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EXPONENTIAL SUMS WITH MULTIPLICATIVE COEFFICIENTS WITHOUT THE RAMANUJAN CONJECTURE

YUJIAO JIANG, GUANGSHI LÜ, AND ZHIWEI WANG

ABSTRACT. We study the exponential sum involving multiplicative function f under milder conditions on the range of f , which generalizes the work of Montgomery and Vaughan. As an application, we prove cancellation in the sum of additively twisted coefficients of automorphic L -function on GL_m ($m \geq 4$), uniformly in the additive character.

1. INTRODUCTION

Let $\mathcal{M}$ be the class of all complex valued multiplicative functions. For $f \in \mathcal{M}$, the exponential sum involving multiplicative function f is defined by

$$S(N, \alpha) := \sum_{n \leq N} f(n) e(n\alpha), \quad (1.1)$$

where $e(t) = e^{2\pi it}$.

The problem of estimating $S(N, \alpha)$ has attracted several mathematicians. Daboussi [8] first studied a class of 1-bounded multiplicative functions $f \in \mathcal{F}$, where $\mathcal{F} \subseteq \mathcal{M}$ denotes the set of those multiplicative functions f with $|f(n)| \leq 1$. He proved that if $|\alpha - a/q| \leq 1/q^2$ for some $(a, q) = 1$ and $3 \leq q \leq (N/\log N)^{1/2}$, then one has

$$S(N, \alpha) \ll \frac{N}{(\log \log N)^{1/2}} \quad (1.2)$$

uniformly for $f \in \mathcal{F}$. An immediate corollary from Daboussi's result (1.2) is that

$$\lim_{N \rightarrow \infty} \frac{1}{N} S(N, \alpha) = 0$$

holds uniformly for all $f \in \mathcal{F}$.

A related problem is to characterize those functions f such that for every irrational α we have

$$\frac{1}{N} S(N, \alpha) = o\left(\frac{1}{N} \left| \sum_{n \leq N} f(n) \right| \right).$$

We refer the reader to the excellent works of Dupain, Hall and Tenenbaum [10], Fouvry and Tenenbaum [12] for details.

On the other hand, Montgomery and Vaughan [30] studied a more general class of multiplicative functions. More precisely, suppose that multiplicative function f satisfies the following two conditions:

$$|f(p)| \leq A, \quad \text{for all primes } p \quad (1.3)$$

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and

$$\sum_{n \leq N} |f(n)|^2 \leq A^2 N, \quad \text{for all natural numbers } N, \quad (1.4)$$

where A is an arbitrary constant with $A \geq 1$. They proved that if $|\alpha - a/q| \leq 1/q^2$ for some $(a, q) = 1$ and $2 \leq R \leq q \leq N/R$, then

$$S(N, \alpha) \ll \frac{N}{\log N} + \frac{N}{R^{1/2}} (\log R)^{3/2} \quad (1.5)$$

uniformly for f satisfying the conditions (1.3) and (1.4).

As an application of (1.5), they obtained the following celebrated result concerning the upper bound of character sum: under the generalized Riemann hypothesis, one has

$$\sum_{n \leq x} \chi(n) \ll q^{1/2} \log \log q$$

for any non-principal character χ modulo q . This estimate is essentially best possible.

Later, Bachman [1] studied the upper bound of $S(N, \alpha)$ at various contexts. In particular, he improved the upper bound in (1.5) with $(\log R)^{3/2}$ replaced by $(\log R \log \log R)^{1/2}$ for 1-bounded multiplicative function f .

In this paper, we consider a more general class of multiplicative functions f than those in the work of Montgomery and Vaughan [30]. More precisely, f satisfies the following three conditions:

(C.1) The second power-moment of f is bounded by

$$\sum_{n \leq N} |f(n)|^2 \ll N.$$

(C.2) The second power-moment of f at prime values is bounded by

$$\sum_{p \leq N} |f(p)|^2 \log p \ll N.$$

(C.3) f satisfies the following upper bound estimate condition in sieve theory:

$$\sum_{\substack{p \leq N \\ p+h \text{ is prime}}} |f(p)f(p+h)| \ll \frac{h}{\varphi(h)} \cdot \frac{N}{(\log N)^2},$$

where h is any positive integer.

Remark 1.1. It is easy to see that Conditions (C.2) and (C.3) are immediate consequences of Montgomery and Vaughan's bounded condition on prime values $|f(p)| \leq A$ in (1.3), by prime number theory and sieve method (see for example [14, Theorem 3.11]) respectively.

Remark 1.2. Observe that the right-hand side term $(h/\varphi(h))(N/(\log N)^2)$ in Condition (C.3) just coincides with the upper bound of the problems of Twin Prime Conjecture type: there exist infinitely many primes p such that $p+h$ is also a prime number for any even integer h . In fact, many interesting multiplicative functions do have such an upper bound in Condition (C.3).

Remark 1.3. By the Cauchy-Schwarz inequality, we can deduce from Conditions (C.1) and (C.2) that

$$\sum_{n \leq N} |f(n)| \ll N, \quad \sum_{p \leq N} |f(p)| \log p \ll N. \quad (1.6)$$

We continue to use the notation $S(N, \alpha)$ in (1.1) for the exponential sum with multiplicative coefficient $f(n)$, where f satisfies the above three conditions (C.1), (C.2) and (C.3). Using the method employed by Montgomery and Vaughan [30], we first have the following result.

Theorem 1.1. *For any real number α , we suppose that*

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2} \quad (1.7)$$

with $|\theta| \leq 1$, $1 \leq q \leq N$ and $(a, q) = 1$. Then for any multiplicative function f satisfying the three conditions (C.1), (C.2), (C.3), we have

$$S(N, \alpha) \ll \frac{N}{\log N} + \frac{N}{\varphi(q)^{1/2}} + (qN)^{1/2} \left(\log \left(\frac{N}{q} \right) \right)^{3/2}.$$

Remark 1.4. In Theorem 1.1, we relax the bounded condition $|f(p)| \leq A$ for all primes p of Montgomery and Vaughan [30] by a weaker condition (C.2). However, we impose an extra sieve condition (C.3) in order to estimate the sum over primes and shifted primes. As we have pointed out in Remark 1.2, many interesting multiplicative functions have the property in Condition (C.3), and therefore we may apply Theorem 1.1 to more general multiplicative functions.

A distinctive feature of Theorem 1.1 is that it applies to any multiplicative functions without the Ramanujan conjecture. One of the most representative examples is the coefficients of automorphic L -functions. Let $m \geq 2$ be an integer and π be an automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$ with unitary central character. Denote by $\lambda_\pi(n)$ the Dirichlet coefficients of automorphic L -function $L(s, \pi)$ attached to π . We shall be concerned with obtaining estimates for

$$S_\pi(x, \alpha) = \sum_{n \leq x} \lambda_\pi(n) e(n\alpha), \quad (1.8)$$

which are uniform in α . The Ramanujan conjecture asserts that

$$|\lambda_\pi(p)| \leq m$$

for all primes p , which agrees with the condition (1.3). However, this conjecture is yet unsettled and rather farther out of reach at the present time. Thus, the remarkable result of Montgomery and Vaughan in [30] is not available to investigate the sum (1.8). Instead, the coefficients $\lambda_\pi(n)$ may fit in with the conditions of our theorem 1.1.

The uniform bounds for (1.8) have some connections with many important problems, such as the shifted convolution sums, the subconvexity problem and moments of L -functions. For instance, it was encountered in the works of Harcos [15], Jutila [22] and Munshi [31] on the shifted sum for cusp forms. It also appeared when Blomer [3] and Munshi [32] studied the

subconvexity bound of the twisted L -functions on different aspects. Moreover, as discussed by Miller in [27], the uniform bound for (1.8) is closely related to the second moment

$$\int_{-T}^T \left| L\left(\frac{1}{2} + it, \pi\right) \right|^2 dt.$$

The study of the sum (1.8) is well understood when the $\lambda_\pi(n)$ are the normalized Fourier coefficients of a modular or Maass form on the upper half plane, i.e. an automorphic form on GL_2 . In the case of cusp forms, one has the classical estimate (see, for example, [19, Theorem 8.1])

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll_\pi x^{\frac{1}{2}} \log x \quad (1.9)$$

for any $\alpha \in \mathbb{R}$. The estimate above is quite sharp, and the exponent $1/2$ cannot be reduced for all α in view of the L^2 -norm of $S_\pi(x, \alpha)$

$$\int_0^1 \left| S_\pi(x, \alpha) \right|^2 d\alpha = \sum_{n \leq x} |\lambda_\pi(n)|^2.$$

Note that the bound in (1.9) holds uniformly in α , which allows us to draw some interesting consequences. For example, by way of additive characters we can get the same bound for the sum of the Fourier coefficients restricted to any arithmetic progression. Moreover, the proof for this case is fairly straightforward, depending only on an estimate for the size of the form.

