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THE N-MEASURE FOR ALGEBRAIC INTEGERS HAVING ALL THEIR CONJUGATES IN A SECTOR.

Abstract

Let α be a nonzero algebraic integer of degree d whose all conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ lie in a sector $|\arg z| \leq \theta, 0 \leq \theta \leq 90^\circ$. We define the *N-measure* of α by $N(\alpha) = \prod_{i=1}^d (|\alpha_i| + 1/|\alpha_i|)$ and the *absolute N-measure* of α by $\nu(\alpha) = N(\alpha)^{1/\deg(\alpha)}$. Firstly, we consider the case $\theta = 0$. We prove that $N(\alpha) \in \mathbb{N}$ and that, if α is a reciprocal algebraic integer, $N(\alpha)$ is a square. Then, we study the set $\mathcal{N}$ of the quantities $\nu(\alpha)$. We prove that there exists a number l such that $\mathcal{N}$ is dense in (l, ∞) . Finally, using the method of auxiliary functions, we find the seven smallest points of $\mathcal{N}$ in $(2, l)$. In case of $0 < \theta \leq 90^\circ$, we compute the greatest lower bound $c(\theta)$ of the absolute N-measure of α , for α belonging to eight subintervals of $]0, \theta[$. Among these subintervals, two are complete. These computations are also done by using auxiliary functions. The polynomials involved in the auxiliary functions are found by our recursive algorithm.

1 Introduction

Let α be a nonzero algebraic integer of degree d . We define the *N-measure* of α by:

$$N(\alpha) = \prod_{i=1}^d (|\alpha_i| + 1/|\alpha_i|).$$

The *absolute N-measure* of α is the quantity $\nu(\alpha) = N(\alpha)^{1/d}$.

If P is the minimal polynomial of α , then $N(P) = N(\alpha)$ and $\nu(P) = \nu(\alpha)$.

It is obvious that $\nu(\alpha) \geq 2$ and the equality holds if and only if α is a root of unity. Indeed, if for $i = 1, \dots, d$, we have $|\alpha_i| + 1/|\alpha_i| = 2$ then for all $i = 1, \dots, d$, $|\alpha_i| = 1$. By the theorem of Kronecker, we deduce that α is a root of unity.

1.1 The case $\theta = 0$

Suppose first that $\theta = 0$. It means α is a *totally positive algebraic integer* : α and all its conjugates are positive real numbers.

Theorem 1. *For all nonzero totally positive algebraic integers α whose minimal polynomial P satisfies $|P(0)| = 1$,*

$$N(\alpha) \in \mathbb{N}.$$

Moreover, if α is reciprocal i.e., α and $1/\alpha$ are conjugates, then $N(\alpha)$ is a square.

In order to study the structure of the set $\mathcal{N}$ of the quantities $\nu(\alpha)$, we show the following

Theorem 2. *$\mathcal{N}$ is dense in (l, ∞) where $l = \lim_{n \rightarrow \infty} \nu(\beta_n^2) = 3.513182\dots$*

The β_n^2 were defined by C.J. Smyth [Sm1] as follows:

$$\begin{cases} \beta_0^2 &= 1 \\ \beta_n^2 &= \beta_{n+1}^2 + \beta_{n+1}^{-2} - 2 \end{cases}$$

β_n^2 is a totally positive algebraic integer of degree 2^n .

Towards determining the structure of $\mathcal{N}$ in the gap $(2, l)$, we prove the following

Theorem 3. *If α is a nonzero totally positive algebraic integer whose minimal polynomial has a constant term of modulus 1 and is different from $x - 1$, $x^2 - 3x + 1$, $x^4 - 7x^3 + 13x^2 - 7x + 1$, $x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1$, $x^6 - 11x^5 + 41x^4 - 63x^3 + 41x^2 - 11x + 1$, $x^8 - 15x^7 + 84x^6 - 225x^5 + 311x^4 - 225x^3 + 84x^2 - 15x + 1$ and $x^{16} - 31x^{15} + 413x^{14} - 3141x^{13} + 15261x^{12} - 50187x^{11} + 115410x^{10} - 189036x^9 + 222621x^8 - 189036x^7 + 115410x^6 - 50187x^5 + 15261x^4 - 3141x^3 + 413x^2 - 31x + 1$ then we have:*

