FLUCTUATIONS OF THE STIELTJES TRANSFORM OF THE EMPIRICAL SPECTRAL DISTRIBUTION OF SELFADJOINT POLYNOMIALS IN WIGNER AND DETERMINISTIC DIAGONAL MATRICES

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SERBAN BELINSCHI, MIREILLE CAPITAINE, SANDRINE DALLAPORTA, AND MAXIME FEVRIER

Abstract. We investigate the fluctuations around the mean of the Stieltjes transform of the empirical spectral distribution of any selfadjoint noncommutative polynomial in a Wigner matrix and a deterministic diagonal matrix. We obtain the convergence in distribution to a centred complex Gaussian process whose covariance is expressed in terms of operator-valued subordination functions.

1. Introduction

From the pioneering work of Wigner to the latest developments, properties of eigenvalues of Wigner matrices have been a major subject in Random Matrix Theory. The celebrated Wigner Theorem states that the empirical spectral distribution of a Wigner matrix – which means a complex Hermitian or real symmetric matrix whose entries are centered, with variance σ^2, and independent up to the symmetry condition – weakly converges in probability to the semicircle law μ_{σ^2} with density $(2\pi\sigma^2)^{-1}\sqrt{4\sigma^2 - x^2}\mathbf{1}_{[-2\sigma,2\sigma]}(x)$. It is then straightforward to deduce the convergence of linear statistics $N^{-1}\sum_{i=1}^{N} f(\lambda_i)$ of eigenvalues $\lambda_1, \dots, \lambda_N$ of $N \times N$ Wigner matrices associated to bounded continuous test functions $f : \mathbb{R} \to \mathbb{R}$ towards $\int_{\mathbb{R}} f d\mu_{\sigma^2}$.

Among the questions that have been addressed, fluctuations of linear statistics have attracted some attention. Initiated in the mid-nineties by investigations on traces of resolvents of real Wigner matrices, central limit theorems for linear spectral statistics of Wigner matrices progressively emerged. Sinai and Soshnikov [SS98], by the method of moments, and Bai and Yao [BY05] (see also Bao and Xie [BX16]), by applying a central limit theorem for martingale differences to the trace of the resolvent, obtained the fluctuations of linear spectral statistics associated to analytic test functions. These central limit theorems have been progressively extended to functions with less regularity: by Pastur and Lytova [LP09b, LP09a] using Fourier analysis, for functions with sufficiently fast decaying Fourier transform, by Bai, Wang and Zhou [BWZ09] for $\mathcal{C}^4$ functions, making use of Bernstein polynomials, by Shcherbina [Shc11], Sosoe and Wong [SW13] for $\mathcal{H}^s$ functions by a density argument, and Kopel [Kop15] by precise computations on complex Gaussian Wigner matrices. Recently, Bao and He [BH21] provided a near optimal rate of convergence for these central limit theorems in Kolmogorov-Smirnov distance.

Gaussian fluctuations with different scale, mean and variance also hold for linear spectral statistics when the entries of the Wigner matrix have an infinite fourth moment ([BGM16]; see also [BGGM14] for the case of non square integrable entries, in which case Wigner's Theorem fails to hold [BAG08]). When entries of the Wigner matrix are not identically distributed in such a way that their variances differ (these matrices are called band matrices or sometimes Wigner matrices with variance profile), fluctuations of linear spectral statistics have also been described (see [AJS19] and references therein). Fluctuations of linear spectral statistics were also investigated at the mesoscopic scale. In this type of study, the object of interest is $\sum_{i=1}^{N} f\big(N^\alpha(E - \lambda_i)\big)$, where $\alpha \in (0,1)$ and $E \in (-2\sigma, 2\sigma)$, see [BK99, LS15, HK17] for Wigner matrices and [LX21] for Wigner matrices with variance profile.

When a Wigner matrix is deformed by an additive (random or deterministic) perturbation, it is a natural question to characterize the effect of the perturbation on fluctuations of linear spectral statistics. This question was raised very early by Khorunzhy [Kho94], who proved in the case of deformed real Gaussian Wigner matrices that the fluctuations were still Gaussian, but without providing an explicit covariance kernel. After contributions by Dembo, Guionnet and Zeitouni [DGZ03] still in the Gaussian case by a dynamical approach, and by Su [Su13] in the case of a random diagonal perturbation on another scale, the topic has been recently revived. Motivated by the analysis of spherical Sherrington-Kirkpatrick

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model or by the problem of statistical detection of noisy signals, the case of a deterministic rank one perturbation has been considered by Baik and Lee [BL17], Baik, Lee and Wu [BLW18], Chung and Lee [CL19], Jung, Chung and Lee [CJL20, CJL21]. Diagonal perturbations with general rank were further investigated by Ji and Lee [JL19], Dallaporta and Fevrier [DF19]. Recently, Li, Schnelli and Xu provided the fluctuations of the linear spectral statistics for deformed Wigner matrices at mesoscopic scale [LSX20].

These papers make naturally use of free probability theory. Indeed, in an influential paper, Voiculescu gave evidence that the noncommutative probability theory he had previously introduced, called free probability theory, was a convenient framework for dealing with the convergence of the process of traces of noncommutative polynomials in several complex Gaussian Wigner matrices [Voi91]. Dykema proved then that polynomials in more general Wigner and deterministic matrices also fit in this framework [Dyk93]. See also the book [MS10] and the paper [BC17]. An analog of the free probability framework for dealing with fluctuations of the process of traces of noncommutative polynomials in Gaussian Wigner and deterministic matrices was designed in a series of papers [MS06, MSS07, CMSS07], building on the work of Mingo and Nica [MN04] (see also the dynamical approach to a close question by Cabanal-Duvillard and Guionnet [CD01, Gui02, CG01, DGZ03]). However, this so-called second order free probability theory does not seem to be the relevant framework to describe fluctuations of the process of traces of noncommutative polynomials in more general Wigner and deterministic matrices, as witnessed by the recent combinatorial analysis by Male, Mingo, Péché and Speicher [MMPS20].

In this paper, we tackle the slightly different question of fluctuations of the Stieltjes transform of the empirical spectral distribution of general polynomials in a Wigner matrix and a diagonal deterministic matrix. Less is known on this question beyond the case of linear polynomials, which is equivalent to that of deformed Wigner matrices. Using a well-known linearization trick to convert our initial general noncommutative polynomial with complex coefficients into a linear polynomial with matrix coefficients and then adapting the strategy used by Bai and Yao [BY05] for a single Wigner matrix and upgraded independently by Ji and Lee [JL19] and Dallaporta and Fevrier [DF19] to deal with deformed Wigner matrices, we establish a central limit theorem for the analytic process of traces of the resolvent. Since complex coefficients are replaced by matrix coefficients, one has to rely on the operator-valued version of free probability to express and analyze the limiting covariance kernel. The latter involves the logarithm of an operator and a highly non-trivial first task is to prove that this logarithm is well defined, making use of the contractivity of analytic self-maps on hyperbolic domains. Moreover, to adapt the strategy previously used for deformed Wigner matrices to general polynomials, many commutativity issues arise and require very technical preliminary results and new approaches. A typical example of these difficulties is the study of the second and third terms in the so-called hook process (see Sections 7.3.2 and 7.3.3): it requires a new trick consisting in writing these terms as the trace of the sum of images of matrix tensors by fit operators and in proving that each of these matrix tensors satisfies an approximated fixed point equation; then, a non-trivial study of spectral radius of operators is still necessary.

Besides the Introduction, the paper is organized as follows. Section 2 introduces the random matrices considered in this work whereas Section 3 presents our main results. Section 4 is devoted to basic background on linearization trick and operator-valued free probability theory that are central in our approach. In Section 5, we prove that the limiting covariance kernel involved in our central limit theorem is well defined and Section 6 gathers numerous preliminary results (bounds, concentration bounds, convergence results...) used in its proof. The proof of the convergence of finite-dimensional distributions of the process of Stieltjes transforms is detailed in Section 7. In Section 8, we establish the tightness of this sequence of random analytic functions. Three appendices conclude the paper: the first one gathers tools from elementary linear algebra, random quadratic forms, martingale theory and complex analysis used in the proofs; the second one establishes results on moments and norm of Wigner matrices that are used throughout the paper; the last one details the truncation argument allowing to assume that entries of Wigner matrices considered in this paper are almost surely bounded by a sequence slowly converging to 0.

2. Presentation of the model

The complex algebra $\mathbb{C}\langle t_1, \ldots, t_n \rangle$ of polynomials with complex coefficients in n noncommuting indeterminates $t_1, \ldots, t_n$ becomes a $*$-algebra by anti-linear extension of

$$(t_{i_1} t_{i_2} \cdots t_{i_l})^* = t_{i_l} \cdots t_{i_2} t_{i_1}, \quad i_1, \ldots, i_l = 1, \ldots, n, \; l \in \mathbb{N}.$$

We consider, on a probability space, a sequence of random matrices

$$X_N := P(W_N, D_N), \quad N \in \mathbb{N},$$

where :

(1) $P \in \mathbb{C}\langle t_1, t_2\rangle$ is a selfadjoint polynomial in two noncommuting indeterminates;
(2) entries $\{W_{ij}\}_{1\le i\le j\le N}$ of the $N \times N$ Hermitian matrix W_N are independent random variables;
(3) off-diagonal entries $\{W_{ij}\}_{1\le i<j\le N}$ of W_N are identically distributed complex random variables such that, for some $\varepsilon > 0$, $\mathbb{E}[|\sqrt{N}W_{ij}|^{6(1+\varepsilon)}] \le C_6$. We assume that $\mathbb{E}[W_{ij}] = 0$ and that
$$\sigma_N^2 := \mathbb{E}[|W_{ij}|^2] \ge 0, \quad \theta_N := \mathbb{E}[W_{ij}^2] \in \mathbb{C}, \quad \kappa_N := \mathbb{E}[|W_{ij}|^4] - 2\sigma_N^4 - |\theta_N|^2 \in \mathbb{R}.$$
satisfy
$$\lim_{N\to+\infty} N\sigma_N^2 = \sigma^2 > 0, \quad \lim_{N\to+\infty} N\theta_N = \theta \in \mathbb{R}, \quad \lim_{N\to+\infty} N^2\kappa_N = \kappa \in \mathbb{R}.$$
The assumption $\theta \in \mathbb{R}$ means that correlations between the real and imaginary parts of off-diagonal entries of W_N are negligible.
(4) diagonal entries $\{W_{ii}\}_{1\le i\le N}$ of W_N are identically distributed real random variables such that, for some $\varepsilon > 0$, $\mathbb{E}[|\sqrt{N}W_{ii}|^{4(1+\varepsilon)}] \le C_4$. We assume that $\mathbb{E}[W_{ii}] = 0$ and that $\tilde{\sigma}_N^2 := \mathbb{E}[W_{ii}^2] \ge 0$ satisfies $\lim_{N\to+\infty} N\tilde{\sigma}_N^2 = \tilde{\sigma}^2 > 0$;
(5) D_N is a $N \times N$ deterministic real diagonal matrix. We assume that $\sup_{N\in\mathbb{N}} \|D_N\| < \infty$ and, for some Borel probability measure ν on $\mathbb{R}$,
$$\nu_N := \frac{1}{N} \sum_{\lambda\in\mathrm{sp}(D_N)} \delta_\lambda \Rightarrow \nu.$$

Here and below, we use the notation $\mathrm{sp}(A)$ for the (multi)set of eigenvalues (counted with their algebraic multiplicity) of a square matrix A. We will also assume that all entries of W_N are almost surely bounded by δ_N, where $(\delta_N)_{N\in\mathbb{N}}$ is a sequence of positive numbers slowly converging to 0 (at rate less than $N^{-\epsilon}$ for any $\epsilon > 0$); this may be assumed without loss of generality, as proved in Appendix C. We will use the notation $m_N := \mathbb{E}[|W_{ij}|^4] = \kappa_N + 2\sigma_N^4 + |\theta_N|^2$. In Assumptions (3) and (4), we ask the entries to be identically distributed. This assumption does not seem to be necessary for our main result to hold, but leads to a simplification of truncation-centering-homogeneization arguments. Therefore, for the readibility of the paper, we will not pursue the task to relax this assumption.

We are interested in the empirical spectral measure μ_N of X_N, defined by:

$$\mu_N := \frac{1}{N} \sum_{\lambda\in\mathrm{sp}(X_N)} \delta_\lambda.$$

More precisely, we study the fluctuations of the Stieltjes transform $\mathbb{C} \setminus \mathbb{R} \ni z \mapsto \int_{\mathbb{R}} (z - x)^{-1}\mu_N(dx)$ of μ_N around its mean.

3. Main result

Before stating our main result, we introduce the necessary material.

3.1. **Definitions.**

3.1.1. *Free probability.* Let $\mathcal{A}$ be a complex algebra with a unit $1_{\mathcal{A}}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ be a linear functional satisfying $\varphi(1_{\mathcal{A}}) = 1$. One usually calls $(\mathcal{A}, \varphi)$ a noncommutative probability space and its elements noncommutative random variables.

We say that a noncommutative random variable $a \in (\mathcal{A}, \varphi)$ is distributed according to a Borel probability measure μ on $\mathbb{R}$ when $\varphi(a^n) = \int_{\mathbb{R}} t^n \mu(dt)$ holds for all $n \in \mathbb{N}$. In particular, a semicircular element with mean 0 and variance σ^2 is a noncommutative random variable distributed according to the absolutely continuous probability measure with density $(2\pi\sigma^2)^{-1}\sqrt{4\sigma^2 - t^2}\mathbf{1}_{[-2\sigma;2\sigma]}(t)$ in some noncommutative probability space.

Two noncommutative random variables s, d in a noncommutative probability space $(\mathcal{A}, \varphi)$ are said to be freely independent if the following holds : for each $n \in \mathbb{N}$ and any polynomials with complex coefficients $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in \mathbb{C}[t]$,

$$\varphi((P_1(s) - \varphi(P_1(s))1_{\mathcal{A}})(Q_1(d) - \varphi(Q_1(d))1_{\mathcal{A}}) \cdots (P_n(s) - \varphi(P_n(s))1_{\mathcal{A}})(Q_n(d) - \varphi(Q_n(d))1_{\mathcal{A}})) = 0.$$

3.1.2. *Linearization.* A powerful tool to deal with noncommutative polynomials in random matrices or in operators is the so-called "linearization trick" that goes back to Haagerup and Thorbjørnsen [HT05, HST06] in the context of operator algebras and random matrices (see [MS10]). We use the procedure introduced in [And13, Proposition 3].

Given a polynomial $P \in \mathbb{C}\langle t_1, \dots, t_n\rangle$, we call *linearization* of P any

$$L_P := \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle t_1, \dots, t_n\rangle,$$

where

(1) $m \in \mathbb{N}$,
(2) $Q \in M_{m-1}(\mathbb{C}) \otimes \mathbb{C}\langle t_1, \dots, t_n\rangle$ is invertible,
(3) u is a row vector and v is a column vector, both of size $m-1$ with entries in $\mathbb{C}\langle t_1, \dots, t_n\rangle$,
(4) the polynomial entries in Q, u and v all have degree ≤ 1,
(5) $P = -uQ^{-1}v$.

Given a selfadjoint polynomial $P \in \mathbb{C}\langle t_1, \dots, t_n\rangle$, it is described in Section 4 of [BC17], from Anderson [And13] (see also [Mai17]), how to build a particular selfadjoint linearization $L_P \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle t_1, \dots, t_n\rangle$ of P. We call this particular linearization the *canonical* linearization of P.

3.2. **Statement of results.** Let $P \in \mathbb{C}\langle t_1, t_2\rangle$ be the selfadjoint polynomial in two noncommuting indeterminates involved in the definition of X_N (see (1) in Section 2). Let $L_P = \gamma_0 \otimes 1 + \gamma_1 \otimes t_1 + \gamma_2 \otimes t_2$ be the canonical linearization of P ; γ_0, γ_1, γ_2 are selfadjoint matrices in $M_m(\mathbb{C})$.

Let s be a semicircular element with mean 0 and variance σ^2 which is freely independent from a noncommutative variable d distributed according to the probability measure ν (see (5) in Section 2) in some noncommutative probability space $(\mathcal{A}, \varphi)$. We denote $\mathrm{id}_m : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ the identity map and define a map ω by

$$\omega(b) = b - \sigma^2 \gamma_1 (\mathrm{id}_m \otimes \varphi)\left[(b \otimes 1_{\mathcal{A}} - \gamma_1 \otimes s - \gamma_2 \otimes d)^{-1} \right] \gamma_1,$$

for any $b \in M_m(\mathbb{C})$ such that $b \otimes 1_{\mathcal{A}} - \gamma_1 \otimes s - \gamma_2 \otimes d$ is invertible in $M_m(\mathbb{C}) \otimes \mathcal{A}$. As explained in Lemma 4 below, ω is well-defined on $\{ze_{11} - \gamma_0, z \in \mathbb{C} \setminus \mathbb{R}\}$. Section 4.3 will describe the occurence of this so-called subordination map.

We denote, for $\beta_1, \beta_2 \in \{\beta \in M_m(\mathbb{C}), \omega(\beta) \otimes I_m - \gamma_2 \otimes d \text{ is invertible}\}$, by $T_{\{\beta_1,\beta_2\}}$ the operator defined on $M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$ by

$$T_{\{\beta_1,\beta_2\}}(x) = \int_{\mathbb{R}} ((\omega(\beta_1) - t\gamma_2)^{-1}\gamma_1 \otimes I_m) x (I_m \otimes \gamma_1(\omega(\beta_2) - t\gamma_2)^{-1}) d\nu(t), \quad x \in M_m(\mathbb{C}) \otimes M_m(\mathbb{C}).$$

The limiting distribution of our central limit theorem involves the logarithms of some operators of the form

$$\mathrm{id}_m \otimes \mathrm{id}_m - T : M_m(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_m(\mathbb{C}),$$

with T some scalar multiples of $T_{\{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0\}}$. Thus, our first result consists in proving that these logarithms are well-defined. Since $\log(\mathrm{id}_m \otimes \mathrm{id}_m - T)$ is well defined by the convergent series expansion

$$\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - T\right] = -\sum_{k=1}^{\infty} \frac{1}{k} T^k \tag{1}$$

as soon as the spectral radius of T is less than 1 (see (6.5.11) in [HJ91]), one proves the following proposition.

Proposition 1. *For any $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$, the spectrum of the operator*

$$T_{\{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0\}} : M_m(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$$

is included in the open disk of radius σ^{-2}.

Denote by $\mathcal{H}(\mathbb{C}\setminus\mathbb{R})$ the space of complex analytic functions on $\mathbb{C}\setminus\mathbb{R}$, endowed with the uniform topology on compact sets. The space $\mathcal{H}(\mathbb{C} \setminus \mathbb{R})$ is equipped with the (topological) Borel σ-field $\mathcal{B}(\mathcal{H}(\mathbb{C} \setminus \mathbb{R}))$.
We are now in position to state our central limit theorem.

Theorem 2. *Let X_N be the random matrix introduced in Section 2. For any $z \in \mathbb{C} \setminus \mathbb{R}$, set*

$$\xi_N(z) := \operatorname{Tr}\left((zI_N - X_N)^{-1}\right) - \mathbb{E}\left[\operatorname{Tr}\left((zI_N - X_N)^{-1}\right)\right].$$

The sequence of $\mathcal{H}(\mathbb{C}\setminus\mathbb{R})$-valued random variables $(\xi_N)_{N\in\mathbb{N}}$ converges in distribution to a centred complex Gaussian process $\{\mathcal{G}(z), z \in \mathbb{C} \setminus \mathbb{R}\}$ determined by $\overline{\mathcal{G}(z)} = \mathcal{G}(\bar{z})$ and

$$\mathbb{E}\left(\mathcal{G}(z_1)\mathcal{G}(z_2)\right) = \Gamma(z_1, z_2) := \frac{\partial^2}{\partial z_1 \partial z_2}\gamma(z_1, z_2), \quad z_1, z_2 \in \mathbb{C} \setminus \mathbb{R},$$

$$\begin{aligned}\gamma(z_1, z_2) = &- \operatorname{Tr} \otimes \operatorname{Tr}\left\{\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - \sigma^2 T_{\{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0\}}\right](I_m \otimes I_m)\right\} \\ &- \operatorname{Tr} \otimes \operatorname{Tr}\left\{\log\left[\mathrm{id}_m \otimes \mathrm{id}_m - \theta T_{\{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0\}}\right](I_m \otimes I_m)\right\} \\ &+ (\tilde{\sigma}^2 - \sigma^2 - \theta) \operatorname{Tr} \otimes \operatorname{Tr}\{T_{\{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0\}}(I_m \otimes I_m)\} \\ &+ \kappa/2 \operatorname{Tr} \otimes \operatorname{Tr}\{T^2_{\{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0\}}(I_m \otimes I_m)\}\end{aligned}$$

4. Review of background

In this section, we gather properties which will be used several times in the sequel.

4.1. **Generalized resolvent.** Let $\mathcal{M}$ be a unital C^*-algebra and $\mathcal{B}$ be a unital C^*-subalgebra of $\mathcal{M}$. For $A \in \mathcal{M}$, we denote by $\mathcal{R}A = (A + A^*)/2$ and $\mathcal{I}A = (A - A^*)/2i$ the real and imaginary parts of A, so $A = \mathcal{R}A + i\mathcal{I}A$. For a selfadjoint operator $A \in \mathcal{M}$, we write $A \geq 0$ if the spectrum of A is contained in $[0, +\infty)$ and $A > 0$ if the spectrum of A is contained in $(0, +\infty)$. The operator upper half-plane of $\mathcal{B}$ is the set $\mathbb{H}^+(\mathcal{B}) = \{b \in \mathcal{B} \colon \mathcal{I}b > 0\}$.

The generalized resolvent of an element $A \in \mathcal{M}$ in this context is the analytic map R defined on the open subset $\{b \in \mathcal{B}, b - A \text{ is invertible in } \mathcal{M}\}$ of $\mathcal{B}$ by $R(b) := (b - A)^{-1}$. Note that the generalized resolvent of a selfadjoint element is in particular defined on the operator upper half-plane $\mathbb{H}^+(\mathcal{B})$ and satisfies

$$\|R(b)\| \leq \|(\mathcal{I}b)^{-1}\|, \quad b \in \mathbb{H}^+(\mathcal{B}). \tag{2}$$

In particular, when $\mathcal{M} = M_n(\mathbb{C})$ and $\mathcal{B} = \mathbb{C}I_n \simeq \mathbb{C}$, the resolvent $R : z \mapsto (zI_n - A)^{-1}$ of a $n \times n$ Hermitian matrix with complex entries $A \in M_n(\mathbb{C})$ is defined on $\mathbb{C} \setminus \mathbb{R}$ and satisfies

$$\|R(z)\| \leq |\mathcal{I}z|^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{3}$$

The following Lemma is elementary but useful.

Lemma 3 (Resolvent identity). *Let A_1 and A_2 be elements of $\mathcal{M}$ and denote by R_1 and R_2 their respective resolvents. Then, for all b_1, b_2 in the respective domains of R_1 and R_2,*

$$R_1(b_1) - R_2(b_2) = R_1(b_1)\big(b_2 - b_1 + A_1 - A_2\big)R_2(b_2).$$

4.2. **Linearization.** In this Section, we collect a few properties of linearizations of polynomials in non-commuting indeterminates introduced above.

Lemma 4. *Let $P = P^* \in \mathbb{C}\langle t_1, \ldots, t_n\rangle$ and let $L_P \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle t_1, \ldots, t_n\rangle$ be a linearization of P with the properties outlined above. Let $y = (y_1, \ldots, y_n)$ be a n-tuple of selfadjoint operators in a unital C^*-algebra $\mathcal{A}$. Then, for any $z \in \mathbb{C}$, $ze_{11} \otimes 1_{\mathcal{A}} - L_P(y)$ is invertible if and only if $z1_{\mathcal{A}} - P(y)$ is invertible and we have*

$$(ze_{11} \otimes 1_{\mathcal{A}} - L_P(y))^{-1} = \begin{pmatrix} (z1_{\mathcal{A}} - P(y))^{-1} & \star \\ \star & \star \end{pmatrix}. \tag{4}$$

Beyond the equivalence described above, we will use the following bound.

Lemma 5. [BBC19] *Let $z \in \mathbb{C}$ be such that $z1_{\mathcal{A}} - P(y)$ is invertible. There exist two polynomials Q_1 and Q_2 in n commuting indeterminates, depending only on L_P, such that*

$$\left\|(ze_{11} \otimes 1_{\mathcal{A}} - L_P(y))^{-1}\right\| \leq Q_1\left(\|y_1\|, \ldots, \|y_n\|\right)\left\|(z1_{\mathcal{A}} - P(y))^{-1}\right\| + Q_2\left(\|y_1\|, \ldots, \|y_n\|\right).$$

4.3. **Operator-valued free probability theory.** There exists an extension of free probability theory, operator-valued free probability theory, which still shares the basic properties of free probability but is much more powerful because of its wider domain of applicability. The concept of freeness with amalgamation and some of the relevant analytic transforms were introduced by Voiculescu in [Voi95].

Definition 6. *Let $\mathcal{M}$ be a unital complex algebra and $\mathcal{B} \subset \mathcal{M}$ be a unital subalgebra. A linear map $E : \mathcal{M} \to \mathcal{B}$ is a conditional expectation if $E(b) = b$ for all $b \in \mathcal{B}$ and $E(b_1 A b_2) = b_1 E(A) b_2$ for all $A \in \mathcal{M}$ and $b_1, b_2 \in \mathcal{B}$. Then $(\mathcal{M}, E)$ is called a $\mathcal{B}$-valued probability space. If in addition $\mathcal{M}$ is a C^*-algebra (von Neumann algebra), $\mathcal{B}$ is a C^*-subalgebra (von Neumann subalgebra) of $\mathcal{M}$, then we have a $\mathcal{B}$-valued C^*-probability space ($\mathcal{B}$-valued W^*-probability space).*

In our applications, the algebra $\mathcal{B}$ is (isomorphic to) $M_m(\mathbb{C})$ for some $m \in \mathbb{N}$. More precisely, let $\mathcal{A}$ be a von Neumann algebra endowed with a normal faithful tracial state τ, and let $m \in \mathbb{N}$. Then $M_m(\mathbb{C})$ can be identified with the subalgebra $M_m(\mathbb{C}) \otimes 1_{\mathcal{A}}$ of $M_m(\mathcal{A}) = M_m(\mathbb{C}) \otimes \mathcal{A}$. Moreover, the von Neumann algebra $M_m(\mathcal{A})$ is endowed with the normal faithful tracial state $m^{-1} \operatorname{Tr} \otimes \tau$, and $\mathrm{id}_m \otimes \tau$ is the trace-preserving conditional expectation from $M_m(\mathcal{A})$ to $M_m(\mathbb{C})$. In other words, $(M_m(\mathcal{A}), \mathrm{id}_m \otimes \tau)$ is a $M_m(\mathbb{C})$-valued W^*-probability space. As mentioned in Section 3 of [BBC19], the distributional limits of random matrices as considered in our models are realized in II_1-factors, with respect to their unique normal faithful tracial states, so that there is no loss of generality in assuming $(\mathcal{A}, \tau)$ to be a von Neumann algebra endowed with a normal faithful tracial state (also called W^*-probability space).

Definition 7. *Let $(\mathcal{M}, E)$ be a $\mathcal{B}$-valued probability space. The $\mathcal{B}$-valued distribution of a noncommutative random variable $A \in \mathcal{M}$ is given by all $\mathcal{B}$-valued moments $E(Ab_1Ab_2 \cdots b_{n-1}A)$, $n \in \mathbb{N}, b_1, \ldots, b_{n-1} \in \mathcal{B}$.*

If s is a (scalar-valued) semicircular element with mean 0 and variance σ^2 in some W^*-probability space $(\mathcal{A}, \tau)$, then, for any Hermitian matrix $\gamma \in M_m(\mathbb{C})$, $\gamma \otimes s \in M_m(\mathcal{A})$ is a $M_m(\mathbb{C})$-valued semicircular element in the $M_m(\mathbb{C})$-valued probability space $(M_m(\mathcal{A}), \mathrm{id}_m \otimes \tau)$, in the sense of [Spe98]. The $\mathcal{B}$-valued distribution of a general centred $\mathcal{B}$-valued semicircular element $S \in \mathcal{M}$ is uniquely determined by its operator-valued variance $\eta \colon b \mapsto E(SbS)$; a characterization in terms of moments and cumulants via η is provided by Speicher in [Spe98]. The operator-valued variance of the $M_m(\mathbb{C})$-valued semicircular element $\gamma \otimes s$ is $\eta \colon b \mapsto \sigma^2 \gamma b \gamma$.

As in scalar-valued free probability, one defines [Voi95] *freeness with amalgamation* over $\mathcal{B}$ via an algebraic relation similar to freeness, but involving E and noncommutative polynomials with coefficients in $\mathcal{B}$.

Definition 8. *Let $(\mathcal{M}, E)$ be a $\mathcal{B}$-valued probability space. Let $(\mathcal{A}_i)_{i \in I}$ be a family of subalgebras with $\mathcal{B} \subset \mathcal{A}_i$ for all $i \in I$. The subalgebras $(\mathcal{A}_i)_{i \in I}$ are free with respect to E or free with amalgamation over $\mathcal{B}$ if $E(A_1 \cdots A_n) = 0$ whenever $A_j \in \mathcal{A}_{i_j}$, $i_j \in I$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and $E(A_j) = 0$, for all j.*
Noncommutative random variables in $\mathcal{M}$ or subsets of $\mathcal{M}$ are free with amalgamation over $\mathcal{B}$ if the algebras generated by $\mathcal{B}$ and the variables or the algebras generated by $\mathcal{B}$ and the subsets, respectively, are so.

The following result from [NSS02] explains why the particular case $\mathcal{B} = M_m(\mathbb{C})$, $\mathcal{M} = M_m(\mathcal{A})$, $E = \mathrm{id}_m \otimes \tau$, where $(\mathcal{A}, \tau)$ is a W^*-probability space, is relevant in our work using linearizations of polynomials.

Proposition 9. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, let $a_1, \ldots, a_n \in (\mathcal{A}, \varphi)$ be freely independent noncommutative random variables and let $m \in \mathbb{N}$. Then the map $\mathrm{id}_m \otimes \varphi \colon M_m(\mathcal{A}) \to M_m(\mathbb{C})$ is a conditional expectation, and $\alpha_1 \otimes a_1, \ldots, \alpha_n \otimes a_n$ are free with amalgamation over $M_m(\mathbb{C})$ for any $\alpha_1, \ldots, \alpha_n \in M_m(\mathbb{C})$.*

The analytic subordination phenomenon for free convolutions was first noted by Voiculescu and Biane in the scalar case and later approached from an abstract coalgebra point of view by Voiculescu in [Voi00] and this approach extends the results to the operator-valued case. In [BMS17], Belinschi, Mai and Speicher develop an analytic theory.

Proposition 10. [Voi00],[BMS17]*(see Theorem 5 p 259* [MS10]*) Let $(\mathcal{M}, E)$ be a $\mathcal{B}$-valued C^*-probability space. Let $A_1, A_2 \in \mathcal{M}$ be selfadjoint noncommutative random variables which are free with amalgamation over $\mathcal{B}$.*
There exists a unique pair of Fréchet analytic maps $\omega_1, \omega_2 \colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ such that, for all $b \in \mathbb{H}^+(\mathcal{B})$,

(1) *$\mathcal{I}\omega_j(b) \geq \mathcal{I}b$, $\omega_j(b^*) = \omega_j(b)^*$, $j = 1, 2$;*

(2) $E[(b-(A_1+A_2))^{-1}] = E[(\omega_1(b)-A_1)^{-1}] = E[(\omega_2(b)-A_2)^{-1}]$;
(3) $E[(b-(A_1+A_2))^{-1}]^{-1} + b = \omega_1(b) + \omega_2(b)$.

Moreover, if $b \in \mathbb{H}^+(\mathcal{B})$, then $\omega_1(b)$ is the unique fixed point of the map $f_b : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ defined by $f_b(\omega) = h_{A_2}(h_{A_1}(\omega)+b)+b$, where $h_{A_i}(b) = E[(b-A_i)^{-1}]^{-1} - b$ and $\omega_1(b) = \lim_{k\to+\infty} f_b^{\circ k}(w)$, for any $\omega \in \mathbb{H}^+(\mathcal{B})$.

Moreover, if $\mathcal{M}$ is a W^*-probability space and $\mathcal{B} \subset \mathcal{D} \subset \mathcal{M}$ are von Neumann subalgebras (hence unital by definition) such that $A_2 \in \mathcal{D}$ and $\mathcal{D}$ is free with amalgamation over $\mathcal{B}$ from A_1 (with respect to the trace-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{B}$), then the following strengthened result holds:

$$E_{\mathcal{D}}[(b-(A_1+A_2))^{-1}] = (\omega_2(b)-A_2)^{-1}, \quad b \in \mathbb{H}^+(\mathcal{B}),$$

where $E_{\mathcal{D}}$ is the trace-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{D}$.

If, in Proposition 10, A_1 is a centred $\mathcal{B}$-valued semicircular element with operator-valued variance η, the subordination function ω_2 has a more explicit form (see [MS10, Chapter 9] and the end of the proof of Theorem 8.3 in [ABFN13]):

$$\omega_2(b) = b - \eta(E[(b-(A_1+A_2))^{-1}]), \quad b \in \mathbb{H}^+(\mathcal{B}). \tag{5}$$

It follows that ω_2 may be analytically extended to the open subset $\{b \in \mathcal{B}, b-(A_1+A_2) \text{ is invertible in } \mathcal{A}\}$. Moreover, for b in the connected component of $\{b \in \mathcal{B}, b-(A_1+A_2) \text{ is invertible in } \mathcal{A}\}$ containing $\mathbb{H}^+(\mathcal{B})$,

$$E_{\mathcal{D}}[(b-(A_1+A_2))^{-1}] = (\omega_2(b)-A_2)^{-1} \tag{6}$$

holds. Note also that $\omega_2(b)$ satisfies the fixed point equation

$$\omega_2(b) = b - \eta(E[(\omega_2(b)-A_2)^{-1}]). \tag{7}$$

5. Definition of the limiting object: proof of Proposition 1

Our strategy of proof for Proposition 1 is the following : in Section 5.1, one proves that the operator $\sigma^2 T_{\{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0\}} : M_m(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$ has the same eigenvalues as the operator $u_{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0}$, where $u_{\beta_1,\beta_2} : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ is defined by

$$b \mapsto \sigma^2(\mathrm{id}_m \otimes \tau)\left[(\omega(\beta_1)\otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}((\gamma_1 b \gamma_1)\otimes 1_{\mathcal{A}})(\omega(\beta_2)\otimes 1_{\mathcal{A}} - \gamma_2\otimes d)^{-1}\right].$$

The key point of this section is then to prove the following Proposition 11.

Proposition 11. *Let $s, d \in (\mathcal{A}, \tau)$ be freely independent selfadjoint noncommutative random variables in a W^*-probability space $(\mathcal{A}, \tau)$. Assume that s is a semicircular element with mean 0 and variance σ^2. Let also $\gamma_0, \gamma_1, \gamma_2$ be non-zero selfadjoint matrices in $M_m(\mathbb{C})$ such that the lower right $(m-1)\times(m-1)$ corner of $\gamma_0 \otimes 1_{\mathcal{A}} + \gamma_1 \otimes s + \gamma_2 \otimes d$ is invertible in $M_{m-1}(\mathcal{A})$. For any $(z_1, z_2) \in \mathbb{C}^2$ such that $(z_je_{11}-\gamma_0)\otimes 1_{\mathcal{A}} - \gamma_1\otimes s - \gamma_2 \otimes d$ is invertible in $M_m(\mathcal{A})$, $j = 1, 2$, the spectrum of the linear operator $u_{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0} : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ defined by*

$$b \mapsto \sigma^2(\mathrm{id}_m \otimes \tau)\left[(\omega(z_1e_{11}-\gamma_0)\otimes 1_{\mathcal{A}} - \gamma_2\otimes d)^{-1}((\gamma_1 b\gamma_1)\otimes 1_{\mathcal{A}})(\omega(z_2e_{11}-\gamma_0)\otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}\right]$$

is included in the open unit disk. Recall that

$$\omega(\beta) = \beta - \sigma^2\gamma_1(\mathrm{id}_m\otimes\tau)\left[(\beta\otimes 1_{\mathcal{A}} - \gamma_1\otimes s - \gamma_2\otimes d)^{-1}\right]\gamma_1.$$

Note that the conclusion of Proposition 11 holds for any $(z_1, z_2) \in (\mathbb{C}\backslash\mathbb{R})^2$ as explained at the beginning of the proof.

The first step in the proof of Proposition 11, detailed in Section 5.2, consists in proving that the spectrum of u_{β_1,β_2} is included in the open unit disk when $\beta_1, \beta_2 \in M_m(\mathbb{C})$ have positive-definite or negative-definite imaginary parts. Then, in a second step in Section 5.3, by using the maximum principle for plurisubharmonic functions, we deduce from the first step the result for $\beta_1 = z_1e_{11}-\gamma_0$, $\beta_2 = z_2e_{11}-\gamma_0$, as required.

5.1. **An equality of spectra.** The aim of this section is to prove that the operators $\sigma^2 T_{\{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0\}}$ and $u_{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0}$ have the same eigenvalues (not counting multiplicities).

Proposition 12. *The operators* $\sigma^2 T_{\{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0\}} : M_m(\mathbb{C})\otimes M_m(\mathbb{C}) \to M_m(\mathbb{C})\otimes M_m(\mathbb{C})$ *and* $u_{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0} : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ *have the same eigenvalues.*

Proof. We will use the following identifications between algebras $M_n(\mathbb{C})\otimes M_n(\mathbb{C})$ and $\mathcal{L}(M_n(\mathbb{C}))$: define an isomorphism of algebras $M_n(\mathbb{C})\otimes M_n(\mathbb{C}) \to \mathcal{L}(M_n(\mathbb{C}))$ by requiring that the image of $A\otimes B$ is the operator $X \mapsto AXB^T$ for any $A, B \in M_n(\mathbb{C})$. Using these identifications, one may observe that the following equalities hold in $M_m(\mathbb{C})^{\otimes 4}$:

$$\begin{aligned}
&\sigma^2 T_{\{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0\}}\\
&\quad= \sigma^2 \int_{\mathbb{R}} (\omega(z_1e_{11}-\gamma_0)-t\gamma_2)^{-1}\gamma_1 \otimes I_m \otimes I_m \otimes (\omega(z_2e_{11}-\gamma_0)-t\gamma_2)^{-1}\gamma_1 d\nu(t)\\
&\quad= F\otimes F(I_m \otimes \sigma^2 \int_{\mathbb{R}} (\omega(z_1e_{11}-\gamma_0)-t\gamma_2)^{-1}\gamma_1 \otimes (\omega(z_2e_{11}-\gamma_0)-t\gamma_2)^{-1}\gamma_1 d\nu(t)\otimes I_m)\\
&\quad= F\otimes F(I_m \otimes u_{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0} \otimes I_m),
\end{aligned}$$

where $F : M_m(\mathbb{C})\otimes M_m(\mathbb{C}) \to M_m(\mathbb{C})\otimes M_m(\mathbb{C})$ is the automorphism of the algebra $M_m(\mathbb{C})\otimes M_m(\mathbb{C})$ determined by $F(A\otimes B) = B\otimes A$. It follows that $\sigma^2 T_{\{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0\}} : M_m(\mathbb{C})\otimes M_m(\mathbb{C}) \to M_m(\mathbb{C})\otimes M_m(\mathbb{C})$ on the one hand and $u_{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0} : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ on the second hand have the same minimal polynomial, hence the same eigenvalues (but not with the same multiplicities though). □

5.2. **First step in the proof of Proposition 11.** Consider an arbitrary C^*-algebra $\mathcal{B}$, a completely positive map $\eta\colon \mathcal{B}\to\mathcal{B}$, and a centred $\mathcal{B}$-valued semicircular element S with operator-valued variance η. Recall that a completely positive map η is automatically completely bounded: $\|\eta\|_{\mathrm{cb}} := \sup_{m\in\mathbb{N}} \|\mathrm{id}_m \otimes \eta\| < \infty$. Assume that $D = D^*$ is free from S with amalgamation over $\mathcal{B}$ with respect to the conditional expectation E. As in (5), we may write

$$\omega(b) = b - \eta\left(E\left[(b-S-D)^{-1}\right]\right) = b - \eta\left(E\left[(\omega(b)-D)^{-1}\right]\right).$$

Proposition 13. *For any* β_1, β_2 *in* $\mathcal{B}$ *such that either* $\{\beta_1, \beta_2 \in \mathbb{H}^+(\mathcal{B})\}$, *or* $\{-\beta_1, -\beta_2 \in \mathbb{H}^+(\mathcal{B})\}$, *or* $\{\beta_1, -\beta_2 \in \mathbb{H}^+(\mathcal{B})\}$, *or* $\{-\beta_1, \beta_2 \in \mathbb{H}^+(\mathcal{B})\}$, *the spectrum of the linear operator* u_{β_1,β_2} *on* $\mathcal{B}$ *defined by*

$$v \mapsto E\left[(\omega(\beta_1)-D)^{-1}\eta(v)(\omega(\beta_2)-D)^{-1}\right]$$

is included in the open unit disk.

The context that will be of interest in our paper corresponds to $\mathcal{B} = M_m(\mathbb{C})$, $S = \gamma_1\otimes s, D = \gamma_2\otimes d$ for an arbitrary selfadjoint noncommutative random variable d, freely independent from the semicircular element s with mean 0 and variance σ^2 in some W^*-probability space $(\mathcal{A}, \tau)$. In that case $\eta : b \mapsto \sigma^2\gamma_1 b\gamma_1$ and $E = \mathrm{id}_m\otimes\tau$. The proof of Proposition 13 given above would benefit little in terms of simplification from the assumption that $\mathcal{B}$ is finite dimensional, so there is no reason not to give it in full generality. The idea of the proof is to make use of the contractivity of analytic self-maps on hyperbolic domains.

Proof. The cases $\{\beta_1, \beta_2 \in \mathbb{H}^+(\mathcal{B})\}$ and $\{-\beta_1, -\beta_2 \in \mathbb{H}^+(\mathcal{B})\}$ are covered in [Bel19, Proposition 4.1]. Thus, we will focus exclusively on the case $\{-\beta_1, \beta_2 \in \mathbb{H}^+(\mathcal{B})\}$, the case $\{\beta_1, -\beta_2 \in \mathbb{H}^+(\mathcal{B})\}$ being identical. However, the reader will find that the methods we employ in our proof below cover the other two cases with virtually no modification.

