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► To cite this version:

Aurelien Genot. Aluminum combustion instabilities: Dimensionless numbers controlling the instability in solid rocket motors. Combustion and Flame, 2021, 232, pp.111563. 10.1016/j.combustflame.2021.111563 . hal-03284251

HAL Id: hal-03284251

<https://hal.science/hal-03284251>

Submitted on 13 Jul 2021

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Aluminum combustion instabilities in solid rocket motors: vortex shedding contribution

Aurelien Genot^{a,*}

^aONERA / DMPE, Université Paris Saclay, 6 Chemin de la Vauve aux Granges, 91120 Palaiseau, France

Abstract

This article is focused on the effect of vortices within the aluminum combustion zone on the aluminum combustion instabilities that can occur in solid rocket motors. As this instability is a thermoacoustic instability, the aluminum combustion dynamics is evaluated considering the rayleigh source term with or without vortices. This term is computed by postprocessing numerical simulations of a generic solid rocket motor, forced acoustically.

Keywords:

aluminum combustion, thermoacoustic instabilities, solid rocket motors, parietal vortex shedding

*Corresponding author.

Email address: aurelien.genot@onera.fr (Aurelien Genot)

1. Introduction: State of the art

Unsteady behaviors in complex systems are usually the most difficult aspects to predict and to control. To design new engines, various small-scale experiments and numerical simulations are executed without real capacities to predict the working of this new design. In addition to the significant cost represented by all of these tests or simulations, they do not make it possible to control unsteady and complex phenomena like combustion instabilities. Moreover, the physics observable in the new system designed remains very poorly understood. Theoretical approaches, despite assumptions sometimes too restrictive, allow a global vision (at the scale of the system) over the entire operating range.

Combustion instabilities are defined as the manifestation of a coupling between flow and combustion generating flow instabilities which can be problematic for the system [1, 2]: premature engine shutdown induced by flame blowoff [3] or structural damage due to pressure oscillations produced by flame instabilities [4]. Many couplings between combustion and flow are possible and they are complex to understand and control. For example, in solid rocket combustion chambers, several phenomena, which can produce and amplify pressure oscillations, can be identified [5–7]: vortex shedding, indirect-combustion noise (entropy inhomogeneities through contractions), vortex-nozzle interactions, unsteady combustion of the solid propellant and the combustion dynamics of the aluminum droplets (released in the chamber from an aluminized solid propellant). Also, these phenomena are suspected to interact with each other [8]. One of these phenomena in a generic solid rocket motor is deeply studied by theoretical developments in this work: the aluminum dynamics which can drive an instability called thermoacoustic instability by the French solid propulsion community [8–12] because the aluminum combustion can respond to acoustic oscillations and amplify them. The other sources of sound are neglected.

Aluminum combustion instability is a quite complex physical phenomenon because aluminum droplet combustion modeling is still an open question [13–17]. This distributed combustion (individual burning droplets [11]) takes place in an acoustic boundary layer [5, 12, 18] close to the solid propellant surface (where they come from, see Fig. 1).

In the state of the art, most of the works focused on a deep analysis of the aluminum droplet combustion dynamics induced by acoustics in solid rockets assumed a very rustic aluminum combustion model which is a combination of the classical D^2 -law with a Heaviside

function to stop the droplet combustion [11, 12]. It should be noted that this very rustic model may make sense considering the latest experimental results [17] that show a very spheric and symmetric droplet combustion (which can be modeled by a D^2 -law) and a sudden quenching of the droplet combustion (a Heaviside function). In addition, this modeling has been successfully used for instability predictions in actual solid rocket motors [8, 10, 19].

With this rustic model, Gallier and Godfroy [10] developed an analytical model quantifying the combustion dynamics, assuming that the heat release rate fluctuations from the combustion of aluminum droplets result mainly from the longitudinal velocity fluctuations of the flow around the droplets. In their analysis, the Ranz-Marshall correlation [20] has been used to model the coupling between the droplet evaporation and the surrounding gas flow. This assumption has been checked in [21]. However, the analytical model of Gallier and Godfroy [10] was not compared with any simulations. Based on this unvalidated model, Casalis *et al.* [22] derived dimensionless numbers controlling the thermoacoustic source. In this new article, improvements and analyses are proposed based on recent state-of-the-art advancements.

Genot *et al.* [11] compared the analytical model of Gallier and Godfroy [10] to numerical simulations and demonstrated that some aspects of the aluminum combustion dynamics were underestimated in their analytical model. They completed this analytical model considering that the heat release rate fluctuations induced by the aluminum combustion dynamics are essentially controlled by the gaseous flow dynamics along the droplet trajectories taking in account the droplet memory and the instant when the droplets are injected (from the solid propellant surface). Consequently, both velocity fluctuations and droplet diameter fluctuations need to be considered to determine the correct fluctuations of the evaporation rate from the droplets and the correct fluctuations of the heat released by the unsteady droplet combustion. A series of analytical models was derived in [11] linking these quantities with different approximations for the gaseous flow dynamics around the droplets. These models were shown to well reproduce the heat release rate fluctuations observed in numerical flow simulations of a generic solid rocket motor (Fig. 1) over the entire combustion region [11, 12].

These heat release rate fluctuations can be split in two contributions [11, 23, 24]. First, it was shown that the combustion dynamics of the individual aluminum droplets, close to the solid propellant surface, within the acoustic boundary layer of the flow, drive heat re-

lease rate fluctuations within the volume of the reactive droplet cloud. A second source of heat release rate fluctuations is also found at the reactive droplet cloud boundary where droplet combustion ceases. The origin of these heat release rate fluctuations was further examined in [24]. It was shown that they were associated with the dynamics of the burning droplets just before quenching (here modeled by a Heaviside function) and with the droplet lifetime oscillations. An analytical model was also derived for this contribution to heat release rate fluctuations when the combustion is abruptly quenched for some droplet residue diameter.

These models for the heat release rate fluctuations are illustrated in Figure 2. The model of Gallier and Godfroy [10] corresponds to the left subfigure of Figure 2 and the model of Genot *et al.* [11] to the right subfigure of Figure 2. Droplets are injected radially in $r = R$ all along the chamber of length L . They are submitted to longitudinal fluctuations (transverse to their trajectory). The red zone highlights the reactive droplet cloud and the white zone is an inert zone after combustion with residual droplets (alumina droplets). In the right subfigure, model improvements are highlighted by a blue square around the variable (here the droplet velocity fluctuations $\hat{u}_p$) and new models are highlighted by variables in blue (here the droplet diameter fluctuations $\hat{D}$ and the droplet cloud boundary motion $\hat{r}_c$ driven by droplet lifetime oscillations [24]). The mathematical formulation of the model of Genot *et al.* [11] for the heat release rate fluctuations is derived below.