In the case of GL_3 , Miller [27] first gave an impressive result and showed that

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll_\pi x^{\frac{3}{4} + \varepsilon},$$

where the implied constant depends only on π and ε . The key tools used in this proof are the Voronoï summation for GL_3 developed by Miller and Schmid [28] and Weil's estimate for Kloosterman sums. Moreover, Miller [27] pointed out that “Future improvements would necessarily obtain cancellation in sums of products of the a_n with Kloosterman sums. We are unable to prove any interesting statements for GL_m , $m > 3, \dots$ ”

Recently, the first and second authors [21] generalized the Bourgain-Sarnak-Ziegler criterion (see [4] or [7]), which plays an important role in proving many cases of Sarnak's disjointness conjecture. As an application, they considered the additively twisted sums (1.8) on GL_m ($m \geq 4$) and showed that under Hypothesis H

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll x \frac{\log \log x}{\log x},$$

where the implied constant depends only on π . Here Hypothesis H posed by Rudnick and Sarnak [35] is a technical hypothesis concerning the coefficients of $L(s, \pi)$ and remains open except for some several cases.

Our goal here is to unconditionally establish the following stronger result.

Theorem 1.2. *Suppose π is an automorphic irreducible cuspidal representation of GL_m with $m \geq 4$ over $\mathbb{Q}$ with unitary central character. Then we have*

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll \frac{x}{\log x} \quad (1.10)$$

uniformly in $\alpha \in \mathbb{R}$, where the constant in the $\ll$ -symbol depends only on π .

Remark 1.5. One can check from the arguments given below that the dependence in the conductor $C(\pi)$ is at most polynomial, which may be crucial for certain applications. More precisely, we obtain that if $x \gg C(\pi)^{72m^2}$, then

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll \exp \left(c_0 m \frac{\log C(\pi)}{\log \log C(\pi)} \right) \frac{x}{\log x}, \quad (1.11)$$

where $c_0 > 0$ is a constant depending on m only. With a few efforts, one could optimize the exponent $72m^2$. Moreover, we emphasize that two estimates play an important role in the proof of (1.11). One is Li's upper bound on the Rankin-Selberg L -function (see [23])

$$\kappa = \operatorname{Res}_{s=1} L(s, \pi \times \tilde{\pi}) \ll \exp \left(cm \frac{\log C(\pi \times \tilde{\pi})}{\log \log C(\pi \times \tilde{\pi})} \right),$$

where $c > 0$ is a constant depending on m only. The other is the special case of [39, Theorem 2.4], which states that if $x \gg C(\pi \times \tilde{\pi})^{36m^2}$, then one has

$$\sum_{n \leq x} \Lambda(n) a_{\pi \times \tilde{\pi}}(n) \ll_m x.$$

Theorem 1.2 states that there exist cancellations among the sequence $\{\lambda_\pi(n) e(n\alpha)\}_{n \geq 1}$ in terms of all these elements by taking absolute values, which means that

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) = o \left(\sum_{n \leq x} |\lambda_\pi(n)| \right). \quad (1.12)$$

To illustrate this, We shall present some evidences as follows:

(1) Let $\lambda_\pi(n)$ be the Fourier coefficients of any holomorphic cusp form. Deligne [9] proved the Ramanujan conjecture, which asserts

$$|\lambda_\pi(p)| \leq 2$$

holds for $p \nmid N_\pi$, where N_π is the arithmetic conductor of π . Moreover, the Sato-Tate conjecture predicts the sequence $\{\lambda_\pi(p)\}_{\text{primes } p}$ is equidistributed on $[-2, 2]$ with respect to the Sato-Tate measure, which is successfully proved by Barnet-Lamb, Geraghty, Harris, and Taylor [2]. Then these immediately lead to the asymptotic estimate

$$\sum_{n \leq x} |\lambda_\pi(n)| \asymp \frac{x}{(\log x)^{1 - \frac{8}{3\pi}}}. \quad (1.13)$$

Note that $1 - \frac{8}{3\pi} \approx 0.1512$, which is much less than the exponent 1 of the factor “ $\log x$ ” in Theorem 1.2.

(2) For the general GL_m case, the Ramanujan conjecture and the Sato-Tate conjecture are both open. If we assume the Ramanujan conjecture is true, then one has, for any $(n, N_\pi) = 1$,

$$|\lambda_\pi(n)| \leq d_m(n).$$

This upper bound, together with the prime number theorem for Rankin-Selberg L -functions

$$\sum_{p \leq x} |\lambda_\pi(p)|^2 \log p = (1 + o(1))x,$$

gives

$$\sum_{p \leq x} |\lambda_\pi(p)| \log p \geq \left(\frac{1}{m} + o(1) \right) x.$$

Next we require the celebrated result of Wirsing [40] on mean values of multiplicative functions: if a non-negative multiplicative arithmetic function $f(n)$ satisfies the following properties: (i) $\sum_{p \leq x} (\log p/p) f(p) \sim \kappa \log x$ ($\kappa > 0$); (ii) $f(p) \ll 1$; (iii) $\sum_{p, \nu \geq 2} f(p^\nu) p^{-\nu} < \infty$; (iv) $\sum_{p, \nu \geq 2; p^\nu \leq x} f(p^\nu) \ll x/\log x$ for $\kappa \leq 1$, then we have the estimate

$$\sum_{n \leq x} f(n) \asymp \frac{x}{(\log x)^{1-\kappa}}. \quad (1.14)$$

Under the Ramanujan conjecture, it is clear that the function

$$f(n) = \begin{cases} |\lambda_\pi(n)| & \text{if } (n, N_\pi) = 1, \\ 0 & \text{otherwise} \end{cases}$$

satisfies the above conditions (i)-(iv) with some $\kappa > 1/m$. Hence we have

$$\sum_{n \leq x} |\lambda_\pi(n)| \geq \sum_{\substack{n \leq x \\ (n, N_\pi) = 1}} |\lambda_\pi(n)| \gg \frac{x}{(\log x)^{1-\frac{1}{m}}}, \quad (1.15)$$

which implies that the estimate (1.10) in Theorem 1.2 can give the cancellation (1.12).

Further, we assume that the Ramanujan conjecture and the Sato-Tate conjecture both hold for GL_m , we can do get a better exponent than $1 - 1/m$ in (1.15). Let $\alpha_{1,\pi}(p), \dots, \alpha_{m,\pi}(p)$ denote the Satake parameters associated to π at the finite place p . We parametrize the circle as e^{it} , $0 \leq t < 2\pi$, and write $\alpha_{1,\pi}(p) = e^{i\theta_1}, \dots, \alpha_{m,\pi}(p) = e^{i\theta_m}$ with $\theta_1 + \dots + \theta_m = 0$. The Sato-Tate measure is given in [36] by

$$d\mu(\theta_1, \dots, \theta_{m-1}) = \frac{1}{m!(2\pi)^{m-1}} \prod_{1 \leq j < k \leq m} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_{m-1}.$$

Since $\lambda_\pi(p)$ can be described in terms of the Satake parameters by

$$\lambda_\pi(p) = \alpha_{1,\pi}(p) + \dots + \alpha_{m,\pi}(p),$$

we get

$$\sum_{p \leq x} |\lambda_\pi(p)| \sim \kappa_m \frac{x}{\log x},$$

where the constant κ_m is defined by

$$\kappa_m := \int_{\substack{[0, 2\pi)^{m-1} \\ \theta_1 + \dots + \theta_m = 0}} |e^{i\theta_1} + \dots + e^{i\theta_m}| d\mu(\theta_1, \dots, \theta_{m-1}).$$

By Wirsing's result (1.14), it follows

$$\sum_{n \leq x} |\lambda_\pi(n)| \asymp \frac{x}{(\log x)^{1-\kappa_m}}.$$

Using Matlab, we can compute the constant κ_m case by case. Here $\kappa_2 = \frac{8}{3\pi} \approx 0.8488$, which is consistent with that in (1.13), and $\kappa_3 \approx 0.8911$, $\kappa_4 \approx 0.8853$, $\kappa_5 \approx 0.8863$,

To summarize, under the Ramanujan conjecture,

$$\sum_{n \leq x} |\lambda_\pi(n)| \gg \frac{x}{\log x} (\log x)^{\frac{1}{m}},$$

and in addition, under the Sato-Tate conjecture,

$$\sum_{n \leq x} |\lambda_\pi(n)| \asymp \frac{x}{\log x} (\log x)^{\kappa_m}. \quad (1.16)$$

It seems impossible to obtain the same bound for $\sum_{n \leq x} |\lambda_\pi(n)|$ as in (1.10). And our uniform estimate (1.10) saves a power κ_m of the logarithm in comparison with the order of the sum in (1.16) under the Ramanujan conjecture and the Sato-Tate conjecture.