We assume without loss of generality that $\mathcal{I}\beta_1 < 0, \mathcal{I}\beta_2 > 0$. Consider the convex set

$$\mathcal{H} = \left\{\begin{bmatrix} w_1 & v\\ 0 & w_2\end{bmatrix} : \begin{bmatrix} -w_1 & v\\ 0 & w_2\end{bmatrix} \in \mathbb{H}^+(M_2(\mathcal{B}))\right\}.$$

Note that for any $a = a^*$, b and $c = c^*$ in $\mathcal{B}$,

$$\begin{pmatrix} a & b\\ b^* & c\end{pmatrix} = \begin{pmatrix} 1 & bc^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix} (a-bc^{-1}b^*) & 0\\ 0 & c\end{pmatrix}\begin{pmatrix} 1 & 0\\ c^{-1}b^* & 1\end{pmatrix}$$

holds. Thus, $\begin{pmatrix} a & b\\ b^* & c\end{pmatrix} > 0$ if and only if $c > 0$ and $a - bc^{-1}b^* > 0$, that is if and only if $c > 0$, $a > 0$, and $a^{-1/2}bc^{-1}b^*a^{-1/2} < 1$.

It is interesting to note that $\mathcal{I}\begin{bmatrix} w_1 & v\\ 0 & w_2\end{bmatrix} = \begin{bmatrix} \mathcal{I}w_1 & \frac{v}{2i}\\ \left(\frac{v}{2i}\right)^* & \mathcal{I}w_2\end{bmatrix}$ is invertible in $M_2(\mathcal{B})$. Indeed, the two diagonal entries are assumed to be invertible, and the Schur complement formula tells us that as long as we can guarantee that $\mathcal{I}w_1 - \frac{v}{2i}(\mathcal{I}w_2)^{-1}\left(\frac{v}{2i}\right)^*$ is invertible, we are done. But $\mathcal{I}w_1 < 0$, so that $0 > \mathcal{I}w_1 \geq \mathcal{I}w_1 - \frac{1}{4}v(\mathcal{I}w_2)^{-1}v^*$ makes $\mathcal{I}w_1 - \frac{v}{2i}(\mathcal{I}w_2)^{-1}\left(\frac{v}{2i}\right)^* = \mathcal{I}w_1 - \frac{1}{4}v(\mathcal{I}w_2)^{-1}v^*$ invertible *regardless of the size of* v! Trivially, so is any element $\mathfrak{w} \in \mathcal{H}$. The maps

$$(\mathrm{id}_2 \otimes E)\left[\left(\begin{bmatrix} w_1 & v \\ 0 & w_2 \end{bmatrix} - \begin{bmatrix} S+D & 0 \\ 0 & S+D \end{bmatrix}\right)^{-1}\right] = \begin{bmatrix} E\left[(w_1-S-D)^{-1}\right] & -E\left[(w_1-S-D)^{-1}v(w_2-S-D)^{-1}\right] \\ 0 & E\left[(w_2-S-D)^{-1}\right] \end{bmatrix}$$

and

$$(\mathrm{id}_2 \otimes E)\left[\left(\begin{bmatrix} w_1 & v \\ 0 & w_2 \end{bmatrix} - \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\right)^{-1}\right] = \begin{bmatrix} E\left[(w_1-D)^{-1}\right] & -E\left[(w_1-D)^{-1}v(w_2-D)^{-1}\right] \\ 0 & E\left[(w_2-D)^{-1}\right] \end{bmatrix}$$

are well-defined (this is a trivial observation) and moreover are the unique extensions through the set $\{b \in M_2(\mathcal{B}) \colon \|b^{-1}\| < \frac{1}{\|S\|+\|D\|}\}$ of the usual operator-valued Cauchy transforms defined on $\mathbb{H}^{\pm}(M_2(\mathcal{B}))$. Of course, $\mathcal{H}$ is not open in $M_2(\mathcal{B})$. However, as the set of invertible bounded operators on a Hilbert space is open, it is clear that for each $\begin{bmatrix} w_1 & v \\ 0 & w_2 \end{bmatrix} \in \mathcal{H}$ one finds a norm-neighborhood V of this point in $M_2(\mathcal{B})$ (and depending on this point) such that both $\mathfrak{w} - \begin{bmatrix} S+D & 0 \\ 0 & S+D \end{bmatrix}$ and $\mathfrak{w} - \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$ are invertible for all $\mathfrak{w} \in V$. Thus, the above defined extensions are indeed unique by the identity principle for analytic functions.

While not open in $M_2(\mathcal{B})$, the space $\mathcal{H}$ is nevertheless an analytic space, open in the complex algebra of upper triangular matrices in $M_2(\mathcal{B})$, so that we may define analytic functions on it and apply analytic function theory results to them. To begin with, observe that both

$$(\mathrm{id}_2 \otimes E)\left[\left(\begin{bmatrix} w_1 & v \\ 0 & w_2 \end{bmatrix} - \begin{bmatrix} S+D & 0 \\ 0 & S+D \end{bmatrix}\right)^{-1}\right], \quad (\mathrm{id}_2 \otimes E)\left[\left(\begin{bmatrix} w_1 & v \\ 0 & w_2 \end{bmatrix} - \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\right)^{-1}\right]$$

send $\mathcal{H}$ to $-\mathcal{H}$. Indeed, for any selfadjoint noncommutative random variable Y

$$\begin{aligned} (\mathrm{id}_2 \otimes E)\left[\left(\begin{bmatrix} w_1 & v \\ 0 & w_2 \end{bmatrix} - \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}\right)^{-1}\right] &= (\mathrm{id}_2 \otimes E)\begin{bmatrix} (w_1 - Y)^{-1} & -(w_1 - Y)^{-1}v(w_2 - Y)^{-1} \\ 0 & (w_2 - Y)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} E[(w_1 - Y)^{-1}] & -E[(w_1 - Y)^{-1}v(w_2 - Y)^{-1}] \\ 0 & E[(w_2 - Y)^{-1}] \end{bmatrix}. \end{aligned}$$

Now $\mathcal{I}w_1 < 0 \implies \mathcal{I}E[(w_1 - Y)^{-1}] > 0$, and $\mathcal{I}w_2 > 0 \implies \mathcal{I}E[(w_2 - Y)^{-1}] < 0$. On the other hand,

$$\begin{aligned} \mathcal{I}(\mathrm{id}_2 \otimes E)\left[\left(\begin{bmatrix} -w_1 & v \\ 0 & w_2 \end{bmatrix} - \begin{bmatrix} -Y & 0 \\ 0 & Y \end{bmatrix}\right)^{-1}\right] &= \mathcal{I}(\mathrm{id}_2 \otimes E)\begin{bmatrix} -(w_1 - Y)^{-1} & (w_1 - Y)^{-1}v(w_2 - Y)^{-1} \\ 0 & (w_2 - Y)^{-1} \end{bmatrix} \\ &= \mathcal{I}\begin{bmatrix} -E[(w_1 - Y)^{-1}] & E[(w_1 - Y)^{-1}v(w_2 - Y)^{-1}] \\ 0 & E[(w_2 - Y)^{-1}] \end{bmatrix} \\ &< 0. \end{aligned}$$

This shows that $(\mathrm{id}_2 \otimes E)\left[\left(\begin{bmatrix} w_1 & v \\ 0 & w_2 \end{bmatrix} - \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}\right)^{-1}\right] \in -\mathcal{H}$.

In addition, η is completely positive, so $(\mathrm{id}_2 \otimes \eta)(\mathcal{H}) \subseteq \mathcal{H}$. Thus, the map

$$f \colon \mathcal{H} \times \mathcal{H} \to \mathcal{H}, \quad f_{\boldsymbol{\beta}}(\mathfrak{w}) = \boldsymbol{\beta} - (\mathrm{id}_2 \otimes \eta) \circ (\mathrm{id}_2 \otimes E)\left[\left(\mathfrak{w} - \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\right)^{-1}\right]$$

is a well-defined map, and for each $\boldsymbol{\beta} \in \mathcal{H}$, $f_{\boldsymbol{\beta}}$ is an analytic self-map of $\mathcal{H}$.

Considering the level one relation from (5) for our given β_1, β_2, we automatically have

$$
\begin{aligned}
(\mathrm{id}_2\otimes E)\left[\left(\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}-\begin{bmatrix}S+D & 0\\ 0 & S+D\end{bmatrix}\right)^{-1}\right] &= \begin{bmatrix}E\left[(\beta_1-S-D)^{-1}\right] & 0\\ 0 & E\left[(\beta_2-S-D)^{-1}\right]\end{bmatrix}\\
&= \begin{bmatrix}E\left[(\omega(\beta_1)-D)^{-1}\right] & 0\\ 0 & E\left[(\omega(\beta_2)-D)^{-1}\right]\end{bmatrix}\\
&= (\mathrm{id}_2\otimes E)\left[\left(\begin{bmatrix}\omega(\beta_1) & 0\\ 0 & \omega(\beta_2)\end{bmatrix}-\begin{bmatrix}D & 0\\ 0 & D\end{bmatrix}\right)^{-1}\right].
\end{aligned}
$$

This guarantees that, for $\boldsymbol{\beta}=\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}$, we have

$$
\begin{aligned}
f_{\boldsymbol{\beta}}\left(\begin{bmatrix}\omega(\beta_1) & 0\\ 0 & \omega(\beta_2)\end{bmatrix}\right) &= \boldsymbol{\beta}-\begin{bmatrix}(\eta\circ E)\left[(\omega(\beta_1)-D)^{-1}\right] & 0\\ 0 & (\eta\circ E)\left[(\omega(\beta_2)-D)^{-1}\right]\end{bmatrix}\\
&= \begin{bmatrix}\beta_1-(\eta\circ E)\left[(\omega(\beta_1)-D)^{-1}\right] & 0\\ 0 & \beta_2-(\eta\circ E)\left[(\omega(\beta_2)-D)^{-1}\right]\end{bmatrix}\\
&= \begin{bmatrix}\omega(\beta_1) & 0\\ 0 & \omega(\beta_2)\end{bmatrix},
\end{aligned}
$$

so that $\begin{bmatrix}\omega(\beta_1) & 0\\ 0 & \omega(\beta_2)\end{bmatrix}$ is a fixed point of $f_{\boldsymbol{\beta}}$. Let us limit ourselves now to the subset

$$
\mathcal{H}_{\beta_1,\beta_2,R}=\left\{\mathfrak{w}=\begin{bmatrix}w_1 & v\\ 0 & w_2\end{bmatrix}\in\mathcal{H}\colon \mathcal{I}\begin{bmatrix}-w_1 & v\\ 0 & w_2\end{bmatrix}>\frac{1}{2}\mathcal{I}\begin{bmatrix}-\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}, \|\mathfrak{w}\|<R\right\}.
$$

It is clear that for $R>0$ sufficiently large, $\mathcal{H}_{\beta_1,\beta_2,R}$ is a nonempty open connected subset of $\mathcal{H}$ (in fact convex). According to what precedes, we have

$$
\begin{aligned}
\mathcal{I}\left(\begin{bmatrix}-\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}-\begin{bmatrix}-(\eta\circ E)\left[(w_1-D)^{-1}\right] & -(\eta\circ E)\left[(w_1-D)^{-1}v(w_2-D)^{-1}\right]\\ 0 & (\eta\circ E)\left[(w_2-D)^{-1}\right]\end{bmatrix}\right) &\geq \mathcal{I}\begin{bmatrix}-\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}\\
&> \frac{1}{2}\mathcal{I}\begin{bmatrix}-\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\|f_{\boldsymbol{\beta}}(\mathfrak{w})\| &= \left\|\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}-\begin{bmatrix}(\eta\circ E)\left[(w_1-D)^{-1}\right] & -(\eta\circ E)\left[(w_1-D)^{-1}v(w_2-D)^{-1}\right]\\ 0 & (\eta\circ E)\left[(w_2-D)^{-1}\right]\end{bmatrix}\right\|\\
&\leq \left\|\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}\right\|+\|\eta\|_{\mathrm{cb}}\left\|\begin{bmatrix}(w_1-D)^{-1} & -(w_1-D)^{-1}v(w_2-D)^{-1}\\ 0 & (w_2-D)^{-1}\end{bmatrix}\right\|\\
&= \left\|\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}\right\|+\|\eta\|_{\mathrm{cb}}\left\|\begin{bmatrix}w_1-D & v\\ 0 & w_2-D\end{bmatrix}^{-1}\right\|\\
&= \left\|\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}\right\|+\|\eta\|_{\mathrm{cb}}\left\|\left(\begin{bmatrix}w_1-D & v\\ 0 & w_2-D\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\right)^{-1}\right\|\\
&= \left\|\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}\right\|+\|\eta\|_{\mathrm{cb}}\left\|\begin{bmatrix}-w_1+D & v\\ 0 & w_2-D\end{bmatrix}^{-1}\right\|\\
&= \left\|\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}\right\|+\|\eta\|_{\mathrm{cb}}\left\|\left(\begin{bmatrix}-w_1 & v\\ 0 & w_2\end{bmatrix}-\begin{bmatrix}-D & 0\\ 0 & D\end{bmatrix}\right)^{-1}\right\|\\
&\leq \|\beta_1\|+\|\beta_2\|+\|\eta\|_{\mathrm{cb}}\left\|\left(\mathcal{I}\begin{bmatrix}-w_1 & v\\ 0 & w_2\end{bmatrix}\right)^{-1}\right\|\\
&< \|\beta_1\|+\|\beta_2\|+2\|\eta\|_{\mathrm{cb}}\left\|\begin{bmatrix}-\mathcal{I}\beta_1 & 0\\ 0 & \mathcal{I}\beta_2\end{bmatrix}^{-1}\right\|\leq \|\beta_1\|+\|\beta_2\|+2\|\eta\|_{\mathrm{cb}}\left[\|(\mathcal{I}\beta_1)^{-1}\|+\|(\mathcal{I}\beta_2)^{-1}\|\right].
\end{aligned}
$$

(We have used (2) and the hypothesis on elements in $\mathcal{H}_{\beta_1,\beta_2,R}$.) Observe that the majorization of $\|f_{\boldsymbol{\beta}}(\mathfrak{w})\|$ is independent of R and $\mathfrak{w}$. Thus, if we choose $R=2\|\beta_1\|+\|\beta_2\|+2\|\eta\|_{\mathrm{cb}}\left[\|(\mathcal{I}\beta_1)^{-1}\|+\|(\mathcal{I}\beta_2)^{-1}\|\right]$, then we are guaranteed that $f_{\boldsymbol{\beta}}(\mathcal{H}_{\beta_1,\beta_2,R})$ is at positive norm-distance from $\mathcal{H}\setminus\mathcal{H}_{\beta_1,\beta_2,R}$. Theorem 74 guarantees that $f_{\boldsymbol{\beta}}$ has a unique *attracting* fixed point in $\mathcal{H}$, which, unsurprisingly, belongs to $\mathcal{H}_{\beta_1,\beta_2,R}$, and that $f_{\boldsymbol{\beta}}$ is a strict contraction in the hyperbolic metric on $\mathcal{H}$. Since $f_{\boldsymbol{\beta}}$ maps $\mathcal{H}_{\beta_1,\beta_2,R}$ strictly inside itself, of course $f_{\boldsymbol{\beta}}$ is a strict contraction in the hyperbolic metric of $\mathcal{H}_{\beta_1,\beta_2,R}$ itself, hence any hyperbolic ball around the fixed point $\begin{bmatrix}\omega(\beta_1) & 0\\ 0 & \omega(\beta_2)\end{bmatrix}$ is mapped strictly inside itself. In particular, given an arbitrary finite-radius hyperbolic ball $B\subsetneq\mathcal{H}_{\beta_1,\beta_2,R}$ around the fixed point, $f_{\boldsymbol{\beta}}^{\circ n}(\mathfrak{w})\to\begin{bmatrix}\omega(\beta_1) & 0\\ 0 & \omega(\beta_2)\end{bmatrix}$ as $n\to\infty$ for all $w\in B$, uniformly on B. Since on any subset at positive distance from the complement of $\mathcal{H}_{\beta_1,\beta_2,R}$ the norm topology and the hyperbolic topology are equivalent, there exists $r>0$ such that $\begin{bmatrix}\omega(\beta_1) & v\\ 0 & \omega(\beta_2)\end{bmatrix}\in B$ for all $v\in\mathcal{B},\|v\|\le r$.

Now assume towards contradiction that the spectral radius of the linear completely bounded map $U\colon\mathcal{B}\ni v\mapsto(\eta\circ E)\left[(\omega(\beta_1)-D)^{-1}v(\omega(\beta_2)-D)^{-1}\right]\in\mathcal{B}$ is greater than or equal to one. According to the spectral radius formula, this forces $\lim_{n\to\infty}\|U^n\|^{\frac1n}=\inf_{n\ge1}\|U^n\|^{\frac1n}\ge1$. However, by direct computation, with $\boldsymbol{\beta}=\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}$, we obtain

$$
\begin{aligned}
f_{\boldsymbol{\beta}}^{\circ2}\left(\begin{bmatrix}\omega(\beta_1) & v\\ 0 & \omega(\beta_2)\end{bmatrix}\right) &= f_{\boldsymbol{\beta}}\left(f_{\boldsymbol{\beta}}\left(\begin{bmatrix}\omega(\beta_1) & v\\ 0 & \omega(\beta_2)\end{bmatrix}\right)\right)\\
&= f_{\boldsymbol{\beta}}\left(\begin{bmatrix}\beta_1-(\eta\circ E)\left[(\omega(\beta_1)-D)^{-1}\right] & (\eta\circ E)\left[(\omega(\beta_1)-D)^{-1}v(\omega(\beta_2)-D)^{-1}\right]\\ 0 & \beta_2-(\eta\circ E)\left[(\omega(\beta_2)-D)^{-1}\right]\end{bmatrix}\right)\\
&= f_{\boldsymbol{\beta}}\left(\begin{bmatrix}\omega(\beta_1) & Uv\\ 0 & \omega(\beta_2)\end{bmatrix}\right)=\begin{bmatrix}\omega(\beta_1) & U^2v\\ 0 & \omega(\beta_2)\end{bmatrix},\quad\text{so that}\\
f_{\boldsymbol{\beta}}^{\circ n}\left(\begin{bmatrix}\omega(\beta_1) & v\\ 0 & \omega(\beta_2)\end{bmatrix}\right) &= \begin{bmatrix}\omega(\beta_1) & U^nv\\ 0 & \omega(\beta_2)\end{bmatrix},\quad n\in\mathbb{N}.
\end{aligned}
$$

Since, as mentioned above, the norm and hyperbolic topologies coincide locally and $f_{\boldsymbol{\beta}}$ is a strict hyperbolic contraction, there exists an $n_0\in\mathbb{N}$ such that $f_{\boldsymbol{\beta}}^{\circ n}\left(\begin{bmatrix}\omega(\beta_1) & v\\ 0 & \omega(\beta_2)\end{bmatrix}\right)=\begin{bmatrix}\omega(\beta_1) & U^nv\\ 0 & \omega(\beta_2)\end{bmatrix}\in\left\{\begin{bmatrix}\omega(\beta_1) & \xi\\ 0 & \omega(\beta_2)\end{bmatrix}\colon\|\xi\|\le r/2\right\}$ for all $v,\|v\|\le r,n\ge n_0$. Thus, $\|U^{n_0}v\|\le\frac r2$ for all v, $\|v\|\le r$, so that $\|U^{n_0}\|\le\frac12$. Of course, this means that $\lim_{n\to\infty}\|U^n\|^{\frac1n}=\lim_{m\to\infty}\|U^{mn_0}\|^{\frac{1}{mn_0}}\le\lim_{m\to\infty}\|U^{n_0}\|^{\frac{m}{mn_0}}=\frac{1}{\sqrt[n_0]{2}}<1$, contradicting our hypothesis. Thus, the spectral radius of U is strictly less than one.

Since U is the composition of η with $E\left[(\omega(\beta_1)-D)^{-1}\ \cdot\ (\omega(\beta_2)-D)^{-1}\right]$ and u_{β_1,β_2} is the composition of $E\left[(\omega(\beta_1)-D)^{-1}\ \cdot\ (\omega(\beta_2)-D)^{-1}\right]$ with η, the spectral radius of U coincides with the spectral radius of u_{β_1,β_2}. This concludes the proof of our proposition. □

Remark 14. (1) *The proof given above works with almost no modification for the case* $\{\beta_1,\beta_2\in\mathbb{H}^+(\mathcal{B})\}$, *case already covered by* [Bel19, Proposition 4.1].
(2) *Although not directly useful in our paper, we mention that another side benefit of Proposition 13 is that it allows the subordination function* $\omega(\boldsymbol{\beta})$ *to be extended to a neighborhood of* $\mathcal{H}$ as a fixed point *of* $f_{\boldsymbol{\beta}}(\cdot)$.

5.3. **Second step in the proof of Proposition 11.** The following theorem will be useful in order to deduce Proposition 11 from Proposition 13.

Theorem 15. *Assume that* $\Omega\subset\mathbb{C}^d$ *is an open connected set,* $\mathcal{A}$ *is a Banach algebra, and* $f\colon\Omega\to\mathcal{A}$ *is analytic. Denote by* $\rho(x)$ *the spectral radius of the element* $x\in\mathcal{A}$. *The function* $\Omega\ni\mathbf{z}\mapsto\rho(f(\mathbf{z}))\in[0,+\infty)$ *is plurisubharmonic.*

The result is well-known (see for instance [Cha20, Section 4.1]), but we provide a brief sketch of a proof. First, recall that the function $\mathcal{A}\ni x\mapsto\rho(x)\in[0,+\infty)$ is upper semicontinuous [Mü03, I.1,Theorem 31]. Obviously, $\Omega\ni\mathbf{z}\mapsto\|f(\mathbf{z})\|\in[0,+\infty)$ is continuous. We claim it is also plurisubharmonic. Indeed, in any Banach space, the norm of an element is equal to the value at it of a certain norm-one linear functional defined on the Banach space. In particular, for any $x\in\mathcal{A}$, there exists a norm-one continuous linear functional $x^\star\colon \mathcal{A}\to\mathbb{C}$ such that $\|x\| = x^\star(x)$. Thus, for any given $\mathbf{z}_0\in\Omega$, one may find a norm-one linear functional $\phi_{\mathbf{z}_0}\colon \mathcal{A}\to\mathbb{C}$ such that $\phi_{\mathbf{z}_0}(f(\mathbf{z}_0)) = \|f(\mathbf{z}_0)\|$. The function $\Omega\ni\mathbf{z}\mapsto\phi_{\mathbf{z}_0}(f(\mathbf{z}))\in\mathbb{C}$ is analytic because of the continuity of the linear functional $\phi_{\mathbf{z}_0}$ and the assumption of analyticity imposed on f. According to [GZ17, Proposition 1.29], the map $\Omega\ni\mathbf{z}\mapsto|\phi_{\mathbf{z}_0}(f(\mathbf{z}))|^\alpha\in[0,+\infty)$ becomes then a continuous plurisubharmonic function satisfying $|\phi_{\mathbf{z}_0}(f(\mathbf{z}_0))|^\alpha = \|f(\mathbf{z}_0)\|^\alpha, |\phi_{\mathbf{z}_0}(f(\mathbf{z}))|^\alpha \le \|f(\mathbf{z})\|^\alpha$, $\mathbf{z}\in\Omega,\alpha>0$. If we let $\Omega^\star = \{\phi_{\mathbf{z}}\colon \mathbf{z}\in\Omega\}\subseteq\{\phi\colon\mathcal{A}\to\mathbb{C}\colon \phi \text{ linear, continuous, } \|\phi\|=1\}$, then for any $\alpha>0$,

$$\|f(\mathbf{z})\|^\alpha = \max\{|\phi_{\mathbf{w}}(f(\mathbf{z}))|^\alpha\colon \phi_{\mathbf{w}}\in\Omega^\star,\mathbf{w}\in\Omega\},\quad \mathbf{z}\in\Omega,$$

so that, according to [GZ17, Proposition 1.40], it is indeed plurisubharmonic.

Finally, as already mentioned, the spectral radius formula is given as

$$\rho(f(\mathbf{z})) = \lim_{n\to\infty}\|f(\mathbf{z})^n\|^{\frac{1}{n}} = \inf_{n\ge1}\|f(\mathbf{z})^n\|^{\frac{1}{n}},$$

that is, the upper semicontinuous function $\Omega\ni\mathbf{z}\mapsto\rho(f(\mathbf{z}))\in[0,+\infty)$ is the infimum of a family of plurisubharmonic functions, hence itself plurisubharmonic (see [GZ17, Proposition 1.28.(2)]).

The property useful for our purposes of plurisubharmonic functions is that they satisfy a maximum principle [GZ17, Corollary 1.37]: if $\mathbf{z}_0$ is a local maximum for $\rho(f(\mathbf{z}))$, then $\mathbf{z}\mapsto\rho(f(\mathbf{z}))$ is constant on a neighborhood of $\mathbf{z}_0$ in Ω.

Remark 16. *Consider an open connected set $\Omega\subseteq\mathbb{C}^d$ for some integer $d\ge1$, a Banach algebra $\mathcal{A}$, an analytic function $f\colon\Omega\to\mathcal{A}$, and a number $M>0$. Assume that $\rho(f(\mathbf{z}))\le M$ for all $\mathbf{z}\in\Omega$. Then either $\rho(f(\mathbf{z}))\equiv M$ for all $\mathbf{z}\in\Omega$, or $\rho(f(\mathbf{z}))<M$ for all $\mathbf{z}\in\Omega$. In the second case, there exists a sequence $\mathbf{z}_n\in\Omega$ such that $\mathbf{z}_n\to\infty$ as $n\to\infty$, $\rho(f(\mathbf{z}_n))<\rho(f(\mathbf{z}_{n+1})), n\in\mathbb{N}$, and $M\ge\sup\{\rho(f(\mathbf{z}))\colon\mathbf{z}\in\Omega\} = \lim_{n\to\infty}\rho(f(\mathbf{z}_n))$. By $\mathbf{z}_n\to\infty$ we mean that for any compact $K\subset\Omega$ there exists $n_K\in\mathbb{N}$ such that $\mathbf{z}_n\notin K$ for all $n\ge n_K$.*

Observe that our boundedness hypothesis automatically excludes the possibility that f is *not* constant and simultaneously $\Omega=\mathbb{C}^d$.

Proof. The statements are trivial: the first one is a consequence of the maximum principle for plurisubharmonic functions [GZ17, Corollary 1.37] and Theorem 15, and the second of the very definition of supremum. ☐

Proof of Proposition 11. First observe that $(ze_{11}-\gamma_0)\otimes1_{\mathcal{A}}-\gamma_1\otimes s-\gamma_2\otimes d$ is invertible for all $z\in\mathbb{C}^+\cup\mathbb{C}^-\cup(\mathbb{R}\setminus\sigma((\gamma_0)_{1,1}\cdot1_{\mathcal{A}}+(\gamma_1)_{1,1}s+(\gamma_2)_{1,1}d+u^*Q^{-1}u))$. Indeed, splitting this element in four blocks

$$(ze_{11}-\gamma_0)\otimes1_{\mathcal{A}}-\gamma_1\otimes s-\gamma_2\otimes d = \begin{bmatrix}((ze_{11}-\gamma_0)\otimes1_{\mathcal{A}}-\gamma_1\otimes s-\gamma_2\otimes d)_{1,1} & u^*\\ u & Q\end{bmatrix}$$

as in Section 4.2, where $Q\in M_{m-1}(\mathcal{A})$ is assumed to be invertible. By our choice, u and Q do not depend on z, but only on s,d, and the γ's. The Schur complement formula guarantees that the above is invertible in $M_m(\mathcal{A})$ whenever $((ze_{11}-\gamma_0)\otimes1_{\mathcal{A}}-\gamma_1\otimes s-\gamma_2\otimes d)_{1,1}-u^*Q^{-1}u$ is invertible in $\mathcal{A}$. Since $(u^*Q^{-1}u)^* = u^*Q^{-1}u$ and $(\gamma_0\otimes1_{\mathcal{A}}+\gamma_1\otimes s+\gamma_2\otimes d)_{1,1} = (\gamma_0)_{1,1}\cdot1_{\mathcal{A}}+(\gamma_1)_{1,1}s+(\gamma_2)_{1,1}d$ is also selfadjoint in $\mathcal{A}$, it follows that for any $z\in\mathbb{C}^+\cup\mathbb{C}^-\cup(\mathbb{R}\setminus\sigma((\gamma_0)_{1,1}\cdot1_{\mathcal{A}}+(\gamma_1)_{1,1}s+(\gamma_2)_{1,1}d+u^*Q^{-1}u))$ – a connected set which is also a neighborhood of infinity – the random variable $(ze_{11}-\gamma_0)\otimes1_{\mathcal{A}}-\gamma_1\otimes s-\gamma_2\otimes d$ is invertible, as claimed.

Direct computation shows that points $(ze_{11}-\gamma_0)$, $z\in\mathbb{C}$, belong to the topological closure of $\mathbb{H}^+(M_m(\mathbb{C}))\cup\mathbb{H}^-(M_m(\mathbb{C}))$. Since the set of invertible elements in a Banach algebra is open in the norm topology, it follows immediately that if z is such that $(ze_{11}-\gamma_0)\otimes1_{\mathcal{A}}-\gamma_1\otimes s-\gamma_2\otimes d$ is invertible in $M_m(\mathcal{A})$, then there is a small enough neighborhood V in $M_m(\mathbb{C})$ such that $w\otimes1_{\mathcal{A}}-\gamma_1\otimes s-\gamma_2\otimes d$ is invertible in $M_m(\mathcal{A})$ for all $w\in V$ and, of course, $V\cap(\mathbb{H}^+(M_m(\mathbb{C}))\cup\mathbb{H}^-(M_m(\mathbb{C})))\neq\varnothing$.

This guarantees in particular that $\omega(w) = w-\sigma^2\gamma_1(\mathrm{id}_m\otimes\tau)\left[(w\otimes1_{\mathcal{A}}-\gamma_1\otimes s-\gamma_2\otimes d)^{-1}\right]\gamma_1$ extends analytically to a neighborhood of $\{ze_{11}-\gamma_0\colon z\in\mathbb{C}^+\cup\mathbb{C}^-\cup(\mathbb{R}\setminus\sigma((\gamma_0)_{1,1}\cdot1_{\mathcal{A}}+(\gamma_1)_{1,1}s+(\gamma_2)_{1,1}d+u^*Q^{-1}u))\}$ in $M_m(\mathbb{C})$. We use that the spectral radius of operators on $M_m(\mathbb{C})$ is continuous to conclude thanks to Proposition 13 that for any $z_1,z_2\in\mathbb{C}^+\cup\mathbb{C}^-\cup(\mathbb{R}\setminus\sigma((\gamma_0)_{1,1}\cdot1_{\mathcal{A}}+(\gamma_1)_{1,1}s+(\gamma_2)_{1,1}d+u^*Q^{-1}u))$, we may write

$$1\ge\lim_{\beta_j\to0,\beta_j\in\mathbb{H}^{\bullet_j}(M_m(\mathbb{C}))}\rho\left(u_{z_1e_{11}-\gamma_0+\beta_1,z_2e_{11}-\gamma_0+\beta_2}\right) = \rho\left(u_{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0}\right),$$

where $\bullet_j$ is the sign of the imaginary part of z_j, $j = 1, 2$, and if one or both of z_j are real, then we agree to make the choice $\bullet_j = +$. Since the correspondence $(z_1, z_2) \mapsto u_{z_1 e_{11}-\gamma_0, z_2 e_{11}-\gamma_0}$ is an analytic map from the open subset $\{\mathbb{C}^+ \cup \mathbb{C}^- \cup (\mathbb{R} \setminus \sigma((\gamma_0)_{1,1} \cdot 1_{\mathcal{A}} + (\gamma_1)_{1,1} s + (\gamma_2)_{1,1} d + u^* Q^{-1} u))\}^2$ of $\mathbb{C}^2$ into the Banach algebra of (bounded) linear self-maps of $M_m(\mathbb{C})$, it follows that the correspondence $(z_1, z_2) \mapsto \rho\left(u_{z_1 e_{11}-\gamma_0, z_2 e_{11}-\gamma_0}\right) \in [0, +\infty)$ is plurisubharmonic on the same set, according to Theorem 15. Since plurisubharmonic functions satisfy the maximum principle, it follows (see Remark 16) that either $\rho\left(u_{z_1 e_{11}-\gamma_0, z_2 e_{11}-\gamma_0}\right) \equiv 1$ for all pairs (z_1, z_2) in the above-described domain of this function, or that $\rho\left(u_{z_1 e_{11}-\gamma_0, z_2 e_{11}-\gamma_0}\right) < 1$ for all pairs (z_1, z_2) in this domain. Thus, in order to show that the second case holds, it is enough to find a single such pair in which this spectral radius is strictly less than one. The pair we focus on will be of the form $(z_1, z_2) = (y, y)$ for $y \in \mathbb{R}$ sufficiently large. Note that, for such a pair, $u_{z_1 e_{11}-\gamma_0, z_2 e_{11}-\gamma_0} \colon M_m(\mathbb{C}) \to M_m(\mathbb{C})$ is completely positive. We apply to it Theorem 67. Assume towards contradiction that the completely positive map $u_{ye_{11}-\gamma_0, ye_{11}-\gamma_0}$ has spectral radius equal to one. This map is the composition of two completely positive maps, namely $v \mapsto \sigma^2 \gamma_1 v \gamma_1$ and $v \mapsto (\mathrm{id}_m \otimes \tau)\left[(\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1} (v \otimes 1_{\mathcal{A}}) (\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}\right]$. Since the spectral radius of AB coincides with the spectral radius of BA for any linear maps A, B on a Banach space, it follows that $\rho(u_{ye_{11}-\gamma_0, ye_{11}-\gamma_0}) = 1$ if and only if

(8)
$$\rho\left(\sigma^2 \gamma_1 (\mathrm{id}_m \otimes \tau)\left[(\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1} (\,\cdot\, \otimes 1_{\mathcal{A}}) (\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}\right] \gamma_1\right) = 1.$$

Thus, according to Theorem 67, there exists a matrix $v_0 \neq 0$ such that

$$\sigma^2 \gamma_1 (\mathrm{id}_m \otimes \tau)\left[(\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1} (v_0 \otimes 1_{\mathcal{A}}) (\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}\right] \gamma_1 = v_0.$$

Recalling that $\omega(w) = w - \sigma^2 \gamma_1 (\mathrm{id}_m \otimes \tau)\left[(\omega(w) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}\right] \gamma_1$, it follows that

$$\omega'(w)(c) = c + \sigma^2 \gamma_1 (\mathrm{id}_m \otimes \tau)\left[(\omega(w) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1} \omega'(w)(c) \, (\omega(w) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}\right] \gamma_1,$$

for all $w \in \mathbb{H}^{\pm}(M_m(\mathbb{C}))$. Extending this to our point $ye_{11} - \gamma_0$, we have two options: either $\omega'(ye_{11} - \gamma_0)$ is bijective, and then there exists a $c_y \in M_m(\mathbb{C})$ such that $\omega'(ye_{11} - \gamma_0)(c_y) = v_0$, or there is no such c_y and then $\omega'(ye_{11} - \gamma_0)$ is not bijective, which in finite dimensional spaces means it has a nontrivial kernel. In the first situation, we obtain

$$\begin{aligned}
v_0 &= \omega'(ye_{11} - \gamma_0)(c_y) \\
&= c_y + \sigma^2 \gamma_1 (\mathrm{id}_m \otimes \tau)\left[(\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1} \omega'(ye_{11} - \gamma_0)(c_y) \, (\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}\right] \gamma_1 \\
&= c_y + \sigma^2 \gamma_1 (\mathrm{id}_m \otimes \tau)\left[(\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1} v_0 \, (\omega(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1}\right] \gamma_1 \\
&= c_y + v_0,
\end{aligned}$$

which implies $c_y = 0$, so that $0 = \omega'(ye_{11} - \gamma_0)(c_y) = v_0 \neq 0$, which implies that the first situation cannot occur. If $0 \neq c \in \ker(\omega'(ye_{11} - \gamma_0))$, then $0 = \omega'(ye_{11} - \gamma_0)(c) = c + 0 = c \neq 0$, again a contradiction. Thus, it is impossible that (8) takes place. This concludes the proof of Proposition 11. $\square$

Unlike for the case of Proposition 13, here there were several points where the finite dimensionality of $M_m(\mathbb{C})$ was used. However, we cannot think at this moment of a situation in which Proposition 11 would have an appropriate formulation involving an infinite-dimensional algebra of scalars.

The reader might be concerned by one element in our proof, namely the fact that we have not hesitated to extend analytically ω around $(ye_{11} - \gamma_0)$. This problem has been essentially addressed in [BBC19]. However, the reader can find a simple argument for this extension by recalling Voiculescu's result [Voi00], namely

$$(\omega(w) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d)^{-1} = E_{M_m(\mathbb{C}\langle d \rangle)}\left[(w \otimes 1_{\mathcal{A}} - \gamma_1 \otimes s - \gamma_2 \otimes d)^{-1}\right].$$

As $y \in \mathbb{R}$ is taken so that $(ye_{11} - \gamma_0) \otimes 1_{\mathcal{A}} - \gamma_1 \otimes s - \gamma_2 \otimes d$ is invertible, we are guaranteed that it will remain invertible on a neighborhood of $ye_{11} - \gamma_0$ in $M_m(\mathbb{C})$. On the one hand this guarantees (via (5)) the existence and analyticity of ω on this neighborhood, and on the other, the boundedness of the conditional expectation of $(w \otimes 1_{\mathcal{A}} - \gamma_1 \otimes s - \gamma_2 \otimes d)^{-1}$ on this neighborhood guarantees that $\omega(w) \otimes 1_{\mathcal{A}} - \gamma_2 \otimes d$ stays invertible on the same neighborhood. This shows that taking the derivative of ω at $w = ye_{11} - \gamma_0 \in M_m(\mathbb{C})$ is permissible.

6. Preliminary results

6.1. **Notations.** We start by fixing some notations. We consider the canonical linearization

$$L_P(t_1,t_2) = \gamma_0 \otimes 1 + \gamma_1 \otimes t_1 + \gamma_2 \otimes t_2 \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle t_1, t_2\rangle$$

of the selfadjoint polynomial P with the properties outlined in Section 3. By (4), $\operatorname{Tr}(zI_N - X_N)^{-1}$ is related to the generalized resolvent

$$R_N(\beta) := (\beta \otimes I_N - \gamma_1 \otimes W_N - \gamma_2 \otimes D_N)^{-1},$$

by

$$\operatorname{Tr}(zI_N - X_N)^{-1} = (\operatorname{Tr}\otimes\operatorname{Tr})((e_{11}\otimes I_N)R_N(ze_{11}-\gamma_0)), \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

Denoting by $(E_{ij})_{1\leq i,j\leq N}$ the canonical basis of $M_N(\mathbb{C})$, we define the matrices $R_{ij}(\beta)$, $1 \leq i,j \leq N$, by

$$R_N(\beta) = \sum_{i,j=1}^{N} R_{ij}(\beta) \otimes E_{ij}.$$

Schur inversion formula (Proposition 60 applied to $A = \beta \otimes I_N - \gamma_1 \otimes W_N - \gamma_2 \otimes D_N$ and $I^c = \{k, k+N, \ldots, k+(N-1)m\}$) relates $R_N(\beta)$ to its Schur complements

$$\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*} R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)}, \quad k = 1,\ldots,N,$$

where

$$R^{(k)}(\beta) = (\beta \otimes I_{N-1} - \gamma_1 \otimes W_N^{(k)} - \gamma_2 \otimes D_N^{(k)})^{-1}.$$

Here $W_N^{(k)}, D_N^{(k)}$ denote the $(N-1)\times(N-1)$ matrices obtained respectively from W_N, D_N by deleting the k-th row/column and $C_k^{(k)}$ is the $(N-1)$-dimensional vector obtained from the k-th column of W_N by deleting its k-th component.

An immediate consequence of Lemma 3 is the following relation between R_N and the generalized resolvent

$$R^{(ab)}(\beta) = (\beta \otimes I_N - \gamma_1 \otimes W_N^{(ab)} - \gamma_2 \otimes D_N)^{-1},$$

where $W_N^{(ab)}$ is obtained from W_N by replacing its (a,b) and (b,a) entries by 0:

$$R^{(ab)}(\beta) - R_N(\beta) = -R^{(ab)}(\beta)(1 - \frac{1}{2}\delta_{ab})(\gamma_1 \otimes W_{ab}E_{ab} + \gamma_1 \otimes \overline{W_{ab}}E_{ba})R_N(\beta). \tag{9}$$

One deduces from (9) the following bound:

$$\|R^{(ab)}(\beta) - R_N(\beta)\| \leq 2\delta_N \|\gamma_1\| \|R^{(ab)}(\beta)\| \|R_N(\beta)\|.$$

Analogously, we denote by $R^{(kab)}$ the generalized resolvent

$$R^{(kab)}(\beta) = (\beta \otimes I_N - \gamma_1 \otimes W_N^{(kab)} - \gamma_2 \otimes D_N^{(k)})^{-1},$$

where $W_N^{(kab)}$ is obtained from $W_N^{(k)}$ by replacing its (a,b) and (b,a) entries by 0; and by $R^{(kabcd)}$ the generalized resolvent

$$R^{(kabcd)}(\beta) = (\beta \otimes I_N - \gamma_1 \otimes W_N^{(kabcd)} - \gamma_2 \otimes D_N^{(k)})^{-1},$$

where $W_N^{(kabcd)}$ is obtained from $W_N^{(k)}$ by replacing its $(a,b),(b,a),(c,d)$ and (d,c) entries by 0.

Martingales appearing in this paper will be with respect to the filtration

$$(\mathcal{F}_k := \sigma(W_{ij}, 1 \leq i \leq j \leq k))_{k\geq 1};$$

$\mathbb{E}_{\leq k}$ denotes the conditional expectation on the sigma-field $\mathcal{F}_k$ and $\mathbb{E}_k$ the expectation with respect to the k-th column $\{W_{ik}, 1 \leq i \leq N\}$ of W_N.

We will consider, for each $N \in \mathbb{N}$, a W^*-probability space $(\mathcal{A}_N, \tau_N)$, a semicircular element $s_N \in \mathcal{A}_N$ of mean 0 and variance $N\sigma_N^2$ and a von Neumann subalgebra $\mathcal{D}_N \subset \mathcal{A}_N$ isomorphic to the algebra of $N \times N$ diagonal matrices with complex entries and freely independent from s_N. As we have seen, $\gamma_1 \otimes s_N \in M_m(\mathcal{A}_N)$ is a centered $M_m(\mathbb{C})$-valued semicircular element of variance $\eta_N : b \mapsto N\sigma_N^2\gamma_1 b\gamma_1$ which is free with amalgamation over $M_m(\mathbb{C})$ from $M_m(\mathcal{D}_N)$ in the $M_m(\mathbb{C})$-valued W^*-probability space $(M_m(\mathcal{A}_N), \mathrm{id}_m \otimes \tau_N)$. For $D_N \in \mathcal{D}_N \simeq \mathcal{D}_N(\mathbb{C})$, the generalized resolvent

$$r_N(\beta) := (\beta \otimes 1_{\mathcal{A}_N} - \gamma_1 \otimes s_N - \gamma_2 \otimes D_N)^{-1}$$

and the subordination map

$$\omega_N(\beta) := \beta - \eta_N\big((\mathrm{id}_m \otimes \tau_N)[r_N(\beta)]\big)$$

are related according to (6) by

$$\hat{R}(\beta) := E_{M_m(\mathcal{D}_N)}[r_N(\beta)] = (\omega_N(\beta)\otimes 1_{\mathcal{A}_N} - \gamma_2\otimes D_N)^{-1}. \tag{10}$$

By analogy, one defines

$$\Omega_N(\beta) := \beta - \eta_N\big(\mathbb{E}[(\mathrm{id}_m\otimes N^{-1}\,\mathrm{Tr})(R_N(\beta))]\big). \tag{11}$$

In the sequel, we will use the notation $O(v_N)$ when a quantity depending on $N\in\mathbb{N}$, $z\in\mathbb{C}\setminus\mathbb{R}$, and sometimes on $k\in\{1,\dots,N\}$, satisfies the following : for any compact subset K of $\mathbb{C}\setminus\mathbb{R}$, there exists $N_K\in\mathbb{N}$ and $C_K>0$ such that for any $N\geq N_K$, for any $z\in K$, (for any $k\in\{1,\dots,N\}$,) this quantity is bounded by $C_K v_N$.