The rustic aluminum combustion model is defined as following [10, 11]:

$$\dot{q} = \dot{q}_{D^2} \mathcal{H}(D - D_r) \quad (1)$$

with $\dot{q}$ the heat release rate, $\mathcal{H}$ the Heaviside function which stops the combustion of the droplet when its size reaches the residue diameter D_r , D the droplet diameter and $\dot{q}_{D^2}$ the heat release rate defined by the D^2 -law:

$$\dot{q}_{D^2} = \Delta H_r N_p \pi D \frac{\mu}{\text{Pr}} \ln(1 + B) \text{Sh} \quad (2)$$

where ΔH_r is the reaction enthalpy, N_p the number of droplets per unit volume, μ the gas viscosity, Pr the Prandtl number, B the Spalding number and Sh the Sherwood number modeled by the Ranz-Marshall correlation [20]:

$$\text{Sh} = 2 + 0.6 \text{Re}_p^{1/2} \text{Pr}^{1/3} \quad (3)$$

In the Fourier space, the fluctuations of the heat release rate, following Genot *et al.* [11, 24], can be split in two contributions:

$$\hat{q} = \hat{q}_{D^2} \mathcal{H}(D - D_r)_0 + \hat{q}_{D^2,0} \hat{\mathcal{H}}(D - D_r) = \hat{q}_{v.c.} + \hat{q}_{b.c.} \quad (4)$$

with $\hat{\cdot}$ the fluctuations in the Fourier space, the subscript 0 for time average of the quantity (for instance $\hat{q}_0 = \frac{1}{T} \int_T \dot{q} dt$), *v.c.* the volume contribution (corresponding to the fluctuations of $\dot{q}_{D^2}$) and *b.c.* the boundary contribution (corresponding to the Heaviside function fluctuations). The volume contribution is derived as following:

$$\frac{\hat{q}_{v.c.}}{\hat{q}_0} = \frac{\text{Sh}_0 - 2}{2\text{Sh}_0} \frac{\delta u_{p,0}}{|\delta \mathbf{u}_p|_0^2} (\hat{u}_p - \hat{u}) + \left(1 + \frac{\text{Sh}_0 - 2}{2\text{Sh}_0}\right) \frac{\hat{D}}{D_0} \quad (5)$$

with $\mathbf{u}_p = u_p \mathbf{e}_x + v_p \mathbf{e}_r$ the droplet velocity, $\mathbf{u} = u \mathbf{e}_x + v \mathbf{e}_r$ the gas velocity, $\delta \mathbf{u}_p = \mathbf{u}_p - \mathbf{u}$ and $\delta u_p = u_p - u$. The boundary contribution to the heat release rate fluctuations is equal to [11, 24]:

$$\frac{\hat{q}_{b.c.}}{\hat{q}_0} = -\frac{\dot{q}_{D^2,0}}{\hat{q}_0} \frac{2 \left(1 - \left(\frac{r-r_{c,0}}{|\hat{r}_c|}\right)^2\right)^{1/2}}{\pi |\hat{r}_c|} \hat{r}_c \quad (6)$$

where r is the radial position and r_c the boundary position of the reactive droplet cloud which is unsteady. So the flapping motion $\hat{r}_c$ of the boundary is defined in the Fourier space as following [24]:

$$\hat{r}_c = \frac{2t_{c,0} v_p D_0}{D_i^2 - D_r^2} \hat{D} \quad (7)$$

with v_p the radial droplet velocity, D_i the injection droplet diameter (initial droplet diameter at the solid propellant surface), D_r the residue diameter (after combustion) and t_c the combustion time:

$$t_{c,0} = \frac{\text{Pr} \rho_p (D_i^2 - D_r^2)}{4\mu \ln(1 + B) \text{Sh}_0} \quad (8)$$

where Pr is the Prandtl number, ρ_p the droplet density, μ the gas density and B the Spalding number. The longitudinal droplet velocity fluctuations are given below:

$$\hat{u}_p = \int_R^r \frac{\hat{u}}{v_p \tau_{d,0}} \left(1 + C_{Re} \frac{\delta u_{p,0}^2}{|\delta \mathbf{u}_p|_0^2}\right) e^{\int_r' \left(i\omega \tau_{d,0} + 1 + C_{Re} \frac{\delta u_{p,0}^2}{|\delta \mathbf{u}_p|_0^2}\right) \frac{dr''}{v_p \tau_{d,0}}} dr' \quad (9)$$

where $i^2 = -1$, C_{Re} a constant defined below, $\omega = 2\pi f$ the excited angular frequency, τ_d the drag characteristic time [11]:

$$\tau_d = \frac{1 + B}{1 + 0.15 \text{Re}_p^{0.687}} \frac{\rho_p D^2}{18\mu}, \quad C_{Re} = \frac{0.10305 \text{Re}_{p,0}^{0.687}}{1 + 0.15 \text{Re}_{p,0}^{0.687}} \quad (10)$$

with $\text{Re}_p = \rho_p D |\delta \mathbf{u}_p| / \mu$ the droplet Reynolds number, ρ_p the droplet density. The droplet diameter fluctuations are defined as:

$$\hat{D} = - \int_R^r \frac{\mu \ln(1 + B) (\text{Sh}_0 - 2)}{v_p \text{Pr} \rho_p D_0 |\delta \mathbf{u}_p|_0^2 / \delta u_{p,0}} (\hat{u}_p - \hat{u}) e^{\int_r' \left(\frac{i\omega}{v_p} - \frac{\mu \ln(1 + B) (\text{Sh}_0 + 2)}{v_p \text{Pr} \rho_p D_0^2}\right) \frac{dr''}{v_p \tau_{d,0}}} dr'' \quad (11)$$

The gas flow velocity and the mean flow were taken from the numerical simulation for the verification of this model Eqs. (4–11) [11].

2. Linear thermoacoustic source of the instability: model reduction and verification

In unstable solid rocket motors, longitudinal modes are usually those excited and more particularly the first longitudinal mode [5, 10]. In case of pure thermoacoustic instabilities (without solid propellant response, hydrodynamic instabilities), the acoustic losses can be assumed linear and the aluminum combustion dynamics weakly nonlinear [12]. From these observations, only one acoustic mode will be considered and aluminum combustion dynamics will be linearized.

In the previous section, the state of the art about the aluminum combustion dynamics was reminded. In the theory of the linear stability, the pressure oscillation amplitude grows with a growth rate α such as [5, 25]:

$$\alpha = \alpha_{losses} + \alpha_{alu} \quad (12)$$

with $\alpha_{losses} < 0$ the growth rate due to the acoustic losses [12, 23, 25] and α_{alu} the growth rate induced by the thermoacoustic coupling (aluminum combustion dynamics and chamber acoustics), assuming a 2D-axisymmetric chamber (Fig 1) [11]:

$$\alpha_{alu} = \frac{2(\gamma - 1)}{R^2 \int_0^L \psi^2 dx} \int_0^L \psi \int_0^R \left(\frac{\hat{q}}{\hat{\eta}} \right)^{(\mathcal{R}_e)} r dr dx \quad (13)$$

$\mathcal{R}_e$ means the real part, $\psi \approx \cos(kx)$ is the longitudinal mode (with $k = \omega/a_0$ the wave length and a_0 the sound speed), $\hat{\eta}$ the pressure amplitude (such that the pressure fluctuations in the Fourier space can be defined as $\hat{p} = \hat{\eta}\psi$), R the radius of the chamber, L the length of the chamber.

This growth rate α_{alu} (Eq. 13) is dependent on the heat release rate fluctuations which can be modeled by the analytical model derived in section 1. As the heat release rate can be split into two contributions, the thermoacoustic growth rate can also be split:

$$\hat{q} = \hat{q}_{v.c.} + \hat{q}_{b.c.}, \quad \alpha_{alu} = \alpha_{v.c.} + \alpha_{b.c.} \quad (14)$$

The objective of this paper is to simplify these previous expressions (Eqs. (4–11) and Eq. (13)) removing the integrals to highlight the key parameters or dimensionless numbers which drive this thermoacoustic coupling. As a reminder to readers, all the results presented in this paper are dependent on assumptions: (i) linear fluctuations (and linear combustion dynamics), (ii) a rustic aluminum combustion model and (iii) a 2D-axisymmetric chamber.

2.1. Model reduction of the heat release rate fluctuations

To reduce the expressions of the analytical model for the heat release rate fluctuations derived in section 1 (Eqs. (4–11)) and the corresponding growth rate (Eq. (13)), simplifications of the velocity and diameter fluctuations (Eqs. (9) and (11)) are derived in following. Then, this new simplified model for the heat release rate fluctuations will be compared with the numerical results of Genot *et al.* [11].