$$\nu(\alpha) \geq 3.483671.$$

Therefore, the points of $\mathcal{N}$ in $(2, l)$ are:

$$\begin{aligned} 2 &= \nu(x - 1) = \nu(\beta_0^2), \\ 3 &= \nu(x^2 - 3x + 1) = \nu(\beta_1^2), \\ 3.316624 \dots &= \nu(x^4 - 7x^3 + 13x^2 - 7x + 1) = \nu(\beta_2^2), \\ 3.433631 \dots &= \nu(x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1) = \nu(\beta_3^2), \\ 3.448217 \dots &= \nu(x^3 - 5x^2 + 6x - 1) = \nu(x^3 - 6x^2 + 5x - 1), \\ 3.470100 \dots &= \nu(x^4 - 7x^3 + 14x^2 - 8x + 1) = \nu(x^4 - 8x^3 + 14x^2 - 7x + 1), \\ 3.479784 \dots &= \nu(x^{16} - 31x^{15} + 413x^{14} - 3141x^{13} + 15261x^{12} - 50187x^{11} + 115410x^{10} - 189036x^9 + 222621x^8 - 189036x^7 + 115410x^6 - 50187x^5 + 15261x^4 - 3141x^3 + 413x^2 - 31x + 1) = \nu(\beta_4^2). \end{aligned}$$

1.2 The case $0 < \theta < 90^\circ$

The study of "small points" leads us to extend the search to algebraic integers α whose conjugates lie in a sector $S_\theta = \{z \in \mathbb{C} \text{ such that } |\arg z| \leq \theta\}$ with $0 < \theta < 90^\circ$. We follow here the work of M. Langevin [La] on the absolute Mahler measure $\Omega(\alpha) = (\prod_{i=1}^d \max(1, |\alpha_i|))^{1/d}$ of algebraic integers α having all their conjugates in a sector. He proved that there exists a function $c(\theta)$ on $[0, 180^\circ)$, always > 1 , such that if $\alpha \neq 0$ is not a root of unity, whose conjugates all lie in S_θ , then $\Omega(\alpha) \geq c(\theta)$. Note that this result is not numerical. G. Rhin and C. Smyth [RS] succeeded in finding the exact value of $c(\theta)$ for θ in nine subintervals of $[0, 120^\circ]$ and conjectured that $c(\theta)$ is a "staircase" function of θ , which is constant except for finitely many left discontinuities in any closed subinterval of $[0, 180^\circ)$. Their method uses explicit auxiliary polynomials $A(X) = X^a R(X)$ where R is a reciprocal polynomial in $\mathbb{Z}[X]$, found by heuristic methods. In 2004, thanks to Wu's algorithm [Wu], G. Rhin and Q. Wu [RW] gave the exact value of $c(\theta)$ for four new subintervals of $[0, 140^\circ]$ and extended four existing subintervals. In 2013, the author and G. Rhin [FR] found for the first time a complete subinterval and a fourteenth subinterval. A complete subinterval is an interval on which the function $c(\theta)$ describing the minimum on the sector $|\arg z| \leq \theta$ is constant, with jump discontinuities at each end. These improvements are due to our recursive algorithm.

Here, we prove the following result:

Theorem 4. *There exist a left discontinuous, strictly positive, staircase function g on $[0, 90^\circ)$ and a positive, continuous, monotonically decreasing function f on $[0, 90^\circ)$ such that:
for $\theta \in [0, 90^\circ)$, if α is a nonzero algebraic integer whose conjugates all lie in S_θ then*

$$N(\alpha) \geq \min(f(\theta), g(\theta)).$$

Moreover, the exact value of $c(\theta) = \inf_{|\arg \alpha| \leq \theta} N(\alpha)$ is known on eight subintervals of $[0, 90^\circ)$.

These intervals are given in Table 1.

The function $g(\theta)$ is a decreasing staircase function having left discontinuities. The function $g(\theta)$ is the smallest value of $\nu(\alpha)$ that could be found for α having all its conjugates in $|\arg z| \leq \theta$.

The function $f(\theta)$ is given by $f(\theta) = \max_{1 \leq i \leq 8} (f_i(\theta))$, and the functions $f_i(\theta)$ are defined as follows:

$$f_i(\theta) = \min_{z \in S_\theta} \left(\log(|z| + 1/|z|) - \sum_{1 \leq j \leq J} c_{ij} \log |Q_{ij}(z)| \right)$$

where the polynomials Q_{ij} and the real numbers c_{ij} are read off from Table 3.