Inspired by the work of Elliott-Moreno-Shahidi [11] and Holowinsky [16], one may be interested in estimating the sum $\sum_{n \leq x} |\lambda_\pi(n)|$ and establish some unconditional upper or lower bounds. Their key technique is to construct an inequality between $|\lambda_\pi(p)|$ and its powers $|\lambda_\pi(p)|^{2r}$ with some positive integers r . Then it reduces to estimate the average value of $|\lambda_\pi(n)|^{2r}$ by using analytic properties of the symmetric power L -functions associated with π . Unfortunately, until now, very little is known about the properties of corresponding L -functions for the GL_m case with $m \geq 3$, except that the Rankin-Selberg theory can yield

$$\sum_{n \leq x} |\lambda_\pi(n)|^2 \sim c_\pi x.$$

Here c_π is a constant depending on π . However, this only gives

$$\sum_{n \leq x} |\lambda_\pi(n)| \ll x.$$

So similar to the previous results on GL_2 , it may be very interesting, but difficult to achieve an unconditional result with a quite small saving on GL_m ($m \geq 3$).

Overview of proof. For Theorem 1.1, we first transform the exponential sum over integers n into a bilinear sum over (n, p) by the Möbius inversion formula to the von Mangoldt function. By elementary arguments, the proof of Theorem 1.1 is reduced to estimate the following sum

$$\sum_{pn \leq N} f(p)f(n)e(pn\alpha) \log p. \quad (1.17)$$

To do this, we partition the hyperbolic region $\{(p, n) : pn \leq N\}$ into the disjoint union of three parts

$$\{(p, n) : pn \leq N\} = \left(\bigcup_i \mathcal{M}_i \right) \cup \left(\bigcup_{i,j,k} \mathcal{M}_{ijk} \right) \cup \mathcal{E},$$

where $\mathcal{M}_i, \mathcal{M}_{ijk}$ are rectangle segments and $\mathcal{E}$ is the remaining region close to the boundary hyperbola $\{(p, n) : pn = N\}$. It is not difficult to show that the contribution to (1.17) from $(p, n) \in \mathcal{E}$ is negligible. To estimate the contribution from $\mathcal{M}_i, \mathcal{M}_{ijk}$, we introduce a general lemma (see Lemma 2.3 in Section 2.4), which is essentially to give an upper bound for the sum

$$\sum_{n \in \mathcal{Q}} \left| \sum_{p \in \mathcal{P}} f(p)e(pn\alpha) \log p \right|^2 \quad (1.18)$$

for some intervals $\mathcal{P}, \mathcal{Q}$. We first insert a non-negative weight in (1.18) and then square out the inner sum, so that we could gain a logarithmic saving. Condition (C.3) plays an important role in the proof of this lemma.

To prove Theorem 1.2, we will apply the Hardy-Littlewood circle method, which has much in common with techniques pioneered by Vinogradov in connection with the Goldbach ternary problem. It follows different paths depending on the Diophantine nature of α : whether or not it is near a rational number with the denominator sufficiently small. It is not surprising that good bounds for $S_\pi(x, \alpha)$ can be obtained in the first case, i.e., when α belongs to the so-called *major arcs*. This cancellation is closely related to the analytic continuation and functional equation of the twisted automorphic L -functions $L(s, \pi \times \chi)$, where χ is a Dirichlet character. The main challenge is to provide estimates in the other case, i.e. when α belongs to the so-called *minor arcs*. By applying Theorem 1.1, we could directly deduce a non-trivial estimate for $S_\pi(x, \alpha)$ as long as α cannot be approximated by rationals with small denominators. To this end, we also discuss how the coefficients $\lambda_\pi(n)$ satisfy the three conditions described in the framework of Theorem 1.1. For the first two conditions, they may be deduced from the Rankin-Selberg theory and the prime number theorem of Rankin-Selberg L -functions, respectively. The last condition is the most critical and difficult one. We shall verify it for $\lambda_\pi(n)$ by the technique of the double upper-bound sieve, which is one of the main ingredients in our proof of Theorem 1.2. To employ sieve method, we need to give an asymptotic formula for

$$\sum_{\substack{n \leq x \\ n \equiv 0 \pmod{d} \\ n \equiv a \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n),$$

with a main term plus an error term which is small on average over d, q , and we do this in Lemma 6.4 in Section 6.2.

In this paper, we obtain a “ $\log x$ ” saving relative to the trivial bound for the sum $\sum_{n \leq x} \lambda_\pi(n) e(n\alpha)$. This saving essentially comes from the estimate of (1.18). Taking $f(n) = \lambda_\pi(n)$ in (1.18) and then squaring out the inner sum, one could try to pursue high levels of cancellation in the sum

$$\sum_{n \in \mathcal{Q}} \sum_{p, p' \in \mathcal{P}} \lambda_\pi(p) \overline{\lambda_\pi(p')} e((p - p')n\alpha) (\log p) (\log p').$$

But, like Holowinsky [17] in his contribution to the proof of arithmetic Quantum Unique Ergodicity for holomorphic GL_2 newforms by estimating the shifted sum of multiplicative function $\sum_n |\lambda_f(n) \lambda_f(n + \ell)|$ with a saving $(\log x)^\delta$ for some small $\delta > 0$, we choose not to pursue this. Alternatively, we just expect a logarithmic saving, which seems to be the best that one could hope for with this choice. This is another instance in the theory of automorphic forms that follows from this risky gambit. Especially, Holowinsky’s ideas required the Ramanujan conjecture whereas we do not need that here.

2. PROOF OF THEOREM 1.1

2.1. Reduction. In this section, we aim to transform the origin sum $S(N, \alpha)$ over integers in Theorem 1.1 into a bilinear form of the type $\sum_{pn \leq N}$ over prime numbers and integers. The process is standard.

First, by the Cauchy-Schwarz inequality and Condition (C.1) we have

$$\sum_{n \leq N} f(n) e(n\alpha) \log \left(\frac{N}{n} \right) \leq \left(\sum_{n \leq N} \log^2 \left(\frac{N}{n} \right) \right)^{1/2} \left(\sum_{n \leq N} |f(n)|^2 \right)^{1/2} \ll N,$$

which yields

$$S(N, \alpha) \log N = \sum_{n \leq N} f(n) e(n\alpha) \log N \ll \left| \sum_{n \leq N} f(n) e(n\alpha) \log n \right| + N. \quad (2.1)$$

For the right-hand side sum of (2.1), by the identity $\log n = \sum_{m|n} \Lambda(m)$, we obtain the bilinear form of the sum

$$\sum_{mn \leq N} f(mn) \Lambda(m) e(mn\alpha). \quad (2.2)$$

Therefore it suffices to show that

$$\sum_{mn \leq N} f(mn) \Lambda(m) e(mn\alpha) \ll N + \frac{N}{\varphi(q)^{1/2}} \log N + (qN)^{1/2} \left(\log \left(\frac{N}{q} \right) \right)^{3/2} \log N.$$

Note that the above sum $\sum_{m,n}$ vanishes if m is not of the form $m = p^k$ for any prime p and $k \geq 1$. We then try separating two variables m, n in multiplicative function $f(mn)$ by considering the remainder term

$$\mathcal{R}(N) := \sum_{mn \leq N} |f(mn) - f(m)f(n)| \Lambda(m).$$

For $\mathcal{R}(N)$, we only need to consider the case with $(m, n) > 1$, since $f(mn) = f(m)f(n)$ when $(m, n) = 1$. The condition $(m, n) > 1$ reveals that $p|n$ if $m = p^k, k \geq 1$. In fact, by this condition $(m, n) > 1$ we can deduce that each term in $\mathcal{R}(N)$ contributed by $f(mn)$ or $f(m)f(n)$ is negligible. For example, we consider the sum

$$\mathcal{R}_1(N) := \sum_{\substack{mn \leq N \\ (m, n) > 1}} |f(mn)| \Lambda(m) = \sum_{\substack{p^k n \leq N \\ k \geq 1, p|n}} |f(p^k n)| \log p.$$

If $p^j \parallel p^k n$ with $j - k \geq 1$, then $p^{j-k} \parallel n$. So $\mathcal{R}_1(N)$ is estimated by

$$\sum_{p \geq 2} \sum_{j \geq 2} \sum_{k \leq j-1} |f(p^j)| \log p \sum_{\substack{n' \leq N/p^j \\ p \nmid n'}} |f(n')| \ll N \left(\sum_{p \geq 2} \sum_{j \geq 2} \frac{j^2 (\log p)^2}{p^{3j/4}} \right)^{\frac{1}{2}} \left(\sum_{n \geq 2} \frac{|f(n)|^2}{n^{5/4}} \right)^{\frac{1}{2}} \ll N,$$

where we have used the Cauchy-Schwarz inequality and the first estimate in (1.6). So $\mathcal{R}_1(N)$ is negligible. For $\mathcal{R}_2(N) := \sum_{m,n} |f(m)f(n)| \Lambda(m)$, we have the same upper bound. Furthermore, for terms in the initial sum (2.2) with $m = p^k, k \geq 2$, they are also negligible by the same argument as above. Finally, we are left with showing

$$\sum_{pn \leq N} f(p)f(n) e(pn\alpha) \log p \ll N + \frac{N}{\varphi(q)^{1/2}} \log N + (qN)^{1/2} \left(\log \left(\frac{N}{q} \right) \right)^{3/2} \log N. \quad (2.3)$$

To obtain the estimate in (2.3), we shall first split the sum over p, n into three parts (Section 2.2), and then estimate the contribution from each part separately (Sections 2.3, 2.4 and 2.5).