2.1.1. Velocity fluctuations

Gallier and Godfroy [10] derived a very simplified model for the gas velocity fluctuations $\hat{u}$, very close to the solid propellant surface:

$$\hat{u} = \hat{u}_{ac}(1 - F) \quad (15)$$

with $\hat{u}_{ac}$ the acoustic velocity and F a correcting function to take in account the acoustic boundary layer:

$$F = e^{\frac{i\omega}{v_p}(R-r)} \quad (16)$$

This model is oversimplified and must be used with caution but it roughly models the acoustic boundary layer. Moreover, its expression allows a simplified integration of Eq. (9). Some other approximations are done in following: $C_{Re} \approx 0$, droplets injected at the equilibrium with the gas ($\mathbf{u}_p = v_p \mathbf{e}_r = v_{inj} \mathbf{e}_r = -v_{inj} \mathbf{e}_r$) and the variation of the mean flow, in the combustion zone, is negligible compared to the variation of velocity fluctuations in the radial direction r . With all these assumptions, Eq. (9) finally reduces to:

$$\hat{u}_p - \hat{u} = -\hat{u}_{ac} \frac{\omega \tau_{d,0}}{\omega \tau_{d,0} - i} \left(1 - e^{\frac{i\omega \tau_{d,0} + 1}{v_p \tau_{d,0}}(R-r)} \right) \quad (17)$$

2.1.2. Droplet diameter fluctuations

With the same assumptions and the expression Eq. (17), diameter fluctuations Eq. (11) can be reduced to:

$$\hat{D} = -B_D \hat{u}_{ac} \frac{\omega \tau_{d,0}}{\omega \tau_{d,0} - i} \left(\frac{1 - e^{A_D(R-r)}}{A_D} - \frac{e^{\frac{i\omega \tau_{d,0} + 1}{v_p \tau_{d,0}}(R-r)} - e^{A_D(R-r)}}{A_D - \frac{i\omega \tau_{d,0} + 1}{v_p \tau_{d,0}}} \right) \quad (18)$$

where A_D and B_D equal to:

$$A_D = \frac{i\omega}{v_p} - \frac{\mu \ln(1+B)(Sh_0 + 2)}{v_p Pr \rho_p D_0^2}, \quad B_D = -\frac{\mu \ln(1+B)(Sh_0 - 2)}{v_p Pr \rho_p D_0 |\delta \mathbf{u}_p|_0^2 / \delta \mathbf{u}_{p,0}} \quad (19)$$

2.2. Model verification of the new model for the heat release rate fluctuations

Finally, with these reductions for $\hat{D}$ and $\hat{u}_p - \hat{u}$, the heat release rate fluctuations $\hat{q} = \hat{q}_{v.c.} + \hat{q}_{b.c.}$ can be expressed without integrals (Eqs. (4–7) and Eqs. (17–18)):

$$\frac{\hat{q}_{v.c.}}{\dot{q}_0} = \frac{Sh_0 - 2}{2Sh_0} \frac{\delta u_{p,0}}{|\delta \mathbf{u}_p|_0^2} (\hat{u}_p - \hat{u}) + \left(1 + \frac{Sh_0 - 2}{2Sh_0}\right) \frac{\hat{D}}{D_0} \quad (20)$$

$$\frac{\hat{q}_{b.c.}}{\dot{q}_0} = -\frac{\dot{q}_{D^2,0}}{\dot{q}_0} \frac{2 \left(1 - \left(\frac{r-r_{c,0}}{|\hat{r}_c|}\right)^2\right)^{1/2}}{\pi |\hat{r}_c|} \hat{r}_c \quad (21)$$

$$\hat{r}_c = \frac{2t_{c,0}v_p D_0}{D_i^2 - D_r^2} \hat{D} \quad (22)$$

$$\hat{u}_p - \hat{u} = -\hat{u}_{ac} \frac{\omega \tau_{d,0}}{\omega \tau_{d,0} - i} \left(1 - e^{\frac{i\omega \tau_{d,0} + 1}{v_p \tau_{d,0}}(R-r)}\right) \quad (23)$$

$$\hat{D} = -B_D \hat{u}_{ac} \frac{\omega \tau_{d,0}}{\omega \tau_{d,0} - i} \left(\frac{1 - e^{A_D(R-r)}}{A_D} - \frac{e^{\frac{i\omega \tau_{d,0} + 1}{v_p \tau_{d,0}}(R-r)} - e^{A_D(R-r)}}{A_D - \frac{i\omega \tau_{d,0} + 1}{v_p \tau_{d,0}}} \right) \quad (24)$$

The model of Genot *et al.* [11] ($\hat{q}$ with integrals) has been compared with a numerical simulation. This simulation was carried out by solving the compressible Navier-Stokes equations with the perfect gas law in a twoway coupling Eulerian framework using Marble's two-phase flow equations [10, 11, 26]. The governing equations were discretized and resolved by a finite volume technique adapted to unstructured meshes. The numerical schemes were second-order accurate in space and second-order accurate in time using explicit two-step RungeKutta time stepping. The simulation was carried out in a two-dimensional axisymmetric framework (Fig. 1). A radial injection modeling the solid propellant combustion (gas and aluminum droplets) is imposed in $r = R$ and an exit boundary condition at the end of the nozzle is considered. The solid propellant burning velocity is neglected, and the geometry is fixed in this simulation because the flow velocity is much higher than the solid propellant burning velocity. The computational grid is composed of 360,000 quads with about 600 points in the axial direction and 600 points in the radial direction. The grid is clustered near the propellant burning surface to resolve the aluminum distributed combustion. The region where aluminum combustion reaction takes place is indicated in red in Fig. 1. The smallest grid spacing at the propellant surface is about 0.1 mm, and the mesh is refined in the aluminum combustion region. Outside the aluminum combustion region, particles are inert. Turbulence is not taken into account in these simulations to

focus the analysis on the coupling between the acoustics and unsteady aluminum droplet combustion [11, 27] without dealing with the complexity of interactions with turbulence. Grid convergence has been checked, and no significant differences have been found between results calculated with this grid and a coarser mesh with 172,000 quads. Droplets were injected in equilibrium with the gas, and the aluminum droplets were subjected to forced acoustic oscillations such that their combustion dynamics was linear. The frequency excited corresponds to the first longitudinal mode which is the mode usually observed for unstable cases [10, 23]. More details about the simulations are given in [11].

To check that this new model Eqs. (20–24) (a reduction of the model of Genot *et al.* without integrals) is still close to the numerical simulation results from Genot *et al.* [11], Figures 3 and 4 compare this new model with the model of [11] and the numerical simulation (called 'Simulation' in the Figures). Figures 3 and 4 illustrate the heat release rate fluctuations (modulus and phase angle) at the first quarter of the tube ($L/4 = 1.75$ m) and at the third quarter ($3L/4 = 5.25$ m). The phase angle is computed relatively to the phase angle of the local pressure oscillations. These two positions correspond to the extrema of the thermoacoustic source for the first longitudinal mode [11, 23]. The mean flow and the gas velocity fluctuations $\hat{u}$ come from the simulation [11] to ensure that potential discrepancies are related to models.

Figures 3 and 4 show very close results between the model of Genot *et al.* [11] and the numerical results (as already demonstrated in [11]). The modulus computed by the new model (without integrals) is also very close to the numerical results. The phase angle is less well reproduced by this new model but the results are widely acceptable given the assumptions. This new model is used and integrated in following to go further in the reductions and analyses.

