(-\frac{\gamma+2}{2c}\xi\right)\exp\left(\frac{c^2}{2\gamma^2}A_1^{-\gamma}\left(\exp\left(\frac{2\gamma}{c}\xi\right)-\exp\left(\frac{2\gamma}{c}\xi_\gamma^*\right)\right)\right)\exp\left(\int_{\xi_\gamma^-}^{\xi_\gamma^*}\frac{c}{\bar{a}}\right).
\end{aligned}
$$

Furthermore, using Lemma 2.13, we have

$$
\int_{\xi_\gamma^-}^{\xi_\gamma^*}\frac{c}{\bar{a}} \le C\gamma + \int_{\xi_\gamma^+}^{\xi_\gamma^*}\frac{c}{\bar{a}} \le \eta_1^{-\gamma}
$$

for some constant $\eta_1 \in [0,1]$. Estimate (55) from Lemma 3.4 follows.

For $\xi \to -\infty$, we have using (16) and denoting $\tilde{\lambda} = \sqrt{1+\frac{c^2}{4\gamma^2}} - \frac{c}{2\gamma}$,

$$
\overline{w}_0 = K N^\gamma (N')^2 \exp\left(\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma N^\gamma}\right) \underset{\xi\to-\infty}{\sim} K\tilde{\lambda}^2 P(1-N)^2 \exp\left(\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma P}\right).
$$

Recalling estimates (7), we get

$$
\frac{C}{\sqrt{\gamma}}(\xi-\xi_\gamma^-) \le \int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma P} \le \frac{c}{\gamma}(\xi-\xi_\gamma^-), \tag{77}
$$

and, for $|\xi| \le C$ with C independent of γ

$$
\frac{C^{-1}}{\gamma}\left(1-\frac{C'}{\sqrt{\gamma}}\right)\exp(\xi) \le 1-N \le \frac{C}{\gamma}\left(1-\frac{C'}{\sqrt{\gamma}}\right)\exp\left(\left(1-\frac{C}{\sqrt{\gamma}}\right)\xi\right).
$$

Gathering the terms results in

$$
C^{-1}\frac{K}{\gamma^2}\exp\left(2\left(1+\frac{C}{\sqrt{\gamma}}\right)\xi\right) \le \overline{w}_0 \le C\frac{K}{\gamma^2}\exp\left(2\left(1-\frac{C}{\sqrt{\gamma}}\right)\xi\right),
$$

and we deduce the result announced in Lemma 3.4.

□

Estimate of the constant δ_γ in $\overline{w}$. Let

$$
\bar{\psi} := \bar{\phi}'\bar{a}\overline{w}_0 = -\frac{\delta_\gamma}{\sqrt{\gamma}}\exp\left(\sqrt{\gamma}\xi\right).
$$

Using Lemma 3.4 and recalling (74), we observe that the double exponential growth of $\overline{w}_0$ dominates the growth in ψ as $\xi \to +\infty$. On the other hand, for $\xi \le -C$, we have

$$
\frac{\bar{\psi}(\xi)}{\bar{a}(\xi)\overline{w}_0(\xi)} = -\frac{\delta_\gamma}{\sqrt{\gamma}}\frac{\exp\left(\sqrt{\gamma}\xi\right)}{\gamma(\bar{N}(\xi))^\gamma\overline{w}_0(\xi)} \ge -C\frac{\delta_\gamma}{\gamma}\exp\left(\left(\sqrt{\gamma}-2\left(1+\frac{C}{\sqrt{\gamma}}\right)\right)\xi\right).
$$

Hence, for γ large enough, $\sqrt{\gamma}-3>0$ and $\dfrac{\bar{\psi}(\xi)}{\bar{a}(\xi)\overline{w}_0(\xi)}$ decreases exponentially to 0 as $\xi \to -\infty$. We conclude then to the integrability of $\dfrac{\bar{\psi}}{\bar{a}\overline{w}_0}$ on $\mathbb{R}$.

Let us now study the behavior of ϕ. For that purpose, we analyze separately the different regions according to the value of ξ.