Throughout the paper, C, c denote some positive constants and Q denotes some deterministic polynomial in one or several commuting indeterminates ; they can depend on m, $\gamma_0,\gamma_1,\gamma_2$ and they may vary from line to line.

6.2. **Free probability bounds and convergences.** For $\beta\in\mathbb{H}^+(M_m(\mathbb{C}))$ and $k=1,\dots,N$, as observed in Section 4.1,

$$\|r_N(\beta)\| \leq \|(\mathcal{I}\beta)^{-1}\| \tag{12}$$

and

$$\|(\omega_N(\beta)\otimes 1_{\mathcal{A}_N}-\gamma_2\otimes D_N)^{-1}\| \leq \|(\mathcal{I}\omega_N(\beta))^{-1}\| \leq \|(\mathcal{I}\beta)^{-1}\|.$$

It is a consequence of Lemma 4 that r_N is also defined on $\{ze_{11}-\gamma_0, z\in\mathbb{C}\setminus\mathbb{R}\}$ and it follows from Lemma 5 with $y=(s_N,D_N)$, Assumptions 3, 5 and (3) that, for $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\|r_N(ze_{11}-\gamma_0)\| \leq Q_1(2N^{1/2}\sigma_N,\|D_N\|)\|(zI_N-P(s_N,D_N))^{-1}\| + Q_2(2N^{1/2}\sigma_N,\|D_N\|) = O(1). \tag{13}$$

Lemma 17.

$$\|(\omega_N(ze_{11}-\gamma_0)\otimes 1_{\mathcal{A}_N}-\gamma_2\otimes D_N)^{-1}\| = O(1);$$
$$\left\|\frac{\partial}{\partial z}(\omega_N(ze_{11}-\gamma_0)\otimes 1_{\mathcal{A}_N}-\gamma_2\otimes D_N)^{-1}\right\| = O(1).$$

Proof. For $z\in\mathbb{C}\setminus\mathbb{R}$, it follows from (6) that

$$E_{M_m(\mathcal{D}_N)}\left[((ze_{11}-\gamma_0)\otimes 1_{\mathcal{A}_N}-\gamma_1\otimes s_N-\gamma_2\otimes D_N)^{-1}\right] = (\omega_N(ze_{11}-\gamma_0)\otimes 1_{\mathcal{A}_N}-\gamma_2\otimes D_N)^{-1}$$

holds and therefore

$$\|(\omega_N(ze_{11}-\gamma_0)\otimes 1_{\mathcal{A}_N}-\gamma_2\otimes D_N)^{-1}\| \leq \|(ze_{11}\otimes 1_{\mathcal{A}_N}-L_P(s_N,D_N))^{-1}\|.$$

Since moreover

$$\left\|e_{11}+\eta_N\Big((\mathrm{id}_m\otimes\tau_N)\left[(ze_{11}\otimes 1_{\mathcal{A}_N}-L_P(s_N,D_N))^{-1}(e_{11}\otimes 1_{\mathcal{A}_N})(ze_{11}\otimes 1_{\mathcal{A}_N}-L_P(s_N,D_N))^{-1}\right]\Big)\right\|$$
$$\leq 1+N\sigma_N^2\|\gamma_1\|^2\|(ze_{11}\otimes 1_{\mathcal{A}_N}-L_P(s_N,D_N))^{-1}\|^2,$$

it follows that

$$\|\frac{\partial}{\partial z}(\omega_N(ze_{11}-\gamma_0)\otimes 1_{\mathcal{A}_N}-\gamma_2\otimes D_N)^{-1}\|$$
$$\leq \|(ze_{11}\otimes 1_{\mathcal{A}_N}-L_P(s_N,D_N))^{-1}\|^2(1+N\sigma_N^2\|\gamma_1\|^2\|(ze_{11}\otimes 1_{\mathcal{A}_N}-L_P(s_N,D_N))^{-1}\|^2).$$

Using Lemma 5 with $y=(s_N,D_N)$,

$$\|(ze_{11}\otimes 1_{\mathcal{A}_N}-L_P(s_N,D_N))^{-1}\| \leq Q_1(N^{1/2}\sigma_N,\|D_N\|)|\mathcal{I}z|^{-1}+Q_2(N^{1/2}\sigma_N,\|D_N\|)$$

and we are done. □

For $\beta\in M_m(\mathbb{C})$ such that $\beta=ze_{11}-\gamma_0$ with $z\in\mathbb{C}\setminus\mathbb{R}$, define

$$\hat{R}_k(\beta) = (\beta - N\sigma_N^2\gamma_1(\mathrm{id}_m\otimes\tau)(r_N(\beta))\gamma_1 - D_{kk}\gamma_2)^{-1} = (\omega_N(\beta)-D_{kk}\gamma_2)^{-1}. \tag{14}$$

It readily follows from Lemma 17 that

$$\|\hat{R}_k(ze_{11}-\gamma_0)\| = O(1), \tag{15}$$

$$\left\|\frac{\partial}{\partial z}\hat{R}_k(ze_{11}-\gamma_0)\right\| = O(1), \tag{16}$$

and that

$$\sup_{t\in\mathrm{supp}(\nu_N)}\|(\omega_N(ze_{11}-\gamma_0)-t\gamma_2)^{-1}\|=O(1). \tag{17}$$

Lemma 18. *The map $\mathbb{C}^\pm\times\mathbb{R}\ni(z,t)\mapsto(\omega(ze_{11}-\gamma_0)-t\gamma_2)^{-1}$ is well-defined and analytic. Thus, it is bounded on any compact subset of $\mathbb{C}^\pm\times\mathbb{R}$.*

Proof. Fix $M>\|d\|$ and a compact $K\subset\mathbb{C}^+$. We use the equality $\omega(ze_{11}-\gamma_0)=ze_{11}-\gamma_0-\gamma_1G(ze_{11}-\gamma_0)\gamma_1$, where G denotes the generalized Cauchy transform of $\gamma_1\otimes s+\gamma_2\otimes d$ (i.e. $G(b)=(\mathrm{id}_m\otimes\tau)(r(b))$), together with the structure of the linearization and normal families. The boundedness statement is obvious if $t\in\sigma(d)$, by Equation (6) and the boundedness of evaluation maps. We show that, given $M>0$ (which we assume for convenience sufficiently large so that $\sigma(d)\subset[-M,M]$), there exists an $\mathfrak{r}>0$ (depending on it) so that $\omega_N(ze_{11}-\gamma_0)-t\gamma_2=ze_{11}-\gamma_0-\gamma_1G(ze_{11}-\gamma_0)\gamma_1-t\gamma_2$ is invertible for all $|z|>\mathfrak{r}$. We recall the shape of $G(ze_{11}-\gamma_0)\in M_m(\mathbb{C})$:

$$\begin{aligned}&G(ze_{11}-\gamma_0)\\&=\begin{bmatrix}\tau\left((z-u^*Q^{-1}u)^{-1}\right) & -(\mathrm{id}_{1\times(m-1)}\otimes\tau)\left((z-u^*Q^{-1}u)^{-1}u^*Q^{-1}\right)\\ -(\mathrm{id}_{(m-1)\times1}\otimes\tau)\left(Q^{-1}u(z-u^*Q^{-1}u)^{-1}\right) & (\mathrm{id}_{m-1}\otimes\tau)\left(Q^{-1}+Q^{-1}u(z-u^*Q^{-1}u)^{-1}u^*Q^{-1}\right)\end{bmatrix},\end{aligned}$$

where u,Q,u^* are the (obvous size) constituents of the linearization of P, evaluated in (s,d). It is useful to note that the above matrix can be re-written as

$$\begin{aligned}&G(ze_{11}-\gamma_0)\\&\quad=\begin{bmatrix}0&0\\0&(\mathrm{id}_{m-1}\otimes\tau)\left(Q^{-1}\right)\end{bmatrix}+(\mathrm{id}_m\otimes\tau)\left[\begin{bmatrix}1\\-Q^{-1}u\end{bmatrix}\left(z-u^*Q^{-1}u\right)^{-1}\begin{bmatrix}1&-u^*Q^{-1}\end{bmatrix}\right].\end{aligned}$$

It is shown in [BBC19, Lemma 4.2] that there exist permutation matrices $T_1,T_2\in M_{m-1}(\mathbb{C})$ and a strictly lower triangular matrix $N\in M_{m-1}(\mathbb{C}\langle s,d\rangle)$ such that $Q^{-1}=-T_1(I_{m-1}+N)T_2$. We multiply $G(ze_{11}-\gamma_0)$ left with $\begin{bmatrix}1&0\\0&T_1^{-1}\end{bmatrix}$ and right with $\begin{bmatrix}1&0\\0&T_2^{-1}\end{bmatrix}$ to get

$$\begin{aligned}&\begin{bmatrix}0&0\\0&-I_{m-1}-(\mathrm{id}_{m-1}\otimes\tau)(N)\end{bmatrix}+\\&\qquad(\mathrm{id}_m\otimes\tau)\left[\begin{bmatrix}1\\(I_{m-1}+N)T_2u\end{bmatrix}\left(z-u^*Q^{-1}u\right)^{-1}\begin{bmatrix}1&u^*T_1(I_{m-1}+N)\end{bmatrix}\right]\\&\qquad=\begin{bmatrix}0&0\\0&-I_{m-1}-(\mathrm{id}_{m-1}\otimes\tau)(N)\end{bmatrix}+\mathrm{O}\left(\frac{1}{z}\right).\end{aligned}$$

(The $\mathrm{O}\left(\frac1z\right)$ part can be easily made precise: it is $\frac1z(\mathrm{id}_m\otimes\tau)\left[\begin{bmatrix}1\\(I_{m-1}+N)T_2u\end{bmatrix}\begin{bmatrix}1&u^*T_1(I_{m-1}+N)\end{bmatrix}\right]+\mathrm{O}\left(\frac{1}{z^2}\right)$.) We recall that $Q=P_{m-1}^*(\gamma_0-\gamma_1-\gamma_2)P_{m-1}$, where we have denoted by P_{m-1} the operator that embeds $\mathbb{C}^{m-1}\hookrightarrow(0,\mathbb{C}^{m-1})\subset\mathbb{C}^m$. It is known that $T_2P_{m-1}^*\gamma_0P_{m-1}T_1=-I_{m-1}$, while $T_1P_{m-1}^*(\gamma_1+\gamma_2)P_{m-1}T_1$ is strictly lower triangular (nilpotent). Moreover, the Schur product of γ_1 and γ_2 is known to have all but possibly the $(1,1)$ entry equal to zero, and $T_1P_{m-1}^*\gamma_1P_{m-1}T_1$, $T_1P_{m-1}^*\gamma_2P_{m-1}T_1$ are both strictly lower triangular. In particular, neither the structure of N nor the presence of I_{m-1} is affected if one replaces γ_2 with $t\gamma_2$ for an arbitrary $t\in\mathbb{R}$. Thus,

$$\begin{bmatrix}1&0\\0&T_2\end{bmatrix}(ze_{11}-\gamma_0-\gamma_1-t\gamma_2)\begin{bmatrix}1&0\\0&T_1\end{bmatrix}=\begin{bmatrix}z+\mathrm{const}_t&\mathfrak{x}_t^*T_1\\T_2\mathfrak{x}_t&-I_{m-1}+\tilde N_t\end{bmatrix},$$

where $\tilde N_t$ is a strictly lower triangular matrix, some of whose entries might depend linearly of $t\in\mathbb{R}$, and the same holds for the column vector $\mathfrak{x}_t$. The constant const_t might be affine in t. It is trivial that the above matrix is invertible for all $z\notin\mathbb{R}$, and equally obvious that $-I_{m-1}+\tilde N_t$ is invertible. Writing the Schur complement wrt the $(1,1)$ entry, we obtain that $z+\mathrm{const}_t-\mathfrak{x}_t^*T_1(\tilde N_t-I_{m-1})^{-1}T_2\mathfrak{x}_t$ is invertible. We study how adding $\gamma_1-\gamma_1G(ze_{11}-\gamma_0)\gamma_1$ to $ze_{11}-\gamma_0-\gamma_1-t\gamma_2$ influences this invertibility. We write

$$\begin{aligned}&\begin{bmatrix}1&0\\0&T_2\end{bmatrix}(\gamma_1-\gamma_1G(ze_{11}-\gamma_0)\gamma_1)\begin{bmatrix}1&0\\0&T_1\end{bmatrix}\\&=\begin{bmatrix}1&0\\0&T_2\end{bmatrix}\gamma_1\begin{bmatrix}1&0\\0&T_1\end{bmatrix}-\begin{bmatrix}1&0\\0&T_2\end{bmatrix}\gamma_1\begin{bmatrix}1&0\\0&T_1\end{bmatrix}\begin{bmatrix}1&0\\0&T_1^{-1}\end{bmatrix}G(ze_{11}-\gamma_0)\begin{bmatrix}1&0\\0&T_2^{-1}\end{bmatrix}\begin{bmatrix}1&0\\0&T_2\end{bmatrix}\gamma_1\begin{bmatrix}1&0\\0&T_1\end{bmatrix}.\end{aligned}$$

As seen above, the lower right $(m-1)\times(m-1)$ corner of $\begin{bmatrix}1 & 0\\ 0 & T_2\end{bmatrix}\gamma_1\begin{bmatrix}1 & 0\\ 0 & T_1\end{bmatrix}$ is a sub-matrix of $\tilde{N}_t$, namely the constant entries; the same holds for the vectors above and to the left of this corner, and the $(1,1)$ entry is the constant part of the affine map $t\mapsto$const$_t$, all those with a minus in front. It was seen above that $\begin{bmatrix}1 & 0\\ 0 & T_1^{-1}\end{bmatrix}G(ze_{11}-\gamma_0)\begin{bmatrix}1 & 0\\ 0 & T_2^{-1}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & -I_{m-1}-(\mathrm{id}_{m-1}\otimes\tau)(N)\end{bmatrix}+\mathrm{O}\left(\frac{1}{z}\right)$. It follows that

$$\begin{aligned}&\begin{bmatrix}1 & 0\\ 0 & T_2\end{bmatrix}\gamma_1\begin{bmatrix}1 & 0\\ 0 & T_1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & T_1^{-1}\end{bmatrix}G(ze_{11}-\gamma_0)\begin{bmatrix}1 & 0\\ 0 & T_2^{-1}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & T_2\end{bmatrix}\gamma_1\begin{bmatrix}1 & 0\\ 0 & T_1\end{bmatrix}\\ &=\begin{bmatrix}\star & \star\\ \star & \mathfrak{n}\end{bmatrix}\left(\begin{bmatrix}0 & 0\\ 0 & -I_{m-1}-(\mathrm{id}_{m-1}\otimes\tau)(N)\end{bmatrix}\right.\\ &\quad\left.+(\mathrm{id}_m\otimes\tau)\left[\begin{bmatrix}1\\ (I_{m-1}+N)T_2u\end{bmatrix}\left(z-u^*Q^{-1}u\right)^{-1}\begin{bmatrix}1 & u^*T_1(I_{m-1}+N)\end{bmatrix}\right]\right)\begin{bmatrix}\star & \star\\ \star & \mathfrak{n}\end{bmatrix}.\end{aligned}$$

As already mentioned, $\mathfrak{n}$ in the above is a constant sub-matrix of $\tilde{N}_t$, hence strictly lower triangular, and the stars stand for constant/constant vectors, whose precise identity is irrelevant for our purposes. The second summand in the above is still $\mathrm{O}\left(\frac{1}{z}\right)$, and the lower right corner of the first summand is $-\mathfrak{n}(I_{m-1}+N)\mathfrak{n}=-\mathfrak{n}^2-\mathfrak{n}N\mathfrak{n}$, a sum of products of strictly lower triangular matrices, is strictly lower triangular (in fact guaranteed to have also all entries $(i,i-1)$ equal to zero as well). It follows immediately that

$$\begin{bmatrix}1 & 0\\ 0 & T_2\end{bmatrix}(\omega(ze_{11}-\gamma_0)-t\gamma_2)\begin{bmatrix}1 & 0\\ 0 & T_1\end{bmatrix}=\begin{bmatrix}z+\mathrm{const}_t+\star & \mathfrak{x}_t^*T_1+\star\\ T_2\mathfrak{x}_t+\star & -I_{m-1}+\tilde{N}_t-\mathfrak{n}^2-\mathfrak{n}N\mathfrak{n}\end{bmatrix}+\mathrm{O}\left(\frac{1}{z}\right).$$

Since the lower right corner is invertible for all z,t, and the other t-dependent terms only depend affinely of it, it follows by the Schur complement that for $|z|$ sufficiently large the first term in the right-hand side is invertible. Since the second is $\mathrm{O}\left(\frac{1}{z}\right)$, the same Schur complement guarantees that, by slightly increasing $|z|$ if necessary, the invertibility statement remains valid. As the invertibility of $\omega(ze_{11}-\gamma_0)-t\gamma_2$ is equivalent to the invertibility of $\begin{bmatrix}1 & 0\\ 0 & T_2\end{bmatrix}(\omega(ze_{11}-\gamma_0)-t\gamma_2)\begin{bmatrix}1 & 0\\ 0 & T_1\end{bmatrix}$, we have established the invertibilty of $\omega(ze_{11}-\gamma_0)-t\gamma_2$ for $|z|$ sufficiently large.

Since $\mathcal{I}\omega(ze_{11}-\gamma_0)\geq 0$ for all $z\in\mathbb{C}^+$ and $\gamma_2=\gamma_2^*$, it follows that $\mathcal{I}(\omega(ze_{11}-\gamma_0)-t\gamma_2)^{-1}\leq 0$ whenever $\omega(ze_{11}-\gamma_0)-t\gamma_2$ is invertible. The argument employed in [BBC19, Lemma 5.5] guarantees that $z\mapsto(\omega(ze_{11}-\gamma_0)-t\gamma_2)^{-1}$ extends analytically to $\mathbb{C}^+\cup\mathbb{C}^-$, in addition to the neighborhood of infinity on which we have already shown it is well-defined. Since $M>\|d\|$ is arbitrary, the map $\mathbb{C}^\pm\times\mathbb{R}\ni(z,t)\mapsto(\omega(ze_{11}-\gamma_0)-t\gamma_2)^{-1}$ is well-defined and analytic. Thus, it is bounded on any compact subset of $\mathbb{C}^\pm\times\mathbb{R}$, and in particular on $K\times[-M,M]$. ☐

Lemma 19. *For any compact subset K of $\mathbb{C}\setminus\mathbb{R}$,*

$$\lim_{N\to+\infty}\sup_{z\in K}\sup_{t\in supp(\nu_N)}\|(\omega_N(ze_{11}-\gamma_0)-t\gamma_2)^{-1}-(\omega(ze_{11}-\gamma_0)-t\gamma_2)^{-1}\|=0.$$

Proof. Let $K\subset\mathbb{C}^+$ be a compact set. Let $M>0$ be such that for all large N, support$(\nu_N)\subset[-M,M]$.

$$\begin{aligned}&\sup_{z\in K}\sup_{t\in\mathrm{supp}(\nu_N)}\|(\omega_N(ze_{11}-\gamma_0)-t\gamma_2)^{-1}-(\omega(ze_{11}-\gamma_0)-t\gamma_2)^{-1}\|\\ &\leq\sup_{z\in K,t\in\mathrm{supp}(\nu_N)}\|(\omega_N(ze_{11}-\gamma_0)-t\gamma_2)^{-1}\|\cdot\sup_{z\in K,|t|\leq M}\|(\omega(ze_{11}-\gamma_0)-t\gamma_2)^{-1}\|\\ &\quad\times\sup_{z\in K}\|\omega_N(ze_{11}-\gamma_0)-\omega(ze_{11}-\gamma_0)\|=0.\end{aligned}\tag{18}$$

Thus, Lemma 19 readily follows from Lemma 18, (17) and the uniform convergence of ω_N to ω on compact subsets of $\mathbb{C}^\pm e_{11}-\gamma_0$ (see Remark 29) ☐

6.3. **Concentration bounds on quadratic forms.** Applying Proposition 60 to $\beta\otimes I_N-\gamma_1\otimes W_N-\gamma_2\otimes D_N$ leads to expressions involving random quadratic maps $\gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)}$. It is easy to compute the expectation of such quadratic maps. (Recall that $\mathbb{E}_k$ denotes the expectation with respect to $\{W_{ik},1\leq i\leq N\}$.)

$$\mathbb{E}\Big[\gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)}\Big]=\mathbb{E}\Big[\mathbb{E}_k[\gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)}]\Big]=\eta_N(\mathbb{E}[(\mathrm{id_m}\otimes N^{-1}\operatorname{Tr})(R^{(k)}(\beta))]).$$

Their variance may be deduced from the following Lemma:

Lemma 20. *For $m(N-1)\times m(N-1)$ random matrices $A'=\sum_{i,j\neq k}\alpha'_{ij}\otimes E_{ij}$, $A''=\sum_{i,j\neq k}\alpha''_{ij}\otimes E_{ij}$, $m\times m$ random matrices $\beta',\beta'',\gamma',\gamma''$, all independent of $\{W_{ik}, 1\le i\le N\}$, and $h\in\{1,\dots,N\}$,*

$$\begin{aligned}
&\mathbb{E}_k\Big[\mathbb{E}_{\le h}[\operatorname{Tr}\left((\gamma'\otimes C_k^{(k)*}A'\gamma'\otimes C_k^{(k)}-\gamma'(\mathrm{id}_d\otimes\sigma_N^2\operatorname{Tr})(A')\gamma')\beta'\right)]\\
&\qquad\qquad\times\mathbb{E}_{\le h}[\operatorname{Tr}\left((\gamma''\otimes C_k^{(k)*}A''\gamma''\otimes C_k^{(k)}-\gamma''(\mathrm{id}_d\otimes\sigma_N^2\operatorname{Tr})(A'')\gamma'')\beta''\right)]\Big]\\
&\quad=\sigma_N^4\sum_{i,j\neq k\le h}\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{ij}\gamma'\beta')]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{ji}\gamma''\beta'')]\\
&\quad+|\theta_N|^2\sum_{i,j\neq k\le h}\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{ij}\gamma'\beta')]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{ij}\gamma''\beta'')]\\
&\quad+\kappa_N\sum_{i\neq k\le h}\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{ii}\gamma'\beta')]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{ii}\gamma''\beta'')]+\varepsilon_k,
\end{aligned}$$

where

$$\varepsilon_k=\sum_{i\neq j\neq k\le h}\Big(\mathbb{E}[\overline{W_{ki}}^2]\mathbb{E}[W_{kj}^2]-|\theta_N|^2\Big)\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{ij}\gamma'\beta')]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{ij}\gamma''\beta'')].$$

Moreover, if A' and A'' are bounded in L^2 and $\beta',\beta'',\gamma',\gamma''$ are bounded deterministic, then

$$\sum_{k=1}^N\varepsilon_k\xrightarrow[N\to+\infty]{\mathbb{P}}0.$$

Proof of Lemma 20. It is sufficient to prove the result for $A'=\alpha'\otimes M', A''=\alpha''\otimes M''$. Since

$$\begin{aligned}
&\mathbb{E}_{\le h}\Big[\operatorname{Tr}\left((\gamma'\otimes C_k^{(k)*}\alpha'\otimes M'\gamma'\otimes C_k^{(k)}-\gamma'(\mathrm{id}_d\otimes\sigma_N^2\operatorname{Tr})(\alpha'\otimes M')\gamma')\beta'\right)\Big]\\
&\quad=\mathbb{E}_{\le h}[\operatorname{Tr}\left((C_k^{(k)*}M'C_k^{(k)}-\sigma_N^2\operatorname{Tr}(M'))\gamma'\alpha'\gamma'\beta'\right)]\\
&\quad=\sum_{i'\neq j'\neq k\le h}\overline{W_{ki'}}W_{kj'}\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'M'_{i'j'}\alpha'\gamma'\beta')]+\sum_{i'\neq k\le h}(|W_{ki'}|^2-\sigma_N^2)\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'M'_{i'i'}\alpha'\gamma'\beta')]\\
&\quad=\sum_{i'\neq j'\neq k\le h}\overline{W_{ki'}}W_{kj'}\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{i'j'}\gamma'\beta')]+\sum_{i'\neq k\le h}(|W_{ki'}|^2-\sigma_N^2)\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{i'i'}\gamma'\beta')]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{\le h}\Big[\operatorname{Tr}\left((\gamma''\otimes C_k^{(k)*}\alpha''\otimes M''\gamma''\otimes C_k^{(k)}-\gamma''(id\otimes\sigma_N^2\operatorname{Tr})(\alpha''\otimes M'')\gamma'')\beta''\right)\Big]\\
&\quad=\sum_{i''\neq j''\neq k\le h}\overline{W_{ki''}}W_{kj''}\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{i''j''}\gamma''\beta'')]+\sum_{i''\neq k\le h}(|W_{ki''}|^2-\sigma_N^2)\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{i''i''}\gamma''\beta'')],
\end{aligned}$$

it follows that

$$\begin{aligned}
&\mathbb{E}_k\Big[\mathbb{E}_{\le h}[\operatorname{Tr}\left((\gamma'\otimes C_k^{(k)*}A'\gamma'\otimes C_k^{(k)}-\gamma'(\mathrm{id}_d\otimes\sigma_N^2\operatorname{Tr})(A')\gamma')\beta'\right)]\\
&\qquad\qquad\times\mathbb{E}_{\le h}[\operatorname{Tr}\left((\gamma''\otimes C_k^{(k)*}A''\gamma''\otimes C_k^{(k)}-\gamma''(\mathrm{id}_d\otimes\sigma_N^2\operatorname{Tr})(A'')\gamma'')\beta''\right)]\Big]\\
&\quad=\sum_{i\neq j\neq k\le h}\mathbb{E}[|W_{ki}|^2]\mathbb{E}[|W_{kj}|^2]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{ij}\gamma'\beta')]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{ji}\gamma''\beta'')]\\
&\quad+\sum_{i\neq j\neq k\le h}\mathbb{E}[\overline{W_{ki}}^2]\mathbb{E}[W_{kj}^2]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{ij}\gamma'\beta')]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{ij}\gamma''\beta'')]\\
&\quad+\sum_{i\neq k\le h}\mathbb{E}\big[(|W_{ki}|^2-\sigma_N^2)^2\big]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma'\alpha'_{ii}\gamma'\beta')]\mathbb{E}_{\le h}[\operatorname{Tr}(\gamma''\alpha''_{ii}\gamma''\beta'')].
\end{aligned}$$

Note that, for $i\neq j$, $\mathbb{E}[\overline{W_{ki}}^2]\mathbb{E}[W_{kj}^2]=|\theta_N|^2$ if i and j are both smaller than k or greater than k. If it is not the case, this term equals θ_N^2 or $\bar\theta_N^2$, according to whether $i<j$ or $i>j$. By Assumption 3 and the fact that $\lim N\theta_N \in \mathbb{R}$, $\mathcal{I}\theta_N = o(N^{-1})$ and $\theta_N^2 = |\theta_N|^2 + o(N^{-2})$ (and so does $\bar{\theta}_N^2$). Therefore

$$\begin{aligned}\mathbb{E}_k\big[\mathbb{E}_{\leq h}[\operatorname{Tr}\big((\gamma' \otimes C_k^{(k)*} A'\gamma' \otimes C_k^{(k)} - \gamma'(id \otimes \sigma_N^2 \operatorname{Tr})(A')\gamma')\beta'\big)]\\ \times \mathbb{E}_{\leq h}[\operatorname{Tr}\big((\gamma'' \otimes C_k^{(k)*} A''\gamma'' \otimes C_k^{(k)} - \gamma''(id \otimes \sigma_N^2 \operatorname{Tr})(A'')\gamma'')\beta''\big)]\big]\\ = \sigma_N^4 \sum_{i,j\neq k\leq h} \mathbb{E}_{\leq h}[\operatorname{Tr}(\gamma'\alpha'_{ij}\gamma'\beta')]\mathbb{E}_{\leq h}[\operatorname{Tr}(\gamma''\alpha''_{ji}\gamma''\beta'')]\\ + |\theta_N|^2 \sum_{i,j\neq k\leq h} \mathbb{E}_{\leq h}[\operatorname{Tr}(\gamma'\alpha'_{ij}\gamma'\beta')]\mathbb{E}_{\leq h}[\operatorname{Tr}(\gamma''\alpha''_{ij}\gamma''\beta'')]\\ + \kappa_N \sum_{i\neq k\leq h} \mathbb{E}_{\leq h}[\operatorname{Tr}(\gamma'\alpha'_{ii}\gamma'\beta')]\mathbb{E}_{\leq h}[\operatorname{Tr}(\gamma''\alpha''_{ii}\gamma''\beta'')] + \varepsilon_k,\end{aligned}$$

where ε_k has the expected expression.

In order to show that $\sum_{k=1}^N \varepsilon_k \to 0$ in probability, we consider the L^1 norm $\|\varepsilon_k\|_1$. We denote by ε_{ijk} the quantity $\mathbb{E}[\overline{W_{ki}}^2]\mathbb{E}[W_{kj}^2] - |\theta_N|^2$.

$$\begin{aligned}\|\varepsilon_k\|_1 &\leq d^2\|\gamma'\|^2\|\gamma''\|^2\|\beta'\|\|\beta''\| \sup_{i\neq j\neq k\leq h} |\varepsilon_{ijk}| \sum_{i\neq j\neq k\leq h} \mathbb{E}\big[\|\alpha'_{ij}\|\|\alpha''_{ij}\|\big]\\ &\leq \frac{d^2}{2}\|\gamma'\|^2\|\gamma''\|^2\|\beta'\|\|\beta''\| \sup_{i\neq j\neq k\leq h} |\varepsilon_{ijk}| \sum_{i\neq j\neq k\leq h} \mathbb{E}\big[\|\alpha'_{ij}\|^2 + \|\alpha''_{ij}\|^2\big]\\ &\leq \frac{d^3}{2}\|\gamma'\|^2\|\gamma''\|^2\|\beta'\|\|\beta''\| \sup_{i\neq j\neq k\leq h} |\varepsilon_{ijk}|(N-1)\mathbb{E}\big[\|A'\|^2 + \|A''\|^2\big]\end{aligned}$$

using Lemma 64. Recall that $\|\gamma'\|$, $\|\gamma''\|$, $\|\beta'\|$, $\|\beta''\|$, $\mathbb{E}[\|A'\|^2]$ and $\mathbb{E}[\|A''\|^2]$ are bounded. Together with $\sup_{i\neq j\neq k\leq h}|\varepsilon_{ijk}| = o(N^{-2})$, it leads to $\|\varepsilon_k\|_1 = o(N^{-1})$, uniformly in k. Therefore, as claimed,

$$\sum_{k=1}^N \varepsilon_k \xrightarrow[N\to+\infty]{\mathbb{P}} 0.$$

□

6.4. **Bounds on R_N and $R^{(k)}$.** For $\beta \in \mathbb{H}^+(M_m(\mathbb{C}))$ and $k = 1, \ldots, N$, by a direct application of (2),

$$\|R_N(\beta)\| \leq \|(\mathcal{I}\beta)^{-1}\|, \quad \|R^{(k)}(\beta)\| \leq \|(\mathcal{I}\beta)^{-1}\|. \tag{19}$$

It is a consequence of Lemma 4 that R_N and $R^{(k)}$ are also defined on $\{ze_{11} - \gamma_0, z \in \mathbb{C}\setminus\mathbb{R}\}$.

Lemma 21. *For $p \geq 1$, $\mathbb{E}[\|R_N(ze_{11} - \gamma_0)\|^p] = O(1)$.*

Proof. Using Lemma 5 with $y = (W_N, D_N)$, one gets

$$\|R_N(ze_{11} - \gamma_0)\| \leq Q_1(\|W_N\|, \|D_N\|)\|(zI_N - X_N)^{-1}\| + Q_2(\|W_N\|, \|D_N\|).$$

Then, using Assumption 5 and the bound (3), there is a polynomial Q such that

$$\|R_N(ze_{11} - \gamma_0)\| \leq (1 + |\mathcal{I}z|^{-1})Q(\|W_N\|).$$

It follows from Proposition 78 that $Q(\|W_N\|)$is bounded in all L^p, $p \geq 1$. □

Remark 22. *The same argument with $y = (W_N^{(k)}, D_N^{(k)})$ (and the observation that $\|W_N^{(k)}\| \leq \|W_N\|$ and $\|D_N^{(k)}\| \leq \|D_N\|$) proves that $\mathbb{E}[\|R^{(k)}(ze_{11} - \gamma_0)\|^p] = O(1)$. With $y = (W_N^{(kab)}, D_N^{(k)})$ (and the observation that $\|W_N^{(kab)}\| \leq \|W_N\| + 2\delta_N$ and $\|D_N^{(k)}\| \leq \|D_N\|$), we get that $\mathbb{E}[\|R^{(kab)}(ze_{11} - \gamma_0)\|^p] = O(1)$, uniformly in $a, b \neq k$.*

By Proposition 60, for $z \in \mathbb{C}\setminus\mathbb{R}$, $ze_{11} - \gamma_0 - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*}R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)}$ is invertible,

$$(ze_{11} - \gamma_0 - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*}R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)})^{-1} = R_{kk}(ze_{11} - \gamma_0)$$

and therefore

$$\|(ze_{11} - \gamma_0 - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*}R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)})^{-1}\| \leq \|R_N(ze_{11} - \gamma_0)\|. \tag{20}$$

Note that the same argument implies the bound

$$\|(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*} R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)})^{-1}\| \leq \|R_N(\beta)\| \leq \|(\mathcal{I}\beta)^{-1}\|$$

for $\beta \in \mathbb{H}^+(M_m(\mathbb{C}))$.

Lemma 23. *For $p \in [2, 4(1+\varepsilon)]$,*

$$\mathbb{E}[\|\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)} - \sigma_N^2 \gamma_1(\mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0))\gamma_1]\|^p] = O(N^{-p/2}).$$

Proof. The result easily follows from Lemmas 69, 77 and Remark 22. ☐

Lemma 24. *For $p \geq 1$,*

$$\mathbb{E}\left[\|\mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0) - \mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)\|^p\right] = O(1),$$

$$\mathbb{E}\left[\|\frac{\partial}{\partial z}(\mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0) - \mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0))\|^p\right] = O(1). \tag{21}$$

Proof. For $z \in \mathbb{C} \setminus \mathbb{R}$, by Proposition 60, since $\mathrm{id}_m \otimes \operatorname{Tr} R = \mathrm{id}_m \otimes \operatorname{Tr}(R_I + R_{I^c})$ with $I^c = \{k, k+N, \ldots, k+(N-1)m\}$,
$\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0) - \mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0)$

$$= s_k^{-1} + \mathrm{id}_m \otimes \operatorname{Tr}(R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)} s_k^{-1} \gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11} - \gamma_0)),$$

where $s_k = ze_{11} - \gamma_0 - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)}$. Observe that

$$\|\mathrm{id}_m \otimes \operatorname{Tr}(R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)} s_k^{-1} \gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11} - \gamma_0))\| \leq m^4 \|\gamma_1\|^2 \|R^{(k)}(ze_{11} - \gamma_0)\|^2 \|C_k^{(k)}\|^2 \|s_k^{-1}\|,$$

so that

$$\begin{aligned}
&\|\mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0) - \mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)\| \\
&\qquad \leq \|s_k^{-1}\|(1 + m^4\|\gamma_1\|^2 \|R^{(k)}(ze_{11} - \gamma_0)\|^2 \|C_k^{(k)}\|^2) \\
&\qquad \leq \|R_N(ze_{11} - \gamma_0)\|\|(1 + m^4\|\gamma_1\|^2 \|R^{(k)}(ze_{11} - \gamma_0)\|^2 \|W_N\|^2)
\end{aligned}$$

is bounded in all L^p, $p \geq 1$, by Lemma 21, Remark 22 and Proposition 78. Similarly one can prove (21). ☐

Lemma 25. *For any even integer $p \in \mathbb{N}$,*

$$\mathbb{E}[\|\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0) - \mathbb{E}[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)]\|^p] = O(N^{p-1}),$$

$$\mathbb{E}[\|\frac{\partial}{\partial z}[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0) - \mathbb{E}(\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0))]\|^p] = O(N^{p-1}).$$

Proof. Observe that $M_k := \mathbb{E}_{\leq k}[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)]$, $k \geq 0$, satisfies

$$\begin{aligned}
M_k - M_{k-1} &= (\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)] \\
&= (\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0) - \mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0)]
\end{aligned}$$

so that, using Jensen's inequality,

$$\mathbb{E}[\|M_k - M_{k-1}\|^p] \leq 2^p \mathbb{E}[\|\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0) - \mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0)\|^p] = O(1)$$

by Lemma 24. Apply then Lemma 71 to the martingale $(M_k = \mathbb{E}_{\leq k}[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)])_{k \geq 0}$ to obtain the first statement. Similarly, one can obtain the second statement by considering $M_k := \mathbb{E}_{\leq k}[\frac{\partial}{\partial z}\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)]$. ☐

Remark 26. *Using Lemma 24, one may deduce from the Lemma 25 that, for any even integer $p \in \mathbb{N}$,*

$$\mathbb{E}[\|\mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0) - \mathbb{E}[\mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0)]\|^p] = O(N^{p-1}).$$

6.5. **Qualitative asymptotic freeness.** We first prove the asymptotic freeness of W_N and D_N.

Lemma 27. *For any polynomial in two noncommuting indeterminates $H \in \mathbb{C}\langle t_1, t_2\rangle$,*

$$\mathbb{E}[N^{-1}\operatorname{Tr}(H(W_N, D_N))] \longrightarrow_{N\to\infty} \tau(H(s,d)). \tag{22}$$

Proof. One may assume, without loss of generality, that $\sigma_N^2 = \sigma^2 N^{-1}$. Define a $N\times N$ random matrix A_N by $A_{ij} = W_{ij}$ when $i \neq j$ and $A_{ii} = \sigma_N \tilde{\sigma}_N^{-1} W_{ii}$. It is straightforward that

$$\|W_N - A_N\| \le |1 - \sigma_N \tilde{\sigma}_N^{-1}|\delta_N \to_{N\to+\infty} 0.$$

Since $\sup_N \|D_N\| < +\infty$ and, by Proposition 78, $(\|W_N\|)_N$ and $(\|A_N\|)_N$ are bounded in all L^p, $p \ge 1$, it follows that

$$\begin{aligned}|\mathbb{E}[N^{-1}\operatorname{Tr}(H(W_N, D_N))] &- \mathbb{E}[N^{-1}\operatorname{Tr}(H(A_N, D_N))]| \\ &\le \mathbb{E}[\|H(W_N, D_N) - H(A_N, D_N)\|] \longrightarrow_{N\to+\infty} 0.\end{aligned}$$

Since $\mathbb{E}[A_{ij}] = 0$, $\mathbb{E}[|A_{ij}|^2] = \sigma^2 N^{-1}$ and, for any $m > 2$, $\sup_{i,j\le N}\mathbb{E}[|A_{ij}|^m] = o(N^{-1})$, one can apply Theorem 1 of [Rya98], and obtain that $\mathbb{E}[N^{-1}\operatorname{Tr} H(A_N, D_N)]$ converges towards $\tau(H(s,d))$. ☐

The following result is a consequence of the above asymptotic freeness.

Lemma 28. *For $z \in \mathbb{C}\setminus\mathbb{R}$,*

$$\mathbb{E}[\big(\mathrm{id}_m \otimes N^{-1}\operatorname{Tr}\big)(R_N(ze_{11} - \gamma_0))] - (\mathrm{id}_m \otimes \tau_N)(r_N(ze_{11} - \gamma_0)) \longrightarrow_{N\to+\infty} 0.$$

$$\frac{\partial}{\partial z}\left(\mathbb{E}[\big(\mathrm{id}_m \otimes N^{-1}\operatorname{Tr}\big)(R_N(ze_{11} - \gamma_0))] - (\mathrm{id}_m \otimes \tau_N)(r_N(ze_{11} - \gamma_0))\right) \longrightarrow_{N\to+\infty} 0.$$

Proof. Let $L_P = \begin{pmatrix} 0 & u \\ v & Q\end{pmatrix}$ be the canonical linearization of P. Remember that the entries of the row vector u, the column vector v and the matrix Q^{-1} are all polynomials and that $P = -uQ^{-1}v$. It readily follows from Proposition 60 applied to $z - L_P$, that for each $p, q \in \{1,\dots,m\}$, there exist two polynomials $H_1^{(p,q)}$ and $H_2^{(p,q)}$ such that, for any $z \in \mathbb{C}\setminus\mathbb{R}$, for any N, the entry (p,q) of the $m\times m$ matrix $(\mathrm{id}_m\otimes\tau_N)[r_N(ze_{11} - \gamma_0)]$ (respectively of the $m\times m$ matrix $\big(\mathrm{id}_m \otimes N^{-1}\operatorname{Tr}\big)[R_N(ze_{11}-\gamma_0)]$) is equal to $\tau_N\Big((z1_{\mathcal{A}_N} - P(s_N, D_N))^{-1}H_1^{(p,q)}(s_N, D_N) + H_2^{(p,q)}(s_N, D_N)\Big)$ (respectively of the $m\times m$ matrix $N^{-1}\operatorname{Tr}\Big((zI_N - P(W_N, D_N))^{-1}H_1^{(p,q)}(W_N, D_N) + H_2^{(p,q)}(W_N, D_N)\Big)$).