2.2. Proof of (2.3), Preliminary partition.

Lemma 2.1. *We partition the region of points (p, n) with $pn \leq N$ as*

$$\{(p, n) : pn \leq N\} = \left(\bigcup_i \mathcal{M}_i \right) \cup \left(\bigcup_{i,j,k} \mathcal{M}_{ijk} \right) \cup \mathcal{E}$$

where $\mathcal{M}_i, \mathcal{M}_{ijk}$ are rectangular segments of the form

$$\mathcal{M}_i = (0, 2^i] \times \left(\frac{N}{2^{i+1}}, \frac{N}{2^i} \right], \quad \mathcal{M}_{ijk} = \left(\frac{2^j}{k} \cdot 2^i, \frac{2^{j+1}}{2k-1} \cdot 2^i \right] \times \left(\frac{k-1}{2^j} \cdot \frac{N}{2^i}, \frac{2k-1}{2^{j+1}} \cdot \frac{N}{2^i} \right]$$

for $0 \leq i \leq \log N / \log 2$, $1 \leq j \leq J_i$, $2^{j-1} < k \leq 2^j$ with J_i defined by (2.4) below, and $\mathcal{E}$ is the complement of the unions of $\mathcal{M}_i$ and $\mathcal{M}_{ijk}$.

Proof. First, in the region $\{(p, n) : pn \leq N\}$, we take

$$\mathcal{M}_i = (0, 2^i] \times (N/2^{i+1}, N/2^i]$$

with $0 \leq i \leq \log N / \log 2$. Thus the remaining regions are

$$\mathcal{R}_i = \{(x, y) : xy \leq N, x > 2^i, N/2^{i+1} < y \leq N/2^i\}.$$

Let

$$J_i := \min \left\{ i+1, \lceil \log N / \log 2 \rceil - i + 1, \lceil \log(64N/q) / (2 \log 2) \rceil \right\}. \quad (2.4)$$

This choice of J_i ensures that

$$64N/2^{2J_i} \geq q \quad (2.5)$$

which is a necessary condition in what lies ahead. Then, we pick out a series rectangles $\mathcal{M}_{ijk}$ in $\mathcal{R}_i$ for $j = 1, 2, \dots, J_i$ and $2^{j-1} < k \leq 2^j$, where $\mathcal{M}_{ijk}$ are defined by the iterations of steps: first we set

$$\mathcal{M}_{i12} = \left(2^i, \frac{4}{3} \cdot 2^i \right] \times \left(\frac{1}{2} \cdot \frac{N}{2^i}, \frac{3}{4} \cdot \frac{N}{2^i} \right]$$

for each $0 \leq i \leq \log N / \log 2$. In each $\mathcal{R}_i$, we are left with two regions after discarding the rectangle $\mathcal{M}_{i12}$, and we repeat the same iteration process in these two regions. After $1 \leq j \leq J_i$ iterations, we obtain $1 + 2 + 4 + \dots + 2^{j-1}$ rectangles and each of them is of the form:

$$\mathcal{M}_{ijk} = \left(\frac{2^j}{k} \cdot 2^i, \frac{2^{j+1}}{2k-1} \cdot 2^i \right] \times \left(\frac{k-1}{2^j} \cdot \frac{N}{2^i}, \frac{2k-1}{2^{j+1}} \cdot \frac{N}{2^i} \right].$$

Here, for example, we just provide a figure of partitions for $N = 8$, $i \leq 3$.

FIGURE 1. Hyperbola $pn = 8$ and partitions $\mathcal{M}_{ijk}$, $0 \leq i \leq 3$.

Moreover, one observes that, by the definition of J_i in (2.4)

$$J_i = \begin{cases} i + 1 & \text{if } 2^{2i} \leq N, 2^{2i} \leq 16N/q, \\ \lceil \log N / \log 2 \rceil - i + 1 & \text{if } 2^{2i} > N, 2^{2i} > Nq/16, \\ \lceil \log(64N/q)/(2 \log 2) \rceil & \text{if } 16N/q < 2^{2i} < Nq/16. \end{cases}$$

Hence, we see that each $\mathcal{M}_{ijk}$ is of the form $(P', P''] \times (N', N'']$ with $P'' - P' \geq 1/4$, $N'' - N' \geq 1/4$ and

$$(P'' - P')(N'' - N') \gg \frac{2^{i+j}}{k(2k-1)} \cdot \frac{N}{2^{i+j+1}} \gg \frac{N}{2^{2J_i}} \gg q.$$

Finally, we denote by $\mathcal{E}$ the set of points (p, n) in $\mathcal{R}_i$ after discarding $\mathcal{M}_{ijk}$ for each $0 \leq i \leq \log N / \log 2$, $1 \leq j \leq J_i$ and $2^{j-1} < k \leq 2^j$, i.e.

$$\mathcal{E} := \bigcup_i \left(\mathcal{R}_i / \left(\bigcup_{j,k} \mathcal{M}_{ijk} \right) \right) = \left(\{(p, n) : pn \leq N\} / \bigcup_i \mathcal{M}_i \right) / \left(\bigcup_{i,j,k} \mathcal{M}_{ijk} \right).$$

More precisely, we may see that $\mathcal{E}$ is the union of three regions, i.e., $\mathcal{E} = \bigcup_{l=1,2,3} \mathcal{E}_l$ where $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ is the union of $\mathcal{H}_i = \{(p, n) : (p, n) \in \mathcal{E}, n \in (N/2^{i+1}, N/2^i]\}$ with $J_i = i + 1$, $J_i = \lceil \log N / \log 2 \rceil - i + 1$ and $J_i = \lceil \log(64N/q)/(2 \log 2) \rceil$, respectively. We complete the partition in Lemma 2.1. $\square$

In the following three sections, we will estimate the contribution of points (p, n) from $\mathcal{E}$, $\mathcal{M}_i$, $\mathcal{M}_{ijk}$ to the sum (2.3) separately.

2.3. Proof of (2.3) when $(p, n) \in \mathcal{E}$. We first show that the contribution of $(p, n) \in \mathcal{E}$ to the sum (2.3) is negligible.

Lemma 2.2. For $(p, n) \in \mathcal{E} = \bigcup_{l=1,2,3} \mathcal{E}_l$ with $\mathcal{E}$, $\mathcal{E}_l$ defined above, we have the following upper bound

$$\sum_{(p,n) \in \mathcal{E}} f(p)f(n)e(pn\alpha) \log p \ll N + (qN)^{1/2} \left(\log \left(\frac{N}{q} \right) \right)^{3/2} \log N. \quad (2.6)$$

Proof. It suffices to show that (2.6) holds with $(p, n) \in \mathcal{E}_l$ for $l = 1, 2, 3$, respectively. Since the arguments for $(p, n) \in \mathcal{E}_l$, $l = 1, 2, 3$ are similar, we just consider $(p, n) \in \mathcal{E}_1$, in which case we have $J_i = i + 1$ and $2^{2i} \leq 16N/q$.

For $p \in (2^i, 2^{i+1}]$, after J_i iterations we partition the interval $(2^i, 2^{i+1}]$ into $1 + 1 + 2 + 2^2 + \dots + 2^{J_i-1} = 2^{J_i}$ subintervals. For $(p, n) \in \mathcal{E}_1$ in one of such intervals $(a, b]$, we may encounter two cases: either $(a, b]$ coincides with one of the intervals which is generated by the last J_i -th iteration, i.e. $p \in (a, b] = (2^{i+J_i}/k, 2^{i+J_i+1}/(2k-1)]$ for $2^{J_i-1} < k \leq 2^{J_i}$, or the left endpoint “ a ” of interval $(a, b]$ is just the right endpoint of the intervals which is generated by the last J_i -th iteration, i.e. $a = 2^{i+J_i+1}/(2k-1)$. So for any fixed such p , the number of $(p, n) \in \mathcal{E}_1$ in the first case is at most

$$\frac{N}{p} - \frac{(2k-1)N}{2^{i+J_i+1}} \ll k \frac{N}{2^{i+J_i}} - \left(k - \frac{1}{2}\right) \frac{N}{2^{i+J_i}} \ll \frac{N}{(2^i)^2} \ll \frac{N}{p^2},$$

and in the second case the number of $(p, n) \in \mathcal{E}_1$ is at most

$$\left(k - \frac{1}{2}\right) \frac{N}{2^{i+J_i}} - (k-1) \frac{N}{2^{i+J_i}} \ll \frac{N}{(2^i)^2} \ll \frac{N}{p^2}.$$

Thus, for any fixed p , the number of n with $(p, n) \in \mathcal{E}_1$ is at most $O(N/p^2)$.