- for $\xi > \xi_\gamma^+$: according to the definition of ξ_γ^+ and to Lemma 2.8, $|N_\gamma'| \geq C N_\gamma$ on this interval, and therefore

$$|\bar{\phi}'(\xi)| = -\frac{\bar{\psi}(\xi)}{\bar{a}(\xi)\overline{w}_0(\xi)} \leq C\frac{\delta_\gamma}{\gamma^{3/2}K}\frac{\exp(\sqrt{\gamma}\xi)}{N_\gamma^{2\gamma+2}}\exp\left(-\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma N_\gamma^\gamma}\right).$$

Integrating by parts,

$$\begin{aligned}
&\int_{\xi_\gamma^+}^{\infty}\frac{\exp(\sqrt{\gamma}\xi)}{N_\gamma^{2\gamma+2}}\exp\left(-\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma N_\gamma^\gamma}\right)\\
=\;&\frac{\gamma\exp(\sqrt{\gamma}\xi_\gamma^+)}{cN_\gamma(\xi_\gamma^+)^{\gamma+2}}\exp\left(-\int_{\xi_\gamma^-}^{\xi_\gamma^+}\frac{c}{\gamma N_\gamma^\gamma}\right)+\frac{\gamma}{c}\int_{\xi_\gamma^+}^{\infty}\frac{d}{d\xi}\left(\frac{\exp(\sqrt{\gamma}\xi)}{N_\gamma^{\gamma+2}}\right)\exp\left(-\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma N_\gamma^\gamma}\right)\\
=\;&\frac{\gamma\exp(\sqrt{\gamma}\xi_\gamma^+)}{cN_\gamma(\xi_\gamma^+)^{\gamma+2}}\exp\left(-\int_{\xi_\gamma^-}^{\xi_\gamma^+}\frac{c}{\gamma N_\gamma^\gamma}\right)\\
&+\int_{\xi_\gamma^+}^{\infty}\left[\frac{\gamma^{3/2}}{cN_\gamma^{\gamma+2}}-\frac{\gamma(\gamma+2)N_\gamma'}{cN_\gamma^{\gamma+3}}\right]e^{\sqrt{\gamma}\xi}\exp\left(-\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma N_\gamma^\gamma}\right)
\end{aligned}$$

Using the definition of ξ_γ^+ from Definition 2.10 and the monotony of N_γ' (see also Fig. 3), we get

$$\begin{aligned}
&\int_{\xi_\gamma^+}^{\infty}\frac{\exp(\sqrt{\gamma}\xi)}{N_\gamma^{2\gamma+2}}\exp\left(-\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma N_\gamma^\gamma}\right)\\
\leq\;&\frac{\gamma\exp(\sqrt{\gamma}\xi_\gamma^+)}{cN_\gamma(\xi_\gamma^+)^{\gamma+2}}\exp\left(-\int_{\xi_\gamma^-}^{\xi_\gamma^+}\frac{c}{\gamma N_\gamma^\gamma}\right)\\
&+\int_{\xi_\gamma^+}^{\infty}\left[\frac{\gamma^{3/2}}{cN_\gamma^{\gamma+2}}+\frac{\gamma(\gamma+2)(c-1)}{4c\gamma^2N_\gamma^{2\gamma+2}}\right]e^{\sqrt{\gamma}\xi}\exp\left(-\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma N_\gamma^\gamma}\right).
\end{aligned}$$

We can easily see that the second term in the right-hand side is smaller than the one in the left-hand side. Hence, recalling that $\xi_\gamma^+ = O(\gamma^{-1})$, we deduce that there exists a constant C, independent of γ, such that

$$\int_{\xi_\gamma^+}^{\infty}\frac{\exp(\sqrt{\gamma}\xi)}{N_\gamma^{2\gamma+2}}\exp\left(-\int_{\xi_\gamma^-}^{\xi}\frac{c}{\gamma N_\gamma^\gamma}\right)\leq C\gamma\frac{\exp\left(-\int_{\xi_\gamma^-}^{\xi_\gamma^+}\frac{c}{\gamma N_\gamma^\gamma}\right)}{(N_\gamma(\xi_\gamma^+))^\gamma}\leq\eta_3^{-\gamma},$$

for some constant $\eta_3 \in]0,1[$, so that

$$\int_{\xi_\gamma^+}^{\infty}|\bar{\phi}'(\xi)|\,d\xi \leq \delta_\gamma\eta_3^{-\gamma} \tag{78}$$

for some possibly different constant $\eta_3 \in]0,1[$.