We have for any selfadjoint operators y_1 and y_2, for any $z\in\mathbb{C}\setminus\mathbb{R}$, for any nonzero integer r,

$$(z - P(y_1,y_2))^{-1} = \sum_{k=0}^{r-1} z^{-1}(z^{-1}P(y_1,y_2))^k + (z - P(y_1,y_2))^{-1}\,(z^{-1}P(y_1,y_2))^r. \tag{23}$$

For any $K > 0$, define

$$\mathcal{O}_K = \{z \in \mathbb{C}\setminus\mathbb{R}, \mathcal{I}(z) > K\}.$$

Let $0 < c < 1$. For any $\kappa > 0$, there exists $K = K(\kappa, P, c) > 0$ such that if $z \in \mathcal{O}_K$, for any y_1 and y_2 such that $\|y_1\| \le \kappa$ and $\|y_2\| \le \kappa$ then

$$\|(z^{-1}P(y_1,y_2))\| \le c, \tag{24}$$

so that

$$\sup_{z\in\mathcal{O}_K}\left\|(z - P(y_1,y_2))^{-1}\,(z^{-1}P(y_1,y_2))^r H_1^{(p,q)}(y_1,y_2)\right\| \le c'\frac{c^r}{K} \to_{r\to+\infty} 0. \tag{25}$$

Fix $K > 0$ such that (24) holds for $(y_1,y_2) = (s_N, D_N)$ and $(y_1,y_2) = (W_N, D_N)$ on $E_N = \{\|W_N\| \le C\}$ where C is defined in Proposition 78. Using Proposition 78, it readily follows from (22) that for any polynomial H in two noncommuting indeterminates,

$$\mathbb{E}[\mathbf{1}_{E_N} N^{-1}\operatorname{Tr}(H(W_N, D_N))] \longrightarrow_{N\to+\infty} \tau(H(s,d)). \tag{26}$$

By the convergence in noncommutative distribution of (s_N, D_N) to (s,d), we have that, for any polynomial H in two noncommuting indeterminates,

$$\tau_N(H(s_N, d_N)) \longrightarrow_{N\to+\infty} \tau(H(s,d)). \tag{27}$$

Using (23), (25), (26), (27), letting N and then r go to infinity, we obtain that for any $z \in \mathcal{O}_K$,

$$\mathbb{E}[\mathbf{1}_{E_N}\big(\mathrm{id}_m \otimes N^{-1}\operatorname{Tr}\big)(R_N(ze_{11} - \gamma_0))] - (\mathrm{id}_m\otimes\tau_N)(r_N(ze_{11}-\gamma_0)) \longrightarrow_{N\to+\infty} 0,$$

and then, using Lemma 21 and $\mathbb{P}(E_N^c) \to_{N\to+\infty} 0$, we readily deduce that, for any $z \in \mathcal{O}_K$,

$$\mathbb{E}[\left(\mathrm{id}_m \otimes N^{-1}\,\mathrm{Tr}\right)(R_N(ze_{11}-\gamma_0))] - (\mathrm{id}_m \otimes \tau_N)\,(r_N(ze_{11}-\gamma_0)) \longrightarrow_{N\to\infty} 0.$$

Functions $\Phi_N(z) = \mathbb{E}[\left(\mathrm{id}_m \otimes N^{-1}\,\mathrm{Tr}\right)(R_N(ze_{11}-\gamma_0))] - (\mathrm{id}_m \otimes \tau_N)\,(r_N(ze_{11}-\gamma_0))$, $N \in \mathbb{N}$, are holomorphic on $\mathbb{C}^+$. Moreover, using Lemma 5 and Proposition 78, there exists a polynomial Q such that,

$$||\Phi_N(z)|| \leq Q((\mathcal{I}z)^{-1}), \quad z \in \mathbb{C}^+. \tag{28}$$

It readily follows that $(\Phi_N)_{N\in\mathbb{N}}$ is a bounded sequence in the set of analytic functions on $\mathbb{C}^+$ endowed with the uniform convergence on compact subsets. We can apply Vitali's theorem to conclude that the convergences of $(\Phi_N)_{N\in\mathbb{N}}$ and $(\partial_z\Phi_N)_{N\in\mathbb{N}}$ to 0 hold on $\mathbb{C}^+$. Of course, this convergence similarly holds on $\mathbb{C}^-$. The proof of Lemma 28 is complete. □

Remark 29. *Note that, following the strategy of the proof of Lemma 28, we obtain the uniform convergence of ω_N to ω on compact subsets of $\mathbb{C}^\pm e_{11} - \gamma_0$.*

As a consequence of Lemma 25 and Lemma 28, we obtain the following

Lemma 30. *For $z \in \mathbb{C}\setminus\mathbb{R}$ and $p \geq 1$,*

$$\mathbb{E}[\|\mathrm{id}_m \otimes N^{-1}\,\mathrm{Tr}(R_N(ze_{11}-\gamma_0)) - \mathrm{id}_m \otimes \tau_N(r_N(ze_{11}-\gamma_0))\|^p] \underset{N\to+\infty}{\longrightarrow} 0, \tag{29}$$

$$\mathbb{E}[\|\frac{\partial}{\partial z}\left[\mathrm{id}_m \otimes N^{-1}\,\mathrm{Tr}(R_N(ze_{11}-\gamma_0)) - \mathrm{id}_m \otimes \tau_N(r_N(ze_{11}-\gamma_0))\right]\|^p] \underset{N\to+\infty}{\longrightarrow} 0. \tag{30}$$

Lemma 31.

$$\|(\Omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N)^{-1}\| = O(1)$$

Proof. It follows from Lemma 17 that $\omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N$ is invertible for every $z \in \mathbb{C}\setminus\mathbb{R}$ and

$$\|(\omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N)^{-1}\| = O(1).$$

One deduces from Lemma 28 that

$$\|(\Omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N) - (\omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N)\| \underset{N\to+\infty}{\longrightarrow} 0.$$

Note that this convergence is uniform on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Hence, it follows from Lemma 62 that, for any compact subset K of $\mathbb{C}\setminus\mathbb{R}$, $\Omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N$, $z \in K$, are all invertible for large enough N and

$$\|(\Omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N)^{-1}\| = O(1).$$

□

Remark 32. *By a similar argument based on Lemma 62, one may deduce from Lemma 31 and Lemma 24 that*

$$\left\|\left((ze_{11}-\gamma_0-\eta_N(\mathbb{E}[(\mathrm{id}_m\otimes N^{-1}\,\mathrm{Tr})(R^{(k)}(ze_{11}-\gamma_0))])\otimes I_N - \gamma_2\otimes D_N\right)^{-1}\right\| = O(1).$$

Lemma 33. *For $z \in \mathbb{C}\setminus\mathbb{R}$ and $p \geq 1$,*

$$\max_{k=1,\ldots,N}\mathbb{E}[\|R_{kk}(ze_{11}-\gamma_0) - \hat{R}_k(ze_{11}-\gamma_0)\|^p] \underset{N\to+\infty}{\longrightarrow} 0,$$

where $\hat{R}_k$ is defined by (14).

Proof. With the notation $\beta = ze_{11}-\gamma_0$, and

$$\hat{R}_k(\beta) = (\omega_N(\beta) - D_{kk}\gamma_2)^{-1}, \tag{31}$$

by Proposition 60, since for any $k \in \{1,\ldots,N\}$, $R_{kk}(\beta)$ is the submatrix of $R_N(\beta)$ corresponding to rows and columns indexed by $\{k, k+m, \ldots, k+m(N-1)\}$,

$$\begin{aligned} R_{kk}(\beta) &= (\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)})^{-1} \\ &= \hat{R}_k(\beta) + \hat{R}_k(\beta)(W_{kk}\gamma_1 + \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)} - \eta_N((\mathrm{id}_m\otimes\tau_N)(r_N(\beta))))R_{kk}(\beta). \end{aligned}$$

Observe that, for $n \geq 2$,

$$\begin{aligned}3^{-n+1}&\mathbb{E}[\|W_{kk}\gamma_1 + \gamma_1 \otimes C_k^{(k)*} R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)} - \eta_N((\mathrm{id}_m \otimes \tau_N)(r_N(\beta)))\|^n] \\ &\leq \|\gamma_1\|^n \mathbb{E}[|W_{kk}|^n] + \mathbb{E}[\|\gamma_1 \otimes C_k^{(k)*} R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)} - \eta_N((\mathrm{id}_m \otimes N^{-1}\operatorname{Tr})(R^{(k)}(\beta)))\|^n] \\ &\quad + \mathbb{E}[\|\eta_N((\mathrm{id}_m \otimes N^{-1}\operatorname{Tr})(R^{(k)}(\beta))) - \eta_N((\mathrm{id}_m \otimes \tau_N)(r_N(\beta)))\|^n].\end{aligned}$$

The first term is bounded by $\|\gamma_1\|^n \delta_N^{n-1}\tilde{\sigma}_N$ by assumption; the second term asymptotically vanishes by Lemma 69; the third term asymptotically vanishes by Lemma 30 and Lemma 24. Then, choosing $q, r \geq 1$ such that $\frac{1}{q} + \frac{1}{r} = 1$ and $pq \geq 2$,

$$\begin{aligned}&\mathbb{E}[\|R_{kk}(\beta) - \hat{R}_k(\beta)\|^p] \\ &\leq \|\hat{R}(\beta)\|^p \mathbb{E}\big[\|W_{kk}\gamma_1 + \gamma_1 \otimes C_k^{(k)*} R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)} - \eta_N((\mathrm{id}_m \otimes \tau_N)(r_N(\beta)))\|^{pq}\big]^{1/q} \mathbb{E}\big[\|R_N(\beta)\|^{pr}\big]^{1/r}\end{aligned}$$

vanishes uniformly in k by using Lemma 21 and Lemma 17. ☐

Remark 34. *Similarly,* $\mathbb{E}[\|R_{ll}^{(k)}(\beta) - \hat{R}_l(\beta)\|^p]$ *vanishes uniformly in* $1 \leq k \neq l \leq N$*. Indeed, using in this Remark only the notation* $R^{(kl)}(\beta) = (\beta \otimes I_{N-2} - \gamma_1 \otimes W_N^{(kl)} - \gamma_2 \otimes D_N^{(kl)})^{-1}$*, where* $W_N^{(kl)}, D_N^{(kl)}$ *denote the* $(N-2) \times (N-2)$ *matrices obtained respectively from* W_N, D_N *by deleting the* k*-th and* l*-th rows/columns,*

$$\begin{aligned}R_{ll}^{(k)}(\beta) &= (\beta - W_{ll}\gamma_1 - D_{ll}\gamma_2 - \gamma_1 \otimes C_l^{(kl)*} R^{(kl)}(\beta)\gamma_1 \otimes C_l^{(kl)})^{-1} \\ &= \hat{R}_l(\beta) + \hat{R}_l(\beta)(W_{ll}\gamma_1 + \gamma_1 \otimes C_l^{(kl)*} R^{(kl)}(\beta)\gamma_1 \otimes C_l^{(kl)} - \eta_N((\mathrm{id}_m \otimes \tau_N)(r_N^{(k)}(\beta))))R_{ll}^{(k)}(\beta).\end{aligned}$$

Observe that, for $n \geq 2$*,*

$$\begin{aligned}3^{-n+1}&\mathbb{E}\big[\|W_{ll}\gamma_1 + \gamma_1 \otimes C_l^{(kl)*} R^{(kl)}(\beta)\gamma_1 \otimes C_l^{(kl)} - \eta_N((\mathrm{id}_m \otimes \tau_N)(r_N(\beta)))\|^n\big] \\ &\leq \|\gamma_1\|^n \mathbb{E}\big[|W_{ll}|^n\big] + \mathbb{E}\big[\|\gamma_1 \otimes C_l^{(kl)*} R^{(kl)}(\beta)\gamma_1 \otimes C_l^{(kl)} - \eta_N((\mathrm{id}_m \otimes N^{-1}\operatorname{Tr})(R^{(kl)}(\beta)))\|^n\big] \\ &\quad + \mathbb{E}\big[\|\eta_N((\mathrm{id}_m \otimes N^{-1}\operatorname{Tr})(R^{(kl)}(\beta))) - \eta_N((\mathrm{id}_m \otimes \tau_N)(r_N(\beta)))\|^n\big].\end{aligned}$$

The first term is bounded by $\|\gamma_1\|^n \delta_N^{n-1}\tilde{\sigma}_N$ *by assumption; the second term asymptotically vanishes by Lemma 69; the third term asymptotically vanishes by Lemma 30 and Lemma 24. Then, choosing* $q, r \geq 1$ *such that* $\frac{1}{q} + \frac{1}{r} = 1$ *and* $pq \geq 2$*,*

$$\begin{aligned}&\mathbb{E}\big[\|R_{ll}^{(k)}(\beta) - \hat{R}_l(\beta)\|^p\big] \\ &\leq \|\hat{R}(\beta)\|^p \mathbb{E}\big[\|W_{ll}\gamma_1 + \gamma_1 \otimes C_l^{(kl)*} R^{(kl)}(\beta)\gamma_1 \otimes C_l^{(kl)} - \eta_N((\mathrm{id}_m \otimes \tau_N)(r_N(\beta)))\|^{pq}\big]^{1/q} \mathbb{E}\big[\|R^{(k)}(\beta)\|^{pr}\big]^{1/r}\end{aligned}$$

vanishes uniformly in $l \neq k$ *by using Remark 22 and Lemma 17.*

Remark 35. *One may also prove that* $\mathbb{E}[\|R_{ll}^{(kil)}(\beta) - \hat{R}_l(\beta)\|^p]$ *vanishes uniformly in* $1 \leq i \neq k \neq l \leq N$*, using* $\mathbb{E}[\|R_{ll}^{(k)}(\beta) - R_{ll}^{(kil)}(\beta)\|^p] = o(1)$*.*

6.6. **Concentration bounds on R_N and $R^{(k)}$.**

Lemma 36. *For any* $c > 0$*,*

$$\mathbb{E}[\|\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0) - \mathbb{E}[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)]\|^4] = O(N^{2+c}).$$

Proof. Observe as in the proof of Lemma 25 that $M_k := \mathbb{E}_{\leq k}[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)]$, $k \geq 0$, satisfies

$$\begin{aligned}M_k - M_{k-1} &= (\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0)] \\ &= (\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0) - \mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0)]\end{aligned}$$

and write as in the proof of Lemma 24: for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned}&\mathrm{id}_m \otimes \operatorname{Tr} R_N(ze_{11} - \gamma_0) - \mathrm{id}_m \otimes \operatorname{Tr} R^{(k)}(ze_{11} - \gamma_0) \\ &\qquad = s_k^{-1} + \mathrm{id}_m \otimes \operatorname{Tr}(R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)} s_k^{-1}\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11} - \gamma_0)),\end{aligned}$$

where $s_k = ze_{11} - \gamma_0 - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)}$. It follows from Remark 32 that for any compact subset K of $\mathbb{C} \setminus \mathbb{R}$, $\mathbb{E}[s_k]$, $z \in K$, are all invertible for N large enough and $\|\mathbb{E}[s_k]^{-1}\| = O(1)$. Observe that, for such N,

$$(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[\mathbb{E}[s_k]^{-1} + \mathbb{E}_k[\mathrm{id}_m \otimes \operatorname{Tr}(R^{(k)}(ze_{11} - \gamma_0)\gamma_1 \otimes C_k^{(k)}\mathbb{E}[s_k]^{-1}\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11} - \gamma_0))]] = 0$$

so that

$$
\begin{aligned}
-(\mathbb{E}_{\le k} - \mathbb{E}_{\le k-1})&[\mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0) - \mathrm{id}_m \otimes \mathrm{Tr}\, R_N(ze_{11}-\gamma_0)] \\
&= (\mathbb{E}_{\le k} - \mathbb{E}_{\le k-1})[s_k^{-1} - \mathbb{E}[s_k]^{-1}] \\
&+ (\mathbb{E}_{\le k} - \mathbb{E}_{\le k-1})[\mathrm{id}_m \otimes \mathrm{Tr}(R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}(s_k^{-1} - \mathbb{E}[s_k]^{-1})\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11}-\gamma_0))] \\
&+ (\mathbb{E}_{\le k} - \mathbb{E}_{\le k-1})[\mathrm{id}_m \otimes \mathrm{Tr}(R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}\mathbb{E}[s_k]^{-1}\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11}-\gamma_0)) \\
&- \mathbb{E}_k[\mathrm{id}_m \otimes \mathrm{Tr}(R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}\mathbb{E}[s_k]^{-1}\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11}-\gamma_0))]]. \qquad (32)
\end{aligned}
$$

From

$$
\begin{aligned}
&s_k^{-1} - \mathbb{E}[s_k]^{-1} \\
&= s_k^{-1}(W_{kk}\gamma_1 + \gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)} - \mathbb{E}[\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}])\mathbb{E}[s_k]^{-1},
\end{aligned}
$$

Hölder's inequality, together with (20), Lemmas 5, 77, 23, 21, Remarks 22 and 32, deduce that, for large N, for any k, for small enough $t > 0$,

$$
\begin{aligned}
&\mathbb{E}[\|s_k^{-1} - \mathbb{E}[s_k]^{-1}\|^{4(1+t)}] \\
&\le \mathbb{E}[\|s_k^{-1}\|^{4(1+t)} \|W_{kk}\gamma_1 + \gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)} \\
&\qquad\qquad - \mathbb{E}[\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}]\|^{4(1+t)}]\|\mathbb{E}[s_k]^{-1}\|^{4(1+t)} \\
&\le O(1)\mathbb{E}[\|s_k^{-1}\|^{4(1+t)(2+1/t)}]^{t/(1+2t)} \Big\{ (\|\gamma_1\|\sigma_N)^{8(1+2t)} \\
&\qquad \times \mathbb{E}\Big[\big\| \mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0) - \mathbb{E}[\mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0)] \big\|^{4(1+2t)} \Big] \\
&\qquad + O(N^{-2(1+2t)})) \Big\}^{(1+t)/(1+2t)} \\
&\le O(1)\mathbb{E}[\|R_N(ze_{11}-\gamma_0)\|^{4(1+t)(2+1/t)}]^{t/(1+2/t)} \Big\{ (\|\gamma_1\|\sigma_N)^{8(1+2t)} \\
&\qquad \times \mathbb{E}\Big[(2N\|R^{(k)}(ze_{11}-\gamma_0)\|)^{9t} \big\| \mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0) - \mathbb{E}\big(\mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0)\big) \big\|^{4-t} \Big] \\
&\qquad + O(N^{-2(1+2t)}) \Big\}^{(1+t)/(1+2t)} \\
&\le O(1) \Big\{ N^{9t-4(1+2t)} \mathbb{E}\big(\big\| \mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0) - \mathbb{E}\big(\mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0)\big) \big\|^4 \big)^{1-t/4} \\
&\qquad + O(N^{-2(1+2t)}) \Big\}^{(1+t)/(1+2t)} \\
&= \Big\{ O(N^{-1+t/4}) + O(N^{-2(1+2t)}) \Big\}^{(1+t)/(1+2t)} \\
&= O(N^{(-1+t/4)(1+t)/(1+2t)}) \qquad (33)
\end{aligned}
$$

where we use Remark 26 in the last line.

It follows that

$$
\begin{aligned}
\mathbb{E}[\|s_k^{-1} - \mathbb{E}[s_k]^{-1}\|^4] &\le \mathbb{E}[\|s_k^{-1} - \mathbb{E}[s_k]^{-1}\|^{4(1+t)}]^{1/(1+t)} \\
&= O(N^{(t/4-1)/(1+2t)}) \qquad (34)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}[\|\mathrm{id}_m \otimes \mathrm{Tr}(R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}(s_k^{-1} - \mathbb{E}[s_k]^{-1})\gamma_1 \otimes C_k^{(k)*} R^{(k)}(ze_{11}-\gamma_0))\|^4] \\
&\le m^{16}\|\gamma_1\|^8 \mathbb{E}[\|R^{(k)}(ze_{11}-\gamma_0)\|^8 \|C_k^{(k)}\|^8 \|s_k^{-1} - \mathbb{E}[s_k]^{-1}\|^4] \\
&\le m^{16}\|\gamma_1\|^8 \mathbb{E}[\|R^{(k)}(ze_{11}-\gamma_0)\|^{8(1+1/t)} \|C_k^{(k)}\|^{8(1+1/t)}]^{t/(1+t)} \mathbb{E}[\|s_k^{-1} - \mathbb{E}[s_k]^{-1}\|^{4(1+t)}]^{1/(1+t)} \\
&= O(N^{(t/4-1)/(1+2t)}), \qquad (35)
\end{aligned}
$$

by using (33), Remark 22 and Proposition 78.
Finally, if $R^{(k)} = \sum_{i,j\neq k} R^{(k)}_{ij} \otimes E_{ij}$ and $R^{(k)pT} = \sum_{i,j\neq k} R^{(k)}_{ij} \otimes E_{ji}$, then

$$\begin{aligned}
&\mathrm{id}_m \otimes \mathrm{Tr}(R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}\mathbb{E}[s_k]^{-1}\gamma_1 \otimes C_k^{(k)*}R^{(k)}(ze_{11}-\gamma_0)) \\
&\qquad = \sum_{a,b,c\neq k} R^{(k)}_{ab}\gamma_1\mathbb{E}[s_k]^{-1}\gamma_1 R^{(k)}_{ca} W_{bk}\overline{W}_{ck} \\
&\qquad = I_m \otimes \overline{C_k^{(k)}}^{*}\Sigma I_m \otimes \overline{C_k^{(k)}},
\end{aligned}$$

where the fourth moment of the norm of

$$\Sigma = \sum_{a,b,c\neq k} R^{(k)}_{ab}\gamma_1\mathbb{E}[s_k]^{-1}\gamma_1 R^{(k)}_{ca}\otimes E_{bc} = R^{(k)}(ze_{11}-\gamma_0)^{pT}(\gamma_1\mathbb{E}[s_k]^{-1}\gamma_1\otimes I_{N-1})R^{(k)}(ze_{11}-\gamma_0)^{pT}$$

is $O(1)$ by using Proposition 61, Remarks 22 and 32. It follows from Lemma 69 that

$$\begin{aligned}
&\mathbb{E}[\|\mathrm{id}_m \otimes \mathrm{Tr}(R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}\mathbb{E}[s_k]^{-1}\gamma_1 \otimes C_k^{(k)*}R^{(k)}(ze_{11}-\gamma_0)) \\
&\qquad - \mathbb{E}_k[\mathrm{id}_m \otimes \mathrm{Tr}(R^{(k)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_k^{(k)}\mathbb{E}[s_k]^{-1}\gamma_1 \otimes C_k^{(k)*}R^{(k)}(ze_{11}-\gamma_0))]\|^4] \\
&\le K_{m,4}\mathbb{E}[\|\Sigma\|^4]((Nm_N)^2 + O(N^{-3})N) = O(N^{-2}), \qquad (36)
\end{aligned}$$

where we use Lemma 77 in the last line. (32), (34), (35) and (36) yield

$$\mathbb{E}[\|(\mathbb{E}_{\le k} - \mathbb{E}_{\le k-1})[\mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0) - \mathrm{id}_m \otimes \mathrm{Tr}\, R_N(ze_{11}-\gamma_0)]\|^4] = O(N^{-1+c}),$$

by choosing $t > 0$ small enough. We conclude by Lemma 71. □

Remark 37. *Using Lemma 24, one may deduce from Lemma 36 that, for $c > 0$,*

$$\mathbb{E}[\|\mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0) - \mathbb{E}[\mathrm{id}_m \otimes \mathrm{Tr}\, R^{(k)}(ze_{11}-\gamma_0)]\|^4] = O(N^{2+c}).$$

Lemma 38. *For $p \in [1,4)$ and $c > 0$*

$$\mathbb{E}[\|R_{kk}(ze_{11}-\gamma_0) - (\Omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N)^{-1}_{kk}\|^p] = O(N^{(c-2)p/4}).$$

Proof. Recall that Ω_N was defined in (11) by

$$\Omega_N(\beta) = \beta - \eta_N\big(\mathbb{E}[(\mathrm{id}_m \otimes N^{-1}\,\mathrm{Tr})(R_N(\beta))]\big).$$

According to Lemma 31 and its proof, for any compact subset K of $\mathbb{C}\setminus\mathbb{R}$, for large enough N, $\Omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N$, $z \in K$, are all invertible with uniformly bounded inverses. For such N and any $k = 1,\ldots,N$,

$$\begin{aligned}
&R_{kk}(ze_{11}-\gamma_0) - (\Omega_N(ze_{11}-\gamma_0) - D_{kk}\gamma_2)^{-1} \\
&\quad = R_{kk}(ze_{11}-\gamma_0)\Big(W_{kk}\gamma_1 + \gamma_1\otimes C_k^{(k)*}R^{(k)}(ze_{11}-\gamma_0)\gamma_1\otimes C_k^{(k)} \\
&\qquad\qquad - \eta_N\big(\mathbb{E}\big[(\mathrm{id}_m\otimes N^{-1}\,\mathrm{Tr})(R_N(ze_{11}-\gamma_0))\big]\big)\Big)\big(\Omega_N(ze_{11}-\gamma_0) - D_{kk}\gamma_2\big)^{-1}.
\end{aligned}$$

Then, using Lemmas 77, 23, 24 and 36, for $p < 4$,

$$\begin{aligned}
&\mathbb{E}\big[\|R_{kk}(ze_{11}-\gamma_0) - (\Omega_N(ze_{11}-\gamma_0) - D_{kk}\gamma_2)^{-1}\|^p\big] \\
&\quad \le \mathbb{E}\big[\|R_{kk}(ze_{11}-\gamma_0)\|^{4p/(4-p)}\big]^{1-p/4}\big(O(N^{-2}) + O(N^{c-2})\big)^{p/4}O(1) \\
&\quad = O(N^{(c-2)p/4}).
\end{aligned}$$

□

6.7. **Quantitative asymptotic freeness.**

Lemma 39. *The family of operators defined on $M_m(\mathbb{C})$ by*

$$u_N(z_1,z_2) : b \mapsto \sigma_N^2 \sum_{i=1}^N \hat{R}_i(z_1e_{11}-\gamma_0)\gamma_1 b\gamma_1\hat{R}_i(z_2e_{11}-\gamma_0), \quad z_1,z_2\in\mathbb{C}\setminus\mathbb{R},\ N\in\mathbb{N},$$

satisfies the following: for any compact subset K of $\mathbb{C}\setminus\mathbb{R}$, $\limsup_{N\to+\infty}\sup_{z_1,z_2\in K}\rho(u_N(z_1,z_2)) < 1$ and $\limsup_{N\to+\infty}\sup_{z_1,z_2\in K}\|(\mathrm{id}_m - u_N(z_1,z_2))^{-1}\| < +\infty$.

Proof. By Proposition 11, we know that the supremum over $z_1, z_2 \in K$ of the spectral radii of operators $u_{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0} : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ defined by

$$b \mapsto \sigma^2 \mathrm{id}_m \otimes \tau \left((\omega(z_1e_{11}-\gamma_0) - \gamma_2 \otimes d)^{-1} (\gamma_1 b \gamma_1) \otimes 1_{\mathcal{A}} (\omega(z_2e_{11}-\gamma_0) - \gamma_2 \otimes d)^{-1} \right), \quad z_1, z_2 \in \mathbb{C} \setminus \mathbb{R},$$

is strictly smaller than 1. Since $(D_N)_{N\in\mathbb{N}}$ converges in $*$-moments towards d, $\lim_{N\to+\infty} N\sigma_N^2 = \sigma^2$, we can easily deduce (using Lemmas 17, 18 and Lemma 19) that the family of operators defined for all $N \in \mathbb{N}$ and for all z_1, z_2 in $\mathbb{C}\setminus\mathbb{R}$, by

$$u_N(z_1, z_2) :$$
$$b \mapsto \sigma_N^2 \mathrm{id}_m \otimes \tau_N \left((\omega_N(z_1e_{11}-\gamma_0) - \gamma_2 \otimes D_N)^{-1} (\gamma_1 b \gamma_1) \otimes 1_{\mathcal{A}} (\omega_N(z_2e_{11}-\gamma_0) - \gamma_2 \otimes D_N)^{-1} \right),$$

satisfies $\sup_{z_1,z_2\in K} \|u_N(z_1,z_2) - u_{z_1e_{11}-\gamma_0, z_2e_{11}-\gamma_0}\| \longrightarrow_{N\to+\infty} 0$. The first assertion of the lemma readily follows by continuity of the spectral radius in finite dimension and the second one follows from the continuity of $X \mapsto X^{-1}$ and of the norm on $\mathbb{G}L_{m^2}(\mathbb{C})$. □

Corollary 40. *For any z_1, z_2 in $\mathbb{C}\setminus\mathbb{R}$, the sequence of operators defined on $M_m(\mathbb{C}) \otimes M_m(\mathbb{C})$ by*

$$T_N(z_1,z_2) : b_1 \otimes b_2 \mapsto N^{-1} \sum_{i=1}^N \hat{R}_i(z_1e_{11}-\gamma_0)\gamma_1 b_1 \otimes b_2 \gamma_1 \hat{R}_i(z_2e_{11}-\gamma_0), \quad N \in \mathbb{N},$$

satisfies $\limsup_{N\to+\infty} \rho(T_N(z_1,z_2)) < \sigma^{-2}$.

Lemma 41. *For $q \in [2, 4(1+\varepsilon))$,*

$$\mathbb{E}[\|\mathrm{id}_m \otimes N^{-1}\operatorname{Tr}(R_N(ze_{11}-\gamma_0)) - \mathrm{id}_m \otimes \tau(r_N(ze_{11}-\gamma_0))\|^q] = O(N^{-\min(2-\varepsilon, q/2)}).$$

Proof. Define $\hat{G} = \sum_{i=1}^N E_{ii} \otimes \hat{R}_i(ze_{11}-\gamma_0)$, and $G = \sum_{i=1}^N E_{ii} \otimes R_{ii}(ze_{11}-\gamma_0)$. Recall from (14) that $\hat{R}_i(ze_{11}-\gamma_0) = (\omega_N(ze_{11}-\gamma_0) - D_{ii}\gamma_2)^{-1}$. We have by (7)

$$\begin{aligned} \omega_N(ze_{11}-\gamma_0) - D_{ii}\gamma_2 &= ze_{11} - \gamma_0 - N\sigma_N^2 \gamma_1 N^{-1} \sum_{k=1}^N (\omega_N(ze_{11}-\gamma_0) - D_{kk}\gamma_2)^{-1} \gamma_1 - D_{ii}\gamma_2 \\ &= ze_{11} - \gamma_0 - N\sigma_N^2 \gamma_1 N^{-1} \sum_{k=1}^N \hat{R}_k(ze_{11}-\gamma_0)\gamma_1 - D_{ii}\gamma_2. \end{aligned}$$

Thus,

$$\hat{G}^{-1} = I_N \otimes \left[ze_{11} - \gamma_0 - N\sigma_N^2 \gamma_1 N^{-1} \operatorname{Tr} \otimes \mathrm{id}_m(\hat{G})\gamma_1 \right] - D_N \otimes \gamma_2. \tag{37}$$

On the other hand, by Schur formula,

$$\begin{aligned} (R_{ii}(ze_{11}-\gamma_0))^{-1} &= ze_{11} - \gamma_0 - W_{ii}\gamma_1 - D_{ii}\gamma_2 - \gamma_1 \otimes C_i^{(i)*} R^{(i)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_i^{(i)} \\ &= ze_{11} - \gamma_0 - D_{ii}\gamma_2 - N\sigma_N^2 \gamma_1 N^{-1} \operatorname{Tr} \otimes \mathrm{id}_m (G)\gamma_1 + \Delta_i, \end{aligned}$$

where

$$\begin{aligned} \Delta_i &= -W_{ii}\gamma_1 - \gamma_1 \otimes C_i^{(i)*} R^{(i)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_i^{(i)} + N\sigma_N^2 \gamma_1 N^{-1} \operatorname{Tr} \otimes \mathrm{id}_m (G)\gamma_1 \\ &= -W_{ii}\gamma_1 - \gamma_1 \otimes C_i^{(i)*} R^{(i)}(ze_{11}-\gamma_0)\gamma_1 \otimes C_i^{(i)} + N\sigma_N^2 \gamma_1 (\mathrm{id}_m \otimes N^{-1}\operatorname{Tr})(R_N(ze_{11}-\gamma_0))\gamma_1. \end{aligned}$$

Define $\Delta = \sum_{i=1}^N E_{ii} \otimes \Delta_i$. Thus,

$$G^{-1} = I_N \otimes \left[ze_{11} - \gamma_0 - N\sigma_N^2 \gamma_1 N^{-1} \operatorname{Tr} \otimes \mathrm{id}_m (G)\gamma_1 \right] - D_N \otimes \gamma_2 + \Delta. \tag{38}$$

Set

$$\Xi_N = N^{-1}\operatorname{Tr} \otimes \mathrm{id}_m \left(\hat{G} - G \right) = \mathrm{id}_m \otimes \tau_N(r_N(ze_{11}-\gamma_0)) - \mathrm{id}_m \otimes N^{-1}\operatorname{Tr}(R_N(ze_{11}-\gamma_0)).$$

Note that, by Lemma 5, all moments of $\|\Xi_N\|$ are $O(1)$. By substracting (37) from (38),

$$\begin{aligned} \hat{G} - G &= G\left(I_N \otimes N\sigma_N^2 \gamma_1 \Xi_N \gamma_1\right)\hat{G} + G\Delta\hat{G} \\ &= \hat{G}\left(I_N \otimes N\sigma_N^2 \gamma_1 \Xi_N \gamma_1\right)\hat{G} + \left(G - \hat{G}\right)\left(I_N \otimes N\sigma_N^2 \gamma_1 \Xi_N \gamma_1\right)\hat{G} + G\Delta\hat{G}. \end{aligned}$$

Hence

$$\begin{aligned}\Xi_N &= N^{-1}\sum_{k=1}^{N}\hat{R}_k(ze_{11}-\gamma_0)N\sigma_N^2\gamma_1\Xi_N\gamma_1\hat{R}_k(ze_{11}-\gamma_0)\\ &\quad + N^{-1}\operatorname{Tr}\otimes\mathrm{id}_m\left[\left(G-\hat{G}\right)\left(I_N\otimes N\sigma_N^2\gamma_1\Xi_N\gamma_1\right)\hat{G}+G\Delta\hat{G}\right]\\ &= u_N(z,z)\Xi_N + N^{-1}\operatorname{Tr}\otimes\mathrm{id}_m\left[\left(G-\hat{G}\right)\left(I_N\otimes N\sigma_N^2\gamma_1\Xi_N\gamma_1\right)\hat{G}+G\Delta\hat{G}\right]\end{aligned}$$

According to Lemma 39, for large enough N, the operators $\mathrm{id}_m - u_N(z,z)$ on $M_m(\mathbb{C})$ are invertible with uniformly bounded inverses. Therefore

$$\Xi_N = (\mathrm{id}_m - u_N(z,z))^{-1}\left\{N^{-1}\operatorname{Tr}\otimes\mathrm{id}_m\left[\left(G-\hat{G}\right)\left(I_N\otimes N\sigma_N^2\gamma_1\Xi_N\gamma_1\right)\hat{G}+G\Delta\hat{G}\right]\right\}$$

and

$$\|\Xi_N\| \leq \|(\mathrm{id}_m - u_N(z,z))^{-1}\|\left\|N^{-1}\operatorname{Tr}\otimes\mathrm{id}_m\left[\left(G-\hat{G}\right)\left(I_N\otimes N\sigma_N^2\gamma_1\Xi_N\gamma_1\right)\hat{G}+G\Delta\hat{G}\right]\right\|. \tag{39}$$

We have

$$\begin{aligned}&N^{-1}\operatorname{Tr}\otimes\mathrm{id}_m\Big[\left(G-\hat{G}\right)\left(I_N\otimes N\sigma_N^2\gamma_1\Xi_N\gamma_1\right)\hat{G}\Big]\\ &\qquad = N^{-1}\sum_{i=1}^{N}(R_{ii}(ze_{11}-\gamma_0)-\hat{R}_i(ze_{11}-\gamma_0))N\sigma_N^2\gamma_1\Xi_N\gamma_1\hat{R}_i(ze_{11}-\gamma_0).\end{aligned}$$

Thus (39) yields

$$\begin{aligned}\|\Xi_N\| &\leq \|(\mathrm{id}_m - u_N(z,z))^{-1}\|\Big[\|\gamma_1\|^2\big\|\hat{G}\big\|\sigma_N^2\Big\{\sum_{i=1}^{N}\big\|R_{ii}(ze_{11}-\gamma_0)-\hat{R}_i(ze_{11}-\gamma_0)\big\|\Big\}\|\Xi_N\|\\ &\qquad + \big\|N^{-1}\operatorname{Tr}\otimes\mathrm{id}_m(G\Delta\hat{G})\big\|\Big].\end{aligned}$$

Therefore, if $\sigma_N^2\big\{\sum_{i=1}^N \big\|R_{ii}(ze_{11}-\gamma_0)-\hat{R}_i(ze_{11}-\gamma_0)\big\|\big\} \leq (2\|\gamma_1\|^2\big\|\hat{G}\big\|\|(\mathrm{id}_m-u_N(z,z))^{-1}\|)^{-1}$, then by using Lemma 5,

$$\begin{aligned}\|\Xi_N\| &\leq 2\|(\mathrm{id}_m - u_N(z,z))^{-1}\|\|N^{-1}\operatorname{Tr}\otimes\mathrm{id}_m(G\Delta\tilde{G})\|\\ &\leq 2\|(\mathrm{id}_m - u_N(z,z))^{-1}\|\|N^{-1}\sum_{i=1}^{N}R_{ii}(ze_{11}-\gamma_0)\Delta_i\hat{R}_i(ze_{11}-\gamma_0)\|\\ &\leq 2\|(\mathrm{id}_m - u_N(z,z))^{-1}\|\|R_N(ze_{11}-\gamma_0)\|\|\hat{R}(ze_{11}-\gamma_0)\|\{N^{-1}\sum_{i=1}^{N}\|\Delta_i\|\}.\end{aligned}$$

Set $E = \left\{\sigma_N^2\sum_{i=1}^{N}\big\|R_{ii}(ze_{11}-\gamma_0)-\hat{R}_i(ze_{11}-\gamma_0)\big\| > (2\|\gamma_1\|^2\big\|\hat{G}\big\|\|(\mathrm{id}_m-u_N(z,z))^{-1}\|)^{-1}\right\}$. For $q \in (0, 4(1+\varepsilon))$, using Lemmas 21 and 39, we have

$$\begin{aligned}\mathbb{E}\left(\|\Xi_N\|^q\right) &= \mathbb{E}\left(\|\Xi_N\|^q\mathbf{1}_E\right)+\mathbb{E}\left(\|\Xi_N\|^q\mathbf{1}_{E^c}\right)\\ &\leq O(1)\mathbb{P}\left(E\right)^{1/p} + O(1)\mathbb{E}\Big(\Big\{N^{-1}\sum_{i=1}^{N}\big\|\Delta_i\big\|\Big\}^{4(1+\varepsilon)}\Big)^{q/4(1+\varepsilon)}\end{aligned}$$

where $p \geq 1$ will be choosen later on.
Note that, for $q \in [2, 4(1+\varepsilon))$,

$$\begin{aligned}\mathbb{E}\Big(\Big\{N^{-1}\sum_{i=1}^{N}\big\|\Delta_i\big\|\Big\}^{4(1+\varepsilon)}\Big)^{q/4(1+\varepsilon)} &\leq \max_{i=1,\dots,N}\mathbb{E}\Big(\big\|\Delta_i\big\|^{4(1+\varepsilon)}\Big)^{q/4(1+\varepsilon)}\\ &= O\Big(N^{-2(1+\varepsilon)}\Big)^{q/4(1+\varepsilon)} = O\Big(N^{-q/2}\Big),\end{aligned}$$

where we use the convexity of $x \mapsto x^{4(1+\varepsilon)}$ and Lemmas 23, 24 and 77. Now, according to Lemma 28 and Lemma 62,

$$\|\hat{R}(ze_{11}-\gamma_0) - (\Omega_N(ze_{11}-\gamma_0)\otimes I_N - \gamma_2\otimes D_N)^{-1}\| \underset{N\to+\infty}{\longrightarrow} 0.$$

For those N for which

$$N\sigma_N^2\|\hat{R}(ze_{11}-\gamma_0)-(\Omega_N(ze_{11}-\gamma_0)\otimes I_N-\gamma_2\otimes D_N)^{-1}\|\leq(4\|\gamma_1\|^2\|\hat{G}\|\|(\mathrm{id}_m-u_N(z,z))^{-1}\|)^{-1},$$

the probability of the event E satisfies, for $\varepsilon'\leq\varepsilon$:

$$\begin{aligned}\mathbb{P}(E)&\leq\mathbb{P}\Big(\sigma_N^2\sum_{i=1}^N\big\|R_{ii}(ze_{11}-\gamma_0)-(\Omega_N(ze_{11}-\gamma_0)\otimes I_N-\gamma_2\otimes D_N)_{ii}^{-1}\big\|\\&\qquad\qquad>(4\|\gamma_1\|^2\big\|\hat{G}\big\|\|(\mathrm{id}_m-u_N(z,z))^{-1}\|)^{-1}\Big)\\&\leq\inf_{x\in[1,4)}(4\|\gamma_1\|^2\big\|\hat{G}\big\|\|(\mathrm{id}_m-u_N(z,z))^{-1}\|N\sigma_N^2)^x\\&\qquad\times\max_{i=1,\dots,N}\mathbb{E}\Big[\big\|R_{ii}(ze_{11}-\gamma_0)-(\Omega_N(ze_{11}-\gamma_0)\otimes I_N-\gamma_2\otimes D_N)_{ii}^{-1}\big\|^x\Big]\\&=O(N^{-2+\varepsilon'})\end{aligned}$$

by Lemma 38.

Finally choosing $p=\frac{2-\varepsilon'}{2-\varepsilon}$, we obtain

$$\mathbb{E}\left(\|\Xi_N\|^q\right)=O(N^{-(2-\varepsilon)})+O(N^{-q/2})=O(N^{-\min(2-\varepsilon;q/2)}).$$

□

7. Proof of convergence in finite-dimensional distributions in Theorem 2

In this section, we will give a proof of the convergence in finite-dimensional distributions of the complex process $(\xi_N(z)=\mathrm{Tr}(zI_N-X_N)^{-1}-\mathbb{E}\left[\mathrm{Tr}(zI_N-X_N)^{-1}\right],\ z\in\mathbb{C}\setminus\mathbb{R})$ to the centred complex Gaussian process $\mathcal{G}$, based on Theorem 72.

7.1. Reduction of the problem.

One has to prove that any linear combination of $\xi_N(z)$, $z\in\mathbb{C}\setminus\mathbb{R}$, converges in distribution to a complex Gaussian variable $Z\sim\mathcal{N}_{\mathbb{C}}(0,V,W)$. In the following, we prove the convergence of $(\xi_N(z))_{N\in\mathbb{N}}$. The case of a general linear combination does not need any additional argument and is left to the reader. Notice that

$$\xi_N(z)=\sum_{k=1}^N(\mathbb{E}_{\leq k}-\mathbb{E}_{\leq k-1})\big[\,\mathrm{Tr}(zI_N-X_N)^{-1}\big].\tag{40}$$

For each $N\in\mathbb{N}$, the random variable $\mathrm{Tr}(zI_N-X_N)^{-1}$ being bounded, $(\mathbb{E}_{\leq k}[\mathrm{Tr}(zI_N-X_N)^{-1}])_{k\geq1}$ is a square integrable complex martingale, hence $(\mathbb{E}_{\leq k}[\mathrm{Tr}(zI_N-X_N)^{-1}]-\mathbb{E}_{\leq k-1}[\mathrm{Tr}(zI_N-X_N)^{-1}])_{k\geq1}$ is a martingale difference. Our strategy is to apply the central limit theorem for sums of martingale differences. More precisely, we will decompose $(\mathbb{E}_{\leq k}-\mathbb{E}_{\leq k-1})[\mathrm{Tr}(zI_N-X_N)^{-1}]$ in two parts and apply Theorem 72 to the first part.