Then for $(p, n) \in \mathcal{E}_1$ with $n \in (N/2^{i+1}, N/2^i]$, the interval $(N/2^{i+1}, N/2^i]$ is also partitioned into 2^{J_i} subintervals, and we also encounter two cases. By the same argument, we can deduce that for any fixed such n , the number of $(p, n) \in \mathcal{E}_1$ in the first case is at most

$$\frac{N}{n} - \frac{2^{i+J_i+1}}{(2k-1)} \ll \frac{2^{i+J_i}}{k-1} - \frac{2 \cdot 2^{i+J_i}}{2k-1} \ll \frac{(2^i)^2}{k^2} \ll 1$$

for $n \in ((k-1)N/2^{i+J_i}, (2k-1)N/2^{i+J_i+1}]$, $2^{J_i-1} < k \leq 2^{J_i}$, $J_i = i + 1$, and in the second case the number of $(p, n) \in \mathcal{E}_1$ is at most

$$\frac{2^{i+J_i+1}}{(2k-1)} - \frac{2^{i+J_i}}{k} \ll \frac{(2^i)^2}{k^2} \ll 1.$$

By applying Conditions (C.1), (C.2), the Cauchy-Schwarz inequality and partial summation, we derive that the contribution of $(p, n) \in \mathcal{E}_1$ is estimated by

$$\left(\sum_{(p,n)} |f(n)|^2 \right)^{\frac{1}{2}} \left(\sum_{(p,n)} |f(p) \log p|^2 \right)^{\frac{1}{2}} \ll \left(\sum_{n \leq N} |f(n)|^2 \right)^{\frac{1}{2}} \left(N \sum_{p \leq N} \frac{|f(p) \log p|^2}{p^2} \right)^{\frac{1}{2}} \ll N.$$

Similarly, for $(p, n) \in \mathcal{E}_l$, $l = 2, 3$, we have the upper bounds

$$\sum_{(p,n) \in \mathcal{E}_2} |f(p)f(n)| \log p \ll N$$

and

$$\sum_{(p,n) \in \mathcal{E}_3} |f(p)f(n)| \log p \ll \left(qN(\log q)^2 + qN \log(2N/q) \log q \log \log q \right)^{\frac{1}{2}}$$

with $q \leq N$. This completes the proof of Lemma 2.2. $\square$

2.4. **Proof of (2.3) when $(p, n) \in \bigcup \mathcal{M}_i$.** To obtain the bound in (2.3) on the contribution from $(p, n) \in \bigcup \mathcal{M}_i$, we first introduce a more general lemma.

Lemma 2.3. *For $1 \leq k \leq K$, we define a series of rectangles by $\mathcal{M}(k) := \mathcal{P}(k) \times \mathcal{Q}(k)$, where $\mathcal{P}(k), \mathcal{Q}(k)$ are disjoint intervals of the form*

$$\mathcal{P}(k) = (P'(k), P''(k)], \quad \mathcal{Q}(k) = (Q'(k), Q''(k)].$$

Further, we impose the conditions on these intervals:

$$\begin{aligned} (P'(k), P''(k)] &\subset (0, P), \quad P''(k) - P'(k) \leq X, \\ (Q'(k), Q''(k)] &\subset (0, Q), \quad Q''(k) - Q'(k) \leq Y \end{aligned}$$

for some P, Q, X, Y . Define the exponential sum by

$$I := \sum_{k \leq K} \sum_{(p, n) \in \mathcal{M}(k)} f(p)f(n)e(pn\alpha) \log p,$$

where α satisfies the condition (1.7) in Theorem 1.1. Then for $(a, q) = 1$, $q \leq XY$, we have the estimate

$$I \ll \left(PQY \log(2P) + PQXY/\varphi(q) + PQX + PQq \log(2XY/q) \right)^{\frac{1}{2}}.$$

Proof. Let $\mathcal{M} = \mathcal{P} \times \mathcal{Q} = (P', P''] \times (Q', Q'']$ be one of the above rectangles $\mathcal{M}(k)$. First, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{(p, n) \in \mathcal{M}} f(p)f(n)e(pn\alpha) \log p &\ll \left(\sum_{n \in \mathcal{Q}} |f(n)|^2 \right)^{\frac{1}{2}} \left(\sum_{n \in \mathcal{Q}} \left| \sum_{p \in \mathcal{P}} f(p)e(pn\alpha) \log p \right|^2 \right)^{\frac{1}{2}} \\ &:= \mathcal{S}_1^{\frac{1}{2}} \mathcal{S}_2^{\frac{1}{2}}. \end{aligned}$$

For $\mathcal{S}_1$, it is estimated by Condition (C.1) directly. For $\mathcal{S}_2$, we introduce the weight

$$w(n) = \max \left(0, 2 - \frac{|2n - 2Q' - Y|}{Y} \right) \quad (2.7)$$

as Montgomery and Vaughan do in [30], so that we can save a power of the logarithm compared to the classical upper bound. Observe that $w(n) \geq 1$ for $n \in \mathcal{Q} = (Q', Q'']$, $Q'' - Q' \leq Y$ by the definition of $w(n)$. So we have

$$\begin{aligned} \mathcal{S}_2 &\leq \sum_n w(n) \left| \sum_{p \in \mathcal{P}} f(p)e(pn\alpha) \log p \right|^2 \\ &= \sum_{p, p' \in \mathcal{P}} f(p)\overline{f(p')} (\log p)(\log p') \sum_n w(n) e((p - p')n\alpha). \end{aligned}$$

For the inner sum over n , by the Euler-Maclaurin formula (see for example [20, Corollary 4.3]) and partial summation, it is estimated by

$$\begin{aligned}
\sum_n w(n)e(n\alpha') &= \int_{Q'-Y/2}^{Q'+3Y/2} w(t)(e(t\alpha') + e(t\alpha' + t))dt + O\left(\int_{Q'-Y/2}^{Q'+3Y/2} \frac{dt}{Y(1+\|t\|)}\right) \\
&\ll \frac{1}{|\alpha'|} \int_{Q'-Y/2}^{Q'+3Y/2} (w(t))' e(t\alpha') dt + O(1) \\
&\ll \frac{1}{Y|\alpha'|^2} \int_{Q'-Y/2}^{Q'+3Y/2} d(e(t\alpha')) + O(1) \\
&\ll \frac{1}{Y\|\alpha'\|^2} + O(1)
\end{aligned}$$

with $\alpha' = (p - p')\alpha$. Combining with the trivial bound for $\sum_n w(n)e(n\alpha')$, we obtain

$$\mathcal{S}_2 \ll \sum_{p, p' \in \mathcal{P}} |f(p)f(p')| (\log p)(\log p') \min\left(Y, \frac{1}{Y\|(p - p')\alpha\|^2}\right).$$

Hence, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
I &\ll \left(\sum_{k \leq K} \sum_{n \in \mathcal{Q}(k)} |f(n)|^2\right)^{\frac{1}{2}} \left(\sum_{k \leq K} \sum_{n \in \mathcal{Q}(k)} \left|\sum_{p \in \mathcal{P}(k)} f(p)e(pn\alpha) \log p\right|^2\right)^{\frac{1}{2}} \\
&\ll Q^{\frac{1}{2}} \left(\sum_{k \leq K} \sum_{p, p' \in \mathcal{P}(k)} |f(p)f(p')| \log p \log p' \min\left(Y, \frac{1}{Y\|(p - p')\alpha\|^2}\right)\right)^{\frac{1}{2}} \\
&\ll Q^{\frac{1}{2}} \left(PY \log P + \log^2 P \sum_{0 < h \leq X} \sum_{\substack{p \leq P \\ p+h=p'}} |f(p)f(p')| \min\left(Y, \frac{1}{Y\|h\alpha\|^2}\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

For the innermost sum over p, p' above, we apply Condition (C.3) on the upper bound of sieve theory:

$$\begin{aligned}
I &\ll \left(PQY \log P + PQ \sum_{0 < h \leq X} \frac{h}{\varphi(h)} \min\left(Y, \frac{1}{Y\|h\alpha\|^2}\right)\right)^{\frac{1}{2}} \\
&\ll \left(PQY \log P + PQ \sum_{m \leq X} \frac{1}{m} \sum_{\ell \leq X/m} \min\left(Y, \frac{1}{Y\|m\ell\alpha\|^2}\right)\right)^{\frac{1}{2}},
\end{aligned} \tag{2.8}$$

where we have used the fact $h/\varphi(h) \ll \sum_{m|h} 1/m$. By the expression of α