2.3. Final reduction of the thermoacoustic source: dimensionless number dependency

The growth rate $\alpha_{alu} = \alpha_{v.c.} + \alpha_{b.c.}$ linked to the thermoacoustic source is split into two contributions (Eq. (13)):

$$\alpha_{v.c.} = \frac{2(\gamma - 1)}{R^2 \int_0^L \psi^2 dx} \int_0^L \psi \int_0^R \left(\frac{\hat{q}_{v.c.}}{\hat{\eta}} \right)^{(\mathcal{R}_e)} r dr dx \quad (25)$$

$$\alpha_{b.c.} = \frac{2(\gamma - 1)}{R^2 \int_0^L \psi^2 dx} \int_0^L \psi \int_0^R \left(\frac{\hat{q}_{b.c.}}{\hat{\eta}} \right)^{(\mathcal{R}_e)} r dr dx \quad (26)$$

with the heat release rate fluctuations $\hat{q}$ which can be modeled by Eqs. (20–24).

In this subsection, both expressions $\alpha_{v.c.}$ and $\alpha_{b.c.}$ are simplified by the radial integration. To do this, some assumptions are proposed.

For the volume contribution $\alpha_{v.c.}$, the mean flow will be simplified as following:

$$X_0(r, x) \sim X_0(x) \quad (27)$$

In other words, its radial variation is assumed negligible compared to the radial variation of the velocity fluctuations (because aluminum combustion takes place in the acoustic boundary layer [11, 18]). For the mean diameter D_0 , as the droplet diameter is following the D^2 -law, its mean value in the r -direction is:

$$D_0 = \sqrt{\frac{D(r=R)^2 + D(r=r_c)^2}{2}} = \sqrt{\frac{D_i^2 + D_r^2}{2}} \quad (28)$$

with D_i the injection diameter and D_r the residue diameter.

For the boundary contribution $\alpha_{b.c.}$, as it is located to the boundary position $r = r_c$, the mean quantities will be considered at this position. For the mean diameter, this assumption gives:

$$D_0(r = r_c) = D_r \quad (29)$$

Reductions of the contributions ($\alpha_{v.c.}$, $\alpha_{b.c.}$) to the thermoacoustic source are proposed below.

2.3.1. Volume contribution

The growth rate of the volume contribution to the thermoacoustic source is defined (Eq. (25)):

$$\alpha_{v.c.} = \frac{2(\gamma - 1)}{R^2} \int_0^L \psi^2 dx \int_0^R \psi \int_0^R \left(\frac{\hat{q}_{v.c.}}{\hat{\eta}} \right)^{(\mathcal{R}_c)} r dr dx \quad (30)$$

Considering a solid rocket motor like a tube with closed-closed acoustic boundaries, the acoustic longitudinal mode ψ can be modeled [10]:

$$\psi = \cos(kx) \quad (31)$$

with $k = \omega/a_0$, a_0 the sound speed, $\omega = 2\pi f$ the angular frequency and f the acoustic frequency. After many steps to calculate the radial integral of the previous expression, one finds:

$$\alpha_{v.c.} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \dot{q}_0 \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{S_{tk}}{a_0 \rho} \frac{S_{tk}}{(S_{tk}^2 + 1) S_r^2} \mathbf{e}_r = v(r = R) = -v_i) \cdot \left[A_{v.c.} + e^{-S_{tk} S_c} B_{v.c.} + e^{\frac{S_{tk} + 2}{4 S_{tk}} \frac{D_i^2 - D_r^2}{D_0^2}} C_{v.c.} + e^{-S_c / S_{tk}} D_{v.c.} \right] dx \quad (32)$$

where D_0 is the mean droplet diameter given by Eq. (28), Sh_0 the Sherwood number (which represents the ratio between the convective mass transfer and the diffusive mass transport from the droplet), S_{tk} is the Stokes number (ratio between the acoustic and droplet drag characteristic times) and S_c the combustion Strouhal (ratio between the acoustic and droplet combustion characteristic times):

$$S_{tk} = \omega \tau_{d,0}, \quad S_c = \omega t_{c,0}, \quad Sh_0 = 2 + 0.6 Re_{p,0}^{1/2} Pr^{1/3} \quad (33)$$

with Pr the Prandtl number. The functions $A_{v.c.}$, $B_{v.c.}$, $C_{v.c.}$ and $D_{v.c.}$ are dependent on these three dimensionless numbers and on another one, the flow Strouhal number S_r (ratio between the radial flow and acoustic characteristic times):

$$S_r = \frac{\omega R}{v_i} \quad (34)$$

The functions $A_{v.c.}$, $B_{v.c.}$ and $C_{v.c.}$ are defined as fol-

lows:

$$\begin{aligned}
A_{v.c.} = & \frac{Sh_0 - 2}{2Sh_0} \left(\frac{S_c}{2} (2S_r - S_c) - \frac{1}{S_{tk}^2 + 1} \right) \\
& + \frac{\left(1 + \frac{Sh_0 - 2}{2Sh_0} \right)}{S_r \left(\left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 + 1 \right)} \left[\frac{S_c S_{tk}}{2} (2S_r - S_c) + \frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right. \\
& \left(\frac{S_c S_{tk}}{2} (2S_r - S_c) + \frac{2S_{tk} + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 - 1}{1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2} \right. \\
& \left. \left. S_r \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} + S_{tk} \right) \left(1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 \right) \right) \right. \\
& \left. - \frac{1}{1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2} \right] \\
& + \frac{S_r \left(1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 \right) \left(1 - S_{tk} \frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)}{1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2} \\
& + \frac{S_{tk} \left(\left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 - 1 \right) - 2}{1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2} \left[\right. \\
& + \frac{1 + \frac{Sh_0 - 2}{2Sh_0}}{S_r \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} + \frac{1}{S_{tk}} \right)} \left[\frac{S_r S_{tk} (S_{tk}^2 - 1) - S_{tk}^2 - 1}{(1 + S_{tk}^2)^2} \right. \\
& \left. S_r \left(1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 \right) \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} + S_{tk} \right) \right. \\
& \left. + \frac{\left(\left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 + 1 \right)^2}{\left(\left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 - 1 + \frac{Sh_0 + 2}{2Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} S_{tk} \right)} \right] \quad (35)
\end{aligned}$$

$$\begin{aligned}
B_{v.c.} = & \frac{Sh_0 - 2}{2Sh_0(1 + S_{tk}^2)^2} \left[\cos(S_c)(1 - S_{tk})(1 + S_{tk}^2) \right. \\
& \left. + \sin(S_c) \left((S_c - S_r)(1 - S_{tk}^2)^2 + S_{tk}^3 + (1 - S_{tk})^2 \right) \right] \quad (36)
\end{aligned}$$

$$\begin{aligned}
C_{v.c.} = & - \frac{1 + \frac{Sh_0 - 2}{2Sh_0}}{S_r \left(\left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 + 1 \right)^2} \left[(a_{Cvc} - S_{tk} b_{Cvc}) \cos(S_c) \right. \\
& \left. + (b_{Cvc} + S_{tk} a_{Cvc}) \sin(S_c) \right] \quad (37)
\end{aligned}$$

with a_{Cvc} and b_{Cvc} :

$$\begin{aligned}
a_{Cvc} = & \frac{\frac{Sh_0 - 2}{2Sh_0} \left(\frac{Sh_0 + 2}{2Sh_0} \right)^2 \left(1 + (S_r - S_c) \frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right) - 1}{1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2} \\
& + \frac{\left(\frac{Sh_0 + 2}{2Sh_0} \right)^2 \left(\frac{S_c - S_r}{S_c} \right) \frac{D_i^2 - D_r^2}{D_0^2} + \left(\left(\frac{Sh_0 + 2}{2Sh_0} \right)^2 - 1 \right) (S_r - S_c)}{1 + \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2} \\
& + \frac{\left(1 + (S_r - S_c) \frac{D_i^2 - D_r^2}{D_0^2} \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \right) \right) \left(1 - \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 \right)}{\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} + \frac{1}{S_{tk}}} \quad (38)
\end{aligned}$$

$$\begin{aligned}
b_{Cvc} = & \frac{\frac{Sh_0 + 2}{4Sh_0^2} (3Sh_0 - 2) + (S_c - S_r) \left(\frac{Sh_0 + 2}{2Sh_0} \right)^2 \left(\frac{3Sh_0 - 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} - \frac{Sh_0 - 2}{2Sh_0} \right)}{\left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 + 1} \\
& + \frac{(S_c - S_r) \left(1 - \frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right) + 1}{\left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 + 1} \\
& + \frac{\frac{Sh_0 + 2}{2Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} - (S_c - S_r) \left(\left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} \right)^2 + 1 \right)}{\frac{1}{S_{tk}} + \frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2}} \quad (39)
\end{aligned}$$