Proposition 42. *For $z\in\mathbb{C}\setminus\mathbb{R}$ and $k\in\{1,\dots,N\}$,*

$$(\mathbb{E}_{\leq k}-\mathbb{E}_{\leq k-1})\big[\,\mathrm{Tr}(zI_N-X_N)^{-1}\big]=\Delta_k^{(N)}+\varepsilon_k^{(N)},$$

where

$$\Delta_k^{(N)}=\mathbb{E}_{\leq k}\Big[-\frac{\partial}{\partial z}\,\mathrm{Tr}\left((W_{kk}\gamma_1+\Phi_k(ze_{11}-\gamma_0))\hat{R}_k(ze_{11}-\gamma_0)\right)\Big],$$

with

$$\Phi_k(ze_{11}-\gamma_0):=\gamma_1\otimes C_k^{(k)*}R^{(k)}(ze_{11}-\gamma_0)\gamma_1\otimes C_k^{(k)}-\gamma_1(\mathrm{id}_m\otimes\sigma_N^2\,\mathrm{Tr})\big(R^{(k)}(ze_{11}-\gamma_0)\big)\gamma_1,$$

and $\sum_{k=1}^N\varepsilon_k^{(N)}\underset{N\to+\infty}{\longrightarrow}0$ *in probability.*

Proof. By (4) and then Proposition 60, with the notation $\beta = ze_{11} - \gamma_0$,

$$\begin{aligned}\mathrm{Tr}(zI_N - X_N)^{-1} &= (\mathrm{Tr}\otimes\mathrm{Tr})\big((e_{11}\otimes I_N)R_N(\beta)\big)\\ &= (\mathrm{Tr}\otimes\mathrm{Tr})\big((e_{11}\otimes I_{N-1})R^{(k)}(\beta)\big)\\ &+ \mathrm{Tr}\left(e_{11}\big(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - (\gamma_1\otimes C_k^{(k)*})R^{(k)}(\beta)(\gamma_1\otimes C_k^{(k)})\big)^{-1}\right)\\ &+ (\mathrm{Tr}\otimes\mathrm{Tr})\Big((e_{11}\otimes I_{N-1})R^{(k)}(\beta)(\gamma_1\otimes C_k^{(k)})\\ &\times\big(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)}\big)^{-1}(\gamma_1\otimes C_k^{(k)*})R^{(k)}(\beta)\Big).\end{aligned}$$

By traciality, the third term of the right-hand side rewrites

$$\mathrm{Tr}(\gamma_1\otimes C_k^{(k)*}R^{(k,1)}(\beta)\gamma_1\otimes C_k^{(k)}(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)})^{-1}),$$

where $R^{(k,1)}(\beta) := R^{(k)}(\beta)(e_{11}\otimes I_{N-1})R^{(k)}(\beta) = -\frac{\partial}{\partial z}R^{(k)}(\beta)$ and combines with the second term to get

$$\begin{aligned}&\mathrm{Tr}(zI_N - X_N)^{-1}\\ &\quad= (\mathrm{Tr}\otimes\mathrm{Tr})((e_{11}\otimes I_{N-1})R^{(k)}(\beta))\\ &\quad+ \mathrm{Tr}\left((e_{11} + \gamma_1\otimes C_k^{(k)*}R^{(k,1)}(\beta)\gamma_1\otimes C_k^{(k)})(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)})^{-1}\right).\end{aligned}$$

In the second term of the right-hand side, decompose

$$e_{11} + \gamma_1\otimes C_k^{(k)*}R^{(k,1)}(\beta)\gamma_1\otimes C_k^{(k)} = (e_{11} + \gamma_1(\mathrm{id}_m\otimes\sigma_N^2\,\mathrm{Tr})(R^{(k,1)}(\beta))\gamma_1) - \frac{\partial}{\partial z}\Phi_k(\beta)$$

and

$$\begin{aligned}(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - &\gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)})^{-1}\\ &= \hat R_k(\beta) + \hat R_k(\beta)\big(W_{kk}\gamma_1 + \Psi_k(\beta)\big)\hat R_k(\beta)\\ &+ \hat R_k(\beta)\big(W_{kk}\gamma_1 + \Psi_k(\beta)\big)\hat R_k(\beta)\big(W_{kk}\gamma_1 + \Psi_k(\beta)\big)\\ &\qquad\times\left(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)}\right)^{-1},\end{aligned}$$

where $\Psi_k(\beta) = \Phi_k(\beta) + \gamma_1((\mathrm{id}_m\otimes\sigma_N^2\,\mathrm{Tr})(R^{(k)}(\beta)) - N\sigma_N^2(\mathrm{id}_m\otimes\tau)(r_N(\beta)))\gamma_1$ so that $\mathrm{Tr}(zI_N - X_N)^{-1}$ is the sum of seven terms. Observe that the first two terms satisfy

$$\begin{aligned}(\mathbb{E}_{\le k} - \mathbb{E}_{\le k-1})\Big[&(\mathrm{Tr}\otimes\mathrm{Tr})\big((e_{11}\otimes I_{N-1})R^{(k)}(\beta)\big)\\ &+ \mathrm{Tr}\left((e_{11} + \gamma_1(\mathrm{id}_m\otimes\sigma_N^2\,\mathrm{Tr})(R^{(k,1)}(ze_{11}-\gamma_0))\gamma_1)\hat R_k(\beta)\right)\Big] = 0\end{aligned}$$

and that the following two terms combine to get

$$\begin{aligned}\mathrm{Tr}\,&\left((e_{11} + \gamma_1(\mathrm{id}_m\otimes\sigma_N^2\,\mathrm{Tr})(R^{(k,1)}(\beta))\gamma_1)\hat R_k(\beta)(W_{kk}\gamma_1 + \Psi_k(\beta))\hat R_k(\beta) - \frac{\partial}{\partial z}\Phi_k(\beta)\hat R_k(\beta)\right)\\ &= -\frac{\partial}{\partial z}\mathrm{Tr}((W_{kk}\gamma_1 + \Phi_k(\beta))\hat R_k(\beta)) + \varepsilon_{k,4}^{(N)} + \varepsilon_{k,5}^{(N)},\end{aligned}$$

where

$$\varepsilon_{k,4}^{(N)} := -\mathrm{Tr}(\gamma_1\frac{\partial}{\partial z}((\mathrm{id}_m\otimes\sigma_N^2\,\mathrm{Tr})(R^{(k)}(\beta)) - N\sigma_N^2(\mathrm{id}_m\otimes\tau)(r_N(\beta)))\gamma_1\hat R_k(\beta)(W_{kk}\gamma_1 + \Phi_k(\beta))\hat R_k(\beta)),$$

and

$$\begin{aligned}\varepsilon_{k,5}^{(N)} := \mathrm{Tr}\,\Big(&(e_{11} + \gamma_1(\mathrm{id}_m\otimes\sigma_N^2\,\mathrm{Tr})(R^{(k,1)}(\beta))\gamma_1)\\ &\times\hat R_k(\beta)\big(\gamma_1((\mathrm{id}_m\otimes\sigma_N^2\,\mathrm{Tr})(R^{(k)}(\beta)) - N\sigma_N^2(\mathrm{id}_m\otimes\tau)(r_N(\beta)))\gamma_1\big)\hat R_k(\beta)\Big).\end{aligned}$$

Note that

$$\mathbb{E}_{\le k-1}\Big[-\frac{\partial}{\partial z}\mathrm{Tr}\left((W_{kk}\gamma_1 + \Phi_k(\beta))\hat R_k(\beta)\right)\Big] = 0.$$

It remains to prove that the last three terms

$$\varepsilon_{k,1}^{(N)} := \operatorname{Tr}\Big((e_{11} + \gamma_1(\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R^{(k,1)}(\beta))\gamma_1)\hat{R}_k(\beta)(W_{kk}\gamma_1 + \Psi_k(\beta))$$
$$\times \hat{R}_k(\beta)(W_{kk}\gamma_1 + \Psi_k(\beta))(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)})^{-1}\Big),$$
$$\varepsilon_{k,2}^{(N)} := \operatorname{Tr}(-\frac{\partial}{\partial z}\Phi_k(\beta)\hat{R}_k(\beta)(W_{kk}\gamma_1 + \Psi_k(\beta))\hat{R}_k(\beta)),$$
$$\varepsilon_{k,3}^{(N)} := \operatorname{Tr}\Big(-\frac{\partial}{\partial z}\Phi_k(\beta)\hat{R}_k(\beta)(W_{kk}\gamma_1 + \Psi_k(\beta))\hat{R}_k(\beta)(W_{kk}\gamma_1 + \Psi_k(\beta))$$
$$\times (\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)})^{-1}\Big),$$

are such that

$$\varepsilon_k^{(N)} = (\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})\left[\varepsilon_{k,1}^{(N)} + \varepsilon_{k,2}^{(N)} + \varepsilon_{k,3}^{(N)} + \varepsilon_{k,4}^{(N)} + \varepsilon_{k,5}^{(N)}\right]$$

satisfies $\sum_{k=1}^N \varepsilon_k^{(N)} \underset{N\to+\infty}{\longrightarrow} 0$ in probability. This is the object of Lemma 45 below (assuming that the entries of W_N are bounded by δ_N). $\square$

Lemma 43. *For all $p \geq 2$,*

$$\mathbb{E}\Big(\Big\|\frac{\partial}{\partial z}\Phi_k(\beta)\Big\|^p\Big) = O\big(N^{-\min(p/2, 2+3\varepsilon)}\big).$$

Proof. For $\beta = ze_{11} - \gamma_0$, $z \in \mathbb{C}\setminus\mathbb{R}$, by Lemma 69, for $p \geq 2$,

$$\mathbb{E}\Big(\Big\|\frac{\partial}{\partial z}\Phi_k(\beta)\Big\|^p\Big) \leq C\Big[\Big(\mathbb{E}[|W_{12}|^4]N\mathbb{E}\Big(\big\|R^{(k)}(\beta)\big\|^4\Big)\Big)^{p/2} + N\mathbb{E}\Big(\big\|R^{(k)}(\beta)\big\|^{2p}\Big)\mathbb{E}[|W_{12}|^{2p}]\Big].$$

As $\mathbb{E}[|W_{12}|^4] = O(N^{-2})$,

$$\Big(\mathbb{E}[|W_{12}|^4]N\mathbb{E}\Big(\big\|R^{(k)}(\beta)\big\|^4\Big)\Big)^{p/2} = O(N^{-p/2}).$$

Now, if $p \geq 3(1+\varepsilon)$,

$$\mathbb{E}[|W_{12}|^{2p}] = O(\delta_N^{2p-6(1+\varepsilon)}N^{-3(1+\varepsilon)}) = O\big(N^{-3(1+\varepsilon)}\big).$$

Therefore,

$$N\mathbb{E}\Big(\big\|R^{(k)}(\beta)\big\|^{2p}\Big)\mathbb{E}[|W_{12}|^{2p}] = O\big(N^{-(2+3\varepsilon)}\big).$$

As a consequence, for $p \geq 3(1+\varepsilon)$,

$$\mathbb{E}\Big(\Big\|\frac{\partial}{\partial z}\Phi_k(\beta)\Big\|^p\Big) = O\big(N^{-\min(p/2, 2+3\varepsilon)}\big).$$

If $2 \leq p < 3(1+\varepsilon)$, $\mathbb{E}[|W_{12}|^{2p}] = O(N^{-p})$ and

$$\mathbb{E}\Big(\Big\|\frac{\partial}{\partial z}\Phi_k(\beta)\Big\|^p\Big) = O\big(N^{-p/2}\big).$$

Note that in this case, $\min(p/2, 2+3\varepsilon) = p/2$, which concludes the proof. $\square$

Lemma 44. *$\forall q \in [2, 4(1+\varepsilon))$,*

$$\mathbb{E}\Big(\big\|\Psi_k(\beta)\big\|^q\Big) = O\big(N^{-\min(2-\varepsilon, q/2)}\big).$$

Proof. For $q \geq 1$,

$$\|\Psi_k(\beta)\|_{L^q} \leq \|\Phi_k(\beta)\|_{L^q} + \|\gamma_1\|^2\|(\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R^{(k)}(\beta)) - \mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R_N(\beta))\|_{L^q}$$
$$+ \|\gamma_1\|^2\|(\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R_N(\beta)) - N\sigma_N^2(\mathrm{id}_m \otimes \tau)(r_N(\beta))\|_{L^q}.$$

Thus, for $q \in [2, 4(1+\varepsilon))$, Lemmas 23, 24 and 41 readily yield that

$$\|\Psi_k(\beta)\|_{L^q} = O\big(N^{-\min(1/2, (2-\varepsilon)/q)}\big),$$

which concludes the proof. $\square$

Lemma 45.

$$\Big\|\sum_{k\geq 1}\varepsilon_k^{(N)}\Big\|_{L^2} \underset{N\to+\infty}{\longrightarrow} 0.$$

Proof. Using Hölder's inequality with $q \in [1, 1+\varepsilon)$, $p, r \geq 1$ such that $p^{-1} + q^{-1} + r^{-1} = 1$,

$$\begin{aligned}
\mathbb{E}[|\varepsilon_{k,1}^{(N)}|^2] &\leq m^2 \|\hat{R}_k(\beta)\|^4 \mathbb{E}\Big[\|e_{11} + \gamma_1(\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R^{(k,1)}(\beta))\gamma_1\|^2 \|W_{kk}\gamma_1 + \Psi_k(\beta)\|^4 \|R_N(\beta)\|^2\Big] \\
&\leq m^2 \|\hat{R}_k(\beta)\|^4 \mathbb{E}\big[\|e_{11} + \gamma_1(\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R^{(k,1)}(\beta))\gamma_1\|^{2p}\big]^{1/p} \\
&\qquad \times \mathbb{E}\big[\|W_{kk}\gamma_1 + \Psi_k(\beta)\|^{4q}\big]^{1/q} \mathbb{E}\big[\|R_N(\beta)\|^{2r}\big]^{1/r} \\
&\leq m^2 \|\hat{R}_k(\beta)\|^4 \mathbb{E}[\|R_N(\beta)\|^{2r}]^{1/r} (1 + C\|\gamma_1\|^2 \mathbb{E}[\|R_N(\beta)\|^{4p}]^{1/2p})^2 \\
&\qquad \times \big(\|\gamma_1\| \|W_{kk}\|_{L^{4q}} + \|\Psi_k(\beta)\|_{L^{4q}}\big)^4.
\end{aligned}$$

Therefore, by Lemmas 17, 21, 77 and 44, $\mathbb{E}[|\varepsilon_{k,1}^{(N)}|^2] = o(N^{-1})$, uniformly in k.

Using Hölder's inequality with $q \in [1, 1+\varepsilon)$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\mathbb{E}[|\varepsilon_{k,2}^{(N)}|^2] \leq m^2 \|\hat{R}_k(\beta)\|^4 \Big[\mathbb{E}[\|\frac{\partial}{\partial z}\Phi_k(\beta)\|^{2p}]\Big]^{1/p} (\|\gamma_1\| \|W_{kk}\|_{L^{2q}} + \|\Psi_k(\beta)\|_{L^{2q}})^2$$

Therefore, by Lemmas 17, 43, 77 and 44 $\mathbb{E}[|\varepsilon_{k,2}^{(N)}|^2] = o(N^{-1})$, uniformly in k.

Using Hölder's inequality with $q \in [1, 1+\varepsilon)$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, by Lemmas 17, 21, 77 and 44,

$$\begin{aligned}
\mathbb{E}[|\varepsilon_{k,3}^{(N)}|^2] &\leq m^2 \|\hat{R}_k(\beta)\|^4 \Big[\mathbb{E}[\|\frac{\partial}{\partial z}\Phi_k(\beta)\|^{2p}]\Big]^{1/p} (\|\gamma_1\| \|W_{kk}\|_{L^{4q}} + \|\Psi_k(\beta)\|_{L^{4q}})^4 \mathbb{E}[\|R_N(\beta)\|^{2r}]^{1/r} \\
&= \Big(O\big(N^{-\min(p, 1+\varepsilon)}\big)\Big)^{1/p} \Big(O\big(N^{-\frac{1}{2q} + \frac{\varepsilon}{4q}}\big)\Big)^4
\end{aligned}$$

so that, noticing that $\frac{2-\varepsilon}{q} > 1$ for any $\varepsilon < 1/2$, $\mathbb{E}[|\varepsilon_{k,3}^{(N)}|^2] = o(N^{-1})$, uniformly in k.
Finally, by Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}[|\varepsilon_{k,4}^{(N)}|^2] \\
&\leq m^2 \|\gamma_1\|^4 \|\hat{R}_k(\beta)\|^4 \mathbb{E}[\|\frac{\partial}{\partial z}((\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R^{(k)}(\beta)) - N\sigma_N^2 (\mathrm{id}_m \otimes \tau)(r_N(\beta)))\|^2 \|W_{kk}\gamma_1 + \Phi_k(\beta)\|^2] \\
&\leq 2^4 m^2 \|\gamma_1\|^4 \|\hat{R}_k(\beta)\|^4 \Big\{\mathbb{E}(\|\frac{\partial}{\partial z}((\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R^{(k)}(\beta) - (\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R_N(\beta))\|^4) \\
&\qquad + \mathbb{E}(\|\frac{\partial}{\partial z}((\mathrm{id}_m \otimes \sigma_N^2 \operatorname{Tr})(R_N(\beta) - N\sigma_N^2 (\mathrm{id}_m \otimes \tau)(r_N(\beta)))\|^4\Big\}^{1/2} \\
&\qquad\qquad \times (\|\gamma_1\|^2 \mathbb{E}[\|W_{kk}\|^4]^{1/2} + \mathbb{E}[\|\Phi_k(\beta)\|^4]^{1/2}) \\
&= o(N^{-1}),
\end{aligned}$$

where we use Lemmas 17, 23, 24, 77 and 30 in the last line.
Noticing that $(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[\varepsilon_{k,5}^{(N)}] = 0$, by Lemma 70 and Jensen's inequality,

$$\begin{aligned}
&\mathbb{E}\big[\big|\sum_{k=1}^N (\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})\big[\varepsilon_{k,1}^{(N)} + \varepsilon_{k,2}^{(N)} + \varepsilon_{k,3}^{(N)} + \varepsilon_{k,4}^{(N)} + \varepsilon_{k,5}^{(N)}\big]\big|^2\big] \\
&\qquad = \sum_{k=1}^N \mathbb{E}\big[\big|(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})\big[\varepsilon_{k,1}^{(N)} + \varepsilon_{k,2}^{(N)} + \varepsilon_{k,3}^{(N)} + \varepsilon_{k,4}^{(N)}\big]\big|^2\big] \\
&\qquad \leq \sum_{k=1}^N 8 \sum_{j=1}^4 \big(\mathbb{E}\big[\big|(\mathbb{E}_{\leq k-1}\big[\varepsilon_{k,j}^{(N)}\big]\big|^2\big] + \mathbb{E}\big[\big|(\mathbb{E}_{\leq k}\big[\varepsilon_{k,j}^{(N)}\big]\big|^2\big]\big) \\
&\qquad \leq 16 \sum_{k=1}^N \mathbb{E}\big[\big|\varepsilon_{k,1}^{(N)}\big|^2 + \big|\varepsilon_{k,2}^{(N)}\big|^2 + \big|\varepsilon_{k,3}^{(N)}\big|^2 + \big|\varepsilon_{k,4}^{(N)}\big|^2\big] = o(1).
\end{aligned}$$

so that $\Big\|\sum_{k=1}^N \varepsilon_k^{(N)}\Big\|_{L^2} = o(1)$. ☐

As announced at the beginning of the section, the strategy is now to apply Theorem 72 to

$$\begin{aligned}\Delta_k^{(N)} &= \mathbb{E}_{\leq k}\Big[-\frac{\partial}{\partial z}\operatorname{Tr}((W_{kk}\gamma_1+\Phi_k(ze_{11}-\gamma_0))\hat{R}_k(ze_{11}-\gamma_0))\Big]\\ &= (\mathbb{E}_{\leq k}-\mathbb{E}_{\leq k-1})\Big[-\frac{\partial}{\partial z}\operatorname{Tr}((W_{kk}\gamma_1+\Phi_k(ze_{11}-\gamma_0))\hat{R}_k(ze_{11}-\gamma_0))\Big].\end{aligned}$$

7.2. **Verification of Lyapounov condition.** To check condition (69), one first uses Markov inequality to get, for $p\in(2,4(1+\varepsilon))$:

$$L(\varepsilon,N)\leq\varepsilon^{2-p}\sum_{k=1}^{N}\|\Delta_k^{(N)}\|_{L^p}^p.$$

It is therefore sufficient to prove that

$$\|\Delta_k^{(N)}\|_{L^p}=O(N^{-\frac{1}{2}}). \tag{41}$$

By Jensen's and triangular inequalities, for $1\leq k\leq N$,

$$\begin{aligned}\|\Delta_k^{(N)}\|_{L^p} &= \Big\|\mathbb{E}_{\leq k}[-\frac{\partial}{\partial z}\operatorname{Tr}((W_{kk}\gamma_1+\Phi_k(ze_{11}-\gamma_0))\hat{R}_k(ze_{11}-\gamma_0))]\Big\|_{L^p}\\ &\leq \Big\|\operatorname{Tr}(\frac{\partial}{\partial z}\Phi_k(ze_{11}-\gamma_0))\hat{R}_k(ze_{11}-\gamma_0))\Big\|_{L^p}\\ &\quad+\ \Big\|\operatorname{Tr}((W_{kk}\gamma_1+\Phi_k(ze_{11}-\gamma_0))\frac{\partial}{\partial z}\hat{R}_k(ze_{11}-\gamma_0))\Big\|_{L^p}\\ &\leq m\big\|\hat{R}_k(ze_{11}-\gamma_0)\big\|\big\|\frac{\partial}{\partial z}\Phi_k(ze_{11}-\gamma_0))\big\|_{L^p}\\ &\quad+m\big\|\frac{\partial}{\partial z}\hat{R}_k(ze_{11}-\gamma_0)\big\|\big(\|W_{kk}\|_{L^p}\|\gamma_1\|+\big\|\Phi_k(ze_{11}-\gamma_0)\big\|_{L^p}\big).\end{aligned}$$

One deduces (41) by using (15), (16), Lemmas 23, 77 and 43.

7.3. **Convergence of the hook process.** By bilinearity, the verification of conditions (70) and (71) is equivalent to the convergence in probability of the *hook process*:

$$\Gamma_N(z_1,z_2):=\sum_{k=1}^{N}\mathbb{E}_{\leq k-1}\Big[\mathbb{E}_{\leq k}\big[\frac{\partial}{\partial z}\operatorname{Tr}((W_{kk}\gamma_1+\Phi_k(\beta_1))\hat{R}_k(\beta_1))\big]\mathbb{E}_{\leq k}\big[\frac{\partial}{\partial z}\operatorname{Tr}((W_{kk}\gamma_1+\Phi_k(\beta_2))\hat{R}_k(\beta_2))\big]\Big],$$

$$\beta_1=z_1e_{11}-\gamma_0,\beta_2=z_2e_{11}-\gamma_0,z_1,z_2\in\mathbb{C}\setminus\mathbb{R}.$$

Proposition 46. *For all $z_1,z_2\in\mathbb{C}\setminus\mathbb{R}$*

$$\Gamma_N(z_1,z_2)\underset{N\to+\infty}{\longrightarrow}\Gamma(z_1,z_2)$$

in probability, where

$$\Gamma(z_1,z_2):=\frac{\partial^2}{\partial z_1\partial z_2}\gamma(z_1,z_2),$$

$$\begin{aligned}\gamma(z_1,z_2):=&-\operatorname{Tr}\otimes\operatorname{Tr}\left\{\log\left[\mathrm{id}_m\otimes\mathrm{id}_m-\sigma^2T_{\{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0\}}\right](I_m\otimes I_m)\right\}\\ &-\operatorname{Tr}\otimes\operatorname{Tr}\left\{\log\left[\mathrm{id}_m\otimes\mathrm{id}_m-\theta T_{\{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0\}}\right](I_m\otimes I_m)\right\}\\ &+(\tilde{\sigma}^2-\sigma^2-\theta)\operatorname{Tr}\otimes\operatorname{Tr}\{T_{\{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0\}}(I_m\otimes I_m)\}\\ &+\kappa/2\operatorname{Tr}\otimes\operatorname{Tr}\{T^2_{\{z_1e_{11}-\gamma_0,z_2e_{11}-\gamma_0\}}(I_m\otimes I_m)\}.\end{aligned}$$

In what follows, we focus on the convergence in probability of

$$\gamma_N(z_1,z_2):=\sum_{k=1}^{N}\mathbb{E}_{\leq k-1}\Big[\mathbb{E}_{\leq k}\big[\operatorname{Tr}((W_{kk}\gamma_1+\Phi_k(\beta_1))\hat{R}_k(\beta_1))\big]\mathbb{E}_{\leq k}\big[\operatorname{Tr}((W_{kk}\gamma_1+\Phi_k(\beta_2))\hat{R}_k(\beta_2))\big]\Big]. \tag{42}$$

This will be enough to establish the convergence of the hook process, as explained in Subsection 7.3.5. Using the independence of Φ_k and W_{kk}, one can easily see that

$$\sum_{k=1}^{N} \mathbb{E}_{\leq k-1}\Big[\mathbb{E}_{\leq k}\big[\operatorname{Tr}((W_{kk}\gamma_1 + \Phi_k(\beta_1))\hat{R}_k(\beta_1))\big]\mathbb{E}_{\leq k}\big[\operatorname{Tr}((W_{kk}\gamma_1 + \Phi_k(\beta_2))\hat{R}_k(\beta_2))\big]\Big]$$

(43)

$$= \tilde{\sigma}_N^2 \sum_{k=1}^{N} \operatorname{Tr}(\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \hat{R}_k(\beta_2)) + \sum_{k=1}^{N} \mathbb{E}_{\leq k-1}\Big[\operatorname{Tr}(\mathbb{E}_{\leq k}\big[\Phi_k(\beta_1)\big]\hat{R}_k(\beta_1)) \operatorname{Tr}(\mathbb{E}_{\leq k}\big[\Phi_k(\beta_2)\big]\hat{R}_k(\beta_2))\Big].$$

Thus, by Lemma 20,

$$\begin{aligned}
(44) \qquad \gamma_N(z_1,z_2) &= \tilde{\sigma}_N^2 \sum_{k=1}^{N} \operatorname{Tr}(\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \hat{R}_k(\beta_2)) \\
&+ \sigma_N^4 \sum_{k=1}^{N} \sum_{i,j<k} \operatorname{Tr}\Big(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)\Big) \operatorname{Tr}\Big(\gamma_1 \mathbb{E}_{\leq k}[R_{ji}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2)\Big) \\
&+ |\theta_N|^2 \sum_{k=1}^{N} \sum_{i,j<k} \operatorname{Tr}\Big(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)\Big) \operatorname{Tr}\Big(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2)\Big) \\
&+ \kappa_N \sum_{k=1}^{N} \sum_{i<k} \operatorname{Tr}\Big(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)\Big) \operatorname{Tr}\Big(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2)\Big) + \sum_{k=1}^{N} \varepsilon_k,
\end{aligned}$$

where

$$\sum_{k=1}^{N} \varepsilon_k \underset{N\to+\infty}{\overset{\mathbb{P}}{\longrightarrow}} 0.$$

Therefore, what remains to study is the sum of four terms. They will be studied separately in the following paragraphs. The first and the fourth terms are studied very easily, whereas the second and third ones need quite long computations, making repeated use of linear algebra properties which were collected in Section 4. These second and third terms are very similar.

7.3.1. *Contribution of the first term of* (44).

Lemma 47. *Define $f(\omega, t) := \operatorname{Tr}(\gamma_1(\omega - t\gamma_2)^{-1})$ for $\omega \in M_m(\mathbb{C}), t \in \mathbb{R}$ such that $\omega - t\gamma_2$ is invertible in $M_m(\mathbb{C})$. Then*

$$\int_{\mathbb{R}} f(\omega_N(\beta_1), t) f(\omega_N(\beta_2), t) \nu_N(dt) \underset{N\to+\infty}{\longrightarrow} \int_{\mathbb{R}} f(\omega(\beta_1), t) f(\omega(\beta_2), t) \nu(dt).$$

Proof.

$$\begin{aligned}
\Big| \int_{\mathbb{R}} & f(\omega_N(\beta_1), t) f(\omega_N(\beta_2), t) \nu_N(dt) - \int_{\mathbb{R}} f(\omega(\beta_1), t) f(\omega(\beta_2), t) \nu(dt) \Big| \\
&= \Big| \int_{\mathbb{R}} f(\omega_N(\beta_1), t) f(\omega_N(\beta_2), t) - f(\omega(\beta_1), t) f(\omega(\beta_2), t) \nu_N(dt) \\
&\qquad\qquad + \int_{\mathbb{R}} f(\omega(\beta_1), t) f(\omega(\beta_2), t) [\nu_N(dt) - \nu(dt)] \Big| \\
&\leq \int_{\mathbb{R}} \Big| f(\omega_N(\beta_1), t) f(\omega_N(\beta_2), t) - f(\omega(\beta_1), t) f(\omega(\beta_2), t) \Big| \nu_N(dt) \\
&\qquad\qquad + \Big| \int_{\mathbb{R}} f(\omega(\beta_1), t) f(\omega(\beta_2), t) [\nu_N(dt) - \nu(dt)] \Big| \\
&\leq \sup_{t \in \operatorname{supp}(\nu_N)} \Big| f(\omega_N(\beta_1), t) f(\omega_N(\beta_2), t) - f(\omega(\beta_1), t) f(\omega(\beta_2), t) \Big| \\
&\qquad\qquad + \Big| \int_{\mathbb{R}} f(\omega(\beta_1), t) f(\omega(\beta_2), t) [\nu_N(dt) - \nu(dt)] \Big|.
\end{aligned}$$

The first summand tends to zero as $N \to \infty$ by ((17), Lemma 18 and) Lemma 19, and the second summand tends to zero according to the hypothesis that ν_N weakly converges to ν, the function $t \mapsto$

$f(\omega(\beta_1),t)f(\omega(\beta_2),t)$ being bounded continuous on a compact set containing the supports of all ν_N, $N\in\mathbb{N}$ (by Lemma 18). $\square$

Hence, since $N^{-1}\sum_{k=1}^N \operatorname{Tr}(\gamma_1\hat{R}_k(\beta_1))\operatorname{Tr}(\gamma_1\hat{R}_k(\beta_2)) = \int_{\mathbb{R}} f(\omega_N(\beta_1),t)f(\omega_N(\beta_2),t)\nu_N(dt)$ (see (31)), it follows that

$$\tilde{\sigma}_N^2\sum_{k=1}^N \operatorname{Tr}(\gamma_1\hat{R}_k(\beta_1))\operatorname{Tr}(\gamma_1\hat{R}_k(\beta_2)) \underset{N\to+\infty}{\longrightarrow} \tilde{\sigma}^2\int_{\mathbb{R}} f(\omega(\beta_1),t)f(\omega(\beta_2),t)\nu(dt).$$

7.3.2. *Contribution of the second term of* (44). Denote by $T_{N,k,t}(z_1,z_2)$ the operator defined on $M_m(\mathbb{C})\otimes M_m(\mathbb{C})$ by

$$\begin{aligned}T_{N,k,t}(z_1,z_2)(b_1\otimes b_2) &= N^{-1}\sum_{i<k}\hat{R}_i(z_1e_{11}-\gamma_0)\gamma_1 b_1\otimes b_2\gamma_1\hat{R}_i(z_2e_{11}-\gamma_0)\\ &\quad + N^{-1}t\hat{R}_k(z_1e_{11}-\gamma_0)\gamma_1 b_1\otimes b_2\gamma_1\hat{R}_k(z_2e_{11}-\gamma_0).\end{aligned}$$

The key idea is to note that this second term can be rewritten as follows

$$\begin{aligned}\sigma_N^4\sum_{k=1}^N\sum_{i,j<k}\operatorname{Tr}(\gamma_1\mathbb{E}_{\le k}[R_{ij}^{(k)}(\beta_1)]\gamma_1\hat{R}_k(\beta_1))\operatorname{Tr}(\gamma_1\mathbb{E}_{\le k}[R_{ji}^{(k)}(\beta_2)]\gamma_1\hat{R}_k(\beta_2))\\ = N\sigma_N^2\operatorname{Tr}\otimes\operatorname{Tr}T_N(z_1,z_2)(\nabla_k),\end{aligned}$$

where, for any $k\in\{1,\dots,N\}$,

$$\nabla_k = \sigma_N^2\sum_{i,j<k}\mathbb{E}_{\le k}[R_{ij}^{(k)}(\beta_1)]\gamma_1\otimes\gamma_1\mathbb{E}_{\le k}[R_{ji}^{(k)}(\beta_2)]\in M_m(\mathbb{C})\otimes M_m(\mathbb{C})$$

satisfies the approximate equation

$$\nabla_k = N\sigma_N^2 T_{N,k,0}(z_1,z_2)(I_m\otimes I_m+\nabla_k)+o_{L^1}(1).$$

In order to deduce an estimate of ∇_k we need to study the spectral radius of $T_{N,k,0}(z_1,z_2)$.

Proposition 48. *For any $z\in\mathbb{C}\setminus\mathbb{R}$, there exists $K(z)\in(0,1)$ such that for all large N, for any $k\in\{1,\dots,N\}$, for any $t\in[0;1]$, the spectral radius of $N\sigma_N^2T_{N,k,t}(z,\bar{z})$ is smaller than $K(z)$.*

Proof. According to Corollary 40, we know that one may find $K(z)\in(0,1)$ such that, for large $N\in\mathbb{N}$, the spectral radius of $N\sigma_N^2T_N(z,\bar{z})$ is smaller than $K(z)$. Now, note that for any choice of $k\in\{1,\dots,N\}$ and $t\in[0;1]$, the operator $T_{N,k,t}(z,\bar{z})$ satisfies $T_{N,k,t}(z,\bar{z})\le T_N(z,\bar{z})$. Thus, by Lemma 68, the spectral radius of $N\sigma_N^2T_{N,k,t}(z,\bar{z})$ is smaller than $K(z)$ for large N. Corollary 48 follows. $\square$

Lemma 49. *Around any $z_1\in\mathbb{C}\setminus\mathbb{R}$, there is an open set O_{z_1} such that, for $z\in O_{z_1}$, for $N\ge N_{z_1}$, for any $k\le N$, for any $t\in[0,1]$, the spectrum of the operator $N\sigma_N^2T_{N,k,t}(z,\bar{z}_1):M_m(\mathbb{C})\otimes M_m(\mathbb{C})\to M_m(\mathbb{C})\otimes M_m(\mathbb{C})$ is included in the open unit disk and $\{(\mathrm{id}_m\otimes\mathrm{id}_m-N\sigma_N^2T_{N,k,t}(z,\bar{z}_1))^{-1}, z\in O_{z_1}, N\ge N_{z_1}, k\le N, t\in[0,1]\}$ is bounded.*

Without loss of generality, one may assume that $z\in O_{z_1}$ satisfy $\mathcal{I}z\ge c\mathcal{I}z_1$ for some $c>0$.

Proof. By (15), $\{T_{N,k,t}(z_1,\bar{z}_1), N\in\mathbb{N}, k\le N, t\in[0,1]\}$ is included in a centered ball with some radius r_{z_1} in $M_{m^4}(\mathbb{C})$. Moreover, by (15) and Lemma 3, the family of functions $\{z\mapsto T_{N,k,t}(z,\bar{z}_1), N\in\mathbb{N}, k\le N, t\in[0,1]\}$ is equicontinuous on any compact set of $\mathbb{C}\setminus\mathbb{R}$. Therefore, the assertions easily follow from Proposition 48 by using the uniform continuity of the spectral radius and of the norm on compact sets of $M_{m^4}(\mathbb{C})$ and Lemma 63. $\square$

Proposition 50. *Fix $z_1\in\mathbb{C}\setminus\mathbb{R}$ and $z\in O_{z_1}$ (defined in Lemma 49). Set $\beta=ze_{11}-\gamma_0$ and $\beta_1^*=\bar{z}_1e_{11}-\gamma_0$ in $M_m(\mathbb{C})$. The following convergence holds in probability:*

$$\begin{aligned}&\sigma_N^4\sum_{k=1}^N\sum_{i,j<k}\operatorname{Tr}(\gamma_1\mathbb{E}_{\le k}[R_{ij}^{(k)}(\beta)]\gamma_1\hat{R}_k(\beta))\operatorname{Tr}(\gamma_1\mathbb{E}_{\le k}[R_{ji}^{(k)}(\beta_1^*)]\gamma_1\hat{R}_k(\beta_1^*))\\ &\quad\to_{N\to+\infty} -\operatorname{Tr}\otimes\operatorname{Tr}\left\{\log\left[\mathrm{id}_m\otimes\mathrm{id}_m-\sigma^2T_{\{\beta,\beta_1^*\}}\right](I_m\otimes I_m)\right\}\\ &\qquad\qquad -\sigma^2\operatorname{Tr}\otimes\operatorname{Tr}\left\{T_{\{\beta,\beta_1^*\}}(I_m\otimes I_m)\right\}.\end{aligned}$$

Proof. Define for any $k \in \{1, \dots, N\}$,

$$\nabla_k = \sigma_N^2 \sum_{i,j<k} \mathbb{E}_{\le k}[R_{ij}^{(k)}(\beta)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\le k}[R_{ji}^{(k)}(\beta_1^*)] \in M_m(\mathbb{C}) \otimes M_m(\mathbb{C}). \tag{45}$$

Note that

$$\begin{aligned}\sigma_N^4 \sum_{k=1}^N \sum_{i,j<k} \mathrm{Tr}(\gamma_1 \mathbb{E}_{\le k}[R_{ij}^{(k)}(\beta)]\gamma_1 \hat{R}_k(\beta)) \,\mathrm{Tr}(\gamma_1 \mathbb{E}_{\le k}[R_{ji}^{(k)}(\beta_1^*)]\gamma_1 \hat{R}_k(\beta_1^*)) \\ = N\sigma_N^2 \,\mathrm{Tr} \otimes \mathrm{Tr}\, T_{N,k,0}(z, \bar{z}_1)(\nabla_k).\end{aligned} \tag{46}$$

Lemma 51. *With the notation of Proposition 50,*

$$\begin{aligned}\nabla_k = {} & \sigma_N^2 \sum_{i<k} \hat{R}_i(\beta)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\le k-1}\big(R_{ii}^{(k)}(\beta_1^*)\big) \\ & + \sigma_N^2 \sum_{i,j,l<k} W_{il}\hat{R}_i(\beta)\gamma_1 \mathbb{E}_{\le k-1}\big(R_{lj}^{(kil)}(\beta)\big)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\le k-1}\big(R_{ji}^{(k)}(\beta_1^*)\big) + o_{L^2}^{(u)}(1).\end{aligned}$$

Proof. By definition of $R^{(k)}(\beta)$, for all $i, j \neq k$,

$$(\beta_1 - D_{ii}\gamma_2)R_{ij}^{(k)}(\beta) = \delta_{ij} I_m + \sum_{l \neq k} W_{il}\gamma_1 R_{lj}^{(k)}(\beta).$$

We want to remove the dependence between W_{il} and $R^{(k)}(\beta)$, using (9):

$$\begin{aligned}(\beta - D_{ii}\gamma_2)R_{ij}^{(k)}(\beta) = \delta_{ij} I_m + \sum_{l \neq k} \Big\{ & W_{il}\gamma_1 R_{lj}^{(kil)}(\beta) \\ & + (1 - \tfrac{1}{2}\delta_{il})\Big(W_{il}^2 \gamma_1 R_{li}^{(kil)}(\beta)\gamma_1 R_{lj}^{(k)}(\beta) + |W_{il}|^2 \gamma_1 R_{ll}^{(kil)}(\beta)\gamma_1 R_{ij}^{(k)}(\beta)\Big)\Big\}.\end{aligned}$$

Hence, noticing that

$$N^{-1} \sum_{l=1}^N \hat{R}_l(\beta) = \mathrm{id}_m \otimes \tau_N((\beta \otimes 1_{\mathcal{A}} - \gamma_1 \otimes s_N - \gamma_2 \otimes D_N)^{-1}),$$

we can deduce that

$$\begin{aligned}(\omega_N(\beta) - D_{ii}\gamma_2)R_{ij}^{(k)}(\beta) = {} & \delta_{ij} I_m + \sum_{l \neq k} W_{il}\gamma_1 R_{lj}^{(kil)}(\beta) \\ & + \sum_{l \neq k} (1 - \tfrac{1}{2}\delta_{il}) W_{il}^2 \gamma_1 R_{li}^{(kil)}(\beta)\gamma_1 R_{lj}^{(k)}(\beta) \\ & + \sum_{l \neq k} \big[(1 - \tfrac{1}{2}\delta_{il})|W_{il}|^2 \gamma_1 R_{ll}^{(kil)}(\beta)\gamma_1 - \sigma_N^2 \gamma_1 \hat{R}_l(\beta))\gamma_1\big] R_{ij}^{(k)}(\beta) \\ & - \sigma_N^2 \gamma_1 \hat{R}_k(\beta)\gamma_1 R_{ij}^{(k)}(\beta),\end{aligned}$$

and

$$\begin{aligned}R_{ij}^{(k)}(\beta) = {} & \delta_{ij}\hat{R}_i(\beta) + \sum_{l \neq k} W_{il}\hat{R}_i(\beta)\gamma_1 R_{lj}^{(kil)}(\beta) \\ & + \sum_{l \neq k} (1 - \tfrac{1}{2}\delta_{il}) W_{il}^2 \hat{R}_i(\beta)\gamma_1 R_{li}^{(kil)}(\beta)\gamma_1 R_{lj}^{(k)}(\beta) \\ & + \sum_{l \neq k} \hat{R}_i(\beta)\big[(1 - \tfrac{1}{2}\delta_{il})|W_{il}|^2 \gamma_1 R_{ll}^{(kil)}(\beta)\gamma_1 - \sigma_N^2 \gamma_1 \hat{R}_l(\beta)\gamma_1\big] R_{ij}^{(k)}(\beta) \\ & - \sigma_N^2 \hat{R}_i(\beta)\gamma_1 \hat{R}_k(\beta)\gamma_1 R_{ij}^{(k)}(\beta).\end{aligned} \tag{47}$$