Table 1 Intervals $[\theta_i, \theta'_i]$ where $c(\theta)$ is known exactly. Here $c(\theta) = c(\theta_i) = \nu(P)$ for $\theta \in [\theta_i, \theta'_i]$. The angles are given in degrees.

i	$c(\theta)$	θ_i	θ'_i	P
1	2.5	0	23.15	$x - 2$
2	2.455652	26.267699	27.23	$x^4 - 7x^3 + 19x^2 - 23x + 11$
3	2.309401	30	37.17	$x^2 - 3x + 3$
4	2.265128	39.985729	45.309985	$x^4 - 5x^3 + 11x^2 - 11x + 5$
5	2.211739	45.309985	52.6	$x^4 - 4x^3 + 8x^2 - 7x + 3$
6	2.193323	63.681104	67.59	$x^4 - 3x^3 + 7x^2 - 7x + 5$
7	2.12.13.20	69.295189	82.588473	$x^2 - x + 2$
8	2.116245	82.588473	89.99	$x^4 - x^3 + 4x^2 - 2x + 3$

In Section 2, we prove Theorem 1. Section 3 deals with the denseness of the set $\mathcal{N}$. In Section 4, we describe the method of explicit auxiliary functions. We link these functions with the integer transfinite diameter. Then, we explain the recursive algorithm which enables us to obtain the constant of Theorem 3. In Section 5, we deal with the case $0 < \theta \leq 90^\circ$. All the computations were done on a MacBookPro with the languages Pascal and Pari.

2 Proof of Theorem 1

Remember that, if $P \in \mathbb{Z}[x]$ of degree d , the *reciprocal polynomial* of P is $P^* = x^d \cdot P(\frac{1}{x})$ and P is *reciprocal* if $P^* = P$.

Let α be a nonzero totally positive algebraic integer of degree d and let $P = \prod_{i=1}^d (x - \alpha_i)$ be its minimal polynomial satisfying $|P(0)| = 1$.

Put $Q(x) = P(x) \cdot P^*(x)$. Then, Q is a reciprocal polynomial of degree $2d$ in which we do the change of variable $y = x + 1/x - 2$. We get $Q(x) = x^d \cdot R(y)$ where $R \in \mathbb{Z}[y]$ with $\deg(R)=d$. The fact that $|P(0)| = 1$ implies that the polynomial R is monic. The roots of R are the $\beta_i = \alpha_i + 1/\alpha_i - 2$ for $i = 1, \dots, d$.

Thus, $N(P) = \prod_{i=1}^d |\beta_i + 2| = |R(-2)|$ because R is monic and $|R(-2)| \in \mathbb{N}$. Now, if P is a

reciprocal polynomial of degree $2d$, then $P = x^d \cdot R(y)$ and $N(P) = \prod_{i=1}^{2d} |\alpha_i + 1/\alpha_i| = \prod_{i=1}^d (\alpha_i + 1/\alpha_i)^2 = (R(-2))^2$.

3 Denseness of the set $\mathcal{N}$

3.1 Study of the sequence $(\nu(\beta_n^2))_{n \geq 1}$

Remember that the sequence $(\beta_n^2)_{n \geq 0}$ satisfies:

$$\begin{cases} \beta_0^2 &= 1 \\ \beta_n^2 &= \beta_{n+1}^2 + \beta_{n+1}^{-2} - 2 \end{cases}$$

We first prove the following

Lemma 5.

$$\nu(\beta_n^2) = 2 \prod_{i=0}^{n-1} (1 + \lambda_i)^{1/2^i} \quad \text{where } \lambda_0 = \frac{1}{2} \text{ and } \lambda_{i+1} = \frac{\lambda_i}{(1 + \lambda_i)^2} \quad \text{for } i \geq 0.$$

Proof

For $n \geq 0$, we put $\gamma_n = \beta_n^2$, so $\gamma_n = \gamma_{n+1} + \gamma_{n+1}^{-1} - 2$.