And the function $D_{v.c.}$:

$$\begin{aligned}
D_{v.c.} = & \frac{S_{tk}}{S_r(1 + S_{tk}^2)^2 \left(\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} + \frac{1}{S_{tk}} \right)} \left[-\sin(S_c) S_{tk}^2 \right. \\
& \left. + \cos(S_c) \left((1 + S_{tk})(1 + S_{tk}^2)(S_r - S_c) \left(1 + \frac{Sh_0 - 2}{2Sh_0} \right) - S_{tk} \right) \right] \quad (40)
\end{aligned}$$

The very complex expression of $\alpha_{v.c.}$ (Eq. (32–40)) is still easier to use than doing many numerical simulations. It shows a dependency on the acoustic mode $\psi = \cos(kx)$ (and its derivative $\sin(kx)$), the mean heat released during the droplet life (see the assumption Eq. (27)):

$$\dot{q}_0(x) \equiv \frac{2 \int_{r_{c,0}}^R \dot{q}_0(r, x) r dr}{R^2 - r_{c,0}^2} \quad (41)$$

and four dimensionless numbers (a Stokes number S_{tk} , a combustion Strouhal number S_c , a flow Strouhal number S_r and the Sherwood number Sh_0). These theoretical results will be useful for the next section (section 3).

Similar developments are proposed in the next subsection for the boundary contribution to the thermoacoustic source.

2.3.2. Boundary contribution

The growth rate for the boundary contribution can be defined as (Eq. (26)):

$$\alpha_{b.c.} = \frac{2(\gamma - 1)}{R^2 \int_0^L \psi^2 dx} \int_0^L \psi_n \int_0^R \left(\frac{\hat{q}_{b.c.}}{\hat{\eta}} \right)^{(\mathcal{R}_e)} r dr dx \quad (42)$$

By substitution with Eq. (21), it gives:

$$\alpha_{b.c.} = \frac{-2(\gamma - 1)}{R^2 \int_0^L \psi^2 dx} \int_0^L \psi_n \int_0^R \frac{\dot{q}_{D^2,0} 2 \left(1 - \left(\frac{r-r_{c,0}}{|\hat{r}_c|} \right)^2 \right)^{1/2} (\hat{r}_c/\hat{\eta})^{(\mathcal{R}_e)}}{\pi |\hat{r}_c|} r dr dx \quad (43)$$

The flapping boundary zone (see [11, 24], illustrated in the right subfigure of Fig. 2) is radially located in $r \in [r_{c,0} - |\hat{r}_c|, r_{c,0} + |\hat{r}_c|]$ ($r_{c,0}$ is its mean position and $|\hat{r}_c|$ the amplitude of the flapping motion of the boundary). As this flapping boundary zone is thin (linear dynamics implies low-amplitude fluctuations $|\hat{r}_c| \ll r_{c,0}$) in comparison with the reactive cloud thickness, $\dot{q}_{D^2,0}$ (the heat released just before the extinction of the droplet), $|\hat{r}_c|$ and $(\hat{r}_c/\hat{\eta})^{(\mathcal{R}_e)}$ are assumed invariant in this zone ($r \in [r_{c,0} - |\hat{r}_c|, r_{c,0} + |\hat{r}_c|]$). Moreover, seeing that:

$$\frac{2}{\pi |\hat{r}_c|} \int_{r_{c,0}-|\hat{r}_c|}^{r_{c,0}+|\hat{r}_c|} \left(1 - \left(\frac{r-r_{c,0}}{|\hat{r}_c|} \right)^2 \right)^{1/2} r dr = r_{c,0} \quad (44)$$

The expression Eq. (43) can be simplified as:

$$\alpha_{b.c.} = \frac{-(\gamma - 1)}{R^2 \int_0^L \psi^2 dx} \int_0^L \dot{q}_{D^2,0} \psi r_{c,0} \left(\frac{\hat{r}_c}{\hat{\eta}} \right)^{(\mathcal{R}_e)} dx \quad (45)$$

with $\hat{r}_c$ given by Eq. (22). After some theoretical developments, Equation (45) gives:

$$\alpha_{b.c.} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \dot{q}_{D^2,0} \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{S_{tk}}{(S_{tk}^2 + 1)} \left[A_{b.c.} + e^{\frac{Sh_0+2}{4Sh_0} \frac{D_i^2-D_r^2}{D_0^2}} C_{b.c.} + e^{-S_c/S_{tk}} D_{b.c.} \right] dx \quad (46)$$

with D_0 the mean droplet diameter given by Eq. (29), $\dot{q}_{D^2,0}$ the heat released by the combustion of the droplets just before their extinction, $A_{b.c.}$, $C_{b.c.}$ and $D_{b.c.}$ defined as follows:

$$A_{b.c.} = \frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} - S_{tk} \quad (47)$$

$$C_{b.c.} = \cos(S_c) \left[\frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} + S_{tk} - 1 \right] + \sin(S_c) \left[1 + S_{tk} - \frac{Sh_0 + 2}{4Sh_0 S_c} \frac{D_i^2 - D_r^2}{D_0^2} S_{tk} \right] \quad (48)$$

$$D_{b.c.} = \cos(S_c) - S_{tk} \sin(S_c) \quad (49)$$

This new expression for the growth rate of the boundary contribution Eqs. (46–49) has been derived as the growth rate of the volume contribution Eqs. (32–40) to compare them. As the volume contribution, the boundary contribution is dependent on the acoustic mode $\cos(kx)$ (and its derivative $\sin(kx)$) and four dimensionless numbers (a Stokes number S_{tk} , a combustion Strouhal number S_c , a flow Strouhal number S_r and the Sherwood number Sh_0). However, $\alpha_{b.c.}$ is also dependent on the heat release rate just before the extinction $\dot{q}_{D^2,0}(r = r_c)$ and not on the mean heat release rate during the droplet life $\dot{q}_0$. As the diameter droplet decreases during its combustion and $\dot{q} \propto D$ (Eq. (1)), the following relationship can be derived (considering Eqs. (28–29)):

$$\frac{\dot{q}_0}{\dot{q}_{D^2,0}} \sim \sqrt{\frac{(D_i/D_r)^2 + 1}{2}} > 1 \quad (50)$$

In the following section, the thermoacoustic source is analyzed considering the expressions of the growth rates (from the volume contribution Eqs. (32–40) and the boundary contribution Eqs. (46–49)) as functions of the dimensionless numbers. Also, both contributions will be compared between them. Finally, the origin of the instability is explained.