Therefore, for $i,j<k$,

$$
\begin{aligned}
&\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ij}(\beta)\big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\\
&\quad=\delta_{ij}\hat{R}_i(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)+\sum_{l<k}W_{il}\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(kil)}_{lj}(\beta)\big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\\
&\qquad+\sum_{l\neq k}\big(1-\tfrac{1}{2}\delta_{il}\big)\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\leq k-1}\Big(W_{il}^2R^{(kil)}_{li}(\beta_1)\gamma_1R^{(k)}_{lj}(\beta)\Big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\\
&\qquad+\sum_{l\neq k}\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\leq k-1}\Big(\big[(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2-\sigma_N^2\big]\hat{R}_l(\beta)\gamma_1R^{(k)}_{ij}(\beta)\Big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\\
&\qquad+\sum_{l\neq k}\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\leq k-1}\Big(\big(1-\tfrac{1}{2}\delta_{il}\big)|W_{il}|^2\big[R^{(kil)}_{ll}(\beta)-\hat{R}_l(\beta)\big]\gamma_1R^{(k)}_{ij}(\beta)\Big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\\
&\qquad-\sigma_N^2\hat{R}_i(\beta)\gamma_1\hat{R}_k(\beta)\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ij}(\beta)\big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\\
&\quad=\delta_{ij}\hat{R}_i(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)+\sum_{l<k}W_{il}\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(kil)}_{lj}(\beta)\big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\\
&\qquad+I_{k,i,j}+II_{k,i,j}+III_{k,i,j}+IV_{k,i,j}.
\end{aligned}
$$

We are going to prove that the contribution of the last four terms of the right-hand side is negligible. We have

$$\|\sigma_N^2\sum_{i,j<k}I_{k,i,j}\|_{L^2}\leq\sigma_N^2\sum_{i<k}\|\sum_{j<k}I_{k,i,j}\|_{L^2}$$

with

$$
\begin{aligned}
&\|\sum_{j<k}I_{k,i,j}\|^2_{L^2}\\
&\quad=\mathbb{E}\Big(\big\|\sum_{j<k}\sum_{l\neq k}(1-\tfrac{1}{2}\delta_{il})\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\leq k-1}\big(W_{il}^2R^{(kil)}_{li}(\beta)\gamma_1R^{(k)}_{lj}(\beta)\big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\big\|^2\Big).
\end{aligned}
$$

Using the fact that for any matrices A, B and X, we have

$$vec\,[AXB]=\big(B^T\otimes A\big)\,vecX \tag{48}$$

where $vecX=(X_{11},\dots,X_{m1},\dots,X_{1m},\dots,X_{mm})^T$, and that there exist permutation matrices P and Q such that for any matrices A, B,

$$A\otimes B=P(B\otimes A)Q$$

(see [HJ91, Chapter 4]), we have

$$
\begin{aligned}
&\Big\|\sum_{j<k}\sum_{l\neq k}\big(1-\tfrac{1}{2}\delta_{il}\big)\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\leq k-1}\big(W_{il}^2R^{(kil)}_{li}(\beta)\gamma_1R^{(k)}_{lj}(\beta)\big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)\Big\|\\
&\qquad=\sup_{\substack{X\in M_m(\mathbb{C})\\ \mathrm{Tr}(XX^*)=1}}\Big\|\sum_{l\neq k}\sum_{j<k}\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)X\gamma_1^T\\
&\qquad\qquad\times\mathbb{E}_{\leq k-1}\Big((R^{(k)}(\beta)^T)_{jl}(R^{(kil)}(\beta)^T)_{il}(1-\tfrac{1}{2}\delta_{il})W_{il}^2\Big)\gamma_1^T\hat{R}_i(\beta)^T\Big\|_{HS}.
\end{aligned}
$$

Now, for any $X\in M_m(\mathbb{C})$ such that $\mathrm{Tr}(XX^*)=1$, denoting by $\mathbf{P}$ the orthogonal projection onto the subspace generated by the first $k-1$ vectors of the canonical basis of $\mathbb{C}^{N-1}$ and using Lemma 17, Lemma 64 and Lemma 5, we have the following inequalities where $C(z_1)$ denotes a positive constant depending on z_1 and m which may vary from line to line,

$$\begin{aligned}
&\Big\|\sum_{l\neq k}\sum_{j<k}\gamma_1\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)X\gamma_1^T\mathbb{E}_{\leq k-1}\Big((R^{(k)}(\beta)^T)_{jl}(R^{(kil)}(\beta)^T)_{il}(1-\tfrac12\delta_{il})W_{il}^2\Big)\gamma_1^T\hat{R}_i(\beta)^T\Big\|_{HS}\\
&=\Big\|\sum_{l\neq k}\gamma_1\mathbb{E}_{\leq k-1}\Big(\sum_{j<k}\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)X\gamma_1^T(R^{(k)}(\beta)^T)_{jl}(R^{(kil)}(\beta)^T)_{il}(1-\tfrac12\delta_{il})W_{il}^2\Big)\gamma_1^T\hat{R}_i(\beta)^T\Big\|_{HS}\\
&\leq\|\gamma_1\|^2\sum_{l\neq k}\mathbb{E}_{\leq k-1}\Big(\Big\|\sum_{j<k}\mathbb{E}_{\leq k-1}\big(R^{(k)}_{ji}(\beta_1^*)\big)X\gamma_1^T(R^{(k)}(\beta)^T)_{jl}(R^{(kil)}(\beta)^T)_{il}\Big\|(1-\tfrac12\delta_{il})|W_{il}|^2\Big)\\
&\qquad\qquad\times\big\|\hat{R}_i(\beta)^T\big\|_{HS}\\
&=\|\gamma_1\|^2\sum_{l\neq k}\mathbb{E}_{\leq k-1}\Big(\Big\|\Big[\mathbb{E}_{\leq k-1}\big(\Theta(R^{(k)}(\beta_1^*))\big)(X\gamma_1^T\otimes\mathbf{P})R^{(k)}(\beta)^T\Big]_{il}(R^{(kil)}(\beta_1)^T)_{il}\Big\|\big(1-\tfrac12\delta_{il})|W_{il}|^2\Big)\\
&\qquad\qquad\times\big\|\hat{R}_i(\beta)^T\big\|_{HS}\\
&\leq C(z_1)\|\gamma_1\|^2\mathbb{E}_{\leq k-1}\Big\{\Big(\sum_{l\neq k}\Big\|\Big[\mathbb{E}_{\leq k-1}\big(\Theta(R^{(k)}(\beta_1^*))\big)(X\gamma_1^T\otimes\mathbf{P})R^{(k)}(\beta)^T\Big]_{il}\Big\|^2\Big)^{1/2}\\
&\qquad\qquad\times\Big(\sum_{l\neq k}\big\|(R^{(kil)}(\beta)^T)_{il}\big\|^2(1-\tfrac12\delta_{il})^2|W_{il}|^4\Big)^{1/2}\Big\}\\
&\leq C(z_1)\|\gamma_1\|^2\mathbb{E}_{\leq k-1}\Big\{\Big\|\mathbb{E}_{\leq k-1}\big(\Theta(R^{(k)}(\beta_1^*))\big)(X\gamma_1^T\otimes\mathbf{P})R^{(k)}(\beta)^T\Big\|\\
&\qquad\qquad\times\Big(\sum_{l\neq k}\big\|(R^{(kil)}(\beta)^T)_{il}\big\|^2(1-\tfrac12\delta_{il})^2|W_{il}|^4\Big)^{1/2}\Big\}\\
&\leq C(z_1)\mathbb{E}_{\leq k-1}\Big\{\mathbb{E}_{\leq k-1}\Big(Q\big(\frac{1}{|\mathcal{I}z_1|},\|W_N^{(k)}\|\big)\Big)Q\big(\frac{1}{|\mathcal{I}z|},\|W_N^{(k)}\|\big)\\
&\qquad\qquad\times\Big(\sum_{l\neq k}Q\big(\frac{1}{|\mathcal{I}z|},\|W_N^{(kil)}\|\big)^2(1-\tfrac12\delta_{il})^2|W_{il}|^4\Big)^{1/2}\Big\},
\end{aligned}$$

where Θ denotes the partial transpose map on $M_m(\mathbb{C})\otimes M_N(\mathbb{C})$, defined by $\Theta(\sum_k A_k\otimes B_k):=\sum_k A_k\otimes B_k^T$ and we use Proposition 61. Hence

$$\begin{aligned}
\|\sum_{j<k}I_{k,i,j}\|_{L^2}^2&\leq C(z_1)\sum_{l\neq k}\mathbb{E}\Big(\mathbb{E}_{\leq k-1}\Big(\big[Q\big(\frac{1}{|\mathcal{I}z_2|},\|W_N^{(k)}\|\big)\big]^2\Big)\big[Q\big(\frac{1}{|\mathcal{I}z|},\|W_N^{(k)}\|\big)\big]^2Q\big(\frac{1}{|\mathcal{I}z|},\|W_N^{(kil)}\|\big)^2\\
&\qquad\qquad\times(1-\tfrac12\delta_{il})^2|W_{il}|^4\Big)\\
&\leq C(z_1)\sum_{l\neq k}\mathbb{E}\Big(\mathbb{E}_{\leq k-1}\Big(\big[Q\big(\frac{1}{|\mathcal{I}z_2|},\|W_N^{(kil)}\|+2\delta_N\big)\big]^2\Big)\big[Q\big(\frac{1}{|\mathcal{I}z|},\|W_N^{(kil)}\|+2\delta_N\big)\big]^2\\
&\qquad\qquad\times Q\big(\frac{1}{|\mathcal{I}z|},\|W_N^{(kil)}\|\big)^2\big(1-\tfrac12\delta_{il}\big)^2|W_{il}|^4\Big)\\
&=C(z_1)\sum_{l\neq k}\mathbb{E}\Big(\mathbb{E}_{\leq k-1}\Big(\big[Q\big(\frac{1}{|\mathcal{I}z_1|},\|W_N^{(kil)}\|+2\delta_N\big)\big]^2\Big)\big[Q\big(\frac{1}{|\mathcal{I}z|},\|W_N^{(kil)}\|+2\delta_N\big)\big]^2\\
&\qquad\qquad\times Q\big(\frac{1}{|\mathcal{I}z|},\|W_N^{(kil)}\|\big)^2\Big)\big(1-\tfrac12\delta_{il}\big)^2\mathbb{E}\big(|W_{il}|^4\big)\\
&\leq C(z_1)\sum_{l\neq k}\mathbb{E}\Big(\mathbb{E}_{\leq k-1}\Big(\big[Q\big(\frac{1}{|\mathcal{I}z_1|},\|W_N\|+4\delta_N\big)\big]^2\Big)\big[Q\big(\frac{1}{|\mathcal{I}z|},\|W_N\|+4\delta_N\big)\big]^2\\
&\qquad\qquad\times Q\big(\frac{1}{|\mathcal{I}z|},\|W_N\|+2\delta_N\big)^2\Big)\big(1-\tfrac12\delta_{il}\big)^2\mathbb{E}\big(|W_{il}|^4\big)\\
&\leq C(z_1)\sum_{l\neq k}\big(1-\tfrac12\delta_{il}\big)^2\mathbb{E}\big(|W_{il}|^4\big),
\end{aligned}$$

where we use Proposition 78 in the last line. Thus,

$$\|\sigma_N^2 \sum_{i,j<k} I_{k,i,j}\|_{L^2} \leq C(z_1)\sigma_N^2 \sum_{i<k} \Big\{ \sum_{l\neq k} \mathbb{E}\Big(|W_{il}|^4\Big)\Big\}^{1/2} \leq C(z_1)N\sigma_N^2(\delta_N^2\tilde{\sigma}_N^2 + (N-2)m_N)^{1/2} = O(N^{-1/2}).$$

Now, similarly, using again (48),

$$\|\sigma_N^2 \sum_{i,j<k} II_{k,i,j}\|_{L^2} \leq \sigma_N^2 \sum_{i<k} \|\mathcal{E}(i,k)\|_{L^2}$$

where

$$\begin{aligned}\mathcal{E}(i,k) = \sup_{\substack{X\in M_m(\mathbb{C})\\ \mathrm{Tr}(XX^*)=1}} & \Big\|\gamma_1 \mathbb{E}_{\leq k-1}\Big\{\Big[\mathbb{E}_{\leq k-1}\big(\Theta(R^{(k)}(\beta_1^*))\big)(X\gamma_1^T \otimes \mathbf{P})R^{(k)}(\beta)^T\Big]_{ii}\gamma_1^T(\hat{R}_l(\beta))^T \\ & \times \Big(\sum_{l\neq k}[(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2]\Big)\Big\}\gamma_1^T\hat{R}_i(\beta)^T\Big\|_{HS}\end{aligned}$$

$$\begin{aligned}&\leq \|\gamma_1\|^3 \mathbb{E}_{\leq k-1}\Big\{\Big\|\Big[\mathbb{E}_{\leq k-1}\Big(\Theta(R^{(k)}(\beta_1^*))\Big)(X\gamma_1^T\otimes\mathbf{P})R^{(k)}(\beta)^T\Big]_{ii}\Big\|\Big|\sum_{l\neq k}[(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2-\sigma_N^2]\Big|\Big\} \\ &\qquad\qquad \times \big\|(\hat{R}_l(\beta))^T\big\|\big\|\hat{R}_i(\beta)^T\big\|_{HS} \\ &\leq C(z_1)\mathbb{E}_{\leq k-1}\Big\{\mathbb{E}_{\leq k-1}\Big(Q(\frac{1}{|\mathcal{I}z_1|},\|W_N^{(k)}\|)\Big)Q(\frac{1}{|\mathcal{I}z|},\|W_N^{(k)}\|)\Big|\sum_{l\neq k}[(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2-\sigma_N^2]\Big|\Big\}.\end{aligned}$$

Hence, using Hölder's inequality and Proposition 78, we obtain that

$$\big\|\sigma_N^2 \sum_{i,j<k} II_{k,i,j}\big\|_{L^2} \leq C(z_1)\sigma_N^2 \sum_{i<k}\mathbb{E}\Big(\Big|\sum_{l\neq k}[(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2-\sigma_N^2]\Big|^4\Big)^{1/4} = o(1),$$

where, in the last equality, we use that, uniformly in i and l,

$$\mathbb{E}\Big([(1-\frac{1}{2}\delta_{il_1})|W_{il_1}|^2 - \sigma_N^2]\Big) = \delta_{il_1}O(1/N),$$

and for $p = 2,3,4$,

$$\mathbb{E}\Big(\Big\{[(1-\tfrac{1}{2}\delta_{il_1})|W_{il_1}|^2-\sigma_N^2\Big\}^p\Big) = o(1/N).$$

Similarly, using again (48),

$$\begin{aligned}&\|\sum_{j<k} III_{k,i,j}\| \\ &\leq \sup_{\substack{X\in M_m(\mathbb{C})\\ \mathrm{Tr}(XX^*)=1}} \|\gamma_1\|^3\big\|\hat{R}_i(\beta)^T\big\|_{HS} \\ &\qquad \times \mathbb{E}_{\leq k-1}\Big\{\Big\|\Big[\mathbb{E}_{\leq k-1}\big(\Theta(R^{(k)}(\beta_1^*))\big)(X\gamma_1^T\otimes\mathbf{P})R^{(k)}(\beta)^T\Big]_{ii}\Big\|\Big\|\sum_{l\neq k}(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2\Delta_{ilk}^T\Big\|\Big\} \\ &\leq C(z_1)\mathbb{E}_{\leq k-1}\Big\{\mathbb{E}_{\leq k-1}\Big(Q(\frac{1}{|\mathcal{I}z_1|},\|W_N^{(k)}\|)\Big)Q(\frac{1}{|\mathcal{I}z|},\|W_N^{(k)}\|)\sum_{l\neq k}(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2\Big\|\Delta_{ilk}^T\Big\|\Big\},\end{aligned}$$

where $\Delta_{ilk} = R_{ll}^{(kil)}(\beta) - \hat{R}_l(\beta))$. Thus,

$$
\begin{aligned}
&\Big\|\sum_{j<k} III_{k,i,j}\Big\|_{L_2}^2 \\
&\quad\le C(z_1)\mathbb{E}\Big[\mathbb{E}_{\le k-1}\Big(Q^2(\frac{1}{|\mathcal{I}z_1|},\|W_N^{(k)}\|)\Big)Q^2(\frac{1}{|\mathcal{I}z|},\|W_N^{(k)}\|) \\
&\qquad\qquad \times \sum_{l,l'\neq k}(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2(1-\frac{1}{2}\delta_{il'})|W_{il'}|^2\big\|\Delta_{ilk}^T\big\|\big\|\Delta_{il'k}^T\big\|\Big] \\
&\quad\le C(z_1)\sum_{l,l'\neq k}\Big\{\mathbb{E}\Big[\Big(\mathbb{E}_{\le k-1}\Big(Q^2(\frac{1}{|\mathcal{I}z_1|},\|W_N^{(k)}\|)\Big)Q^2(\frac{1}{|\mathcal{I}z|},\|W_N^{(k)}\|)\Big)^p\big\|\Delta_{ilk}^T\big\|^p\big\|\Delta_{il'k}^T\big\|^p\Big]\Big\}^{1/p} \\
&\qquad\qquad \times\Big\{\mathbb{E}\Big((1-\tfrac{1}{2}\delta_{il})^q|W_{il}|^{2q}(1-\frac{1}{2}\delta_{il'})^q|W_{il'}|^{2q}\Big)\Big\}^{1/q},
\end{aligned}
$$

where $q = 1+\varepsilon$ and $p^{-1}+q^{-1}=1$. (3) and (4) readily yield that

$$
\Big\{\mathbb{E}\Big((1-\tfrac{1}{2}\delta_{il})^q|W_{il}|^{2q})(1-\frac{1}{2}\delta_{il'})^q|W_{il'}|^{2q})\Big)\Big\}^{1/q} = O(N^{-2})
$$

uniformly in i,l,l' so that $\|\sum_{j<k} III_{k,i,j}\|_{L_2} = o(1)$ uniformly in i,k by using Remark 35 and Proposition 78. Therefore

$$
\Big\|\sigma_N^2\sum_{i,j<k} III_{k,i,j}\Big\|_{L^2} = o(1).
$$

Finally, similarly,

$$
\begin{aligned}
\Big\|\sum_{j<k} IV_{k,i,j}\Big\| &\le \sigma_N^2 \sup_{\substack{X\in M_m(\mathbb{C})\\ \mathrm{Tr}(XX^*)=1}} \big\|\gamma_1\big\|^3\big\|\hat{R}_i(\beta)\big\|\big\|\hat{R}_k(\beta)^T\big\|_{HS} \\
&\qquad\times\Big\|\Big[\mathbb{E}_{\le k-1}\big(\Theta(R^{(k)}(\beta_1^*))\big)(X\gamma_1^T\otimes\mathbf{P})\mathbb{E}_{\le k-1}\big(R^{(k)}(\beta)^T\big)\Big]_{ii}\Big\| \\
&\le C(z_1)\sigma_N^2\mathbb{E}_{\le k-1}\big(R^{(k)}(\beta_1^*)\big)\mathbb{E}_{\le k-1}\big(R^{(k)}(\beta)\big) \\
&\le C(z_1)\sigma_N^2\mathbb{E}_{\le k-1}\Big(Q(\frac{1}{|\mathcal{I}z_1|},\|W_N^{(k)}\|)\Big)\mathbb{E}_{\le k-1}\Big(Q(\frac{1}{|\mathcal{I}z|},\|W_N^{(k)}\|)\Big),
\end{aligned}
$$

so that by using Proposition 78, $\|\sum_{j<k} IV_{k,i,j}\|_{L^2} = O(\sigma_N^2)$. Thus

$$
\big\|\sigma_N^2\sum_{i,j<k} IV_{k,i,j}\big\|_{L^2} \le \sigma_N^2\sum_{i<k}\|\sum_{j<k} IV_{k,i,j}\|_{L^2} = O(N^{-1}).
$$

Lemma 51 readily follows. □

Lemma 52. *With the notation of Proposition 50,*

$$
\nabla_k = N\sigma_N^2 T_{N,k,0}(z,\bar{z}_1)(I_m\otimes I_m+\nabla_k)+o_{L^1}(1).
$$

Proof. Let us consider the second term of the right hand side of Lemma 51. Using (9), we have

$$
\sigma_N^2\sum_{i,j,l<k} W_{il}\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\le k-1}\left(R_{lj}^{(kil)}(\beta)\right)\gamma_1\otimes\gamma_1\mathbb{E}_{\le k-1}\left(R_{ji}^{(k)}(\beta_1^*)\right)
$$

$$
= \sigma_N^2\sum_{i,j,l<k} W_{il}\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\le k-1}\left(R_{lj}^{(kil)}(\beta)\right)\gamma_1\otimes\gamma_1\mathbb{E}_{\le k-1}\left(R_{ji}^{(kil)}(\beta_1^*)\right) \tag{49}
$$

$$
+\sigma_N^2\sum_{i,j,l<k}(1-\tfrac{1}{2}\delta_{il})W_{il}^2\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\le k}[R_{lj}^{(kil)}(\beta)]\gamma_1\otimes\gamma_1\mathbb{E}_{\le k}[R_{ji}^{(kil)}(\beta_1^*)\gamma_1R_{li}^{(k)}(\beta_1^*)] \tag{50}
$$

$$
+\sigma_N^2\sum_{i,j,l<k}(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\le k}[R_{lj}^{(kil)}(\beta)]\gamma_1\otimes\gamma_1\mathbb{E}_{\le k}[R_{jl}^{(kil)}(\beta_1^*)\gamma_1R_{ii}^{(k)}(\beta_1^*)]. \tag{51}
$$

Set

$$
\alpha_{il} = \sum_{j<k}\gamma_1\mathbb{E}_{\le k-1}\left(R_{lj}^{(kil)}(\beta)\right)\gamma_1\otimes\gamma_1\mathbb{E}_{\le k-1}\left(R_{ji}^{(kil)}(\beta_1^*)\right).
$$

Note that α_{il} and W_{il} are independent. Let us consider the L^2 norm of the term (49):

$$\begin{aligned}\|\sigma_N^2 \sum_{i<k}\sum_{l<k} W_{il}\hat{R}_i(\beta)\alpha_{il}\|_{L^2} &\le \sigma_N^2 \sum_{i<k} \|\hat{R}_i(\beta)\| \|\sum_{l<k} W_{il}\alpha_{il}\|_{L^2} \\ &\le \sigma_N^2 Q(|\mathcal{I}z|^{-1}, \|D_N\|, N^{1/2}\sigma_N) \sum_{i<k}\Big\{ \sum_{l<k, l'<k} \mathbb{E}\left(W_{il}\overline{W_{il'}} \operatorname{Tr} \alpha_{il}\alpha_{il'}^*\right) \Big\}^{1/2} \end{aligned} \tag{52}$$

by using (15). First,

$$\begin{aligned}\sum_{l<k} \mathbb{E}\left(|W_{il}|^2 \operatorname{Tr} \alpha_{il}\alpha_{il}^*\right) &= \sum_{l<k} \mathbb{E}\left(|W_{il}|^2\right) \mathbb{E}\left(\operatorname{Tr} \alpha_{il}\alpha_{il}^*\right) \\ &\le m^2\sigma_N^2 \sum_{l<k, l\neq i} \mathbb{E}\left(\|\alpha_{il}\|^2\right) + m^2\tilde{\sigma}_N^2 \mathbb{E}\left(\|\alpha_{ii}\|^2\right).\end{aligned}$$

Replacing $R^{(kil)}$ by $R^{(k)}$ yields the following.

$$\begin{aligned}\alpha_{il} &= \sum_{j<k} \gamma_1 \mathbb{E}_{\le k-1}\big(R_{lj}^{(k)}(\beta)\big)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\le k-1}\big(R_{ji}^{(k)}(\beta_1^*)\big) \\ &\quad + \sum_{j<k} \gamma_1 \mathbb{E}_{\le k-1}\big(R_{lj}^{(kil)}(\beta) - R_{lj}^{(k)}(\beta)\big)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\le k-1}\big(R_{ji}^{(kil)}(\beta_1^*)\big) \\ &\quad + \sum_{j<k} \gamma_1 \mathbb{E}_{\le k-1}\big(R_{lj}^{(k)}(\beta)\big)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\le k-1}\big(R_{ji}^{(kil)}(\beta_1^*) - R_{ji}^{(k)}(\beta_1^*)\big) \\ &:= I_{il} + II_{il} + III_{il}.\end{aligned}$$

Set $A(k) = (\gamma_1 \otimes I_{N-1})R^{(k)}(\beta)\gamma_1 \otimes I_{N-1}$, $B(k,i,l) = \gamma_1 \otimes I_{N-1}\mathbb{E}_{\le k-1}\left(R^{(kil)}(\beta_1^*) - R^{(k)}(\beta_1^*)\right)$, and $C_{s,t}^{(il)} = \sum_{j<k} A(k)_{sj} \otimes B(k,i,l)_{jt}$. By Lemma 65, we obtain $\|C_{s,t}^{(il)}\| \le \|A(k)\|\|B(k,i,l)\|$ so that, choosing $s = l$ and $t = i$, we have

$$\begin{aligned}\|III_{il}\| &\le \|\gamma_1\|^3 \mathbb{E}_{\le k-1}\left(Q(\|W_N\|, |\mathcal{I}z|^{-1})\right) \mathbb{E}_{\le k-1}\big(\|R^{(kil)}(\beta_1^*) - R^{(k)}(\beta_1^*)\|\big) \\ &\le \|\gamma_1\|^4 \delta_N \mathbb{E}_{\le k-1}\big(Q(\|W_N\|, |\mathcal{I}z|^{-1})\big)\mathbb{E}_{\le k-1}\big(Q(\|W_N\|, |\mathcal{I}z_1|^{-1})\big),\end{aligned}$$

where we use the resolvent identity and Lemma 5. Similarly,

$$\|II_{il}\| \le \|\gamma_1\|^4 \delta_N \mathbb{E}_{\le k-1}\left(Q(\|W_N\|, |\mathcal{I}z|^{-1})\right) \mathbb{E}_{\le k-1}\left(Q(\|W_N\|, |\mathcal{I}z_1|^{-1})\right).$$

Thus, using Proposition 78, we obtain

$$\mathbb{E}\left(\|II_{il}\|^2 + \|III_{il}\|^2\right) = O(\delta_N^2).$$

Now, set $B(k) = \gamma_1 \otimes I_{N-1}\mathbb{E}_{\le k-1}\left(R^{(k)}(\beta_2)\right)$ and $C_{s,t} = \sum_{j<k} A(k)_{sj} \otimes B(k)_{jt}$. By Lemma 65,

$$\Big(\sum_{l\neq k} \|C_{l,i}\|^2\Big)^{1/2} \le \|A(k)\|\|B(k)\|$$

that is

$$\sum_{l\neq k}^{N} \|I_{il}\|^2 \le \|\gamma_1\|^6 \mathbb{E}_{\le k-1}\left(\|R^{(k)}(\beta)\|^2\right) \mathbb{E}_{\le k-1}\left(\|R^{(k)}(\beta_1^*)\|^2\right).$$

It readily follows by Remark 22 that $\sum_{l<k} \mathbb{E}\|\alpha_{il}\|^2 = O(N\delta_N^2) + O(1) = O(N\delta_N^2)$ and then that

$$\sum_{l<k} \mathbb{E}\left(|W_{il}|^2 \operatorname{Tr} \alpha_{il}\alpha_{il}^*\right) = O(\delta_N^2),$$

uniformly in k.

It remains to control the sum of the cross terms $\mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}(\alpha_{il}\alpha_{il'}{}^*)]$, $l \neq l'$, $i<k, l<k, l'<k$. Let us replace $R^{(kil')}$ and $R^{(kil)}$ by $R^{(kil'il)}$:

$$\begin{aligned}\alpha_{il} = &\sum_{j<k}\gamma_1\mathbb{E}_{\leq k-1}\left(R_{lj}^{(kilil')}(\beta)\right)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\left(R_{ji}^{(kilil')}(\beta_1^*)\right)\\ &+\sum_{j<k}\gamma_1\mathbb{E}_{\leq k-1}\left(R_{lj}^{(kil)}(\beta)-R_{lj}^{(kilil')}(\beta)\right)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\left(R_{ji}^{(kil)}(\beta_1^*)\right)\\ &+\sum_{j<k}\gamma_1\mathbb{E}_{\leq k-1}\left(R_{lj}^{(kilil')}(\beta)\right)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\left(R_{ji}^{(kil)}(\beta_1^*)-R_{ji}^{(kilil')}(\beta_1^*)\right)\\ &:= \alpha_{ill'}+\beta_{ill'}+\gamma_{ill'}.\end{aligned}$$

Note that, due to independence properties, the sum vanishes when α_{il} is replaced by $\alpha_{ill'}$ or when $\alpha_{il'}$ is replaced by $\alpha_{ill'}$. It remains to control the four error terms:

$$\mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}(\beta_{ill'}\beta_{il'l}{}^*)], \mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}(\beta_{ill'}\gamma_{il'l}{}^*)], \mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}(\gamma_{ill'}\beta_{il'l}{}^*)] \text{ and } \mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}(\gamma_{ill'}\gamma_{il'l}{}^*)].$$

(9) yields (note that we can remove one $\mathbb{E}_{\leq k-1}$):

$$\begin{aligned}&\mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}(\beta_{ill'}\beta_{il'l}{}^*)] = (1-\tfrac{1}{2}\delta_{il})(1-\frac{1}{2}\delta_{il'})\\ &\quad\times\Bigg\{\mathbb{E}\Big[|W_{il}|^2|W_{il'}|^2\operatorname{Tr}\Big\{\mathbb{E}_{\leq k-1}\Big(\gamma_1R_{li}^{(kilil')}(\beta)\gamma_1\sum_{j<k}R_{l'j}^{(kil)}(\beta))\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\Big(R_{ji}^{(kil)}(\beta_1^*)\Big)\Big)\\ &\qquad\times\Big(\gamma_1R_{l'i}^{(kil'il)}(\beta)\gamma_1\sum_{j<k}R_{lj}^{(kil')}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\Big(R_{ji}^{(kil')}(\beta_1^*)\Big)\Big)^*\Big\}\Big]\end{aligned}\tag{53}$$

$$\begin{aligned}&+\mathbb{E}\Big[W_{il}^2|W_{il'}|^2\operatorname{Tr}\Big\{\gamma_1\mathbb{E}_{\leq k-1}\Big(R_{li}^{(kilil')}(\beta)\gamma_1\sum_{j<k}R_{l'j}^{(kil)}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}[R_{ji}^{(kil)}(\beta_1^*)]\Big)\\ &\qquad\times\Big(\gamma_1R_{l'l}^{(kil'il)}(\beta)\gamma_1\sum_{j'<k}R_{ij'}^{(kil')}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}[R_{j'i}^{(kil')}(\beta_1^*)]\Big)^*\Big\}\Big]\end{aligned}\tag{54}$$

$$\begin{aligned}&+\mathbb{E}\Big[|W_{il}|^2\overline{W_{il'}}^2\operatorname{Tr}\Big\{\gamma_1R_{ll'}^{(kilil')}(\beta)\gamma_1\sum_{j<k}R_{ij}^{(kil)}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}[R_{ji}^{(kil)}(\beta_1^*)]\\ &\qquad\times\Big(\gamma_1\mathbb{E}_{\leq k-1}\Big(R_{l'i}^{(kil'il)}(\beta)\gamma_1\sum_{j'<k}R_{lj'}^{(kil')}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}[R_{j'i}^{(kil')}(\beta_1^*)]\Big)\Big]\Big)^*\Big\}\end{aligned}\tag{55}$$

$$\begin{aligned}&+\mathbb{E}\Big[W_{il}^2\overline{W_{il'}}^2\operatorname{Tr}\Big\{\gamma_1\mathbb{E}_{\leq k-1}\Big(R_{ll'}^{(kilil')}(\beta)\gamma_1\sum_{j<k}R_{ij}^{(kil)}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}[R_{ji}^{(kil)}(\beta_1^*)]\Big)\\ &\qquad\times\Big(\gamma_1R_{l'l}^{(kil'il)}(\beta)\gamma_1\sum_{j'<k}R_{ij'}^{(kil')}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}[R_{j'i}^{(kil')}(\beta_1^*)]\Big)^*\Big\}\Big]\Bigg\}.\end{aligned}\tag{56}$$

Using Lemmas 65 and 5, there exist polynomials Q_1 and Q_2 such that

$$\Big\|\sum_{j<k}R_{l'j}^{(kil)}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\Big(R_{ji}^{(kil)}(\beta_1^*)\Big)\Big\| \leq \|\gamma_1\|^2Q_1(\|W_N\|,|\mathcal{I}z|^{-1})\mathbb{E}_{\leq k-1}\Big(Q_2(\|W_N\|,|\mathcal{I}z_1|^{-1})\Big).$$

Thus, using Hölder's inequality and Proposition 78, the sum over $l\neq l'$ of (53) can be bounded as follows:

$$\begin{aligned}&\Big|\sum_{l\neq l'}(1-\tfrac{1}{2}\delta_{il})(1-\frac{1}{2}\delta_{il'})\Big\{\mathbb{E}\Big[|W_{il}|^2|W_{il'}|^2\operatorname{Tr}\Big\{\gamma_1\mathbb{E}_{\leq k-1}\Big(R_{li}^{(kilil')}(\beta)\gamma_1\sum_{j<k}R_{l'j}^{(kil)}(\beta)\gamma_1\otimes\gamma_1\\ &\qquad\times\mathbb{E}_{\leq k-1}\Big(R_{ji}^{(kil)}(\beta_1^*)\Big)\Big)(\gamma_1R_{l'i}^{(kil'il)}(\beta)\gamma_1\sum_{j<k}R_{lj}^{(kil')}(\beta)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\Big(R_{ji}^{(kil')}(\beta_1^*)\Big))^*\Big\}\Big]\Big|\end{aligned}$$

$$\leq O(1)\sum_{l\neq l'}\mathbb{E}[|W_{il}|^4|W_{il'}|^4\mathbb{E}_{\leq k-1}\Big(\|R^{(kilil')}_{li}(\beta)\|^2\Big)\mathbb{E}_{\leq k-1}\Big(\|R^{(kil'il)}_{l'i}(\beta)\|^2\Big)]^{1/2}$$

$$\leq O(1)N^{-2}\sum_{l\neq l'}\mathbb{E}[\mathbb{E}_{\leq k-1}\Big(\|R^{(kilil')}_{li}(\beta)\|^2\Big)\mathbb{E}_{\leq k-1}\Big(\|R^{(kil'il)}_{l'i}(\beta)\|^2\Big)]^{1/2} \text{ by independence}$$

$$\leq O(1)\Big(N^{-2}\sum_{l\neq l'}\mathbb{E}[\mathbb{E}_{\leq k-1}\Big(\|R^{(kilil')}_{li}(\beta)\|^2\Big)\mathbb{E}_{\leq k-1}\Big(\|R^{(kil'il)}_{l'i}(\beta)\|^2\Big)]\Big)^{1/2}$$

by concavity of $x\mapsto x^{1/2}$

$$\leq O(1)\Big(N^{-2}\sum_{l,l'}\mathbb{E}[\mathbb{E}_{\leq k-1}(\|R^{(k)}(\beta)_{li}\|^2)\mathbb{E}_{\leq k-1}(\|R^{(k)}(\beta)_{l'i}\|^2]+O(\delta_N)\Big)^{1/2}$$

by (9), Lemmas 5 and Proposition 78

$$\leq O(1)(N^{-2}\mathbb{E}[(\sum_{l}\mathbb{E}_{\leq k-1}(\|R^{(k)}(\beta)_{li}\|^2))^2]+O(\delta_N))^{1/2}$$

$$\leq O(1)(N^{-2}\mathbb{E}[(\mathbb{E}_{\leq k-1}(\|R^{(k)}(\beta)\|^2))^2]+O(\delta_N))^{1/2} \text{ by Lemma 64}$$

$$\leq O(1)(O(N^{-2})+O(\delta_N))^{1/2}=O(\delta_N^{1/2}) \text{ using Remark 22,}$$

uniformly in k. The sums over l and l' of the three other terms (54), (55) and (56) can be treated similarly: they are of order $O(\delta_N)$, uniformly in k.

Using again the resolvent identity and very similar computations, the sums over l and l' of the other three error terms $\mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}\beta_{ill'}\gamma_{il'l}{}^*]$, $\mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}\gamma_{ill'}\beta_{il'l}{}^*]$ and $\mathbb{E}[W_{il}\overline{W_{il'}}\operatorname{Tr}\gamma_{ill'}\gamma_{il'l}{}^*]$ are proved to be of order $O(\sqrt{\delta_N})$, uniformly in k. As a consequence, from (52) we can deduce that the first term (49)

$$\sigma_N^2\sum_{i,j,l<k}W_{il}\hat{R}_i(\beta)\gamma_1\mathbb{E}_{\leq k-1}\Big(R^{(kil)}(\beta)_{lj}\Big)\gamma_1\otimes\gamma_1\mathbb{E}_{\leq k-1}\Big(R^{(kil)}(\beta_1^*)_{ji}\Big)=o_{L^2}(1)$$

uniformly in k.

The L^2 norm of the second term (50) is bounded by

$$\sigma_N^2\|\hat{R}(\beta)\|\|\gamma_1\|$$
$$\times\sum_{i<k}\mathbb{E}\Big[\Big|\sum_{l<k}(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2\mathbb{E}_{\leq k}[\|\sum_{j<k}\mathbb{E}_{\leq k}[R^{(kil)}(\beta)_{lj}]\gamma_1\otimes\gamma_1 R^{(kil)}_{ji}(\beta_1^*)\|\|\gamma_1\|\|R^{(k)}_{li}(\beta_1^*)\|]\Big|^2\Big]^{1/2},$$

and then, using Jensen's inequality (with respect to $\mathbb{E}_{\leq k-1}$) and Cauchy-Schwarz inequality (with respect to the l-sum), by

$$\sigma_N^2\|\hat{R}(\beta)\|\|\gamma_1\|^2\sum_{i<k}\mathbb{E}[\sum_{l<k}|W_{il}|^4\|\sum_{j<k}\mathbb{E}_{\leq k}[R^{(kil)}_{lj}(\beta)]\gamma_1\otimes\gamma_1 R^{(kil)}_{ji}(\beta_1^*)\|^2\sum_{l<k}\|R^{(k)}_{li}(\beta_1^*)\|^2]^{1/2}.$$

From Lemma 65 and Lemma 64, and then Lemma 5 one can deduce the following bounds of the L^2 norm of (50):

$$\sigma_N^2\|\hat{R}(\beta)\|\|\gamma_1\|^4\sum_{i<k}\mathbb{E}[\sum_{l<k}|W_{il}|^4\|\mathbb{E}_{\leq k}[R^{(kil)}(\beta)]\|^2\|R^{(kil)}(\beta_1^*)\|^2\|R^{(k)}(\beta_1^*)\|^2]^{1/2}$$

$$\leq\sigma_N^2\|\hat{R}(\beta)\|\|\gamma_1\|^4\sum_{i<k}\mathbb{E}[\sum_{l<k}|W_{il}|^4\|\mathbb{E}_{\leq k}[R^{(kil)}(\beta)]\|^2\|R^{(kil)}(\beta_1^*)\|^2 Q(\|W_N^{(k)}\|,|\mathcal{I}z_1|^{-1})^2]^{1/2}$$

$$\leq\sigma_N^2\|\hat{R}(\beta)\|\|\gamma_1\|^4$$
$$\times\sum_{i<k}\Big\{\sum_{l<k}\mathbb{E}(|W_{il}|^4)\mathbb{E}\Big(\|\mathbb{E}_{\leq k}[R^{(kil)}(\beta)]\|^2\|R^{(kil)}(\beta_1^*)\|^2Q_1(\|W_N^{(kil)}\|+2\delta_N,|\mathcal{I}z_1|^{-1})^2\Big)\Big\}^{1/2}$$

$$\leq\sigma_N^2\|\hat{R}(\beta_1)\|\|\gamma_1\|^4\sum_{i<k}\Big\{\sum_{l<k}\mathbb{E}(|W_{il}|^4)\mathbb{E}\Big(|\mathbb{E}_{\leq k}[Q_2(\|W_N\|,\delta_N,|\mathcal{I}z|^{-1})]|^2Q_3(\|W_N\|,\delta_N,|\mathcal{I}z_1|^{-1})\Big)\Big\}^{1/2}$$

$$=O(N^{-1/2}),$$

where we use Lemma 17, Lemmas 77 and 78 in the last line. Let us consider the last term (51). Replacing successively $(1-\frac{1}{2}\delta_{il})|W_{il}|^2$, $R^{(k)}_{ii}(\beta_1^*)$, $R^{(kil)}_{lj}(\beta)$, $R^{(kil)}_{jl}(\beta_1^*)$ by σ_N^2, $\hat{R}_i(\beta_1^*)$, $R^{(k)}_{lj}(\beta)$, $R^{(k)}_{jl}(\beta_1^*)$, this last term can be written as follows:

$$\begin{aligned}
&\sigma_N^2 \sum_{i,j,l<k} (1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 \hat{R}_i(\beta)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{jl}^{(kil)}(\beta_1^*)\gamma_1 R_{ii}^{(k)}(\beta_1^*)] \\
&\quad = \sigma_N^4 \sum_{i,j,l<k} \hat{R}_i(\beta)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(k)}(\beta)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{jl}^{(k)}(\beta_1^*)]\gamma_1 \hat{R}_i(\beta_1^*) \\
&\qquad + \sigma_N^2 \sum_{i,j,l<k} ((1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2)\hat{R}_i(\beta)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{jl}^{(kil)}(\beta_1^*)\gamma_1 R_{ii}^{(k)}(\beta_1^*)] \\
&\qquad + \sigma_N^4 \sum_{i,j,l<k} \hat{R}_i(\beta)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{jl}^{(kil)}(\beta_1^*)\gamma_1 (R_{ii}^{(k)}(\beta_1^*) - \hat{R}_i(\beta_1^*))] \\
&\qquad + \sigma_N^4 \sum_{i,j,l<k} \hat{R}_i(\beta)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta) - R_{lj}^{(k)}(\beta)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{jl}^{(kil)}(\beta_1^*)]\gamma_1 \hat{R}_i(\beta_1^*) \\
&\qquad + \sigma_N^4 \sum_{i,j,l<k} \hat{R}_i(\beta)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(k)}(\beta)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{jl}^{(kil)}(\beta_1^*) - R_{jl}^{(k)}(\beta_1^*)]\gamma_1 \hat{R}_i(\beta_1^*) \\
&\quad := N\sigma_N^2 T_{N,k,0}(z,\bar{z}_1)(\nabla_k) + h_k^{(N)} + p_k^{(N)} + q_k^{(N)} + r_k^{(N)}.
\end{aligned}$$

By triangular inequality, Jensen's inequality and Cauchy-Schwarz inequality, the L^1 norm of term $h_k^{(N)}$ can be bounded as follows:

$$\begin{aligned}
\|h_k^{(N)}\|_1 &\leq \sigma_N^2 \|\hat{R}(\beta_1)\| \|\gamma_1\| \\
&\qquad \times \sum_{i<k} \||| \sum_{l<k} ((1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2) \sum_{j<k} \mathbb{E}_{\leq k}[R^{(kil)}(\beta)_{lj}]\gamma_1 \otimes \gamma_1 R_{jl}^{(kil)}(\beta_1^*)\gamma_1 \| \| R_{ii}^{(k)}(\beta_1^*) \| \|_1 \\
&\leq \sigma_N^2 \|\hat{R}(\beta)\| \|\gamma_1\| \\
&\qquad \times \sum_{i<k} \| \sum_{l<k} ((1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2) \sum_{j<k} \mathbb{E}_{\leq k}[R^{(kil)}(\beta)_{lj}]\gamma_1 \otimes \gamma_1 R_{jl}^{(kil)}(\beta_1^*)\gamma_1 \|_2 \| R_{ii}^{(k)}(\beta_1^*) \|_2 \\
&\leq \sigma_N^2 \|\hat{R}(\beta)\| \|\gamma_1\| \\
&\qquad \times \|R^{(k)}(\beta_1^*)\|_2 \sum_{i<k} \mathbb{E}[\| \sum_{l<k} ((1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2) \sum_{j<k} \mathbb{E}_{\leq k}[R^{(kil)}(\beta)_{lj}]\gamma_1 \otimes \gamma_1 R_{jl}^{(kil)}(\beta_1^*)\gamma_1 \|^2]^{1/2}
\end{aligned}$$

$$= O\big(\delta_N^{1/2}\big), \tag{57}$$

if one may prove that

$$\mathbb{E}[\| \sum_{l<k} ((1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2) \sum_{j<k} \mathbb{E}_{\leq k}[R^{(kil)}(\beta)_{lj}]\gamma_1 \otimes \gamma_1 R_{jl}^{(kil)}(\beta_1^*)\gamma_1 \|_{HS}^2] = O(\delta_N), \tag{58}$$

uniformly in i,k, using also Remark 22 and Lemma 17.