- for the intermediate region $\xi \in [\xi_\gamma^-, \xi_\gamma^+]$, we write

$$\begin{aligned}
\int_{\xi_\gamma^-}^{\xi_\gamma^+}|\bar{\phi}'(z)|dz &= \frac{\delta_\gamma}{\sqrt{\gamma}}\int_{\xi_\gamma^-}^{\xi_\gamma^+}\frac{e^{\sqrt{\gamma}z}}{\gamma(N(z))^\gamma}\times\frac{1}{K(N(z))^\gamma(N'(z))^2\exp(\int_{\xi_\gamma^-}^{z}\frac{c}{\gamma N^\gamma})}dz\\
&= \frac{\delta_\gamma}{\gamma^{-1/2}K}\int_{\xi_\gamma^-}^{\xi_\gamma^+}\frac{e^{\sqrt{\gamma}z}}{(P'(z))^2(N(z))^2}\exp\left(-\int_{\xi_\gamma^-}^{z}\frac{c}{\gamma N^\gamma}\right)dz.
\end{aligned}$$

Now, using Lemma 2.12 and the definition of ξ_γ^-, we have in this region, for γ large enough,

$$|P'(\xi)| \geq \frac{C}{\gamma}, \qquad N(\xi) \geq N(\xi_\gamma^+) > 1 - \frac{1}{c} - \eta, \qquad (N(\xi))^\gamma \leq P(\xi_\gamma^-) = O\left(\frac{1}{\sqrt{\gamma}}\right),$$

with $\eta \in]0, 1-c^{-1}[$, and thus

$$\begin{aligned}\int_{\xi_\gamma^-}^{\xi_\gamma^+} |\bar{\phi}'(z)| dz &\leq C \frac{\delta_\gamma}{\gamma^{-5/2} K} e^{\sqrt{\gamma}\xi_\gamma^+} \int_{\xi_\gamma^-}^{\xi_\gamma^+} \exp\left(-\frac{C}{\gamma^{1/2}}(z - \xi_\gamma^-)\right) dz \\ &\leq C\gamma^{3/2}\delta_\gamma \left(1 - \exp\left(-C\frac{\xi_\gamma^+ - \xi_\gamma^-}{\sqrt{\gamma}}\right)\right) \\ &\leq C\gamma^{1/2}\delta_\gamma, \end{aligned} \tag{79}$$

where we have used the fact that $\xi_\gamma^+ = O(\gamma^{-1})$, $\xi_\gamma^- = O(\gamma^{-\frac{1}{2}})$ (cf. Lemma 2.12).

- for $\xi < \xi_\gamma^-$, we use Lemma 2.8

$$\sup_{\xi<0} \left|\frac{1-P(\xi)}{P'(\xi)}\right| \leq C,$$

and the control (18)

$$1 - P(\xi) \geq \left(1 - \frac{1}{\gamma}\right) e^\xi \quad \forall\, \xi < 0,$$

to infer that

$$\begin{aligned}-\int_{-\infty}^{\xi_\gamma^-} \phi'(z)dz &= \frac{\delta_\gamma}{\gamma^{-1/2}K} \int_{-\infty}^{\xi_\gamma^-} \frac{e^{\sqrt{\gamma}z}}{(P'(z))^2(N(z))^2} \exp\left(\int_z^{\xi_\gamma^-} \frac{c}{\gamma N^\gamma}\right) dz \\ &\leq C\frac{\delta_\gamma}{\gamma^{-1/2}K} \int_{-\infty}^{\xi_\gamma^-} \frac{e^{\sqrt{\gamma}z}}{(1-P(z))^2} e^{\frac{c}{\sqrt{\gamma}}(\xi_\gamma^- - z)} dz \\ &\leq C\frac{\delta_\gamma}{\gamma^{-1/2}K} \int_{-\infty}^{\xi_\gamma^-} e^{\sqrt{\gamma}z} e^{-2z} e^{\frac{c}{\sqrt{\gamma}}(\xi_\gamma^- - z)} dz \\ &\leq C\frac{\delta_\gamma}{\gamma^{-1/2}K} \int_{-\infty}^{\xi_\gamma^-} e^{\frac{\sqrt{\gamma}}{2}z} dz \\ &\leq C\gamma^{-3/2}\delta_\gamma, \end{aligned} \tag{80}$$

thanks to the fact that $\xi_\gamma^- = O(\gamma^{-1/2})$ (cf. Lemma 2.12). Combining (78)-(79)-(80), there exists $\eta_3 \in\ ]0,1[$, independent of γ, such that for $\delta_\gamma = \eta_3^\gamma$, $\left|\int_{\mathbb{R}} \bar{\phi}'\right| \leq 1$.