Therefore, we can write: $N(\beta_n^2) = N(\gamma_n) = \prod_{i=1}^{2^n} (\gamma_{n,i} + \gamma_{n,i}^{-1}) = \prod_{i=1}^{2^{n-1}} (\gamma_{n,i} + \gamma_{n,i}^{-1})^2$ because $\gamma_{n,i} = \gamma_{2^{n-1}+i}$ for $1 \leq i \leq 2^{n-1}$. (For $1 \leq i \leq 2^n$, $\gamma_{n,i}$ denote the conjugates of γ_n).

Then, we have: $N(\beta_n^2) = \prod_{i=1}^{2^{n-1}} (\gamma_{n,i}^2 + \gamma_{n,i}^{-2} + 2) = \prod_{i=1}^{2^{n-1}} (2 + \gamma_{n-1,i})^2$ because $\gamma_{n+1}^2 + \gamma_{n+1}^{-2} = \gamma_n^2 + 4\gamma_n + 2$.

Finally, we have $N(\beta_n^2) = \left[2^{2^{n-1}} \prod_{i=1}^{2^{n-1}} (1 + 1/2\gamma_{n-1,i}) \right]^2$.

We recall here a more general lemma that we proved in [F]:

Lemma 6. *The notations are the same as previously. Then we have*

$$\prod_{i=1}^{2^n} (1 + \lambda_0 \gamma_{n,i}) = \left(\prod_{i=0}^n (1 + \lambda_i)^{1/2^i} \right)^{2^n} \quad \text{where } \lambda_0 = \frac{1}{2} \text{ and } \lambda_{i+1} = \frac{\lambda_i}{(1 + \lambda_i)^2} \quad \text{for } i \geq 0.$$

Now, we can conclude that $\nu(\beta_n^2) = 2 \prod_{i=0}^{n-1} (1 + \lambda_i)^{1/2^i}$.

A short program, that can be read off from Annex below, allows us to compute with Pari the limit l of the sequence $\nu(\beta_n^2)$ and we obtain $l = 3.513182\dots$. Note that l gives an upper bound for the first accumulation point of $\mathcal{N}$.

3.2 Proof of Theorem 2

The proof and notations follow those of C.J. Smyth in [Sm1]. For a given function $g : [0, \infty) \rightarrow \mathbb{R}$, put $\mathcal{M}(g)$ the set of all means

$$M_g(\alpha) = \frac{1}{d} \sum_{i=1}^d g(|\alpha_i|)$$

for α a totally real algebraic integer, i.e., all its conjugates $\alpha_1 = \alpha, \dots, \alpha_d$ are real numbers. When the limits exist, put $a(g) = \lim_{n \rightarrow \infty} M_g(\beta_n)$ and $c(g) = \lim_{n \rightarrow \infty} M_g(2 \cos(2\pi/n))$.

Here a convenient choice for g is

$$g : x \rightarrow \log(x^2 + \frac{1}{x^2}) \tag{3.1}$$

because then $M_g(\alpha) = \log \nu(\alpha^2)$.

The proof consists in two parts.

3.2.1 First step of the proof

C.J. Smyth [Sm1] proved the following

Theorem 7. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a monotonic increasing function, zero on $[0, 1]$ such that*

$$\lim_{x \rightarrow \infty} g(x+1)/g(x) = 1$$

and the values of $\log_2 g(2k+1) \bmod 1$ ($k=0,1,2,\dots$) are everywhere dense in $(0, 1)$.

Then the limit $a(g)$ exists and $\mathcal{M}(g)$ is dense in $(a(g), \infty)$.

We replace the function g defined by (3.1) by the function g^* which satisfies the hypothesis of Theorem 7:

$$g^*(x) = \begin{cases} g(x) & \text{if } x > 1 \\ 0 & \text{if } 0 \leq x \leq 1 \end{cases} \quad (3.2)$$

Since $M_g(\beta_n) = M_{g^*}(\beta_n)$, the existence of $a(g^*)$ implies those of $a(g)$ and $a(g^*) = a(g)$.

It is easy to see that g^* satisfies the first hypothesis of Theorem 7. So, it is sufficient to study the denseness of the set $\mathcal{F} = \{\log_2 g(2k+1) \bmod 1, k \in \mathbb{N}\}$.