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a, q) = 1, \quad |\theta| \leq 1,$$

for the innermost sum over ℓ in (2.8), it suffices to study the sum

$$S(q_m, Y) := \sum_{|k| < q_m/2} \min\left(Y, \frac{1}{Y\|\frac{k}{q_m} + \frac{\theta_1(k)}{q_m}\|^2}\right), \quad |\theta_1(k)| < 1$$

with $q_m = q/(q, m)$. To estimate $S(q_m, Y)$, we consider two cases. Without loss of generality, we may assume that $q_m \geq 6$. For $0 \leq |k| \leq 3$ and $q_m/2 - 2 < |k| < q_m/2$, we utilize the trivial bound

$$\sum_{0 \leq |k| \leq 3} Y + \sum_{q_m/2 - 2 < |k| < q_m/2} Y \ll Y.$$

For the case $4 \leq |k| \leq q_m/2 - 2$, we have $\|\frac{k}{q_m} + \frac{\theta_1(k)}{q_m}\| = |\frac{k + \theta_1(k)}{q_m}|$, and further deduce

$$\begin{aligned} \sum_{4 \leq |k| \leq q_m/2 - 2} \min\left(Y, \frac{1}{Y \|\frac{k}{q_m} + \frac{\theta_1(k)}{q_m}\|^2}\right) &\ll \sum_{\substack{4 \leq |k| \leq q_m/2 - 2 \\ |\frac{k + \theta_1(k)}{q_m}| < 1/Y}} Y + \sum_{\substack{4 \leq |k| \leq q_m/2 - 2 \\ |\frac{k + \theta_1(k)}{q_m}| \geq 1/Y}} \frac{1}{Y \|\frac{k}{q_m} + \frac{\theta_1(k)}{q_m}\|^2} \\ &\ll q_m + \frac{1}{Y} \sum_{q_m/Y \leq k \leq q_m/2} \frac{q_m^2}{k^2} \ll q_m. \end{aligned}$$

So we get $S(q_m, Y) \ll Y + q_m$ and

$$\sum_{\ell \leq X/m} \min\left(Y, \frac{1}{Y \|m\ell\alpha\|^2}\right) \ll \min\left(\frac{XY}{m}, \left(\frac{X}{mq_m} + 1\right)(Y + q_m)\right). \quad (2.9)$$

Therefore, we have the upper bound

$$\begin{aligned} \sum_{m \leq X} \frac{1}{m} \sum_{\ell \leq X/m} \min\left(Y, \frac{1}{Y \|m\ell\alpha\|^2}\right) &\ll \sum_{\substack{m \leq X \\ XY/m \leq q_m}} \frac{XY}{m^2} + \sum_{\substack{m \leq X \\ XY/m > q_m}} \frac{1}{m} \cdot \left(\frac{XY}{mq_m} + \frac{X}{m} + Y + q_m\right) \\ &\ll \sum_{r|q} \sum_{s > XY/q} \frac{XY}{s^2 r^2} + \sum_{r|q} \sum_s \frac{XY}{s^2 r q} \\ &\quad + X + Y \log X + \sum_{r|q} \sum_{s < XY/q} \frac{q}{s r^2} \\ &\ll \frac{XY}{\varphi(q)} + X + Y \log X + q \log\left(\frac{2XY}{q}\right). \end{aligned}$$

Inserting this upper bound in (2.8), we complete the proof of Lemma 2.3. $\square$

Remark 2.1. Comparing to the classical estimate

$$\sum_{n \leq N} \min\left(Y, \frac{1}{\|n\alpha + \beta\|}\right) \ll \left(\frac{N}{q} + 1\right)(Y + q \log q),$$

we save a logarithmic factor “ $\log q$ ” in (2.9). This is why we introduce the weight $w(n)$ defined by (2.7) in the sum $\mathcal{S}_2$.

With the help of Lemma 2.3, one can show that the bound in (2.3) holds for $(p, n) \in \bigcup \mathcal{M}_i$.

Lemma 2.4. *For $(p, n) \in \bigcup \mathcal{M}_i$ with rectangles $\mathcal{M}_i$ defined in Lemma 2.1, we have*

$$\sum_{(p, n) \in \bigcup_i \mathcal{M}_i} f(p)f(n)e(pn\alpha) \log p \ll N + \frac{N \log N}{\varphi(q)^{1/2}} + \left(qN \log\left(\frac{N}{q}\right)\right)^{1/2} \log N.$$

Proof. Taking $K = 1$, $P = X = 2^i$, $Q = Y = N/2^i$ in Lemma 2.3, we have $XY = N \geq q$. Then the contribution from $\bigcup \mathcal{M}_i$ is at most

$$\begin{aligned} \sum_{0 \leq i \leq \frac{\log N}{\log 2}} \sum_{(p,n) \in \mathcal{M}_i} f(p)f(n)e(pn\alpha) \log p &\ll \sum_{0 \leq i \leq \frac{\log N}{\log 2}} \left(N \left(\frac{i+1}{2^i} \right)^{1/2} + \frac{N}{\varphi(q)^{1/2}} \right. \\ &\quad \left. + (N2^i)^{1/2} + \left(qN \log \left(\frac{N}{q} \right) \right)^{1/2} \right) \\ &\ll N + \frac{N \log N}{\varphi(q)^{1/2}} + \left(qN \log \left(\frac{N}{q} \right) \right)^{1/2} \log N, \end{aligned}$$

which completes the proof. $\square$

2.5. Proof of (2.3) when $(p, n) \in \bigcup \mathcal{M}_{ijk}$. In this case, Lemma 2.3 is also applicable. Arguments similar to Lemma 2.4 yield:

Lemma 2.5. *For $(p, n) \in \bigcup \mathcal{M}_{ijk}$ with rectangles $\mathcal{M}_{ijk}$ defined in Lemma 2.1, we have*

$$\sum_{(p,n) \in \bigcup_{i,j,k} \mathcal{M}_i} f(p)f(n)e(pn\alpha) \log p \ll N + \frac{N \log N}{\varphi(q)^{1/2}} + (qN)^{1/2} \left(\log \left(\frac{N}{q} \right) \right)^{3/2} \log N.$$

Proof. For each $\mathcal{M}_{ijk}$, $2^{j-1} < k \leq 2^j$, we can apply Lemma 2.3 with

$$K = 2^{j-1}, \quad P = 2^{i+1}, \quad Q = N/2^i, \quad X = 2^{i-j+1}, \quad Y = 32N/2^{i+j},$$

since $XY = 64N/2^{2j} \geq 64N/2^{2J_i} \geq q$ by (2.5). Then the upper bound in Lemma 2.5 follows. $\square$

Proof of Theorem 1.1. Combining Lemmas 2.2, 2.4, 2.5, and the estimate (2.3), we conclude the proof of Theorem 1.1. $\square$

3. BACKGROUND ON L -FUNCTIONS

This section is devoted to reviewing some more or less standard facts about L -functions arising from cuspidal automorphic representations and their Rankin-Selberg convolutions. We refer the reader to [35, Section 2, the appendix] for a more detailed overview.

3.1. Standard L -functions. Let $m \geq 1$ be an integer, and let $\mathcal{A}(m)$ be the set of all cuspidal automorphic representations of GL_m over $\mathbb{Q}$. We consider each $\pi = \otimes_p \pi_p \in \mathcal{A}(m)$ to be normalized so that π has unitary central character which is trivial on the positive reals. Here p ranges over primes.

Let N_π denote the conductor of π . For $\pi \in \mathcal{A}(m)$, the standard L -function $L(s, \pi)$ associated to π is of the form

$$L(s, \pi) = \prod_{p < \infty} L_p(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}.$$

The Euler product and Dirichlet series converge absolutely for $\Re(s) > 1$. For each (finite) prime p , the inverse of the local factor $L_p(s, \pi)$ is a polynomial in p^{-s} of degree $\leq m$

$$L_p(s, \pi)^{-1} = \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\right)$$

for suitable complex numbers $\alpha_{j,\pi}(p)$. With this convention, we have $\alpha_{j,\pi}(p) \neq 0$ for all j whenever $p \nmid N_\pi$, and it might be the case that $\alpha_{j,\pi}(p) = 0$ for some j when $p \mid N_\pi$. At the archimedean place of $\mathbb{Q}$, there are m complex Langlands parameters $\mu_\pi(j)$ from which we define

$$L_\infty(s, \pi) = \pi^{-\frac{ms}{2}} \prod_{j=1}^m \Gamma\left(\frac{s + \mu_\pi(j)}{2}\right).$$

For $\pi \in \mathcal{A}(m)$, it is known that there exists a constant

$$\delta_m \in \left[0, \frac{1}{2} - \frac{1}{m^2 + 1}\right] \quad (3.1)$$

such that

$$|\alpha_{j,\pi}(p)| \leq p^{\delta_m} \quad \text{and} \quad \Re(\mu_\pi(j)) \geq -\delta_m \quad (3.2)$$

for all j and p . The bounds follow from the work of Rudnick and Sarnak [35, Proposition A.1] at the unramified finite places and Luo, Rudnick and Sarnak [24, Theorem 1.2] at the infinite place. These bounds are extended to include analogous bounds at the ramified finite places by Müller and Speh [33]. The generalized Ramanujan and Selberg conjectures assert that δ_m may be taken as 0 in (3.1).

