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FEIGENBAUM'S RATIOS OF TWO-DIMENSIONAL AREA PRESERVING MAPS

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We generalize simple renormalization methods to calculate Feigenbaum's ratios for a two-dimensional area preserving map. Our results are in satisfactory agreement with known results and give approximate estimate for the ratio associated to more general cascades like $1, 3, 3^2, \dots, 3^n, \dots$.

Maps of the interval have attracted a lot of attention in the last few years (some recent references can be found in ref. [1]). The interest of physicists for these mathematical models was increased by the work of Feigenbaum [2,3] who discovered that the period-doubling cascades have universal critical properties: for a large class of one-dimensional maps T_μ which depend on a deformation parameter μ , there is a cascade of bifurcations $1 \rightarrow 2 \rightarrow 4 \rightarrow \dots \rightarrow 2^n \rightarrow \dots$. The values μ_n of the parameter where the cycle 2^n becomes unstable and where a cycle 2^{n+1} appears accumulate at some value μ_∞ . Feigenbaum pointed out that for a large class of maps T_μ , one has

$$\lim_{n \rightarrow \infty} \frac{\mu_n - \mu_\infty}{\mu_{n+1} - \mu_\infty} = \delta, \quad (1)$$

δ is a universal number which depends only on the shape of the map T_μ near its maximum. When the maximum of T_μ is quadratic, the value of δ is

$$\delta = 4.66920. \quad (2)$$

As mentioned by Eckmann [4], the same value of δ was obtained by several authors [5–7] when they studied other dissipative systems and it is believed [8] that this value will be soon measured in the cascades observed in Rayleigh Benard experiments.

A renormalization group theory of these period-doubling cascades was proposed by Feigenbaum [2,3] and rigorous results were derived by Collet et al. [9]. For maps like $(x \rightarrow 1 - a|x|^{1+\epsilon})$ it was possible to expand δ for small values of ϵ [7,9]. In our work [7]

with A. Gervois, we developed several approximate renormalization methods similar to real space renormalizations used in statistical physics. They led to rather accurate values of δ . Moreover, in ref. [7], the period doubling phenomenon was generalized to other cascades like $1, 3, 3^2, \dots, 3^n, \dots$. Though for these new cascades, the stability zones of the cycles (values of μ where the cycle is stable) are not contiguous, well defined periods 3^n appear at values μ'_n which accumulate to μ'_∞ in the same way as in period-doubling cascades:

$$\lim_{n \rightarrow \infty} \frac{\mu'_n - \mu'_\infty}{\mu'_{n+1} - \mu'_\infty} = \delta'. \quad (3)$$

The number δ' is also universal and characteristic of the factor 3. For maps with a quadratic maximum, its value is

$$\delta' = 55.247 \dots \quad (4)$$

Again this universal ratio can be recovered approximately by simple renormalization equations.

In this letter, we use the same kind of renormalization equations to calculate the ratios δ and δ' for two-dimensional area preserving maps. The importance of these maps is that they are good examples of conservative dynamical systems. We generalize to the two-dimensional case the method called "equality of slopes" in ref. [7]. The principle of the method is the following: consider a two-dimensional map $(x, y) \rightarrow T_\mu(x, y)$. Here we take for T_μ the Hénon map in the area preserving case [10]

$$T_\mu(x, y) = (1 - \mu x^2 + y, -x). \quad (5)$$

For the simplest cycles (cycles of length 1, 2, 3, 4) one can calculate analytically the coordinates of the points of a cycle as functions of μ . We can also linearize the map $T_\mu^{(n)}$ in the neighbourhood of the points of cycle n . This linearized map is a 2×2 matrix M given by

$$M = \prod_{i=1}^n \begin{pmatrix} -2\mu x_i & 1 \\ -1 & 0 \end{pmatrix}, \quad (6)$$

where the x_i are the coordinates of the points of cycle n . The eigenvalues λ_n of this matrix can be expressed as a function of μ . For $n = 1, 2, 3, 4$, the λ_n are roots of