Let $t \in [0, 1]$ and $\epsilon > 0$. Does there exist $f \in \mathcal{F}$ such that $|f(k) - t| < \epsilon$? We search for n and k satisfying:

$$|\log_2 g^*(2k+1) - t - n| < \epsilon$$

i.e.,

$$|\log g^*(2k+1) - t' - n \log 2| < \epsilon' \quad (3.3)$$

The uniform continuity of the function $\log$ on $[1, \infty)$ gives :

$$\forall \epsilon' > 0, \exists \eta(\epsilon') \text{ such that } \forall x, y > 0, |x - y| < \eta(\epsilon') \Rightarrow |\log x - \log y| < \epsilon'.$$

We choose n such that $2^{-n} < \eta(\epsilon')$ and k such that $|(2k+1) - (g^*)^{-1}(2^n e^{t'})| \leq 1$. As $(g^*)'$ is bounded by 1, the mean value Theorem for g^* on $(1, \infty)$ gives:

$$|g^*(2k+1) - 2^n e^{t'}| \leq 1,$$

i.e.,

$$|2^{-n} g^*(2k+1) - e^{t'}| \leq 2^{-n} < \eta(\epsilon')$$

and the inequality (3.3) follows immediately. Thus, we have proved that $\mathcal{M}(g)$ is dense in $(a(g^*), \infty) = (a(g), \infty) = (\log l, \infty) = (1.256522\dots, \infty)$.

3.2.2 Second step of the proof

C.J. Smyth [Sm1] established the following

Theorem 8. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function such that $\lim_{x \rightarrow \infty} g(x) = \infty$ and which satisfies a Lipschitz condition*

$$|g(x) - g(y)| < B(\lambda)|x - y|$$

for $x, y \in [0, \lambda]$, for each $\lambda > 0$. Then $\mathcal{M}(g)$ is dense on $(c(g), \infty)$, where

$$c(g) = \frac{2}{\pi} \int_0^{\pi/2} g(2 \cos \theta) d\theta.$$

It is easy to see that, for our function g defined by (3.1), the Lipschitz condition is satisfied for $B(\lambda) = 2\lambda \cdot (\lambda^2 + 1)$. Thus, $\mathcal{M}(g)$ is dense on $(c(g), \infty) = (1.465715\dots, \infty)$.

3.2.3 Conclusion

We have shown that $\mathcal{M}(g)$ is dense on $(\min(a(g), c(g)), \infty) = (\log l, \infty)$ which means that $\mathcal{N}$ is dense on (l, ∞) , where $l = \lim_{n \rightarrow \infty} \nu(\beta_n^2) = 3.513182\dots$

3.3 Proof of Theorem 3

3.3.1 The explicit auxiliary function

The auxiliary function involved in Theorem 3 is of the following type:

$$\text{for } x > 0, \quad f(x) = \log(x + 1/x) - \sum_{0 \leq j \leq J} c_j \log |Q_j(x)|$$

where the c_j are positive real numbers and the polynomials Q_j are non zero polynomials in $\mathbb{Z}[x]$.

Let α be a totally positive algebraic integer with conjugates $\alpha_1 = \alpha, \dots, \alpha_d$. Then we have

$$\sum_{i=1}^d f(\alpha_i) \geq md,$$

where m denotes the minimum of the function f . Thus,

$$\log N(\alpha) \geq md + \sum_{1 \leq j \leq J} c_j \log \left| \prod_{i=1}^d Q_j(\alpha_i) \right|.$$

We assume that P does not divide any Q_j , then $\prod_{i=1}^d Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j .

Therefore, if α is not a root of Q_j , we have

$$\nu(\alpha) \geq e^m.$$

To avoid convergence problems with Pascal, we put $y = x + 1/x - 2$. Then, $f(x)$ becomes

$$\text{for } y > 0, \quad g(y) = \log(y + 2) - \sum_{1 \leq j \leq J} c_j \log |U_j(y)| \quad (1).$$

Now, the main problem is to find a good list of polynomials U_j which gives a value of m as large as possible. Thus, we link the auxiliary function with the integer transfinite diameter in order to find the polynomials with our recursive algorithm.

3.3.2 Auxiliary functions and integer transfinite diameter

In this section, we shall need the following definition:

Let K be a compact subset of $\mathbb{C}$.