□

Proof of inequality (68). First, using (76), we find that for $\xi > \xi_\gamma^*$, the inequality follows easily from the fact that $A_2 < 1$ and from the convexity of the exponential.
For $\xi < \xi_\gamma^-$, using (77) together with the estimate of K, we have

$$\begin{aligned}\overline{w}_0 &\geq \frac{K}{\gamma^2}(P')^2 P^{\frac{2}{\gamma}-1} \exp\left(\frac{C}{\sqrt{\gamma}}(\xi - \xi_\gamma^-)\right) \\ &\geq \frac{C}{\sqrt{\gamma}}(P')^2 \exp\left(\frac{C}{\sqrt{\gamma}}(\xi - \xi_\gamma^-)\right). \end{aligned}$$

Recalling (39) together with the sub/super solutions for P from Proposition 2.4, we infer that for $\xi < \xi_\gamma^-$,

$$\overline{w}_0 \geq \frac{C}{\sqrt{\gamma}} \exp\left(\left(2 + \frac{C}{\sqrt{\gamma}}\right)\xi\right).$$

Inequality (68) follows on this zone.
There remains to consider the transition zone $(\xi_\gamma^-, \xi_\gamma^*)$. In this region, using Lemma 2.5, we merely note that

$$\overline{w}_0 \geq C\gamma^{3/2} N^\gamma (N')^2 \geq C\gamma^{3/2} N(\xi_\gamma^*)^\gamma \min\left(N'(\xi_\gamma^-)^2, N'(\xi_\gamma^*)^2\right).$$

By definition of ξ_γ^*, $N'(\xi_\gamma^*)$ is a constant independent of γ. As for $N'(\xi_\gamma^-)$, we have, by definition of ξ_γ^- and recalling Lemma 2.12,

$$N'(\xi_\gamma^-) = \frac{1}{\gamma} P'(\xi_\gamma^-) P(\xi_\gamma^-)^{\frac{1}{\gamma}-1} = \frac{1}{\gamma} P'(\xi_\gamma^-) \left(\frac{c^3}{(c-1)(\gamma+1)}\right)^{\frac{1}{\gamma}-\frac{1}{2}} \leq -C\gamma^{-3/2}.$$

Therefore, since $N(\xi_\gamma^*) \to 1 - c^{-1}$ as $\gamma \to \infty$, we deduce that there exists a constant $\eta \in (0,1)$ such that

$$\min_{(\xi_\gamma^-, \xi_\gamma^*)} \overline{w}_0 \geq \eta^\gamma.$$

Inequality (68) follows easily. □

4.4 Proof of the weighted Poincaré inequality

Proof of Proposition 3.5. To lighten the notations, we forget in what follows the notation $\bar{\cdot}$ when it is clear that we work with functions of variable ξ. Formally, we have the following inequalities, for any $\rho \in \mathcal{C}^2(\mathbb{R})$ and $v \in \mathcal{C}_c^1(\mathbb{R})$

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}} \left(\partial_\xi(v\rho)\right)^2 d\xi \\
&= \int_{\mathbb{R}} \left[\rho^2(\partial_\xi v)^2 + v^2(\partial_\xi \rho)^2\right] d\xi + 2\int_{\mathbb{R}} v\partial_\xi v \rho \partial_\xi \rho d\xi \\
&= \int_{\mathbb{R}} \left[\rho^2(\partial_\xi v)^2 + v^2(\partial_\xi \rho)^2\right] d\xi - \int_{\mathbb{R}} v^2 \partial_\xi(\rho\partial_\xi\rho) d\xi \\
&= \int_{\mathbb{R}} \rho^2(\partial_\xi v)^2 d\xi - \int_{\mathbb{R}} v^2 \rho \partial_\xi^2 \rho \; d\xi.
\end{aligned}$$