Develop the Hilbert-Schmidt norm of the l-sum: the sum of "squares" is bounded by

$$\begin{aligned}
m(m_N - \sigma_N^4) \sum_{l<k, l\neq i} \mathbb{E}\Big[\| \sum_{j<k} \mathbb{E}_{\leq k}[R^{(kil)}(\beta)_{lj}]\gamma_1 \otimes \gamma_1 R_{jl}^{(kil)}(\beta_1^*)\gamma_1 \|^2\Big] \\
+ m(\frac{1}{4}\delta_N^2 \tilde{\sigma}_N^2 - \sigma_N^2 \tilde{\sigma}_N^2 + \sigma_N^4)\mathbb{E}\Big[\| \sum_{j<k} \mathbb{E}_{\leq k}[R^{(kii)}(\beta)_{ij}]\gamma_1 \otimes \gamma_1 R_{ji}^{(kii)}(\beta_1^*)\gamma_1 \|^2\Big]
\end{aligned}$$

which is $O(N^{-1})$, uniformly in i,k, using Lemma 65 and Remark 22.

It remains to control the sum of the cross terms

$$\mathbb{E}[((1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2)((1-\frac{1}{2}\delta_{il'})|W_{il'}|^2 - \sigma_N^2)\langle C_{ll}^{il}, C_{l'l'}^{il'}\rangle],$$

where $C_{ll}^{il} = \sum_{j<k} \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta)]\gamma_1 \otimes \gamma_1 R_{jl}^{(kil)}(\beta_1^*)\gamma_1$ and $\langle A, B\rangle = \operatorname{Tr} AB^*$. Define

$$C_{ll}^{ilil'} = \sum_{j<k} \mathbb{E}_{\leq k}[R_{lj}^{(kilil')}(\beta)]\gamma_1 \otimes \gamma_1 R_{jl}^{(kilil')}(\beta_1^*)\gamma_1.$$

Note that by Lemma 65

$$\|C_{ll}^{il}\| \leq \|\gamma_1\|^2 \mathbb{E}_{\leq k}[\|R^{(kil)}(\beta)\|] \|R^{(kil)}(\beta_1^*)\|,$$
$$\|C_{ll}^{ilil'}\| \leq \|\gamma_1\|^2 \mathbb{E}_{\leq k}[\|R^{(kilil')}(\beta)\|] \|R^{(kilil')}(\beta_1^*)\|.$$

Observe that, by independence, since at least one of $l, l' \neq i$,

$$\begin{aligned}
&|\mathbb{E}[((1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2)((1-\frac{1}{2}\delta_{il'})|W_{il'}|^2 - \sigma_N^2)\langle C_{ll}^{il}, C_{l'l'}^{il'}\rangle]| \\
&= |\mathbb{E}[((1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2)((1-\frac{1}{2}\delta_{il'})|W_{il'}|^2 - \sigma_N^2)(\langle C_{ll}^{il}, C_{l'l'}^{il'}\rangle - \langle C_{ll}^{ilil'}, C_{l'l'}^{ilil'}\rangle)]| \\
&\leq \mathbb{E}[|(1-\tfrac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2|^2]^{1/2} \mathbb{E}[|(1-\frac{1}{2}\delta_{il'})|W_{il'}|^2 - \sigma_N^2|^2]^{1/2} \mathbb{E}[|\langle C_{ll}^{il}, C_{l'l'}^{il'}\rangle - \langle C_{ll}^{ilil'}, C_{l'l'}^{ilil'}\rangle|^2]^{1/2}.
\end{aligned}$$

Now, from (9),

$$\big\|R^{(kil)}(\beta) - R^{(kilil')}(\beta)\big\| \leq 2\delta_N \|\gamma_1\| \big\|R^{(kil)}(\beta)\big\| \big\|R^{(kilil')}(\beta)\big\|. \tag{59}$$

Thus, using Cauchy-Schwarz inequality and Lemma 64, one can easily obtain that

$$\begin{aligned}
\big\|C_{ll}^{il} - C_{ll}^{ilil'}\big\| &\leq O(\delta_N)\Big(\mathbb{E}_{\leq k}[\|R^{(kil)}(\beta)\|^2 \big\|R^{(kilil')}(\beta)\big\|^2]\Big)^{1/2} \|R^{(kil)}(\beta_1^*)\| \\
&+ O(\delta_N)\Big(\mathbb{E}_{\leq k}\big[\|R^{(kilil')}(\beta)\|^2\big]\Big)^{1/2} \|R^{(kil)}(\beta_1^*)\| \big\|R^{(kilil')}(\beta_1^*)\big\|.
\end{aligned}$$

Then Remark 22 readily implies that

$$\mathbb{E}[|\langle C_{ll}^{il}, C_{l'l'}^{il'}\rangle - \langle C_{ll}^{ilil'}, C_{l'l'}^{ilil'}\rangle|^2]^{1/2} = O(\delta_N)$$

uniformly in i, k, l, l' and then that $\mathbb{E}[((1-\frac{1}{2}\delta_{il})|W_{il}|^2 - \sigma_N^2)((1-\frac{1}{2}\delta_{il'})|W_{il'}|^2 - \sigma_N^2)\langle C_{ll}^{il}, C_{l'l'}^{il'}\rangle] = O(\delta_N N^{-2})$ uniformly in i, k, l, l'. There are less than N^2 such cross terms. Therefore, (58) and then (57) are true.

The L^1 norm of $p_k^{(N)}$ can be bounded as follows (using (15), Lemma 65, Remarks 34 and 22:

$$\begin{aligned}
\|p_k^{(N)}\|_1 &\leq \sigma_N^4 \sum_{i,l<k} \|\hat{R}_i(\beta)\| \|\gamma_1\| \mathbb{E}[\| \sum_{j<k} \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta)]\gamma_1 \otimes \gamma_1 R_{jl}^{(kil)}(\beta_1^*)\gamma_1\| \|R_{ii}^{(k)}(\beta_1^*) - \hat{R}_i(\beta_1^*)\|] \\
&= O(N\sigma_N^4 \sum_{i<k} \mathbb{E}[\|R_{ii}^{(k)}(\beta_1^*) - \hat{R}_i(\beta_1^*)\|^p]^{1/p}) = o(1)
\end{aligned}$$

uniformly in k.

Using Lemma 65 and (9),

$$\begin{aligned}
\Big\| \sum_{j<k} \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta) - R^{(k)}(\beta)_{lj}]\gamma_1 \otimes \gamma_1 R_{jl}^{(kil)}(\beta_1^*) \Big\| &\leq \|\gamma_1\|^2 \|\mathbb{E}_{\leq k-1}[R^{(kil)}(\beta) - R^{(k)}(\beta)]\| \|R^{(kil)}(\beta_1^*)\| \\
&\leq 2\delta_N \|\gamma_1\|^3 \mathbb{E}_{\leq k-1}[\|R^{(kil)}(\beta)\| \|R^{(k)}(\beta)\|] \|R^{(kil)}(\beta_1^*)\|.
\end{aligned}$$

Hence, using Remark 22, the L^1 norm of $q_k^{(N)}$ is $O(\delta_N)$, uniformly in k. Similarly, the L^1 norm of $r_k^{(N)}$ is $O(\delta_N)$, uniformly in k. Thus, we have established that (51) is equal to $N\sigma_N^2 T_{N,k,0}(z, \bar{z}_1)(\nabla_k) + o_{L^1}(1)$. Since moreover we also established that (49) and (50) are $o_{L^1}(1)$ uniformly in k, Lemma 51 yields that

$$\nabla_k = \sigma_N^2 \sum_{i<k} \hat{R}_i(\beta)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k-1}\left(R_{ii}^{(k)}(\beta_1^*)\right) + N\sigma_N^2 T_{N,k,0}(z, \bar{z}_1)(\nabla_k) + o_{L^1}(1).$$

Lemma 52 readily follows by using Remark 34.

□

Thus, for any $z_1 \in \mathbb{C} \setminus \mathbb{R}$ and $z \in O_{z_1}$ (see Lemma 49), setting $\beta = ze_{11} - \gamma_0$ and $\beta_1^* = \bar{z}_1 e_{11} - \gamma_0$ in $M_m(\mathbb{C})$, we obtain from Lemma 52 that

$$\begin{aligned}
\nabla_k &= \left(\mathrm{id}_m \otimes \mathrm{id}_m - N\sigma_N^2 T_{N,k,0}(z, \bar{z}_1)\right)^{-1} \left(N\sigma_N^2 T_{N,k,0}(z, \bar{z}_1)(I_m \otimes I_m)\right) + o_{L^1}^{(u)}(1) \\
&= -I_m \otimes I_m + \left(\mathrm{id}_m \otimes \mathrm{id}_m - N\sigma_N^2 T_{N,k,0}(z, \bar{z}_1)\right)^{-1} (I_m \otimes I_m) + o_{L^1}^{(u)}(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
N\sigma_N^2 T_N(z, \bar{z}_1)(\nabla_k) &= -N\sigma_N^2 T_N(z, \bar{z}_1)(I_m \otimes I_m) \\
&+ N\sigma_N^2 T_N(z, \bar{z}_1)\left(\mathrm{id}_m \otimes \mathrm{id}_m - N\sigma_N^2 T_{N,k,0}(z, \bar{z}_1)\right)^{-1} (I_m \otimes I_m) + o_{L^1}(1).
\end{aligned}$$

Thus from (46), the term under study in Proposition 50 can be rewritten as follows

$$\sigma_N^4 \sum_{k=1}^N \sum_{i,j<k} \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta)]\gamma_1 \hat{R}_k(\beta)) \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ji}^{(k)}(\beta_1^*)]\gamma_1 \hat{R}_k(\beta_1^*))$$
$$= -\sigma_N^2 \sum_{k=1}^N \operatorname{Tr}\left(\hat{R}_k(\beta)\gamma_1\right) \operatorname{Tr}\left(\gamma_1 \hat{R}_k(\beta_1^*)\right) + N^{-1}\sum_{k=1}^N f_{k,k,N}(0) + o_{\mathbb{P}}(1),$$

where for $t \in [0;1]$, with the notations of Section 7.3.2,

$$f_{k,k,N}(t) = N\sigma_N^2 \operatorname{Tr}\otimes\operatorname{Tr}\left[\hat{R}_k(\beta)\gamma_1 \otimes I_m \left(\mathrm{id}_m \otimes \mathrm{id}_m - N\sigma_N^2 T_{N,k,t}(z,\bar{z}_1)\right)^{-1} (I_m \otimes I_m)\, I_m \otimes \gamma_1 \hat{R}_k(\beta_1^*)\right].$$

The first term can be analysed as in Section 7.3.1. The second term can be analysed as follows: for any $t \in [0;1]$,

$$f_{k,k,N}(t) - f_{k,k,N}(0)$$
$$= tN\sigma_N^4 \operatorname{Tr}\otimes\operatorname{Tr}\Big[\hat{R}_k(\beta)\gamma_1 \otimes I_m\big(\mathrm{id}_m \otimes \mathrm{id}_m - N\sigma_N^2 T_{N,k,t}(z,\bar{z}_1)\big)^{-1}\hat{R}_k(\beta)\gamma_1 \otimes I_m$$
$$\times \left(\mathrm{id}_m \otimes \mathrm{id}_m - N\sigma_N^2 T_{N,k,0}(z,\bar{z}_1)\right)^{-1}\big(I_m \otimes I_m\big) I_m \otimes \gamma_1 \hat{R}_k(\beta_1^*), I_m \otimes \gamma_1 \hat{R}_k(\beta_1^*)\Big].$$

It readily follows from Lemma 17 and Lemma 49 that there exists some constant $C(z_1) > 0$ such that for any $t \in [0,1]$,

$$|f_{k,k,N}(t) - f_{k,k,N}(0)| \leq C(z_1) N\sigma_N^4.$$

Integrating with respect to $t \in [0,1]$ and summing on k we obtain that

$$\left|N^{-1}\sum_{k=1}^N f_{k,k,N}(0) - N^{-1}\sum_{k=1}^N \int_0^1 f_{k,k,N}(t)dt\right| \leq C(z_1)N\sigma_N^4 = o(1).$$

Now, set for $k = 1, \ldots, N$,

$$A_k : X \mapsto \sigma_N^2 \hat{R}_k(\beta)\gamma_1 \otimes I_m X I_m \otimes \gamma_1 \hat{R}_k(\beta_1^*),$$
$$B_k = N\sigma_N^2 T_{N,k,0}(z,\bar{z}_1),$$

and for any $t \in [0,1]$,

$$g_k(t) = \log(\mathrm{id}_m \otimes \mathrm{id}_m - B_k - tA_k) = -\sum_{p=1}^{+\infty} \frac{1}{p}(B_k + tA_k)^p.$$

Note that $f_{k,k,N}(t) = N(\operatorname{Tr}\otimes\operatorname{Tr})\left[A_k(\mathrm{id}_m \otimes \mathrm{id}_m - B_k - tA_k)^{-1}(I_m \otimes I_m)\right]$.

Lemma 53.

$$(\operatorname{Tr}\otimes\operatorname{Tr})(g_k'(t)(I_m \otimes I_m)) = (\operatorname{Tr}\otimes\operatorname{Tr})\left[-A_k(\mathrm{id}_m \otimes \mathrm{id}_m - B_k - tA_k)^{-1}(I_m \otimes I_m)\right].$$

Proof. First observe that $T \mapsto \operatorname{Tr}\otimes\operatorname{Tr}(T(I_m \otimes I_m))$ is a trace on the algebra generated by $A_1, \ldots, A_N$ (to which belong $B_1, \ldots, B_N$). Note that

$$g_k'(t) = -\sum_{p=1}^{\infty} \frac{1}{p} \sum_{i=0}^{p-1} (B_k + tA_k)^i A_k (B_k + tA_k)^{p-1-i}$$

Hence

$$\operatorname{Tr}\otimes\operatorname{Tr}[g_k'(t)(I_m \otimes I_m)] = -\sum_{p=1}^{\infty} \frac{1}{p} \sum_{i=0}^{p-1} \operatorname{Tr}\otimes\operatorname{Tr}[(B_k + tA_k)^i A_k (B_k + tA_k)^{p-1-i}(I_m \otimes I_m)]$$
$$= -\sum_{p=1}^{\infty} \frac{1}{p} \sum_{i=0}^{p-1} \operatorname{Tr}\otimes\operatorname{Tr}[A_k(B_k + tA_k)^{p-1}(I_m \otimes I_m)]$$
$$= -\operatorname{Tr}\otimes\operatorname{Tr}\Big[\sum_{p=0}^{\infty} A_k(B_k + tA_k)^p (I_m \otimes I_m)\Big]$$
$$= \operatorname{Tr}\otimes\operatorname{Tr}\left[-A_k(\mathrm{id}_m \otimes \mathrm{id}_m - B_k - tA_k)^{-1}(I_m \otimes I_m)\right].$$

□

Lemma 53 readily implies that

$$\int_0^1 (\mathrm{Tr}\otimes\mathrm{Tr})\left[-A_k(\mathrm{id}_m\otimes\mathrm{id}_m - B_k - tA_k)^{-1}(I_m\otimes I_m)\right]\,\mathrm{d}t$$
$$= (\mathrm{Tr}\otimes\mathrm{Tr})\left[\log(\mathrm{id}_m\otimes\mathrm{id}_m - B_k - A_k)(I_m\otimes I_m)\right] - (\mathrm{Tr}\otimes\mathrm{Tr})\left[\log(\mathrm{id}_m\otimes\mathrm{id}_m - B_k)(I_m\otimes I_m)\right].$$

Therefore, noticing that $B_{k+1} = A_k + B_k$, we obtain that

$$N^{-1}\sum_{k=1}^N \int_0^1 f_{k,k,N}(t)dt = (\mathrm{Tr}\otimes\mathrm{Tr})\left[\log\left(\mathrm{id}_m\otimes\mathrm{id}_m - N\sigma_N^2 T_N(z,\bar z_1)\right)(I_m\otimes I_m)\right].$$

Lemma 54. *For any $\beta_1 = z_1 e_{11} - \gamma_0, z_1\in\mathbb{C}\setminus\mathbb{R}$ and any $\beta_2 = z_2 e_{11} - \gamma_0, z_2\in\mathbb{C}\setminus\mathbb{R}$, we have*

$$\lim_{N\to\infty}(\mathrm{Tr}\otimes\mathrm{Tr})\left[\log\left(\mathrm{id}_m\otimes\mathrm{id}_m - N\sigma_N^2 T_N(z_1,z_2)\right)(I_m\otimes I_m)\right]$$
$$= (\mathrm{Tr}\otimes\mathrm{Tr})\left[\log\left(\mathrm{id}_m\otimes\mathrm{id}_m - \sigma^2 T_{\{\beta_1,\beta_2\}}\right)(I_m\otimes I_m)\right].$$

Proof. For any $\alpha\otimes\upsilon\in M_m(\mathbb{C})\otimes M_m(\mathbb{C})$, we have

$$\left[N\sigma_N^2 T_N(z_1,z_2)\right](\alpha\otimes\upsilon)$$
$$= N\sigma_N^2(\mathrm{id}_{m^2}\otimes\tau_N)\left[(\omega_N(\beta_1)\otimes 1 - \gamma_2\otimes D_N)^{-1}(\gamma_1\alpha\otimes 1)\right]\otimes\left[(\upsilon\gamma_1\otimes 1)(\omega_N(\beta_2)\otimes 1 - \gamma_2\otimes D_N)^{-1}\right], \tag{60}$$

where by $(\mathrm{id}_{m^2}\otimes\tau_N)$ we mean that τ_N is applied entrywise to the matrix belonging to $M_{m^2}(C^*\langle D_N\rangle)\simeq M_m(C^*\langle D_N\rangle)\otimes M_m(C^*\langle D_N\rangle)\simeq M_m(\mathbb{C})\otimes C^*\langle D_N\rangle\otimes M_m(\mathbb{C})\otimes C^*\langle D_N\rangle$. Since $(D_N)_{N\in\mathbb{N}}$ converges in $*$-moments towards d, $\lim_{N\to+\infty} N\sigma_N^2 = \sigma^2$ and using Lemma 19, we can easily deduce from (60) (using also Lemmas 17 and 18) that for any z_1, z_2 in $\mathbb{C}\setminus\mathbb{R}$, the sequence of operators $(N\sigma_N^2 T_N(z_1,z_2))_N$ converges in operator norm to $\sigma^2 T_{\{\beta_1,\beta_2\}}$. We know by Corollary 40 that there exists $0<\epsilon_0<1$ such $\limsup_{N\to+\infty}\rho(N\sigma_N^2 T_N(z_1,z_2) < 1-\epsilon_0$. Thanks to the Cauchy formula, for all $x\in\mathbb{C}$ such that $|x|<1-\epsilon_0/2$, for any $k\geq 0$, $x^k = \frac{1}{2i\pi}\int_{|w|=1-\epsilon_0/2}\frac{w^k}{w-x}dw$. Therefore, using the holomorphic functional calculus, we have for all large N,

$$\forall k\geq 0\ ,(N\sigma_N^2 T_N(z_1,z_2))^k = \frac{1}{2i\pi}\int_{|w|=1-\epsilon_0/2} w^k(w\mathrm{id}_m\otimes\mathrm{id}_m - N\sigma_N^2 T_N(z_1,z_2))^{-1}dw,$$

and therefore

$$\forall k\geq 0\ ,\|(N\sigma_N^2 T_N(z_1,z_2))^k\| \leq \sup_{|w|=1-\epsilon_0/2}\|(w\mathrm{id}_m\otimes\mathrm{id}_m - N\sigma_N^2 T_N(z_1,z_2))^{-1}\|(1-\epsilon_0/2)^{k+1}.$$

Now, using Lemmas 17 and 63, there exists $C(m,\epsilon_0)>0$ such that we have for all large N,

$$\sup_{|w|=1-\epsilon_0/2}\|(w\mathrm{id}_m\otimes\mathrm{id}_m - N\sigma_N^2 T_N(z_1,z_2))^{-1}\|\leq C(m,\epsilon_0),$$

and thus

$$\forall k\geq 0\ ,\|(N\sigma_N^2 T_N(z_1,z_2))^k\|\leq C(m,\epsilon_0)\|(1-\epsilon_0/2)^{k+1}.$$

Therefore, using dominated convergence Theorem, it readily follows that

$$\log\left[\mathrm{id}_m\otimes\mathrm{id}_m - N\sigma_N^2 T_N(z_1,z_2)\right] = -\sum_{k=1}^\infty\frac{1}{k}(N\sigma_N^2 T_N(z_1,z_2))^k$$
$$\to_{N\to+\infty} -\sum_{k=1}^\infty\frac{1}{k}(\sigma^2 T_{\{\beta_1,\beta_2\}})^k = \log\left[\mathrm{id}_m\otimes\mathrm{id}_m - \sigma^2 T_{\{\beta_1,\beta_2\}}\right].$$

□

Thus Proposition 50 is proved. □

Proposition 55. *For $\beta_1 = z_1 e_{11} - \gamma_0$ and $\beta_2 = z_2 e_{11} - \gamma_0$, for any $z_1 \in \mathbb{C}\setminus\mathbb{R}, z_2 \in \mathbb{C}\setminus\mathbb{R}$, the following convergence holds in probability:*

$$\begin{aligned}&\sigma_N^4 \sum_{k=1}^{N}\sum_{i,j<k} \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ji}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2))\\ &\qquad \to_{N\to+\infty} -\operatorname{Tr}\otimes\operatorname{Tr}\left\{\log\left[\mathrm{id}_m\otimes\mathrm{id}_m - \sigma^2 T_{\{\beta_1,\beta_2\}}\right](I_m\otimes I_m)\right\}\\ &\qquad\qquad - \sigma^2 \operatorname{Tr}\otimes\operatorname{Tr}\left\{T_{\{\beta_1,\beta_2\}}(I_m\otimes I_m)\right\}.\end{aligned} \tag{61}$$

Proof. Recall that a sequence $(X_N)_{N\geq 1}$ of random variables converges in probability to a random variable X if and only if, from any subsequence extracted from $(X_N)_{N\geq1}$, one can further extract a subsubsequence almost surely converging to X. We will use this criterion in the following argument.

Let C be as in Proposition 78. If $\|W_N\| \leq C$, there exists $M > 0$ such that for any $k \in \{1,\dots,N\}$, $\|P(W_N^{(k)}, D_N^{(k)})\| \leq M$. Let $K > 0$ be such that $\|P(s_N, D_N)\| \leq K$. Thus, for any $z \in \mathbb{R}$ such that $|z| > K$, $z\mathbf{1}_{\mathcal{A}_N} - P(s_N, D_N)$ is invertible and the, according to Lemma 4, $ze_{11}\otimes 1_{\mathcal{A}_N} - L_P(s_N, D_N)$ is invertible.

Let us fix $z_1 \in \mathbb{C}\setminus\mathbb{R}$. We know, using (17), Lemma 65, Remark 22 and Proposition 78, that by Proposition 50, for any $z \in O_{z_1}$,

$$\begin{aligned}f_N^{(z_1)}(z) := \mathbf{1}_{\{\|W_N\|\leq C\}}\sigma_N^4 \sum_{k=1}^{N}\sum_{i,j<k} &\operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(ze_{11}-\gamma_0)]\gamma_1 \hat{R}_k(ze_{11}-\gamma_0))\\ &\times \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ji}^{(k)}(\bar{z}_1 e_{11}-\gamma_0)]\gamma_1 \hat{R}_k(\bar{z}_1 e_{11}-\gamma_0))\end{aligned}$$

converges in probability towards

$$\begin{aligned}f^{(z_1)}(z) = &-\operatorname{Tr}\otimes\operatorname{Tr}\left\{\log\left[\mathrm{id}_m\otimes\mathrm{id}_m - \sigma^2 T_{\{ze_{11}-\gamma_0,\bar{z}_1e_{11}-\gamma_0\}}\right](I_m\otimes I_m)\right\}\\ &- \sigma^2 \operatorname{Tr}\otimes\operatorname{Tr}\left[T_{\{ze_{11}-\gamma_0,\bar{z}_1e_{11}-\gamma_0\}}(I_m\otimes I_m)\right].\end{aligned}$$

For $N \geq 1$, $f_N^{(z_1)}$ is an holomorphic function on $\mathbb{C}\setminus[-\max(M,K);\max(M,K)]$.

Fix an arbitrary subsequence extracted from $(f_N^{(z_1)})_{N\geq1}$. By diagonal extraction from the convergence in probability above, given a countable subset of uniqueness of O_{z_1}, one can extract a subsubsequence, let us say $(f_{\Psi(N)}^{(z_1)})_{N\geq1}$, almost surely converging to $f^{(z_1)}$ pointwise on this subset.

Using Lemma 65 and Lemma 5, $(f_N^{(z_1)})_{N\geq1}$ is a bounded sequence in $\mathcal{H}(\mathbb{C}\setminus[-\max(M,K);\max(M,K)])$. We conclude by Vitali's Theorem that almost surely $(f_{\Psi(N)}^{(z_1)})$ converges towards an holomorphic function on $\mathbb{C}\setminus[-\max(M,K);\max(M,K)]$.
Note that $f^{(z_1)}$ is holomorphic on $\mathbb{C}\setminus\mathbb{R}$. Hence almost surely, for any $z\in\mathbb{C}\setminus\mathbb{R}$, $f_{\Psi(N)}^{(z_1)}(z)$ converges towards $f^{(z_1)}(z)$. Therefore for any $z \in \mathbb{C}\setminus\mathbb{R}$, $f_N^{(z_1)}(z)$ converges in probability towards $f^{(z_1)}(z)$. Proposition 55 readily follows since, by Proposition 78, Remark 22 and Lemma (17),

$$\begin{aligned}\mathbf{1}_{\{\|W_N\|> C\}}\sigma_N^4 \sum_{k=1}^{N}\sum_{i,j<k} &\operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(ze_{11}-\gamma_0)]\gamma_1 \hat{R}_k(ze_{11}-\gamma_0))\\ &\times \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ji}^{(k)}(\bar{z}_1 e_{11}-\gamma_0)]\gamma_1 \hat{R}_k(\bar{z}_1 e_{11}-\gamma_0)) \;= o_{L^1}(1).\end{aligned}$$

□

7.3.3. *Contribution of the third term of* (44).

Proposition 56. *Fix $z_1, z_2 \in \mathbb{C}\setminus\mathbb{R}$ and set $\beta_1 = z_1e_{11}-\gamma_0$ and $\beta_2 = z_2e_{11}-\gamma_0$ in $M_m(\mathbb{C})$. The following convergence holds in probability:*

$$\begin{aligned}&|\theta_N|^2 \sum_{k=1}^{N}\sum_{i,j<k} \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2))\\ &\qquad \to_{N\to+\infty} -\operatorname{Tr}\otimes\operatorname{Tr}\left\{\log\left[\mathrm{id}_m\otimes\mathrm{id}_m - \theta T_{\{\beta_1,\beta_2\}}\right](I_m\otimes I_m)\right\}\\ &\qquad\qquad - \theta\operatorname{Tr}\otimes\operatorname{Tr}\{T_{\{\beta_1,\beta_2\}}(I_m\otimes I_m)\}.\end{aligned}$$

The proof of Proposition 56 is very similar to the proof of Propositions 50 and 55. Therefore, we only notice the main differences. Instead of (45), we define for any $k \in \{1, \dots, N\}$,

$$\tilde{\nabla}_k = |\theta_N| \sum_{i,j<k} \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_1)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_2)] \in M_m(\mathbb{C}) \otimes M_m(\mathbb{C}),$$

and note that

$$|\theta_N|^2 \sum_{k=1}^{N} \sum_{i,j<k} \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ij}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2))$$
$$= N|\theta_N| \operatorname{Tr} \otimes \operatorname{Tr} T_N(z_1, z_2)(\tilde{\nabla}_k).$$

Sticking to the proof of Lemma 51, we can prove the following

Lemma 57.

$$\tilde{\nabla}_k = |\theta_N| \sum_{i<k} \hat{R}_i(\beta_1)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k-1}\left(R_{ii}^{(k)}(\beta_2)\right)$$
$$+ |\theta_N| \sum_{i,j,l<k} W_{il} \hat{R}_i(\beta_1)\gamma_1 \mathbb{E}_{\leq k-1}\left(R_{lj}^{(kil)}(\beta_1)\right)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k-1}\left(R_{ij}^{(k)}(\beta_2)\right) + o_{L^2}^{(u)}(1).$$

We can also establish the following lemma which is an analog of Lemma 52.

Lemma 58.

$$\tilde{\nabla}_k = N|\theta_N| T_{N,k,0}(z_1, z_2)(I_m \otimes I_m + \tilde{\nabla}_k) + o_{L^1}(1).$$

The proof of the last lemma starts as the proof of Lemma 52 by writing

$$|\theta_N| \sum_{i,j,l<k} W_{il} \hat{R}_i(\beta_1)\gamma_1 \mathbb{E}_{\leq k-1}\big(R_{lj}^{(kil)}(\beta_1)\big)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k-1}\big(R_{ij}^{(k)}(\beta_2)\big)$$
$$= |\theta_N| \sum_{i,j,l<k} W_{il} \hat{R}_i(\beta_1)\gamma_1 \mathbb{E}_{\leq k-1}\big(R_{lj}^{(kil)}(\beta_1)\big)\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k-1}\big(R_{ij}^{(kil)}(\beta_2)\big)$$
$$+ |\theta_N| \sum_{i,j,l<k} (1 - \tfrac{1}{2}\delta_{il}) W_{il}^2 \hat{R}_i(\beta_1)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta_1)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(kil)}(\beta_2)\gamma_1 R_{lj}^{(k)}(\beta_2)]$$
$$+ |\theta_N| \sum_{i,j,l<k} (1 - \tfrac{1}{2}\delta_{il}) |W_{il}|^2 \hat{R}_i(\beta_1)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta_1)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{il}^{(kil)}(\beta_2)\gamma_1 R_{ij}^{(k)}(\beta_2)].$$

But now, it is the second term (and note the third one) of the right-hand side that leads to a significant term whereas the other ones are negligible:

$$|\theta_N| \sum_{i,j,l<k} (1 - \tfrac{1}{2}\delta_{il}) W_{il}^2 \hat{R}_i(\beta_1)\gamma_1 \mathbb{E}_{\leq k}[R_{lj}^{(kil)}(\beta_1)]\gamma_1 \otimes \gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(kil)}(\beta_2)\gamma_1 R_{lj}^{(k)}(\beta_2)]$$
$$= |\theta_N| \sum_{i<k} T_{N,k,0}(z_1, z_2)(\tilde{\nabla}_k) + o_{L^1}(1).$$

The rest of the proof of Proposition 56 sticks to the proof of Propositions 50 and 55, using that $0 \leq |\theta_N| \leq \sigma_N^2$ ensuring the invertibility of the involved operators.

7.3.4. *Contribution of the fourth term of* (44). To handle the fourth term, define

$$f^{(1)}(\omega, x, y) = \operatorname{Tr}\left(\gamma_1(\omega - x\gamma_2)^{-1}\gamma_1(\omega - y\gamma_2)^{-1}\right)$$

for $\omega \in M_m(\mathbb{C}), x, y \in \mathbb{R}$ such that $\omega - x\gamma_2$ and $\omega - y\gamma_2$ are invertible. Note that

$$\sum_{k=1}^{N} \kappa_N \sum_{i<k} \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2))$$
$$= \kappa_N \sum_{1 \leq i < k \leq N} f^{(1)}(\omega_N(\beta_1), D_{ii}, D_{kk}) f^{(1)}(\omega_N(\beta_2), D_{ii}, D_{kk}) + I_1 + I_2,$$

where

$$I_1 = \sum_{k=1}^{N} \kappa_N \sum_{i<k} \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_1) - \hat{R}_i(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2))$$

and

$$I_2 = \sum_{k=1}^{N} \kappa_N \sum_{i<k} \operatorname{Tr}(\gamma_1 \hat{R}_i(\beta_1)\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_2) - \hat{R}_i(\beta_2)]\gamma_1 \hat{R}_k(\beta_2)).$$

Using (15), Remark 34, Remark 22 and Proposition 78, we readily obtain that for any $p \geq 1$, $\|I_1 + I_2\|_{L^p} \to_{N\to+\infty} 0$. Now, noticing that $f^{(1)}(\omega, x, y) = f^{(1)}(\omega, y, x)$ and using Lemma 19, one can easily see that

$$\begin{aligned}\kappa_N \sum_{1\leq i<k\leq N} & f^{(1)}(\omega_N(\beta_1), D_{ii}, D_{kk}) f^{(1)}(\omega_N(\beta_2), D_{ii}, D_{kk}) \\ &= \frac{N^2\kappa_N}{2} \iint_{\mathbb{R}^2} f^{(1)}(\omega_N(\beta_1), x, y) f^{(1)}(\omega_N(\beta_2), x, y)(\nu_N \otimes \nu_N)(x, y) + O(N^{-1}) \\ &\underset{N\to+\infty}{\longrightarrow} \frac{\kappa}{2} \iint_{\mathbb{R}^2} f^{(1)}(\omega(\beta_1), x, y) f^{(1)}(\omega(\beta_2), x, y)(\nu \otimes \nu)(x, y)\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{N} \kappa_N \sum_{i<k} & \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_1)]\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \mathbb{E}_{\leq k}[R_{ii}^{(k)}(\beta_2)]\gamma_1 \hat{R}_k(\beta_2)) \\ &= \frac{\kappa}{2} \iint_{\mathbb{R}^2} f^{(1)}(\omega(\beta_1), x, y) f^{(1)}(\omega(\beta_2), x, y)(\nu \otimes \nu)(x, y) + o_{\mathbb{P}}(1).\end{aligned}$$

7.3.5. *Conclusion.* We obtained that for any $z_1, z_2 \in (\mathbb{C}\setminus\mathbb{R})^2$, $\gamma_N(z_1, z_2)$, defined in (42), converges in probability towards $\gamma(z_1, z_2)$. As already observed in (43),

$$\begin{aligned}\gamma_N(z_1, z_2) = \sum_{k=1}^{N} \Big\{ & \tilde{\sigma}_N^2 \operatorname{Tr}(\gamma_1 \hat{R}_k(\beta_1)) \operatorname{Tr}(\gamma_1 \hat{R}_k(\beta_2)) \\ & + \mathbb{E}_k \Big[\mathbb{E}_{\leq k}\big[\operatorname{Tr}(\Phi_k(\beta_1))\hat{R}_k(\beta_1))\big] \mathbb{E}_{\leq k}\big[\operatorname{Tr}(\Phi_k(\beta_2))\hat{R}_k(\beta_2))\big]\Big]\Big\}.\end{aligned}$$

Let C be as in Proposition 78. If $\|W_N\| \leq C$, then there exists $M > 0$ such that for any $k \in \{1, \ldots, N\}$, $\|P(W_N^{(k)}, D_N^{(k)})\| \leq M$. Let $K > 0$ be such that $\|P(x_N, D_N)\| \leq K$. Set $\tilde{\gamma}_N(z_1, z_2) = \gamma_N(z_1, z_2)\mathbf{1}_{\{\|W_N\|\leq C\}}$. Fix $z_1 \in \mathbb{C}\setminus\mathbb{R}$ and set $g_N^{(z_1)}(z) = \tilde{\gamma}_N(z_1, z)$, $g^{(z_1)}(z) = \gamma(z_1, z)$. Fix an arbitrary subsequence extracted from $(g_N^{(z_1)})_{N\geq 1}$. By diagonal extraction from the convergence in probability above, given a countable subset of $\mathbb{C}\setminus\mathbb{R}$, one can extract a subsubsequence, let us say $(g_{\Psi(N)}^{(z_1)})_{N\geq 1}$, almost surely converging to $g^{(z_1)}$ pointwise on this subset. Cauchy-Schwarz inequality (with respect to $\mathbb{E}_{\leq k}$ and then to the sum over k), (17), Lemmas 5 and 69 readily yield that $(g_N^{(z_1)})$ is a bounded sequence in $\mathcal{H}(\mathbb{C}\setminus[-\max(M, K); \max(M, K)])$. We conclude by Vitali's Theorem that almost surely $(g_{\Psi(N)}^{(z_1)})$ converges, uniformly on any compact set of $\mathbb{C}\setminus[-\max(M, K); \max(M, K)]$, towards an holomorphic function on $\mathbb{C}\setminus[-\max(M, K); \max(M, K)]$.
Note that $g^{(z_1)}$ is holomorphic on $\mathbb{C}\setminus\mathbb{R}$. Hence almost surely, $g_{\Psi(N)}^{(z_1)}$ converges, uniformly on any compact set of $\mathbb{C}\setminus[-\max(M, K); \max(M, K)]$, towards $g^{(z_1)}$. This implies that almost surely, $\frac{d}{dz} g_{\Psi(N)}^{(z_1)}$ converges, uniformly on any compact set of $\mathbb{C}\setminus[-\max(M, K); \max(M, K)]$, towards $\frac{d}{dz} g^{(z_1)}$. Thus, we obtain that for any $z_1, z_2 \in (\mathbb{C}\setminus\mathbb{R})^2$, $\frac{\partial}{\partial z_2}\tilde{\gamma}_N(z_1, z_2)$ converges in probability towards $\frac{\partial}{\partial z_2}\gamma(z_1, z_2)$.
Now, fix $z_2 \in \mathbb{C}\setminus\mathbb{R}$ and set $h_N^{(z_2)}(z) = \frac{\partial}{\partial z_2}\tilde{\gamma}_N(z, z_2)$, $h^{(z_2)}(z) = \frac{\partial}{\partial z_2}\gamma(z, z_2)$. The same procedure applied to $h_N^{(z_2)}$ as the one used for $g_N^{(z_1)}$ above yields that for any $z_1, z_2 \in (\mathbb{C}\setminus\mathbb{R})^2$, $\frac{\partial^2}{\partial z_1 \partial z_2}\tilde{\gamma}_N(z_1, z_2)$ converges in probability towards $\frac{\partial^2}{\partial z_1 \partial z_2}\gamma(z_1, z_2)$.
Finally, $\gamma_N(z_1, z_2)$ is bounded in L^2 so that by Lemma 78, $\gamma_N(z_1, z_2)\mathbf{1}_{\{\|W_N\|>C\}} = o_{\mathbb{P}}(1)$. Proposition 46 follows.