If φ is a positive function defined on K , the φ -integer transfinite diameter of K is defined as

$$t_{\mathbb{Z}, \varphi}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{P \in \mathbb{Z}[Y] \\ \deg(P) = n}} \sup_{y \in K} \left(|P(y)|^{\frac{1}{n}} \varphi(y) \right).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A] and is an important tool in the study of rational approximations of logarithms of rational numbers.

Inside the auxiliary function (1), we replace the numbers c_j by rational numbers a_j/q where q is a integer such that $q.c_j$ is an integer for all $0 \leq j \leq J$. Then we can write:

$$\text{for } y > 0, \quad g(y) = \log(y+2) - \frac{t}{r} \log |U(y)| \quad (2)$$

where $U = \prod_{i=0}^J U_j^{a_j} \in \mathbb{Z}[Y]$ is of degree $r = \sum_{i=0}^J a_j \deg(U_j)$ and $t = \sum_{j=0}^J c_j \deg(U_j)$ is a positive real number. We want to get a function whose minimum m is as large as possible. Thus we search a polynomial $U \in \mathbb{Z}[Y]$ such that

$$\sup_{y>0} |U(y)|^{t/r} (y+2)^{-1} \leq e^{-m}.$$

If we suppose that t is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}((0,\infty)) = \liminf_{\substack{r \geq 1 \\ r \rightarrow +\infty}} \inf_{\substack{P \in \mathbb{Z}[Y] \\ \deg(P) = r}} \sup_{y > 0} \left(|P(y)|^{\frac{t}{r}} \varphi(y) \right)$$

where we use the weight $\varphi(y) = (y+2)^{-1}$.

Even if we replace the compact subset K by the infinite interval $(0, \infty)$, the weight φ ensures that the quantity $t_{\mathbb{Z},\varphi}((0, \infty))$ is finite.

3.3.3 Construction of the auxiliary function

The improvement compared with Wu's algorithm is that our polynomials are obtained by induction. Suppose that we have $U_1, U_2, \dots, U_J$. Then we use the semi-infinite linear programming (introduced in number theory by C. J. Smyth [Sm2]) to optimize g for this set of polynomials (i.e., to get the greatest possible m). We obtain the numbers $c_1, c_2, \dots, c_J$ and g in the form (2)

with $t = \sum_{i=1}^J c_j \deg(U_j)$.

For several value of k , we seek a polynomial $R(y) = \sum_{l=0}^k a_l y^l \in \mathbb{Z}[y]$ such that

$$\sup_{y>0} |U(y)R(y)|^{\frac{t}{r+k}} (y+2)^{-1} \leq e^{-m},$$

i.e., such that

$$\sup_{y>0} |U(y)R(y)| (y+2)^{-(r+k)/t}$$

is as small as possible.

We apply the LLL algorithm to the linear forms in $a_0, \dots, a_k$

$$U(y_i)R(y_i)(y_i+2)^{-(r+k)/t}$$

where y_i are control points uniformly distributed in the interval $[0,70]$, including the points where g has its least local minima. We get a polynomial R whose factors R_j are good candidates to enlarge the set of polynomials $(U_1, U_2, \dots, U_J)$. We only keep the polynomials R_j which have a nonzero coefficient c_j in the new optimized auxiliary function g . After optimization, some previous polynomials U_j may have a zero coefficient and are removed.

In order to get the constant of Theorem 3, we take k from 4 to 20 successively.

The polynomials and the coefficients c_j involved in the auxiliary function of Theorem 3 are listed in the Table 2 below.

4 Proof of Theorem 4

Let α be a nonzero algebraic integer all of whose conjugates lie in a sector $S_\theta = \{z \in \mathbb{C} \text{ such that } |\arg z| \leq \theta\}$ with $0 < \theta \leq 90^\circ$. The auxiliary functions f_i , $1 \leq i \leq 8$ are of the following type:

$$\forall z \in S_\theta, f(z) = \log(|z| + 1/|z|) - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)|,$$

where the c_j are positive real numbers and the polynomials Q_j are non zero polynomials in $\mathbb{Z}[z]$.

The function f is harmonic outside a finite set containing the roots of the polynomials Q_j and thus, it takes its minimum on the boundary of S_θ . It is clear that $f(z) \rightarrow \infty$ as $z \rightarrow \infty$. So, since $f(z) = f(\bar{z})$, it is sufficient to search the minimum of f on the upper edge of the sector S_θ where $z = xe^{i\theta}$ with $x > 0$.