With all the local factors defined as above, we can turn to the functional equation. Let $\tilde{\pi}$ denote the contragredient of $\pi \in \mathcal{A}(m)$, which is also an irreducible cuspidal automorphic representation in $\mathcal{A}(m)$. For each $p \leq \infty$, we have

$$\{\alpha_{j,\tilde{\pi}}(p) : 1 \leq j \leq m\} = \{\overline{\alpha_{j,\pi}(p)} : 1 \leq j \leq m\}$$

and

$$\{\mu_{\tilde{\pi}}(j) : 1 \leq j \leq m\} = \{\overline{\mu_\pi(j)} : 1 \leq j \leq m\}.$$

Define the completed L -function

$$\Lambda(s, \pi) = N_\pi^{s/2} L(s, \pi) L(s, \pi_\infty).$$

Thus, $\Lambda(s, \pi)$ extends to an entire function (except in the case of $\zeta(s)$, which has a simple pole at $s = 1$). Moreover, $\Lambda(s, \pi)$ is bounded in vertical strips and satisfies a functional equation of the form

$$\Lambda(s, \pi) = W(\pi) \Lambda(1 - s, \tilde{\pi}),$$

where $W(\pi)$ is a complex number of modulus 1.

Finally, we define the analytic conductor of π to be

$$C(\pi, t) = N_\pi \prod_{j=1}^m (1 + |it + \mu_\pi(j)|), \quad C(\pi) = C(\pi, 0).$$

3.2. Rankin-Selberg L -functions. Now we turn to the Rankin-Selberg L -functions. Let $\pi = \otimes_p \pi_p \in \mathcal{A}(m)$ and $\pi' = \otimes_p \pi'_p \in \mathcal{A}(m')$. The Rankin-Selberg L -function $L(s, \pi \times \pi')$ associated to π and π' is of the form

$$L(s, \pi \times \pi') = \prod_p L_p(s, \pi \times \pi') = \sum_{n=1}^{\infty} \frac{\lambda_{\pi \times \pi'}(n)}{n^s}.$$

The Euler product and Dirichlet series converge absolutely for $\Re(s) > 1$. For each (finite) prime p , the inverse of the local factor $L(s, \pi_p \times \pi'_p)$ is a polynomial in p^{-s} of degree $\leq mm'$

$$L_p(s, \pi \times \pi')^{-1} = \prod_{j=1}^m \prod_{j'=1}^{m'} \left(1 - \frac{\alpha_{j,j',\pi \times \pi'}(p)}{p^s} \right) \quad (3.3)$$

for suitable complex numbers $\alpha_{j,j',\pi \times \pi'}(p)$. With δ_m as in (3.1), we have the pointwise bound

$$|\alpha_{j,j',\pi \times \pi'}(p)| \leq p^{\delta_m + \delta_{m'}}. \quad (3.4)$$

If $p \nmid N_\pi$ or $p \nmid N_{\pi'}$, then we have the equality of sets

$$\{\alpha_{j,j',\pi \times \pi'}(p) : j \leq m, j' \leq m'\} = \{\alpha_{j,\pi}(p)\alpha_{j',\pi'}(p) : j \leq m, j' \leq m'\}. \quad (3.5)$$

At the archimedean place of $\mathbb{Q}$, there are mm' complex Langlands parameters $\mu_{\pi \times \pi'}(j, j')$ from which we define

$$L_\infty(s, \pi \times \pi') = \pi^{-\frac{mm's}{2}} \prod_{j=1}^m \prod_{j'=1}^{m'} \Gamma\left(\frac{s + \mu_{\pi \times \pi'}(j, j')}{2}\right).$$

These parameters satisfy the equality

$$\{\mu_{\tilde{\pi} \times \tilde{\pi}'}(j, j')\} = \{\overline{\mu_{\pi \times \pi'}(j, j')}\}$$

for $1 \leq j \leq m, 1 \leq j' \leq m'$ and the pointwise bound

$$\Re(\mu_{\pi \times \pi'}(j, j')) \geq -\delta_m - \delta_{m'}. \quad (3.6)$$

We refer to [5, Section 1] or [39, Section 2] for a discussion of these bounds in (3.4) and (3.6).

The complete L -function

$$\Lambda(s, \pi \times \pi') = N_{\pi \times \pi'}^{s/2} L(s, \pi \times \pi') L_\infty(s, \pi \times \pi')$$

has a meromorphic continuation and is bounded (away from its poles) in vertical strips. Under our normalization on the central characters, $\Lambda(s, \pi \times \pi')$ is entire if and only if $\tilde{\pi} \neq \pi'$. Moreover, $\Lambda(s, \pi \times \pi')$ satisfies the functional equation

$$\Lambda(s, \pi \times \pi') = W(\pi \times \pi') \Lambda(1-s, \tilde{\pi} \times \tilde{\pi}'), \quad (3.7)$$

where $W(\pi \times \pi')$ is a complex number of modulus 1.

As with $L(s, \pi)$, we define the analytic conductor of $\pi \times \pi'$ to be

$$C(\pi \times \pi', t) = N_{\pi \times \pi'} \prod_{j=1}^m \prod_{j'=1}^{m'} (1 + |it + \mu_{\pi \times \pi'}(j, j')|), \quad C(\pi \times \pi') = C(\pi \times \pi', 0).$$

It will be important to be able to decouple the dependencies of $C(\pi \times \pi', t)$ on π, π' , and t . To this end, we state the combined work of Bushnell and Henniart [6, Theorem 1] and Brumley [18, Lemma A.2] which yields

$$C(\pi \times \pi', t) \leq C(\pi \times \pi')(1 + |t|)^{m'm}, \quad C(\pi \times \pi') \leq e^{O(m+m')} C(\pi)^{m'} C(\pi')^m. \quad (3.8)$$

We are especially interested in the case where $\pi' = \tilde{\pi}$. In this case the Rankin-Selberg L -function $L(s, \pi \times \tilde{\pi})$ has non-negative coefficients $\lambda_{\pi \times \tilde{\pi}}(n)$. Moreover, $L(s, \pi \times \tilde{\pi})$ extends to the complex plane with a simple pole at $s = 1$. Hence, Landau's lemma [26] gives

$$\sum_{n \leq x} \lambda_{\pi \times \tilde{\pi}}(n) = c_{\pi} x + O_{\pi, \varepsilon} \left(x^{\frac{m^2-1}{m^2+1}} \right). \quad (3.9)$$

For our purpose, we need an inequality between the coefficients of the L -function $L(s, \pi)$ and those of the Rankin-Selberg L -function $L(s, \pi \times \tilde{\pi})$. In fact, motivated by Brumley's inequality on coefficients of the logarithmic derivatives of the related L -functions, we could prove the following result, which may be viewed as a complement to Brumley's inequality in [39, Lemma 2.2].

Lemma 3.1. *Let $\pi \in \mathcal{A}(m)$ and $\pi' \in \mathcal{A}(m')$. Then the inequality*

$$|\lambda_{\pi \times \pi'}(n)| \leq \sqrt{\lambda_{\pi \times \tilde{\pi}}(n) \lambda_{\pi' \times \tilde{\pi}'}(n)}$$

holds for any positive integer n . In particular, for any $\pi \in \mathcal{A}(m)$, we have

$$|\lambda_{\pi}(n)|^2 \leq \lambda_{\pi \times \tilde{\pi}}(n). \quad (3.10)$$

Remark 3.1. Let N_{π} be the conductor of π . It is well known that the inequality (3.10) holds when $(n, N_{\pi}) = 1$. This fact has been given by several authors (see, for example, Soundararajan [38], Molteni [29, Proposition 6]). We here not only remove the coprime restriction, but also get a more general inequality.

Proof. In order to prove this lemma, we first introduce a notation $a_{\pi \times \pi'}(n)$, which is given by

$$-\frac{L'}{L}(s, \pi \times \pi') = \sum_{n=1}^{\infty} \frac{\Lambda(n) a_{\pi \times \pi'}(n)}{n^s} = \sum_p \sum_{k=1}^{\infty} \frac{\sum_{j=1}^m \sum_{j'=1}^{m'} \alpha_{j, j', \pi \times \pi'}(p)^k \log p}{p^{ks}} \quad (3.11)$$

for $\Re(s) > 1$. Note that we could set $a_{\pi \times \pi'}(n) = 0$ if n is not a prime power. For all prime powers n , it is known from [39, Lemma 2.2] that

$$|a_{\pi \times \pi'}(n)| \leq \sqrt{a_{\pi \times \tilde{\pi}}(n) a_{\pi' \times \tilde{\pi}'}(n)}. \quad (3.12)$$

The argument is supplied by Brumley. We refer to the Appendix of [39] for more details. This inequality plays a key role in the following proof.