Note that when ρ is positive and strictly convex, we obtain a Poincaré inequality. We want to apply this inequality with $\rho := (\overline{a}\overline{w}_0)^{1/2}$. However, the weight ρ is not convex on $\mathbb{R}$ and we cannot guarantee the sign of the second integral. Let us compute the derivatives of ρ. Using (54), we have

$$\rho = -\sqrt{\gamma K} N' N^\gamma \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{2\gamma N(z)^\gamma}\right),$$

consequently,

$$\rho'(\xi) = \sqrt{\gamma K} \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{2\gamma N(z)^\gamma}\right) \left[-(N'(\xi)N(\xi)^\gamma)' - \frac{c}{2\gamma} N'(\xi)\right].$$

We recall that

$$-cN' - \gamma(N'N^\gamma)' = N(1 - N^\gamma),$$

so that

$$\rho'(\xi) = \sqrt{\gamma K} \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{2\gamma N^\gamma(z)}\right) \left[\frac{1}{\gamma} N(\xi)(1 - N^\gamma(\xi)) + \frac{c}{2\gamma} N'(\xi)\right].$$

Differentiating once again and using the equation on N, we get

$$\rho''(\xi) = \sqrt{\frac{K}{\gamma}} \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{2\gamma N^\gamma(z)} dz \right) \left[N'(1-(\gamma+1)N^\gamma) + \frac{c}{2} N'' + \frac{c}{2\gamma N^{\gamma-1}}(1-N^\gamma) + \frac{c^2}{4\gamma}\frac{N'}{N^\gamma} \right]$$

$$= \sqrt{\frac{K}{\gamma}} \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{2\gamma N^\gamma(z)} dz \right) [N'(1-(\gamma+1)N^\gamma)$$

$$- \frac{c}{2\gamma N^\gamma}\left(cN' + \gamma^2 (N')^2 N^{\gamma-1} + N(1-N^\gamma) \right) + \frac{c}{2\gamma N^{\gamma-1}}(1-N^\gamma) + \frac{c^2}{4\gamma}\frac{N'}{N^\gamma}]$$

$$= -\sqrt{\frac{K}{\gamma}} N' \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{2\gamma N^\gamma(z)} dz \right) \left[(\gamma+1)N^\gamma + \frac{c^2}{4\gamma N^\gamma} + c\gamma \frac{N'}{2N} - 1 \right].$$

Note that $\rho'' \geq 0$ provided the term in brackets is non-negative. The bracketed term is a sum of four terms, among which the first two are positive, and the last two are negative. Furthermore,

$$\frac{\gamma+1}{c} N^\gamma + \frac{c}{4\gamma N^\gamma} = \left(\sqrt{\frac{\gamma+1}{c} N^\gamma} - \sqrt{\frac{c}{4\gamma N^\gamma}} \right)^2 + \sqrt{\frac{\gamma+1}{\gamma}} \geq 1 + \frac{1}{4\gamma}, \qquad \forall \gamma \geq 1,$$

so that

$$\rho'' \geq -\sqrt{\frac{K}{\gamma}} N' \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{2\gamma N^\gamma} \right) \left[(\gamma+1)\left(1 - \frac{1}{c}\right) N^\gamma + \frac{c(c-1)}{4\gamma N^\gamma} + \frac{1}{4\gamma} + c\gamma\frac{N'}{2N} \right]. \tag{81}$$

Thus the only zone where ρ'' is non-positive is the region where the last term in the above bracket is not dominated by the others. Decomposing the domain in three zones, we have

- *Free zone*: In $\xi \geq \xi_\gamma^+$, using the notations of Section 2 and recalling Definition 2.10 and Figure 3, we have

  $$N'(\xi) \geq \tilde{Q}(N) = -\frac{c-1}{4\gamma^2 (N(\xi))^{\gamma-1}},$$

  so that

  $$c\gamma \frac{|N'|}{2N} \leq \frac{c(c-1)}{8\gamma N^\gamma}, \qquad \forall\, \xi \geq \xi^+.$$