The auxiliary function on the half line $R_\theta = \{z \in \mathbb{C}, z = xe^{i\theta}, x > 0\}$ is

$$f(z) = \log(x + 1/x) - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)|.$$

Note that, concerning the integer transfinite diameter, we deal here with the compact K replaced by the half line R_θ . But the weight $\varphi(z) = (|z| + 1/|z|)^{-1}$ ensures that the quantity $t_{\mathbb{Z}, \varphi}(R_\theta)$ is a finite number.

Then, for several values of k , we search a polynomial $R(z) = \sum_{l=0}^k a_l z^l \in \mathbb{Z}[z]$ such that

$$\sup_{x>0} |Q(z)R(z)|^{\frac{t}{r+k}} |x + 1/x|^{-1} \leq e^{-m},$$

Where m is the minimum of the function f . We search

$$\sup_{x>0} |Q(z)R(z)| |x + 1/x|^{-(r+k)/t}$$

is as small as possible.

But, here, $R(z)$ is not a real linear form in the unknown coefficients a_i . Thus we replace it by its real part and its imaginary part. Then, we apply LLL to the linear forms

$$|Q(z_n) \cdot \text{Re}(R(z_n))| \cdot |x_n + 1/x_n|^{-(r+k)/t} \quad \text{and} \quad |Q(z_n) \cdot \text{Im}(R(z_n))| \cdot |x_n + 1/x_n|^{-(r+k)/t}.$$

The numbers x_n are suitable control points in $[0, 50]$, including the points where f has its least local minima. Then we apply our recursive algorithm as described in Section 3.

The polynomials and the coefficients c_j involved in the auxiliary function of Theorem 4 are listed in the Table 3 below.

Annex: The program which computes $\lim_{n \rightarrow \infty} \nu(\beta_n^2)$

```
N=25;
lambda=vector(N,j,0);
lambda[1]=2/9; for(k=2,N,lambda[k]=lambda[k-1]/(1+lambda[k-1])^2);p=1;for(k=1,N,p=p*(1+lambda[k])^(1/2^k);print(2*p*(1+1/2)))
```

The computation gives:

```
3.31662479035540
3.43363162301252
3.47978396444099
3.49878633852976
3.50685371295751
3.51035762903346
3.51190629049069
3.51260021930633
3.51291460021634
3.51305831890415
3.51312451476539
3.51315519816299
3.51316949809199
3.51317619392804
3.51317934210340
3.51318082764469
3.51318153088927
3.51318186475827
3.51318202367455
3.51318209949318
3.51318213574302
3.51318215310816
3.51318216144157
3.51318216544726
3.51318216737561
```

Table 2: The polynomials involved in Theorem 3.

j	c_j	U_j
1	0.19631395	x
2	0.10343181	$x - 1$
3	0.00112589	$x - 2$
4	0.03608644	$x^2 - 3x + 1$
5	0.00182988	$x^2 - 4x + 1$
6	0.00084807	$x^2 - 4x + 2$
7	0.00197955	$2x^2 - 4x + 1$
8	0.00584071	$x^3 - 5x^2 + 6x - 1$
9	0.00091541	$x^3 - 6x^2 + 5x - 1$
10	0.01112510	$x^4 - 7x^3 + 13x^2 - 7x + 1$
11	0.00424427	$x^4 - 7x^3 + 14x^2 - 8x + 1$
12	0.00010144	$x^4 - 8x^3 + 15x^2 - 8x + 1$
13	0.00145858	$x^5 - 9x^4 + 26x^3 - 29x^2 + 11x - 1$
14	0.00050273	$x^6 - 11x^5 + 42x^4 - 67x^3 + 45x^2 - 12x + 1$
15	0.00055594	$x^7 - 13x^6 + 61x^5 - 131x^4 + 136x^3 - 66x^2 + 14x - 1$
16	0.00041956	$x^7 - 13x^6 + 61x^5 - 132x^4 + 138x^3 - 67x^2 + 14x - 1$
17	0.00129057	$x^7 - 13x^6 + 62x^5 - 137x^4 + 147x^3 - 73x^2 + 15x - 1$
18	0.00287103	$x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1$
19	0.00075792	$x^8 - 15x^7 + 85x^6 - 232x^5 + 328x^4 - 242x^3 + 91x^2 - 16x + 1$
20	0.00055993	$2x^8 - 26x^7 + 128x^6 - 308x^5 + 391x^4 - 265x^3 + 94x^2 - 16x + 1$
21	0.00019384	$x^9 - 17x^8 + 116x^7 - 411x^6 + 819x^5 - 934x^4 + 599x^3 - 206x^2 + 34x - 2$