3. Analysis of the thermoacoustic source

The volume contribution of the thermoacoustic source was expressed in the previous section as:

$$\alpha_{v.c.} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \dot{q}_0 \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{S_{tk}}{(S_{tk}^2 + 1) S_r^2} \left[A_{v.c.} + e^{-S_{tk} S_c} B_{v.c.} + e^{\frac{Sh_0+2}{4Sh_0} \frac{D_i^2-D_r^2}{D_0^2}} C_{v.c.} + e^{-S_c/S_{tk}} D_{v.c.} \right] dx \quad (51)$$

and the boundary contribution as:

$$\alpha_{b.c.} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \dot{q}_{D^2,0} \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{S_{tk}}{(S_{tk}^2 + 1) S_r^2} \left[A_{b.c.} + e^{\frac{Sh_0+2}{4Sh_0} \frac{D_i^2-D_r^2}{D_0^2}} C_{b.c.} + e^{-S_c/S_{tk}} D_{b.c.} \right] dx \quad (52)$$

Two functions are introduced for the following study in this section, such as:

$$\mathcal{F}_{v.c.} = \frac{S_{tk} \left[A_{v.c.} + e^{-S_{tk} S_c} B_{v.c.} + e^{\frac{Sh_0+2}{4Sh_0} \frac{D_i^2-D_r^2}{D_0^2}} C_{v.c.} + e^{-S_c/S_{tk}} D_{v.c.} \right]}{(S_{tk}^2 + 1) S_r^2} \quad 2.82 < Sh_0 < 5.20 \quad (60)$$

$$\mathcal{F}_{b.c.} = \frac{2S_c(S_c - S_r)S_{tk}}{S_r} \frac{\left[A_{b.c.} + e^{\frac{Sh_0+2}{4Sh_0} \frac{D_i^2-D_r^2}{D_0^2}} C_{b.c.} + e^{-S_c/S_{tk}} D_{b.c.} \right]}{(S_{tk}^2 + 1) S_r^2} \quad (54)$$

which give:

$$\alpha_{v.c.} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \dot{q}_0 \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{\mathcal{F}_{v.c.}}{a_0 \rho} dx \quad (55)$$

$$\alpha_{b.c.} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \dot{q}_{D^2,0} \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{\mathcal{F}_{b.c.}}{a_0 \rho} dx \quad (56)$$

and for the thermoacoustic growth rate $\alpha_{alu} = \alpha_{v.c.} + \alpha_{b.c.}$:

$$\alpha_{alu} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{(\dot{q}_0 \mathcal{F}_{v.c.} + \dot{q}_{D^2,0} \mathcal{F}_{b.c.})}{a_0 \rho} dx \quad (57)$$

The dependency of the functions $\mathcal{F}_{v.c.}$ and $\mathcal{F}_{b.c.}$ on the dimensionless numbers (S_r , S_c , Sh_0 and S_{tk}) is further analyzed. Then, both contributions $\dot{q}_0 \mathcal{F}_{v.c.}$ and $\dot{q}_{D^2,0} \mathcal{F}_{b.c.}$ are compared. Finally, some aspects of the longitudinal variation of the mean flow are taken in account in the analysis of the thermoacoustic source to explain the instability triggering.

3.1. Dependency on the dimensionless numbers

Casalis *et al.* [22] proposed the following ranges for the dimensionless numbers:

$$S_c \in [0, 20], \quad S_{tk} \in [0, 20], \quad S_r = [80; 100; 150; 300] \quad (58)$$

Also, for a computation case very close to that of Genot *et al.* [11], Casalis *et al.* [22] proposed an expression for the Sherwood number based on numerical simulations of solid rocket motors:

$$Sh_0 = 2.82 + 0.34x \quad (59)$$

This longitudinal variation of the Sherwood number is due to the acceleration of the mean flow all along the tube [5, 28]. The Sherwood number Sh_0 allows to model the external convection around the droplet, accelerating the evaporation [20]. As the chemistry is assumed

infinitely fast [10, 11], an increase of the Sherwood number corresponds to an increase of the flame surface around the droplets. Knowing that $0 < x < L = 7 \text{ m}$ in the combustion chamber (Fig. 1), Equation (59) gives:

And, at the first quarter ($x = L/4$) and at the third quarter ($x = 3L/4$), the Sherwood number equals to:

$$Sh_0(L/4) = 3.415, \quad Sh_0(3L/4) = 4.605 \quad (61)$$

This means that the mean flame surface around the individual burning droplets is smaller at the first quarter of the chamber than at the third quarter as illustrated in Figure 5.

The functions $\mathcal{F}_{v.c.}$ and $\mathcal{F}_{b.c.}$ are illustrated Figures 6–13 as functions of the Stokes number S_{tk} and the combustion Strouhal number S_c for different flow Strouhal numbers $S_r = [80; 100; 150; 300]$. Each figure has two subfigures: the left one corresponds to the behavior of the function $\mathcal{F}$ at the first quarter of the chamber (for $Sh_0 = 3.415$ and $x = L/4$) and the right one to the behavior of the function $\mathcal{F}$ at the third quarter (for $Sh_0 = 4.605$ and $x = 3L/4$). The volume contribution $\mathcal{F}_{v.c.}$ is plotted in Figures 6–9 and the boundary contribution $\mathcal{F}_{b.c.}$ in Figures 10–13. To control the instability, the norm of the functions $|\mathcal{F}_{v.c.}|$ and $|\mathcal{F}_{b.c.}|$ should be minimized.

Figures 6–9 show that $\mathcal{F}_{v.c.}$ is strictly positive whatever the combustion Strouhal numbers S_c , the Stokes number S_{tk} , the flow Strouhal number S_r or the x -position (first quarter or third quarter of the chamber). To minimize the function $\mathcal{F}_{v.c.} = |\mathcal{F}_{v.c.}|$, the combustion Strouhal numbers S_c must be low, the Stokes number S_{tk} close to 1 and the flow Strouhal number S_r high. Moreover, the values of $\mathcal{F}_{v.c.}$ seems very similar at the first quarter or the third quarter of the chamber.

Figures 10–13 highlight the alternative (positive and negative) behavior of the boundary contribution $\mathcal{F}_{b.c.}$ as a function of the combustion Strouhal number S_c . When $\mathcal{F}_{b.c.}$ is negative, it means that the boundary contribution may damp the volume contribution which is strictly positive. Moreover, the boundary contribution is very weakly dependent on the Stokes number S_{tk} but it seems better to minimize its value to minimize the boundary contribution $|\mathcal{F}_{b.c.}|$. The boundary contribution $|\mathcal{F}_{b.c.}|$ decreases with the Strouhal number S_r as the volume contribution.

In following, volume contribution $\mathcal{F}_{v.c.}$ and boundary contribution $\mathcal{F}_{b.c.}$ are compared for $S_r = 100$.