  Recalling the expressions of ρ and $\overline{w}_0$, we infer that for $\xi > \xi^+$

  $$\rho\rho'' \geq C\sqrt{\gamma K}\sqrt{\frac{K}{\gamma}} (N')^2 N^\gamma \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N^\gamma(z)} dz \right) \frac{C}{\gamma N^\gamma}$$

  $$\geq \frac{C}{\gamma N^\gamma} w_0.$$

- *Congested zone*: for $\xi \leq \xi_\gamma^-$, we have $P \geq P(\xi_\gamma^-) = \left(\frac{c^3}{(c-1)(\gamma+1)} \right)^{1/2}$ while $P' \in [-c, 0]$. Hence, we ensure that

  $$-cP' \leq c^2 \leq (\gamma+1)\left(1 - \frac{1}{c}\right) P^2 \qquad \forall\, \xi \leq \xi_\gamma^-,$$

  and therefore

  $$-c\gamma \frac{N'}{2N} \leq \frac{(\gamma+1)}{2}\left(1 - \frac{1}{c}\right) N^\gamma \qquad \forall\, \xi \leq \xi_\gamma^-.$$

  Let us mention that this inequality is precisely the property that lead us to the normalization (2.10) and to the definition of ξ_γ^-. We deduce then

  $$\rho\rho'' \geq K(N')^2 N^{2\gamma} \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N^\gamma(z)} dz \right) \frac{(\gamma+1)}{2}\left(1 - \frac{1}{c}\right)$$

  $$\geq C\gamma N^\gamma w_0.$$

- *Transition zone:* For $\xi \in [\xi_\gamma^-, \xi_\gamma^+]$, we can always bound the negative contribution as follows

$$\rho(\rho'')_- \leq \frac{c\gamma}{2} K|N'|^3 \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N^\gamma(z)} dz\right) N^{\gamma-1} \leq C_\gamma e^{\sqrt{\gamma}\xi},$$

for $\xi \in (\xi_\gamma^-, \xi_\gamma^+)$, where

$$C_\gamma \leq C\gamma^{5/2} \|N'\|^3_{L^\infty(\xi_\gamma^-, \xi_\gamma^+)} \exp\left(\int_{\xi_\gamma^-}^{\xi_\gamma^+} \frac{1}{\gamma N^\gamma}\right).$$

Using Lemma 2.13, we find that $C_\gamma \leq C^\gamma$ for some constant $C > 1$ independent of γ, where the exponential growth stems from $\|N'\|_{L^\infty(\xi_\gamma^-, \xi_\gamma^+)}$.

Gathering all the terms, we obtain

$$\int_{\xi \leq \xi_\gamma^-} v^2 \rho\rho'' + \int_{\xi \geq \xi^+} v^2 \rho\rho'' \leq \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a}\bar{w}_0 + \int_{\xi_\gamma^-}^{\xi^+} v^2 \rho(\rho'')_-.$$

Replacing $\rho\rho''$ by their lower bounds on $(-\infty, 0)$ and on $(\xi^+, +\infty)$, we obtain the inequality announced in the proposition.

□

A List of abscissas

We list below the main abscissas we have introduced throughout the paper and briefly describe their use.

Abscissa	**Size/Order**	**Definition**	**Description**
ξ_γ^-	$O(\gamma^{-1/2})$	Def. 2.10 p. 22	sup. limit of the congested zone in Section 3 - see (81)
ξ_γ^{int}	$O(\gamma^{-1})$	Def. (42) p. 23	point on the left-side of the interface with $P_\gamma' \leq -C < 0$, it is used to characterize the size of ξ_γ^+
ξ_γ^0	$\xi_\gamma^{int} < \xi_\gamma^0 < \xi_\gamma^+$ $O(\gamma^{-1})$	Lem. 2.6 p. 14	abscissa characterizing the change of convexity of N_γ, it corresponds to the minimum of N_γ'
ξ_γ^+	$O(\gamma^{-1})$	Def. 2.10 p. 22	inf. limit of the free zone in Section 3 - see (81)
ξ_γ^*	$\xi_\gamma^+ < \xi_\gamma^*$	Lem. 2.6 p. 14	the value of $N_\gamma'(\xi_\gamma^*)$ corresponds to the value of N_{HS}' on the right-side of the interface, it characterizes the zone where N_γ has an exponential behavior

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