Table 3: The auxiliary functions f_i , $1 \leq i \leq 8$

f_1	$N(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	2.0000000	0.27726826	0	$x - 1$
	2.5000000	0.13135075	0	$x - 2$
	2.6321199	0.00099184	23.749627	$x^4 - 8x^3 + 25x^2 - 35x + 19$
	2.4556520	0.00681346	26.267699	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	2.3094011	0.02113516	30.000000	$x^2 - 3x + 3$
f_2	$N(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	2.0000000	0.28694004	0	$x - 1$
	2.4556520	0.01101937	26.267699	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	2.6832816	0.01253213	26.565051	$x^2 - 4x + 5$
	2.5057826	0.00273581	27.277283	$x^4 - 7x^3 + 19x^2 - 22x + 10$
	2.4304356	0.00124547	29.171773	$x^8 - 13x^7 + 76x^6 - 258x^5 + 554x^4 - 769x^3 + 675x^2 - 344x + 79$
	2.3094011	0.04293284	30.000000	$x^2 - 3x + 3$
f_3	$N(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	2.0000000	0.24253217	0	$x - 1$
	2.3094011	0.01928768	30.000000	$x^2 - 3x + 3$
	2.3420541	0.00307938	37.011218	$x^8 - 11x^7 + 56x^6 - 168x^5 + 323x^4 - 406x^3 + 327x^2 - 155x + 34$
	2.3167326	0.00438622	37.168675	$x^6 - 8x^5 + 29x^4 - 58x^3 + 68x^2 - 44x + 13$
	2.2651285	0.01631844	39.985729	$x^4 - 5x^3 + 11x^2 - 11x + 5$
	2.1213203	0.02800549	45.000000	$x^2 - 2x + 2$
	2.2804413	0.00209584	46.139190	$x^8 - 10x^7 + 47x^6 - 131x^5 + 236x^4 - 281x^3 + 218x^2 - 102x + 23$
f_4	$N(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	2.1213203	0.072660766	45.000000	$x^2 - 2x + 2$
	2.2117398	0.027983260	45.309985	$x^4 - 4x^3 + 8x^2 - 7x + 3$
	2.2613295	0.015502898	45.459748	$x^6 - 6x^5 + 18x^4 - 29x^3 + 28x^2 - 15x + 4$
	2.3694703	0.000822075	47.464280	$x^8 - 9x^7 + 41x^6 - 112x^5 + 200x^4 - 233x^3 + 177x^2 - 80x + 18$
	2.3451610	0.000681118	50.081319	$x^8 - 9x^7 + 40x^6 - 106x^5 + 183x^4 - 207x^3 + 154x^2 - 69x + 16$
	2.0000000	0.041823516	60.000000	$x^2 - x + 1$
f_5	$N(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	2.1213203	0.063690701	45.000000	$x^2 - 2x + 2$
	2.2117398	0.007017163	45.309985	$x^4 - 4x^3 + 8x^2 - 7x + 3$
	2.0000000	0.077288797	60.000000	$x^2 - x + 1$
f_6	$N(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	2.0000000	0.074090899	60.000000	$x^2 - x + 1$
	2.1933236	0.002608550	63.681104	$x^4 - 3x^3 + 7x^2 - 7x + 5$
	2.2473333	0.016313417	68.667414	$x^4 - 2x^3 + 5x^2 - 3x + 2$
	2.1213203	0.060755551	69.295189	$x^2 - x + 2$
f_7	$N(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	2.1213203	0.077121906	69.295189	$x^2 - x + 2$
	2.0000000	0.068834995	90.000000	$x^2 + 1$
f_8	$N(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	2.1162455	0.043532810	82.588473	$x^4 - x^3 + 4x^2 - 2x + 3$
	2.0000000	0.083777873	90.000000	$x^2 + 1$

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