Equation (57) gives an expression for the thermo-

coustic growth rate:

$$\alpha_{alu} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{1}{a_0 \rho} (\dot{q}_0 \mathcal{F}_{v.c.} + \dot{q}_{D^2,0} \mathcal{F}_{b.c.}) dx \quad (62)$$

To compare boundary and volume contribution, the function $\mathcal{G}$ is introduced:

$$\mathcal{G} = \frac{\dot{q}_{D^2,0} \mathcal{F}_{b.c.}}{\dot{q}_0 \mathcal{F}_{v.c.}} \quad (63)$$

in such way that:

$$\alpha_{alu} = \frac{2(\gamma - 1)}{\int_0^L \psi^2 dx} \int_0^L \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{1}{a_0 \rho} \dot{q}_0 \mathcal{F}_{v.c.} (1 + \mathcal{G}) dx \quad (64)$$

If $|\mathcal{G}| > 1$, it means that the boundary contribution is higher than the volume contribution. If $-1 < \mathcal{G} < 0$, it means that the boundary of the droplet cloud can damp pressure oscillations but it is maybe not sufficiently powerful to avoid the instability. If $\mathcal{G} < -1$, the thermoacoustic instability is avoided (in the absence of other acoustic sources). For $0 < \mathcal{G}$, the boundary contribution can drive thermoacoustic instabilities. The ratio between the heat release rates Eq. (50) allows to rewrite the expression of $\mathcal{G}$ (Eq. (63)):

$$\mathcal{G} = \frac{\dot{q}_{D^2,0} \mathcal{F}_{b.c.}}{\dot{q}_0 \mathcal{F}_{v.c.}} \approx \frac{\mathcal{F}_{b.c.}}{\mathcal{F}_{v.c.} \sqrt{\frac{(D_i/D_r)^2 + 1}{2}}} \quad (65)$$

Figure 14 illustrates the variable $\mathcal{G}$ as a function of the Stokes number S_{tk} and the combustion Strouhal number S_c . The flow Strouhal number is imposed to $S_r = 100$ and the ratio between the injection diameter D_i and the residue diameter D_r is fixed arbitrary to $D_i/D_r = 2$ [10, 23]. For $S_r = 100$, $D_i/D_r = 2$ and most of (S_r, S_c) , $\mathcal{G} > -1$, so the thermoacoustic instability may appear. Therefore, its triggering basically depends on the equilibrium between the acoustic losses and the thermoacoustic source. Note that $\lim_{S_{tk} \rightarrow 0} \mathcal{G} = 0$ which means that the boundary contribution can be neglected for low S_{tk} . Finally, the alternative behavior of $\mathcal{G}$ (positive or negative) shows that, for some values of S_c , the thermoacoustic instability can be easier controlled.

In the following subsection, the acoustic mode and the mean flow inside the x -integral of Eq. (57) will be considered. By this way, new explanations on the thermoacoustic instability triggering will be proposed.

3.2. Origin of the thermoacoustic instability

In Genot *et al.* [23], Gallier and Godfroy [10] or Orlandi *et al.* [8], the first mode is usually excited which

corresponds to a wave length $k = \pi/L$. The first longitudinal mode $\psi = \cos(kx)$ is illustrated in Figure 15. From Eq. (57), the product of $\cos(kx)$ with $\sin(kx)$ can be identified. This product is also plotted in Figure 15 all along the combustion chamber. It is maximum at the first quarter and minimum at the third quarter and the following equality can be observed:

$$\sin(3kL/4) \cos(3kL/4) = -\sin(kL/4) \cos(kL/4) \quad (66)$$

The first quarter and the third quarter correspond to the extrema of the thermoacoustic coupling for the first longitudinal mode and the first half of the motor is destabilizing (positive coupling) and the second half is stabilizing (negative coupling) [11, 23]. To observe an instability, the first half (or the first quarter) is more destabilizing than the second half (or the third quarter) stabilizes. Moreover, the function $\cos(kx) \sin(kx)$ at the first quarter and the third quarter strictly compensate each other (see Figure 15 and Eq. (66)). So this function $\cos(kx) \sin(kx)$ cannot explain how the thermoacoustic coupling can drive thermoacoustic instabilities. Also, the results of the functions $\mathcal{F}_{v.c.}$, $\mathcal{F}_{b.c.}$ or $\mathcal{G}$ show quite similar results between the first quarter and the third quarter (Figs. 6–14) of the chamber which means that this imbalance between first quarter and third quarter is induced by other terms of Eq. (62).

To better understand how the thermoacoustic coupling can trigger an instability, Eq. (62) needs to be integrated in the x -direction. The heat release rate $\dot{q}_0$ has been integrated radially (Eq. (41)). As the droplet trajectory can be assumed quasi-radial [11] and the aluminum mass injected is assumed uniform along the chamber [11], the growth rate α_{alu} can be rewritten as following (Eq. (62)):

$$\alpha_{alu} = \frac{2(\gamma - 1) \dot{q}_0}{\int_0^L \psi^2 dx} \int_0^L \frac{(-\delta u_{p,0}) \cos(kx) \sin(kx)}{|\delta \mathbf{u}_p|_0^2} \frac{1}{a_0 \rho} \mathcal{F}_{v.c.} (1 + \mathcal{G}) dx \quad (67)$$

The thermoacoustic growth rate α_{alu} (Eq. (67)) is simplified to the following expression (considering Eq. (66)):

$$\alpha_{alu} \propto \left\{ \frac{-\delta u_{p,0}}{|\delta \mathbf{u}_p|_0^2 a_0 \rho} \mathcal{F}_{v.c.} (1 + \mathcal{G}) \right\}_{x=L/4} - \left\{ \frac{-\delta u_{p,0}}{|\delta \mathbf{u}_p|_0^2 a_0 \rho} \mathcal{F}_{v.c.} (1 + \mathcal{G}) \right\}_{x=3L/4} \quad (68)$$

To estimate the mean flow variation along the chamber (in Eq. (68)), an expression of the Sherwood number Sh_0 was proposed (Eq. (59)). Moreover, the Sherwood number Sh_0 is given by the Ranz-Marshall correlation

[20]:

$$\text{Sh}_0 = 2.82 + 0.34x, \quad \text{Sh}_0 = 2 + 0.6\text{Re}_{p,0}^{1/2}\text{Pr}^{1/3}, \quad \text{Re}_{p,0} = \frac{\rho|\delta\mathbf{u}_p|_0\delta u_{p,0}}{|\delta\mathbf{u}_p|_0^2 a_0 \rho} \mathcal{F}_{v.c.}(1 + \mathcal{G}) \quad (69)$$

With Eq. (69), the equation Eq. (68) can be numerically computed. To do this, some physical quantities should be estimated. Gallier and Godfroy [10] proposed the following values:

$$\text{Pr} = 0.4, \quad \mu = 9.1 \times 10^{-5} \text{ Pa.s}, \quad a_0 = 1073 \text{ m/s}, \quad D_0 \sim 100 \mu\text{m} \quad (70)$$

In addition, the droplets drag in the gas and knowing that they are injected radially at equilibrium with the gas, another assumption can be made:

$$|\delta\mathbf{u}_p|_0 \sim -\delta u_{p,0} \quad (71)$$

which gives:

$$-\frac{\delta u_{p,0}}{|\delta\mathbf{u}_p|_0^2 a_0 \rho} \sim -\frac{1}{\delta u_{p,0} a_0 \rho} \quad (72)$$

and finally, with Eq. (69):

$$-\frac{\delta u_{p,0}}{|\delta\mathbf{u}_p|_0^2 a_0 \rho} \approx \frac{1}{3.46 \times 10^3 + 2.86 \times 10^3 x + 1.54 \times 10^3 x^2} \quad (73)$$

At the first quarter and at the third quarter, it is equal to:

$$\left\{ -\frac{\delta u_{p,0}}{|\delta\mathbf{u}_p|_0^2 a_0 \rho} \right\}_{x=L/4} = 0.076 \times 10^{-3}, \quad \left\{ -\frac{\delta u_{p,0}}{|\delta\mathbf{u}_p|_0^2 a_0 \rho} \right\}_{x=3L/4} = 0.16 \times 10^{-3} \quad (74)$$

In the following, the effect of the mean flow acceleration on the source along the tube on the thermoacoustic source (quantified by Eq. (68)) is highlighted. Note that the combustion Strouhal number is proportional to the inverse of the Sherwood number (Eq. (8) and Eq. (33)):

$$\text{Sc} \propto \frac{1}{\text{Sh}_0}, \quad \text{Sc}(3L/4) = \text{Sc}(L/4) \frac{\text{Sh}_0(L/4)}{\text{Sh}_0(3L/4)} \quad (75)$$

So, the results are plotted as functions of the combustion Strouhal number at the first quarter and the x -position is considered in the computation of the combustion Strouhal number.