$$\lambda_n^2 + 2f_n(\mu)\lambda_n + 1 = 0, \quad (7)$$

with

$$f_1(\mu) = (1 + \mu)^{1/2} - 1, \quad f_2(\mu) = 2\mu - 7,$$

$$f_3(\mu) = 4\mu[1 + (\mu - 1)^{1/2}] - 5 - 3(\mu - 1)^{1/2},$$

$$f_4(\mu) = 8\mu^2 - 16\mu^{3/2} - 1.$$

(Note that the cycle n is elliptic if $-1 \leq f_n(\mu) \leq 1$.)

For period doubling problems, the idea of renormalization is to try to associate at each value of μ , a value μ' such that T_μ looks like $T_{\mu'}^{(2)}$. An approximate way to do so is to say that the linearization of $T_\mu^{(n)}$ around a point of cycle n is identical to the linearization of $T_{\mu'}^{(2n)}$ around a point of cycle $2n$. Therefore:

$$f_n(\mu) = f_{2n}(\mu'). \quad (8)$$

This relation provides an approximate value for μ_∞ (the value of μ where the cascade $1 \rightarrow 2 \rightarrow \dots \rightarrow 2^n \dots$ accumulates) and of δ . μ_∞ is the fixed point of renormalization (8)

$$f_n(\mu_\infty) = f_{2n}(\mu_\infty), \quad (9)$$

whereas δ is given by:

$$\delta = d\mu/d\mu'|_{\mu_\infty}. \quad (10)$$

The choice of small value of n constitutes an approximation which should be improved by increasing n . We have done the calculations for $n = 1$ and $n = 2$ and the results are summarized in table 1. In the same way, one can study the cascade $1, 3, 3^2, \dots, 3^n \dots$ which accumulates at value μ'_∞ with a Feigenbaum ratio δ' . The re-

Table 1

Results of the approximate renormalizations for the Hénon map $(x, y) \rightarrow (1 - \mu x^2 + y, -x)$. The expected values are unknown except δ which has been recently obtained by several authors [11–15].

n	$2n$	μ_∞	δ
1	2	4.13278	9.0623
2	4	4.13621	8.6845
expected values	?	8.72 ...	
n	$3n$	μ'_∞	δ'
1	3	1.18362	22.869
expected values	?	?	

normalization equation (8) becomes in this case:

$$f_n(\mu) = f_{3n}(\mu'), \quad (11)$$

and the procedure remains the same to find μ'_∞ and δ' . We have done in this case the calculations for $n = 1$ and the results are also given in table 1. They agree rather well with recent works [11–15] for the period doubling ratio δ and provide an estimate for the period tripling ratio δ' .

In order to show the accuracy of these calculations, we reproduce in table 2 the results obtained [7] with the same method for the well known one-dimensional map $T_\mu(x \rightarrow 1 - \mu x^2)$. In this case the method is not very different from the one used by May and Oster [1] to calculate approximately δ .

The method we used in this letter gives rather accurate results with a very small amount of calculations. From tables 1 and 2, there is no doubt that the universal ratios δ and δ' are different for conservative systems (like the Hénon map) and dissipative systems (like x

Table 2

Results of the approximate renormalization for the one-dimensional map $x \rightarrow 1 - \mu x^2$.

n	$2n$	μ_∞	δ
1	2	1.39039	5.123
2	4	1.40142	4.614
expected values	1.40115519	4.66920	
n	$3n$	μ'_∞	δ'
1	3	1.78597	61.61
expected values	1.786440255	55.247	

$\rightarrow 1 - \mu x^2$). It would be interesting to know more about the ratios δ one can define for cascades like p , p^2 , p^3 , ..., p^n , ... and to see if the p dependence of these universal ratios is simple. Finally, it seems more difficult in the conservative case to define a map (like $x \rightarrow 1 - a|x|^{1+\epsilon}$ in the dissipative case) where an analytic expansion of δ and δ' can be done.

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