8. Tightness of $\{\xi_N(z), z \in \mathbb{C}\setminus\mathbb{R}\}_{N\in\mathbb{N}}$ in $\mathcal{H}(\mathbb{C}\setminus\mathbb{R})$ and conclusion

For each $N \in \mathbb{N}$, $\xi_N : z \mapsto \operatorname{Tr}((zI_N - P(W_N, D_N)^{-1}) - \mathbb{E}[\operatorname{Tr}((zI_N - P(W_N, D_N))^{-1})]$ is a random analytic function on $\mathbb{C}\setminus\mathbb{R}$. Let K be a compact set in $\mathbb{C}\setminus\mathbb{R}$. According to Lemma 76, there exists $\delta > 0$

such that $\overline{K_\delta} \subset \mathbb{C} \setminus \mathbb{R}$ and for any $r > 0$,

$$\|\xi_N\|_K^r \leq (\pi\delta^2)^{-1} \int_{\overline{K_\delta}} |\xi_N(z)|^r \, m(dz).$$

Therefore

$$\mathbb{E}\left(\|\xi_N\|_K^r\right) \leq (\pi\delta^2)^{-1} \int_{\overline{K_\delta}} \mathbb{E}\left(|\xi_N(z)|^r\right) m(dz) \tag{62}$$

$$\leq (\pi\delta^2)^{-1} \sup_{z \in \overline{K_\delta}} \mathbb{E}\left(|\xi_N(z)|^r\right) m(\overline{K_\delta}). \tag{63}$$

In order to prove the tightness of $\{\xi_N\}_{N\in\mathbb{N}}$ in $\mathcal{H}(\mathbb{C}\setminus\mathbb{R})$, using Proposition 75, (72) and (63), it is sufficient to prove that

$$\mathbb{E}\left(|\xi_N(z)|\right) = O(1). \tag{64}$$

This will readily follows from the following

Proposition 59.

$$\operatorname{Var}[\operatorname{Tr}(R_N(z))] = O(1).$$

Proof. From the decomposition (40), apply Lemma 70 to the martingale $(\mathbb{E}_{\leq k}[\operatorname{Tr}(zI_N - X_N)^{-1}])_{k\geq 1}$ and deduce that

$$\operatorname{Var}[\operatorname{Tr}(zI_N - X_N)^{-1}] = \sum_{k=1}^{N} \mathbb{E}\Big[\big|(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[\operatorname{Tr}(zI_N - X_N)^{-1}]\big|^2\Big].$$

Recall from the preceding section that, setting $\beta := ze_{11} - \gamma_0$,

$$\begin{aligned}
&\operatorname{Tr}(zI_N - X_N)^{-1} \\
&\quad= (\operatorname{Tr}\otimes\operatorname{Tr})((e_{11}\otimes I_{N-1})R^{(k)}(\beta)) \\
&\qquad+ \operatorname{Tr}((e_{11} + \gamma_1\otimes C_k^{(k)*}R^{(k,1)}(\beta)\gamma_1\otimes C_k^{(k)})(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)})^{-1}).
\end{aligned}$$

In the second term of the right-hand side, decompose

$$e_{11} + \gamma_1\otimes C_k^{(k)*}R^{(k,1)}(\beta)\gamma_1\otimes C_k^{(k)} = (e_{11} + \gamma_1(\mathrm{id}_m\otimes\sigma_N^2\operatorname{Tr})(R^{(k,1)}(\beta))\gamma_1) - \frac{\partial}{\partial z}\Phi_k(\beta)$$

and

$$\begin{aligned}
&(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)})^{-1} \\
&\qquad= \hat{R}_k(\beta) + \hat{R}_k(\beta)(W_{kk}\gamma_1 + \Psi_k(\beta))(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)})^{-1},
\end{aligned}$$

so that

$$\begin{aligned}
\operatorname{Tr}(zI_N - X_N)^{-1} &= (\operatorname{Tr}\otimes\operatorname{Tr})\big((e_{11}\otimes I_{N-1})R^{(k)}(\beta)\big) \\
&\quad+ \operatorname{Tr}\Big((e_{11} + \gamma_1(\mathrm{id}_m\otimes\sigma_N^2\operatorname{Tr})(R^{(k,1)}(\beta))\gamma_1)\hat{R}_k(\beta)\Big) - \operatorname{Tr}\Big(\frac{\partial}{\partial z}\Phi_k(\beta)\hat{R}_k(\beta)\Big) \\
&\quad+ \operatorname{Tr}\Big((e_{11} + \gamma_1\otimes C_k^{(k)*}R^{(k,1)}(\beta)\gamma_1\otimes C_k^{(k)})\hat{R}_k(\beta) \\
&\qquad\times (W_{kk}\gamma_1 + \Psi_k(\beta))(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1\otimes C_k^{(k)*}R^{(k)}(\beta)\gamma_1\otimes C_k^{(k)})^{-1}\Big).
\end{aligned}$$

Observe that the first two terms satisfy

$$(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[(\operatorname{Tr}\otimes\operatorname{Tr})((e_{11}\otimes I_{N-1})R^{(k)}(\beta)) + \operatorname{Tr}((e_{11} + \gamma_1(\mathrm{id}_m\otimes\sigma_N^2\operatorname{Tr})(R^{(k,1)}(\beta))\gamma_1)\hat{R}_k(\beta))] = 0$$

and denote by T_1 and T_2 the last two terms. Using Jensen's inequality (with respect to $\mathbb{E}_{\leq k}$) after writing $\mathbb{E}_{\leq k-1} = \mathbb{E}_{\leq k}\mathbb{E}_k$,

$$\begin{aligned}
\mathbb{E}[|(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[\operatorname{Tr}(zI_N - X_N)^{-1}]|^2] &= \mathbb{E}[|(\mathbb{E}_{\leq k} - \mathbb{E}_{\leq k-1})[T_1 + T_2]|^2] \\
&\leq \mathbb{E}[|T_1 + T_2 - \mathbb{E}_k[T_1 + T_2]|^2] \\
&\leq \mathbb{E}[\mathbb{E}_k[|T_1 + T_2 - \mathbb{E}_k[T_1 + T_2]|^2] \\
&\leq \mathbb{E}[\mathbb{E}_k[|T_1 + T_2|^2]] \\
&\leq 2(\mathbb{E}[\mathbb{E}_k[|T_1|^2]] + \mathbb{E}[\mathbb{E}_k[|T_2|^2]]).
\end{aligned}$$

Bound on $\mathbb{E}[\mathbb{E}_k[|T_1|^2]]$:

$$|T_1| \leq m\|\frac{\partial}{\partial z}\Phi_k(\beta)\|\|\hat{R}_k(\beta)\|$$

and deduce from Lemma 43 and (15) that

$$\mathbb{E}[|T_1|^2] = O(N^{-1}). \tag{65}$$

Bound on $\mathbb{E}[\mathbb{E}_k[|T_2|^2]]$: by traciality and using (15),

$$\begin{aligned}
|T_2| = \Big| \operatorname{Tr}\Big(&(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*} R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)})^{-1} \\
&\times (e_{11} + \gamma_1 \otimes C_k^{(k)*} R^{(k,1)}(\beta)\gamma_1 \otimes C_k^{(k)})\hat{R}_k(\beta)(W_{kk}\gamma_1 + \Psi_k(\beta))\Big)\Big| \\
\leq m\big\|&(\beta - W_{kk}\gamma_1 - D_{kk}\gamma_2 - \gamma_1 \otimes C_k^{(k)*} R^{(k)}(\beta)\gamma_1 \otimes C_k^{(k)})^{-1} \\
&\times (e_{11} + \gamma_1 \otimes C_k^{(k)*} R^{(k,1)}(\beta)\gamma_1 \otimes C_k^{(k)})\big\|\|\hat{R}_k(\beta)\|\|W_{kk}\gamma_1 + \Psi_k(\beta)\|
\end{aligned}$$

Then,

$$|T_2|^2 \leq 4m^2\|\hat{R}_k(\beta)\|^2\|R_N(\beta)\|^2(1 + \|\gamma_1\|^4\|C_k^{(k)}\|^4\|R^{(k)}(\beta)\|^4)(|W_{kk}|^2\|\gamma_1\|^2 + \|\Psi_k(\beta)\|^2),$$

and consequently, by Hölder's inequality with $q \in [1, 2(1+\varepsilon))]$ and $p, r \geq 1$ such that $p^{-1}+q^{-1}+r^{-1} = 1$.

$$\begin{aligned}
\mathbb{E}[|T_2|^2] &\leq O(1)\mathbb{E}\big[\|R_N(\beta)\|^{2p}\big]^{1/p}\mathbb{E}\big[(1 + \|\gamma_1\|^4\|C_k^{(k)}\|^4\|R^{(k)}(\beta)\|^4)^r\big]^{1/r} \\
&\qquad \times \mathbb{E}\big[(|W_{kk}|^2\|\gamma_1\|^2 + \|\Psi_k(\beta)\|^2)^q\big]^{1/q} \\
&\leq O(1)\Big(1 + \|\gamma_1\|^4\mathbb{E}\big[\|C_k^{(k)}\|^{8r}\big]^{1/2r}\mathbb{E}\big[\|R^{(k)}(\beta)\|^{8r}\big]^{1/2r}\Big) \\
&\qquad \times \Big(\|\gamma_1\|^2\mathbb{E}\big[|W_{kk}|^{2q}\big]^{1/q} + \mathbb{E}\big[\|\Psi_k(\beta)\|^{2q}\big]^{1/q}\Big) \\
&\leq O(1)(\|\gamma_1\|^2N^{-1} + O(N^{-1})) = O(N^{-1}),
\end{aligned}$$

because of Lemma 21, Remark 22, Lemma 44 and assumptions on entries of W_N. □

Conclusion. It follows from the discussion above that $\{\xi_N\}_{N\in\mathbb{N}}$ is tight in $\mathcal{H}(\mathbb{C}\setminus\mathbb{R})$. Then, according to Theorem 5.1 in [Bil99], it is relatively compact. According to Section 7, the finite dimensional distributions converge towards those of the Gaussian process $\mathcal{G}$ defined in Theorem 2. Since the class of finite dimensional sets $\big\{\{f \in \mathcal{H}(\mathbb{C}\setminus\mathbb{R}), (f(z_1), \dots, f(z_k)) \in B\}, k \in \mathbb{N}, (z_1, \dots, z_k) \in (\mathbb{C}\setminus\mathbb{R})^k, B \in \mathcal{B}(\mathbb{C}^k)\big\}$ is a separating class, we can deduce Theorem 2 by Theorem 2.6 in [Bil99].

Appendix A. Tools

A.1. Linear algebra.

Proposition 60 (Schur inversion formula)**.** *Let $\mathcal{A}$ be a unital complex algebra. For non-empty subsets I, J of $\{1, \dots, n\}$ and $A \in M_n(\mathcal{A})$, we denote by $A_{I\times J}$ the submatrix of A corresponding to rows indexed by I and columns indexed by J. In the particular case where $I = J$, we will use the notation A_I.*
Let I be a non-empty subset of $\{1, \dots, n\}$ and $A \in M_n(\mathcal{A})$ such that A_I is invertible, then A is invertible if and only if $A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c}$ is invertible, in which case the following formulas hold:

$$\begin{aligned}
(A^{-1})_I &= (A_I)^{-1} + (A_I)^{-1}A_{I\times I^c}(A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c})^{-1}A_{I^c\times I}(A_I)^{-1}, \\
(A^{-1})_{I\times I^c} &= -(A_I)^{-1}A_{I\times I^c}(A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c})^{-1}, \\
(A^{-1})_{I^c\times I} &= -(A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c})^{-1}A_{I^c\times I}(A_I)^{-1}, \\
(A^{-1})_{I^c} &= (A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c})^{-1}.
\end{aligned}$$

Proposition 61. *(Theorem 2.9 in* [AS08]*) The transpose map on $M_N(\mathbb{C})$ induces the well-defined linear map Θ on $M_m(\mathbb{C}) \otimes M_N(\mathbb{C})$, called the partial transpose map: for $X = \sum_k A_k \otimes B_k$,*

$$\Theta(X) := \sum_k A_k \otimes B_k^T.$$

For every unitarily invariant norm $\|\cdot\|$,

$$\|\Theta(X)\| \leq \min(m, N)\,\|X\|.$$

Lemma 62. *Assume that an operator A is invertible and $\|A^{-1}\| \leq K$. Then if $\|\Delta\| \leq (2K)^{-1}$, $A+\Delta$ is invertible and $\|(A+\Delta)^{-1}\| \leq 2K$.*

Lemma 63. *Let A be in $M_n(\mathbb{C})$ with spectral radius $\rho(A)$. Then for any $z \in \mathbb{C} \setminus spect(A)$, we have*

$$\|(z-A)^{-1}\| \leq \sum_{p=1}^{n} (d(z, spect(A))^{-p} \left[\|A\| + \rho(A)\right]^{p-1}.$$

Proof. Using Schur decomposition, $A = P(D+\mathcal{N})P^*$, where P is a unitary matrix, D is a diagonal with the same spectrum as A and $\mathcal{N}$ is a strictly upper triangular matrix. Note that for any $z \in \mathbb{C} \setminus \mathrm{spect}(A)$, $(zI_n - D)^{-1}\mathcal{N}$ is a nilpotent matrix so that $[(zI_n - D)^{-1}\mathcal{N}]^n = 0$

$$(zI_n - A)^{-1} = P\Big(\sum_{p=0}^{n-1} [(zI_n - D)^{-1}\mathcal{N}]^p\Big)(zI_n - D)^{-1}P^*$$

Hence

$$\begin{aligned}\|(zI_n - A)^{-1}\| &\leq \sum_{p=1}^{n} \|(zI_n - D)^{-1}\|^p \|\mathcal{N}\|^{p-1} \\ &\leq \sum_{p=1}^{n} (d(z, \mathrm{spect}(A))^{-p} \left[\|A\| + \rho(A)\right]^{p-1}.\end{aligned}$$

where we use $\|\mathcal{N}\| \leq \|P^*AP\| + \|D\| \leq \|A\| + \rho(A)$ in the last line. □

Lemma 64. *(Lemma 8.1 [BC17]) For any matrix $M = \sum_{i,j=1}^n M_{ij} \otimes E_{ij} \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ and for any fixed k,*

$$\sum_{i=1}^{n} \|M_{ik}\|^2 \leq m\|M\|^2; \quad \sum_{j=1}^{n} \|M_{kj}\|^2 \leq m\|M\|^2. \tag{66}$$

Hence,

$$\sum_{i,j=1}^{n} \|M_{ij}\|^2 \leq mn\|M\|^2 \tag{67}$$

Lemma 65. *Let $m \in \mathbb{N}$, $1 \leq k \leq n$ and $A = \sum_{i,j=1}^n A_{ij} \otimes E_{ij}, B = \sum_{i,j=1}^n B_{ij} \otimes E_{ij} \in \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$. Define $C \in \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ by $C = \sum_{i,j=1}^n C_{ij} \otimes E_{ij}$ where $C_{ij} = \sum_{l<k} A_{il} \otimes B_{lj}$, for $1 \leq i,j \leq n$. Then $\|C\| \leq \|A\|\|B\|$. In particular,*

$$\|C_{ij}\| \leq \|A\|\|B\|, \quad 1 \leq i,j \leq n;$$

$$\Big(\sum_{i=1}^{n} \|C_{ij}\|^2\Big)^{1/2} \leq \|A\|\|B\|, \quad 1 \leq j \leq n.$$

Proof. Let $\tilde{A} := \sum_{i,l_1=1}^n A_{il_1} \otimes I_m \otimes E_{il_1}$, $P = \sum_{l<k} I_m \otimes I_m \otimes E_{ll}$ and $\tilde{B} := \sum_{l_2,j=1}^n I_m \otimes B_{l_2j} \otimes E_{l_2j}$. Then

$$\begin{aligned}\tilde{A}P\tilde{B} &= \sum_{i,l_1,l<k,l_2,j} A_{il_1} \otimes B_{l_2j} \otimes E_{il_1}E_{ll}E_{l_2j} \\ &= \sum_{i,j=1}^{n} \sum_{l<k} A_{il} \otimes B_{lj} \otimes E_{ij} \\ &= \sum_{i,j=1}^{n} C_{ij} \otimes E_{ij} \\ &= C.\end{aligned}$$

Hence $\|C\| \leq \|\tilde{A}\|\|P\|\|\tilde{B}\|$. Observe that

$$\|\tilde{B}\| = \|I_m \otimes B\| = \|B\|,$$

$$\|P\| = \|I_m \otimes I_m \otimes \sum_{l<k} E_{ll}\| = \|\sum_{l<k} E_{ll}\| = 1$$

and

$$\|\tilde{A}\| = \|I_m \otimes \sum_{i,l_1=1}^{n} E_{il_1} \otimes A_{il_1}\| = \| \sum_{i,l_1=1}^{n} E_{il_1} \otimes A_{il_1}\| = \|A\|.$$

□

Lemma 66. *If $\varphi : \mathbb{R} \to \mathbb{C}$ is Lipschitz continuous, then, for $n\times n$ Hermitian matrices M_1, M_2 and $p \geq 1$,*

$$|\operatorname{Tr}(\varphi(M_1)) - \operatorname{Tr}(\varphi(M_2))|^p \leq \|\varphi\|^p_{Lip} n^{p-1} \|M_1 - M_2\|^p_{S^p},$$

where $\|A\|_{S^p} = \left(\sum_{\lambda \in sp(A)} |\lambda|^p\right)^{1/p}$ is the Schatten p-norm of the normal matrix A.

Proof. Denote by $\lambda_1 \geq \cdots \geq \lambda_n$ the eigenvalues of M_1 and $\mu_1 \geq \cdots \geq \mu_n$ the eigenvalues of M_2. Then,

$$|\operatorname{Tr}(\varphi(M_1)) - \operatorname{Tr}(\varphi(M_2))| \leq \sum_{i=1}^{n} |\varphi(\lambda_i) - \varphi(\mu_i)| \leq \|\varphi\|_{\mathrm{Lip}} \sum_{i=1}^{n} |\lambda_i - \mu_i|.$$

Using Hölder's and Hoffman-Wielandt inequalities,

$$|\operatorname{Tr}(\varphi(M_1)) - \operatorname{Tr}(\varphi(M_2))|^p \leq \|\varphi\|^p_{\mathrm{Lip}} n^{p-1} \sum_{i=1}^{n} |\lambda_i - \mu_i|^p \leq \|\varphi\|^p_{\mathrm{Lip}} n^{p-1} \|M_1 - M_2\|^p_{S^p}.$$

□

Theorem 67. *(*[EH78]*Theorem 2.5) Let Φ be a positive linear map on a finite-dimensional C^*-algebra $\mathcal{A}$. If ρ is the spectral radius of Φ, there is a non-zero positive element z in $\mathcal{A}$ such that $\Phi(z) = \rho z$.*

Lemma 68. *Let Φ and Ψ be positive linear maps on $M_m(\mathbb{C})$ such that $\Phi \leq \Psi$. Then their spectral radii satisfy $\rho(\Phi) \leq \rho(\Psi)$.*

Proof. The proof we give here follows the proof of Theorem 2.5 in [EH78]. One may assume without loss of generality that Φ and Ψ are irreducible. Indeed, given χ a fixed irreducible positive linear map on $M_m(\mathbb{C})$, $\Phi_n = \Phi + n^{-1}\chi$ and $\Psi_n = \Psi + n^{-1}\chi$ are irreducible positive linear maps on $M_m(\mathbb{C})$ such that $\Phi_n \leq \Psi_n$ and converging respectively to Φ and Ψ in norm. If the result holds for Φ_n and Ψ_n, letting n tend to $+\infty$ in $\rho(\Phi_n) \leq \rho(\Psi_n)$ gives the conclusion by continuity of the spectral radius in finite dimension. Assume then that Φ and Ψ are irreducible. According to Theorem 2.4 in [EH78] and sentences below this Theorem 2.4 in [EH78], the spectral radius of irreducible positive linear maps χ on a finite dimensional C^*-algebra satisfy

$$\rho(\chi) = \max_{y\geq 0} \inf\{\alpha \in \mathbb{R}, \alpha y \geq \chi(y)\}. \tag{68}$$

We have by assumption that for any $y \geq 0$, $\Psi(y) \geq \Phi(y)$. Thus, (68) readily implies that $\rho(\Psi) \geq \rho(\Phi)$. □

A.2. **Concentration bounds for quadratic forms.** One can easily deduce the following result from Lemma 2.7 in [BS98]:

Lemma 69. *Let $m, n \in \mathbb{N}$, $\gamma \in \mathcal{M}_m(\mathbb{C})$ and $A = \sum_{i,j=1}^{m} e_{ij} \otimes A_{ij} \in \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$. Let $p \geq 2$ and $Y = (Y_1, \ldots, Y_n)$ be a n-tuple of independent identically distributed standard complex random variables with finite $2p$-th moment, then*

$$\begin{aligned}&\mathbb{E}\big[\|\gamma \otimes Y^* A \gamma \otimes Y - \gamma \mathrm{id}_m \otimes \operatorname{Tr}(A)\gamma\|^p\big] \\ &\qquad \leq K_{m,p}\|\gamma\|^{2p}\Big(\big(\mathbb{E}\big[|Y_1|^4\big] \max_{i,j} \operatorname{Tr}(A_{ij}A_{ij}^*)\big)^{p/2} + \mathbb{E}\big[|Y_1|^{2p}\big] \max_{i,j} \operatorname{Tr}((A_{ij}A_{ij}^*)^{p/2})\Big).\end{aligned}$$

A.3. **Martingales.** The proofs of our variance bounds and of our CLT rely on martingale theory.

Lemma 70. *Let $(M_k)_{k\in\mathbb{N}}$ be a martingale with values in $\mathbb{C}$ and satisfying $\mathbb{E}[|M_k|^2] < +\infty$, $k \in \mathbb{N}$. Then*

$$\mathbb{E}[|\sum_{k=1}^{N}(M_k - M_{k-1})|^2] = \sum_{k=1}^{N} \mathbb{E}[|(M_k - M_{k-1})|^2], \quad N \in \mathbb{N}.$$

Lemma 71. *Let $(M_k)_{k\in\mathbb{N}}$ be a $M_n(\mathbb{C})$-valued martingale and $p \in \mathbb{N}$ be an even integer. Then*

$$\mathbb{E}[\|M_N - M_0\|^p] \leq n^{p/2} p! \binom{N+p-2}{p-1} \max_{k=1,\ldots,N} \mathbb{E}[\|M_k - M_{k-1}\|^p], \quad N \in \mathbb{N}.$$

Proof. We assume that $\max_{k=1,\dots,N} \mathbb{E}[\|M_k - M_{k-1}\|^p] < +\infty$. Recall that $\|\cdot\| \le \|\cdot\|_{HS} \le n^{1/2}\|\cdot\|$. Observe that

$$\begin{aligned}\|M_N - M_0\|_{HS}^p &= \|\sum_{k=1}^N (M_k - M_{k-1})\|_{HS}^p \\ &= \sum_{i:\{1,\dots,p\}\to\{1,\dots,N\}} \langle M_{i_1} - M_{i_1-1}, M_{i_2} - M_{i_2-1}\rangle \cdots \langle M_{i_{p-1}} - M_{i_{p-1}-1}, M_{i_p} - M_{i_p-1}\rangle.\end{aligned}$$

Note that, using Hölder's inequality,

$$\begin{aligned}\mathbb{E}[|\langle M_{i_1} - M_{i_1-1}, M_{i_2} - M_{i_2-1}\rangle \cdots \langle M_{i_{p-1}} - M_{i_{p-1}-1}, M_{i_p} - M_{i_p-1}\rangle|] \\ \le \mathbb{E}[\|M_{i_1} - M_{i_1-1}\|_{HS} \cdots \|M_{i_p} - M_{i_p-1}\|_{HS}] \\ \le \max_{k=1,\dots,N} \mathbb{E}[\|M_k - M_{k-1}\|_{HS}^p] \\ \le n^{p/2} \max_{k=1,\dots,N} \mathbb{E}[\|M_k - M_{k-1}\|^p].\end{aligned}$$

It follows that

$$\mathbb{E}[\|M_N - M_0\|_{HS}^p] = \sum_{i:\{1,\dots,p\}\to\{1,\dots,N\}} \mathbb{E}[\langle M_{i_1} - M_{i_1-1}, M_{i_2} - M_{i_2-1}\rangle \cdots \langle M_{i_{p-1}} - M_{i_{p-1}-1}, M_{i_p} - M_{i_p-1}\rangle].$$

Consider a term indexed by i such that $i^{-1}(\max i)$ is a singleton. Then

$$\begin{aligned}\mathbb{E}[\langle M_{i_1} &- M_{i_1-1}, M_{i_2} - M_{i_2-1}\rangle \cdots \langle M_{i_{p-1}} - M_{i_{p-1}-1}, M_{i_p} - M_{i_p-1}\rangle] \\ &= \mathbb{E}[\mathbb{E}[\langle M_{i_1} - M_{i_1-1}, M_{i_2} - M_{i_2-1}\rangle \cdots \langle M_{i_{p-1}} - M_{i_{p-1}-1}, M_{i_p} - M_{i_p-1}\rangle | \mathcal{F}_{\max i-1}]] \\ &= \mathbb{E}\Big[\langle M_{i_1} - M_{i_1-1}, M_{i_2} - M_{i_2-1}\rangle \cdots \langle \mathbb{E}[M_{\max i} - M_{\max i-1} | \mathcal{F}_{\max i-1}], *\rangle \\ &\qquad \times \cdots \langle M_{i_{p-1}} - M_{i_{p-1}-1}, M_{i_p} - M_{i_p-1}\rangle\Big] \\ &= 0.\end{aligned}$$

There are at most $p!\binom{N+p-2}{p-1}$ choices of indices i such that $i^{-1}(\max i)$ is not a singleton. Indeed, for a map $i : \{1,\dots,p\} \to \{1,\dots,N\}$, there are $p!$ ways to rank $i_1,\dots,i_p$ in increasing order. Now, since $i^{-1}(\max i)$ is not a singleton, we know that at least the two last values of the increasing sequence are equal; since there are $\binom{N+p-2}{p-1}$ choices of increasing sequence of $p-1$ numbers in $\{1,\dots,N\}$, the result follows. $\square$

The following result may be deduced from its real-valued analogue (Theorem 35.12 in [Bil95]).

Theorem 72. *Suppose that, for all $N \ge 1$, $(M_k^{(N)})_{k\in\mathbb{N}}$ is a square integrable complex martingale and define, for $k \ge 1$, $\Delta_k^{(N)} := M_k^{(N)} - M_{k-1}^{(N)}$. If*

$$\forall \varepsilon > 0,\ L(\varepsilon, N) := \sum_{k=1}^N \mathbb{E}[|\Delta_k^{(N)}|^2 \mathbf{1}_{|\Delta_k^{(N)}|\ge\varepsilon}] \underset{N\to+\infty}{\longrightarrow} 0, \tag{69}$$

$$V_N := \sum_{k=1}^N \mathbb{E}_{\le k-1}[|\Delta_k^{(N)}|^2] \underset{N\to+\infty}{\longrightarrow} v \ge 0, \tag{70}$$

and

$$W_N := \sum_{k=1}^N \mathbb{E}_{\le k-1}[(\Delta_k^{(N)})^2] \underset{N\to+\infty}{\longrightarrow} w \in \mathbb{C} \tag{71}$$

(convergences in (70) *and* (71) *have to be understood in probability), then*

$$\sum_{k=1}^N \Delta_k^{(N)} \Rightarrow_{N\to+\infty} \mathcal{N}_{\mathbb{C}}(0, v, w).$$

A.4. **Complex analysis.**

Theorem 73. *[Vitali, see* [Sch05] *Exercise 1.4.37] Let $D \subset \mathbb{C}^l$ be a domain, $A \subset D$ a set of uniqueness (for instance an open set) and $(f_n)_{n\in\mathbb{N}}$ a bounded sequence in the set $\mathcal{H}(D)$ of holomorphic functions on D (that is $\sup_{n\in\mathbb{N}} \sup_{x\in K} \|f_n(x)\| < +\infty$ for any compact subset $K \subset D$), which converges pointwise on A. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges towards a holomorphic function $f \in \mathcal{H}(D)$.*

Theorem 74. *[Earle-Hamilton] Let D be a nonempty domain in a complex Banach space X and let $f : D \mapsto D$ be a bounded holomorphic function. If $f(D)$ lies strictly inside D, then f is a strict contraction in the Carathéodory-Riffen-Finsler metric and thus has a unique fixed point in D. Furthermore, $(f^n(x_0))_{n\in\mathbb{N}}$ converges in norm, for any $x_0 \in D$, to this fixed point.*

We recall here a criterion of tightness for random analytic functions from [Shi12]. Let $D \subset \mathbb{C}$ be an open set of the complex plane. Denote by $\mathcal{H}(D)$ the space of complex analytic functions on D, endowed with the uniform topology on compact sets. For $f \in \mathcal{H}(D)$ and K a compact set of D, we denote $\|f\|_K = \sup_{z\in K} |f(z)|$. The space $\mathcal{H}(D)$ is equipped with the (topological) Borel σ-field $\mathcal{B}(\mathcal{H}(D))$ and the set of probability measures on $(\mathcal{H}(D); \mathcal{B}(\mathcal{H}(D)))$ is denoted by $\mathcal{P}(\mathcal{H}(D))$. By a random analytic function on D we mean an $\mathcal{H}(D)$-valued random variable on a probability space.

Proposition 75. *(Proposition 2.5. in* [Shi12]*) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of random analytic functions on D. If $\{\|f_n\|_K\}_{n\in\mathbb{N}}$ is tight for any compact set K, then $\{\mathcal{L}(f_n)\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{H}(D))$.*

Using that, by Markov's inequality, for any $C > 0$ and any $r > 0$,

$$\mathbb{P}(\|f_n\|_K > C) \le \frac{1}{C^r}\mathbb{E}(\|f_n\|_K^r), \tag{72}$$

the following lemma turns out to be useful to prove tightness results.

Lemma 76. *(lemma 2.6* [Shi12]*) For any compact set K in D, there exists $\delta > 0$ such that*

$$\|f\|_K^r \le (\pi\delta^2)^{-1} \int_{\overline{K_\delta}} |f(z)|^r \, m(dz), \ f \in \mathcal{H}(D),$$

for any $r > 0$, where $\overline{K_\delta} \subset D$ is the closure of the δ-neighborhood of K and m denotes the Lebesgue measure.

Appendix B. Norm of Wigner matrices

Lemma 77. *Let, for each $N \in \mathbb{N}$, W_N be a $N \times N$ Hermitian matrix such that entries $\{W_{ij}\}_{1\le i\le j\le N}$ are random variables bounded by δ and such that for $i \neq j$, for some $\varepsilon > 0$, $\mathbb{E}[|\sqrt{N}W_{ij}|^{6(1+\varepsilon)}] \le C_6$, and $\mathbb{E}[|\sqrt{N}W_{ii}|^{4(1+\varepsilon)}] \le C_4$. Then*

$$\forall p \in [2; 4(1+\varepsilon)],\ \mathbb{E}[|W_{ij}|^{2p}] = O(N^{-p/2-1}) \text{ and } \mathbb{E}[|W_{ii}|^p] = O(N^{-p/2}).$$

Proof. By Jensen's inequality, for $p \in [2; 4(1+\varepsilon)]$,

$$\mathbb{E}[|W_{ii}|^p] \le \mathbb{E}[|W_{ii}|^{4(1+\varepsilon)}]^{p/4(1+\varepsilon)} \le C_4^{p/4(1+\varepsilon)} N^{-p/2} = O(N^{-p/2}).$$

Similarly, by Jensen's inequality, for $p \in [2; 3(1+\varepsilon)[$,

$$\mathbb{E}[|W_{ij}|^{2p}] \le \mathbb{E}[|W_{ij}|^{6(1+\varepsilon)}]^{p/3(1+\varepsilon)} \le C_6^{p/3(1+\varepsilon)} N^{-p} = O(N^{-p}) = O(N^{-p/2-1}),$$

the last equality following from the fact that $p \ge 1 + p/2$ when $p \ge 2$.
Now, for $p \in [3(1+\varepsilon); 4(1+\varepsilon)]$,

$$\mathbb{E}[|W_{ij}|^{2p}] \le \delta^{2p-6(1+\varepsilon)}\mathbb{E}[|W_{ij}|^{6(1+\varepsilon)}] \le \delta^{2p-6(1+\varepsilon)} C_6 N^{-3(1+\varepsilon)} = O(N^{-3(1+\varepsilon)}) = O(N^{-p/2-1}),$$

the last equality following from the fact that $1 + p/2 \le 3(1+\varepsilon)$ when $p \le 4(1+\varepsilon)$. ☐

Proposition 78. *There exists $C > 0$ such that for every $p \ge 1$, $\mathbb{P}(\|W_N\| > C) = o(N^{-p})$. In particular, the sequence of random variables $(\|W_N\|)_{N\ge 1}$ is bounded in every L^p, $p \ge 1$.*

Proof. By assumption, entries of W_N satisfy

$$\mathbb{E}[\sqrt{N}W_{ij}] = 0, \quad \mathbb{E}[|\sqrt{N}W_{ij}|^2] \le \Sigma^2, \quad \mathbb{E}[|\sqrt{N}W_{ij}|^\ell] \le b(\delta_N\sqrt{N})^{\ell-3}, \ (\ell \ge 3).$$

For example, $\Sigma = \max(\sup_{N\in\mathbb{N}} \sqrt{N}\sigma_N, \sup_{N\in\mathbb{N}} \sqrt{N}\tilde{\sigma}_N)$ and $b = \max(C_6^{1/2(1+\varepsilon)}, C_4^{3/4(1+\varepsilon)})$. It then follows from Remark 5.7 in the book of Bai and Silverstein that $\mathbb{P}(\|W_N\| > C) = o(N^{-p})$ for any $C > 2\Sigma$ and any $p \geq 1$. Then, for such C, p and $N \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}[\|W_N\|^p] &= \mathbb{E}[\|W_N\|^p \mathbf{1}_{\|W_N\|\leq C}] + \mathbb{E}[\|W_N\|^p \mathbf{1}_{\|W_N\|>C}] \\ &\leq C^p + (N\delta_N)^p \mathbb{P}(\|W_N\| > C)\end{aligned}$$

is bounded uniformly in N. ☐

Appendix C. Truncation and centering

Fluctuations of the trace of the resolvent of X_N were studied under the hypothesis that entries of W_N are bounded by δ_N, for a sequence $(\delta_N)_{N\in\mathbb{N}}$ slowly converging to 0.

For any bounded continuous function $\varphi : \mathbb{R} \to \mathbb{C}$, let

$$\mathcal{N}_N(\varphi) := \mathrm{Tr}(\varphi(X_N)) = \sum_{\lambda\in \mathrm{sp}(X_N)} \varphi(\lambda).$$

In this section, we truncate and center the entries of W_N, in order to show that it is sufficient to study the fluctuations of $\mathcal{N}_N(\varphi)$ for matrices W_N with entries bounded by δ_N, where $(\delta_N)_{N\geq 1}$ is a sequence of positive numbers such that $\delta_N \underset{N\to+\infty}{\longrightarrow} 0$ at rate less than $N^{-\epsilon}$ for any $\epsilon > 0$.

Define $\hat{X}_N = P(\hat{W}_N, D_N)$ by

$$\hat{W}_{ij} := W_{ij}\mathbf{1}_{|W_{ij}|\leq\delta_N/2}, \quad 1 \leq i, j \leq N,$$

and accordingly

$$\hat{\mathcal{N}}_N(\varphi) := \mathrm{Tr}(\varphi(\hat{X}_N)).$$

By union bound,

$$\begin{aligned}\mathbb{P}(\hat{\mathcal{N}}_N(\varphi) \neq \mathcal{N}_N(\varphi)) &\leq \mathbb{P}(\hat{W}_N \neq W_N) \\ &\leq \sum_{1\leq i\leq j\leq N} \mathbb{P}(|W_{ij}| > \delta_N/2) \\ &\leq \sum_{1\leq i<j\leq N} (\delta_N/2)^{-6(1+\varepsilon)}\mathbb{E}[|W_{ij}|^{6(1+\varepsilon)}] + \sum_{1\leq i\leq N} (\delta_N/2)^{-4(1+\varepsilon)}\mathbb{E}[W_{ii}^{4(1+\varepsilon)}] \\ &\leq \sum_{1\leq i<j\leq N} (\delta_N/2)^{-6(1+\varepsilon)} \frac{C_6}{N^{3(1+\varepsilon)}} + \sum_{1\leq i\leq N} (\delta_N/2)^{-4(1+\varepsilon)} \frac{C_4}{N^{2(1+\varepsilon)}} \\ &\leq C_6(\delta_N/2)^{-6(1+\varepsilon)}N^{-1-3\varepsilon} + C_4(\delta_N/2)^{-4(1+\varepsilon)}N^{-1-2\varepsilon} = o(N^{-1}).\end{aligned}$$

Using the naive bound $|\hat{\mathcal{N}}_N(\varphi) - \mathcal{N}_N(\varphi)| \leq 2\|\varphi\|_\infty N$ yields:

$$\begin{aligned}\mathbb{E}[|\hat{\mathcal{N}}_N(\varphi) - \mathcal{N}_N(\varphi)|] &= \mathbb{E}[|\hat{\mathcal{N}}_N(\varphi) - \mathcal{N}_N(\varphi)|\mathbf{1}_{\hat{\mathcal{N}}_N(\varphi)\neq\mathcal{N}_N(\varphi)}] \\ &\leq 2\|\varphi\|_\infty N\mathbb{P}(\hat{\mathcal{N}}_N(\varphi) \neq \mathcal{N}_N(\varphi)) \\ &\leq o(1).\end{aligned}$$

Define then $\mathring{X}_N = P(\mathring{W}_N, D_N)$ by $\mathring{W}_N = \hat{W}_N - \mathbb{E}[\hat{W}_N]$ and accordingly

$$\mathring{\mathcal{N}}_N(\varphi) := \mathrm{Tr}(\varphi(\mathring{X}_N)).$$

Note that the entries of $\mathring{W}$ are independent, centred and bounded by δ_N. Furthermore, the off-diagonal entries are independent and identically distributed, as well as entries on the diagonal. Observe that, for $i \neq j$,

$$|\mathbb{E}[\hat{W}_{ij}]| = |\mathbb{E}[W_{ij}\mathbf{1}_{|W_{ij}|\leq\frac{\delta_N}{2}}]| = |\mathbb{E}[W_{ij}\mathbf{1}_{|W_{ij}|>\frac{\delta_N}{2}}]| = O(\delta_N^{-5-6\varepsilon}N^{-3(1+\varepsilon)}).$$

Furthermore,

$$\mathbb{E}[|\mathring{W}_{ij}|^2] = \mathbb{E}[|\hat{W}_{ij} - \mathbb{E}[\hat{W}_{ij}]|^2] = \mathbb{E}[|\hat{W}_{ij}|^2] + R_2,$$

with $|R_2| \leq 3|\mathbb{E}[\hat{W}_{ij}]|^2 = O(\delta_N^{-10-12\varepsilon}N^{-6(1+\varepsilon)})$. Moreover $\mathbb{E}[|\hat{W}_{ij}|^2] = \sigma_N^2 - \mathbb{E}[|W_{ij}|^2\mathbf{1}_{|W_{ij}|>\frac{\delta_N}{2}}]$, and $\mathbb{E}[|W_{ij}|^2\mathbf{1}_{|W_{ij}|>\frac{\delta_N}{2}}] = O(\delta_N^{-4-6\varepsilon}N^{-3(1+\varepsilon)})$. Therefore,

$$\mathbb{E}[|\mathring{W}_{ij}|^2] = \mathring{\sigma}_N^2$$

with $\mathring{\sigma}_N^2 = \sigma_N^2 + O(\delta_N^{-3-4\varepsilon} N^{-3(1+\varepsilon)})$. As a consequence, $N\mathring{\sigma}_N^2 \underset{N\to+\infty}{\longrightarrow} \sigma^2$.

We turn now to $\mathbb{E}[\mathring{W}_{ij}^2]$.

$$\mathbb{E}[\mathring{W}_{ij}^2] = \mathbb{E}[W_{ij}^2 \mathbf{1}_{|W_{ij}|\le \frac{\delta_N}{2}}] - \mathbb{E}[\hat{W}_{ij}]^2 = \theta_N + \tilde{R}_2,$$

with $\tilde{R}_2 = \mathbb{E}[W_{ij}^2 \mathbf{1}_{|W_{ij}|> \frac{\delta_N}{2}}] - \mathbb{E}[\hat{W}_{ij}]^2 = O(\delta_N^{-4-6\varepsilon} N^{-3(1+\varepsilon)})$. Therefore $\mathbb{E}[\mathring{W}_{ij}^2] = \mathring{\theta}_N$, with $\lim_{N\to+\infty} N\mathring{\theta}_N = \lim_{N\to+\infty} N\theta_N = \theta \in \mathbb{R}$. Note that, even if θ_N is supposed to be real for all N, $\mathbb{E}[\mathring{W}_{ij}^2]$ is not real anymore, but its imaginary part is negligible.

Similar bounds may be proved for $\mathbb{E}[|\mathring{W}_{ij}|^4]$, $\mathbb{E}[|\mathring{W}_{ij}|^{6(1+\varepsilon)}]$, $\mathbb{E}[\mathring{W}_{ii}^2]$ and $\mathbb{E}[|\mathring{W}_{ii}|^{4(1+\varepsilon)}]$ from which it may be shown that the entries of $\mathring{W}$ satisfy the same properties as the ones of W_N. In particular, one has for all $i \in \{1, \dots, N\}$, $N\mathbb{E}[\mathring{W}_{ii}^2] \underset{N\to+\infty}{\longrightarrow} \tilde{\sigma}^2$, and, for $1 \le i \ne j \le N$, $N^2(\mathbb{E}[|\mathring{W}_{ij}|^4] - 2\mathring{\sigma}_N^4 - \mathring{\theta}_N^2) = N^2\mathring{\kappa}_N \underset{N\to+\infty}{\longrightarrow} \kappa \in \mathbb{R}$.

Assume now that φ is a Lipschitz function. Note that this will be true in particular for functions in $\mathcal{C}_c^1(\mathbb{R})$. Using first Cauchy-Schwarz inequality then Hoffman-Wielandt inequality (see for example [AGZ10] Section 2.1.5), we get

$$\begin{aligned}
|\mathring{\mathcal{N}}_N(\varphi) - \hat{\mathcal{N}}_N(\varphi)| &= \left| \operatorname{Tr}\left(\varphi(\hat{X}_N) - \varphi(\mathring{X}_N)\right) \right| \\
&\le \sqrt{N} \Big(\sum_{i=1}^N |\varphi(\lambda_i(\hat{X}_N)) - \varphi(\lambda_i(\mathring{X}_N))|^2 \Big)^{1/2} \\
&\le \sqrt{N} \|\varphi\|_{\mathrm{Lip}} \Big(\sum_{i=1}^N |\lambda_i(\hat{X}_N) - \lambda_i(\mathring{X}_N)|^2 \Big)^{1/2} \\
&\le \sqrt{N} \|\varphi\|_{\mathrm{Lip}} \|\hat{X}_N - \mathring{X}_N\|_{HS} \\
&\le N \|\varphi\|_{\mathrm{Lip}} \|\hat{X}_N - \mathring{X}_N\|,
\end{aligned}$$

where we have used in the last line the classical inequality $\|M_N\|_{HS} \le \sqrt{N}\|M_N\|$ for $N \times N$ matrix M_N.

Now, in $\hat{X}_N = P(\hat{W}_N, D_N)$, decompose each $\hat{W}_N = \mathring{W}_N + \mathbb{E}[\hat{W}_N]$, so that $\hat{X}_N - \mathring{X}_N$ is a sum of a bounded number of monomials in $\mathring{W}_N$, $\mathbb{E}[\hat{W}_N]$ and D_N. All these monomials are of positive degree in $\mathbb{E}[\hat{W}_N]$. Recall that $(\|\mathring{W}_N\|)_{N\ge1}$ is bounded in all L^p, $p \ge 1$ (see Proposition 78) and $(\|D_N\|)_{N\ge1}$ is bounded (from Assumption 5). Furthermore, $(\|\mathbb{E}[\hat{W}_N]\|)_{N\ge1}$ is $o(N^{-1})$, by the classical bound $\|\mathbb{E}[\hat{W}_N]\|^2 \le \sum_{i,j} |\mathbb{E}[\hat{W}_{ij}]|^2$. Consequently, we deduce that $\mathbb{E}[\|\hat{X}_N - \mathring{X}_N\|] = o(N^{-1})$. Therefore $\mathbb{E}[|\mathring{\mathcal{N}}_N(\varphi) - \hat{\mathcal{N}}_N(\varphi)|] = o(1)$.

From these controls of $\mathbb{E}[|\mathring{\mathcal{N}}_N(\varphi) - \hat{\mathcal{N}}_N(\varphi)|]$ and $\mathbb{E}[|\mathcal{N}_N(\varphi) - \hat{\mathcal{N}}_N(\varphi)|]$, we conclude that

$$\mathbb{E}[|\mathcal{N}_N(\varphi) - \mathring{\mathcal{N}}_N(\varphi)|] \underset{N\to+\infty}{\longrightarrow} 0.$$

Hence

$$\big(\mathcal{N}_N(\varphi) - \mathbb{E}[\mathcal{N}_N(\varphi)]\big) - \big(\mathring{\mathcal{N}}_N(\varphi) - \mathbb{E}[\mathring{\mathcal{N}}_N(\varphi)]\big) \underset{N\to+\infty}{\overset{\mathbb{P}}{\longrightarrow}} 0.$$

Therefore, by Slutsky's Lemma, assuming that $\mathring{\mathcal{N}}_N(\varphi) - \mathbb{E}[\mathring{\mathcal{N}}_N(\varphi)]$ converges to a Gaussian variable yields that $\mathcal{N}_N(\varphi) - \mathbb{E}[\mathcal{N}_N(\varphi)]$ converges to the same Gaussian variable.

As a consequence, for our purposes, we may suppose that the entries of W_N are bounded almost surely by δ_N, as long as φ is a Lipschitz function.

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