Figure 16 illustrates the following quantities which correspond to the thermoacoustic source (see Eq. (68)):

Left subfigure:

$$\left\{ \frac{-\delta u_{p,0}}{|\delta\mathbf{u}_p|_0^2 a_0 \rho} \right\}_{x=L/4} \left(\{\mathcal{F}_{v.c.}(1 + \mathcal{G})\}_{x=L/4} - \{\mathcal{F}_{v.c.}(1 + \mathcal{G})\}_{x=3L/4} \right) \quad (76)$$

Right subfigure:

$$\frac{\rho|\delta\mathbf{u}_p|_0\delta u_{p,0}}{|\delta\mathbf{u}_p|_0^2 a_0 \rho} \mathcal{F}_{v.c.}(1 + \mathcal{G}) \Big|_{x=L/4} - \left\{ \frac{-\delta u_{p,0}}{|\delta\mathbf{u}_p|_0^2 a_0 \rho} \mathcal{F}_{v.c.}(1 + \mathcal{G}) \right\}_{x=3L/4} \quad (77)$$

In other words, in the left subfigure of Figure 16, the longitudinal variation of the mean flow is neglected for the thermoacoustic source and assumed equal to the mean flow at the first quarter of the tube. For the right subfigure, the mean flow acceleration is considered. The flow Strouhal number is fixed to $\text{Sr} = 100$ and the residue droplet diameter is equal to $D_r = D_i/2$.

The left subfigure highlights an alternative behavior of the thermoacoustic source α_{alu} with negative and positive values for different combustion Strouhal numbers Sc . This means that, for a uniform mean flow in the x -direction, the thermoacoustic coupling can be a source (red zone) or an acoustic loss (blue zone) and the thermoacoustic instability can be easily avoided just changing the combustion Strouhal number Sc . For a mean flow accelerating along the x -direction (which is typical of a solid rocket motor [5]), positive zones (in red) are larger than the negative ones (in blue). The probability of observing thermoacoustic instability is therefore higher. Negative values are related to the contribution of the boundary (see section 3.1 and Fig. 14). Therefore, it shows that the acceleration of the mean flow (which drives the mean flame surface) along the chamber is the origin of the thermoacoustic instability triggering. Note that the thermoacoustic coupling α_{alu} is highly dependent on the combustion Strouhal number Sc . So, to avoid a thermoacoustic instability with an accelerating mean flow, the combustion Strouhal number needs to be chosen with caution to minimize α_{alu} (white and blue zones in the right subfigure of Fig. 16).

4. Conclusion

This paper is fully focused on theoretical developments and model reductions to study the aluminum combustion instability for any Stokes number, flow Strouhal number, combustion Strouhal number, Sherwood number, global heat release or ratio between the initial diameter and the residue diameter. A new model for the heat release rate fluctuations, induced by the aluminum combustion response to acoustic oscillations, was proposed. This model was compared with a more complex model and a numerical simulation published in a previous article. It opens up new perspectives to express analytically the growth rate related to the thermoacoustic coupling which quantifies the thermoacoustic

source. Two contributions to the thermoacoustic source were considered: a volume contribution corresponding to the way the droplets are burnt during their life and a boundary contribution linked to the end-combustion dynamics. After that, a dependency of the thermoacoustic source on parameters was identified: (i) a flow Strouhal number, (ii) a Stokes number, (iii) a combustion Strouhal number, (iv) the ratio between the initial droplet diameter and the droplet diameter after combustion, (v) the excited mode and (vi) the acceleration of the mean flow. Then, some analyses were provided. The thermoacoustic source decreases with the flow Strouhal number. It is weakly dependent on the Stokes number and highly dependent on the combustion Strouhal number. It has been shown that the ratio between the diameters controls the ratio between the volume and the boundary contributions. Finally, considering a stationary half-wave (first longitudinal mode usually observed in a solid rocket motor), it was clearly shown that the thermoacoustic coupling is a source of pressure oscillations due the acceleration of the mean flow along the combustion chamber. Knowing that the mean flow controls the flame surfaces around the individual burning droplets, it leads to say that a lower mean flame surface in the positive thermoacoustic coupling zone can drive a thermoacoustic instability. In perspective to this work, it should be nice to further analyze the dependency of the thermoacoustic source on the aluminum combustion model, to compare the source to the acoustic losses, to develop a low-order tool to optimize a solid rocket motor, to consider a droplet size distribution and to analyze the non-linear effects of the aluminum combustion dynamics.

Acknowledgments

The author would like to thank Thierry Schuller from CNRS/IMFT for fruitful discussions and Ajmal Mohamed from ONERA for his support.

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Figure 1: Geometry sketch of a generic solid rocket motor with aluminum combustion location (red zone) and streamlines of the steady gas flow.

Figure 2: Models of the heat release rate fluctuations in the aluminum combustion zone: model of Gallier and Godfroy on the left, model of Genot *et al.* on the right

Figure 3: Modulus (figure left) and phase angle (figure right) of the heat release rate fluctuations $\hat{q}$ at the first quarter of the chamber $x/L = 1/4$, in the aluminum combustion zone

Figure 4: Modulus (figure left) and phase angle (figure right) of the heat release rate fluctuations $\hat{q}$ at the third quarter of the chamber $x/L = 3/4$, in the aluminum combustion zone

Figure 5: Acceleration of the mean flow which controls the mean flame surface around the burning droplet along the motor

Figure 6: Volume contribution $\mathcal{F}_{v.c.}$ at the first quarter (left) and third quarter (right) for $S_r = 80$

Figure 7: Volume contribution $\mathcal{F}_{v.c.}$ at the first quarter (left) and third quarter (right) for $S_r = 100$

Figure 8: Volume contribution $\mathcal{F}_{v.c.}$ at the first quarter (left) and third quarter (right) for $S_r = 150$

Figure 9: Volume contribution $\mathcal{F}_{v.c.}$ at the first quarter (left) and third quarter (right) for $S_r = 300$

Figure 10: Boundary contribution $\mathcal{F}_{b.c.}$ at the first quarter (left) and third quarter (right) for $S_r = 80$

Figure 11: Boundary contribution $\mathcal{F}_{b.c.}$ at the first quarter (left) and third quarter (right) for $S_r = 100$

Figure 12: Boundary contribution $\mathcal{F}_{b.c.}$ at the first quarter (left) and third quarter (right) for $S_r = 150$

Figure 13: Boundary contribution $\mathcal{F}_{b.c.}$ at the first quarter (left) and third quarter (right) for $S_r = 300$

Figure 14: Boundary contribution relative to the volume contribution $\mathcal{G}$ at the first quarter (left) and third quarter (right) for $S_r = 100$ and $D_i = 2D_r$

Figure 15: Acoustic mode ($\cos(kx)$ with $k = \pi/L$ for the first longitudinal mode) and the longitudinal distribution of the thermoacoustic coupling ($\sin(kx) \cos(kx)$) along the combustion chamber ($0 < x < L$)

Figure 16: Sum of the first-quarter contribution and the third-quarter contribution to the thermoacoustic source without the longitudinal evolution of $\rho \delta u_{p,0}(x) = \rho \delta u_{p,0}(L/4)$ (left) and with the longitudinal evolution of $\rho \delta u_{p,0}(x)$ (right) for $S_r = 100$ and $D_i = 2D_r$