Second, we recall a useful recursion (see, for example [34, Lemma 5.2]), which states as follows: Let $\{\beta_j\}_{j=1}^m$ be a sequence of complex numbers. Define the sequence $\{b_n\}_{n \geq 1}$ by means of the formal identity

$$\sum_{n=0}^{\infty} b_n X^n = \prod_{j=1}^m (1 - \beta_j X)^{-1}.$$

Also, for $\nu \geq 1$, define $c_{\nu} = \sum_{j=1}^m \beta_j^{\nu}$. Then one has, for any $\nu \geq 1$,

$$\nu b_{\nu} = c_1 b_{\nu-1} + c_2 b_{\nu-2} + \cdots + c_{\nu-1} b_1 + c_{\nu}. \quad (3.13)$$

We shall apply this recursion to the following two cases. Comparing the different expressions of local factor $L_p(s, \pi \times \pi')$

$$L_p(s, \pi \times \pi') = \sum_{\nu=0}^{\infty} \frac{\lambda_{\pi \times \pi'}(p^{\nu})}{p^{\nu s}} = \prod_{j=1}^m \prod_{j'=1}^{m'} \left(1 - \frac{\alpha_{j, j', \pi \times \pi'}(p)}{p^s} \right)^{-1},$$

we deduce from (3.11) and (3.13) the following recursion

$$\nu \lambda_{\pi \times \pi'}(p^\nu) = \sum_{l=1}^{\nu} a_{\pi \times \pi'}(p^l) \lambda_{\pi \times \pi'}(p^{\nu-l}) \quad (3.14)$$

for $\nu \geq 1$.

Since each Rankin-Selberg L -function admits the Euler product representation, we find the coefficients $\lambda_{\pi \times \pi'}(n)$, $\lambda_{\pi \times \tilde{\pi}}(n)$ and $\lambda_{\pi' \times \tilde{\pi}'}(n)$ are all multiplicative. It is sufficient to show (3.10) holds for the power of all primes, that is

$$|\lambda_{\pi \times \pi'}(p^\nu)| \leq \sqrt{\lambda_{\pi \times \tilde{\pi}}(p^\nu) \lambda_{\pi' \times \tilde{\pi}'}(p^\nu)} \quad (3.15)$$

holds for all primes p and integers $\nu \geq 1$. We shall prove this by induction on the exponent ν . For $\nu = 1$, it follows from (3.14) and (3.12) that

$$|\lambda_{\pi \times \pi'}(p)|^2 = |a_{\pi \times \pi'}(p)|^2 \leq a_{\pi \times \tilde{\pi}}(p) a_{\pi' \times \tilde{\pi}'}(p) = \lambda_{\pi \times \tilde{\pi}}(p) \lambda_{\pi' \times \tilde{\pi}'}(p).$$

For any $\nu \geq 1$, assume that (3.15) holds for all $k \leq \nu$, that is

$$|\lambda_{\pi \times \pi'}(p^k)|^2 \leq \lambda_{\pi \times \tilde{\pi}}(p^k) \lambda_{\pi' \times \tilde{\pi}'}(p^k) \quad \text{for all } k \leq \nu. \quad (3.16)$$

By (3.14) and the triangle inequality, we have

$$(\nu + 1) |\lambda_{\pi \times \pi'}(p^{\nu+1})| \leq \sum_{l=1}^{\nu+1} |a_{\pi \times \pi'}(p^l) \lambda_{\pi \times \pi'}(p^{\nu+1-l})|. \quad (3.17)$$

Further, by (3.12), the induction hypothesis (3.16) and the Cauchy-Schwarz inequality, the sum on the right-hand side of (3.17) is less than

$$\begin{aligned} & \sum_{l=1}^{\nu+1} \sqrt{a_{\pi \times \tilde{\pi}}(p^l) a_{\pi' \times \tilde{\pi}'}(p^l) \lambda_{\pi \times \tilde{\pi}}(p^{\nu+1-l}) \lambda_{\pi' \times \tilde{\pi}'}(p^{\nu+1-l})} \\ & \leq \left(\sum_{l=1}^{\nu+1} a_{\pi \times \tilde{\pi}}(p^l) \lambda_{\pi \times \tilde{\pi}}(p^{\nu+1-l}) \right)^{\frac{1}{2}} \left(\sum_{l=1}^{\nu+1} a_{\pi' \times \tilde{\pi}'}(p^l) \lambda_{\pi' \times \tilde{\pi}'}(p^{\nu+1-l}) \right)^{\frac{1}{2}}, \end{aligned}$$

which is just equal to $(\nu + 1) \sqrt{\lambda_{\pi \times \tilde{\pi}}(p^{\nu+1}) \lambda_{\pi' \times \tilde{\pi}'}(p^{\nu+1})}$ by (3.14). Thus, we get from (3.17)

$$|\lambda_{\pi \times \pi'}(p^{\nu+1})| \leq \sqrt{\lambda_{\pi \times \tilde{\pi}}(p^{\nu+1}) \lambda_{\pi' \times \tilde{\pi}'}(p^{\nu+1})}.$$

This completes the proof of Lemma 3.1. $\square$

The convexity of the Rankin-Selberg L -functions is important and used several times in our proofs, so we state it as follows.

Lemma 3.2. *Let $\pi \in \mathcal{A}(m)$ and $\pi' \in \mathcal{A}(m')$. For any $\varepsilon > 0$, we have*

$$L\left(\frac{1}{2} + it, \pi \times \pi'\right) \ll C(\pi \times \pi', t)^{\frac{1}{4} + \varepsilon},$$

where the implied constant depends only on ε and m .

Proof. Let $s = \sigma + it$. It is known from [23, Theorem 2] that the Rankin-Selberg L -function $L(s, \pi \times \pi')$ is bounded by $O_m(C(\pi \times \pi', t)^\varepsilon)$ for $\sigma > 1$. Together this with the functional equation (3.7), Stirling's estimate for the gamma function and the Phragmen-Lindelöf convexity principle, the result then immediately follows. $\square$

3.3. Twists. Let χ be a primitive Dirichlet character with conductor q . As is well known, χ corresponds to a Hecke character of the idele class group $\mathbb{A}^\times/\mathbb{Q}^\times$ trivial on $\mathcal{R}_+^\times$, so χ is of the form $\chi = \otimes_p \chi_p$.

We apply the Rankin-Selberg theory described above to the following situation: Fix π in $\mathcal{A}(m)$ with conductor N_π and $m \geq 2$, and let χ be a primitive Dirichlet character modulo q . Take $\pi' = \chi$. The twisted L -function is given by

$$L(s, \pi \times \chi) = \prod_p \prod_{j=1}^m \left(1 - \frac{\alpha_{j, \pi \times \chi}(p)}{p^s} \right)^{-1}$$

for $\Re s > 1$. The corresponding complete L -function

$$\Lambda(s, \pi \times \chi) = N_{\pi \times \chi}^{s/2} L_\infty(s, \pi \times \chi) L(s, \pi \times \chi)$$

has an analytic continuation to the whole complex plane and satisfies some functional equation. Moreover, by (3.5), if $p \nmid q$ then

$$\{\alpha_{j, \pi \times \chi}(p) : 1 \leq j \leq m\} = \{\alpha_{j, \pi}(p) \chi(p) : 1 \leq j \leq m\}.$$

Thus, we should have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda_\pi(n) \chi(n)}{n^s} &= \prod_p \prod_{j=1}^m \left(1 - \frac{\alpha_{j, \pi}(p) \chi(p)}{p^s} \right)^{-1} \\ &= L(s, \pi \times \chi) \prod_{p|q} \prod_{j=1}^m \left(1 - \frac{\alpha_{j, \pi \times \chi}(p)}{p^s} \right). \end{aligned} \tag{3.18}$$

Similarly, if we take $\pi' = \tilde{\pi}(\chi) := \tilde{\pi} \times \chi$, then we have

$$L(s, \pi \times \tilde{\pi}(\chi)) = \prod_p \prod_{j=1}^m \prod_{j'=1}^{m'} \left(1 - \frac{\alpha_{j, j', \pi \times (\tilde{\pi} \times \chi)}(p)}{p^s} \right)^{-1}.$$

The complete L -function

$$\Lambda(s, \pi \times \tilde{\pi}(\chi)) = N_{\pi \times \tilde{\pi}(\chi)}^{s/2} L_\infty(s, \pi \times \tilde{\pi}(\chi)) L(s, \pi \times \tilde{\pi}(\chi))$$

has an analytic continuation to the whole complex plane and satisfies some functional equation. Moreover, if $p \nmid q$ then

$$\{\alpha_{j_1, j_2, \pi \times \tilde{\pi}(\chi)}(p) : 1 \leq j_1 \leq m, 1 \leq j_2 \leq m\} = \{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p) \chi(p) : 1 \leq j_1 \leq m, 1 \leq j_2 \leq m\}.$$

Thus, we should have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda_{\pi \times \tilde{\pi}}(n) \chi(n)}{n^s} &= \prod_p \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p) \chi(p)}{p^s} \right)^{-1} \\ &= L(s, \pi \times \tilde{\pi}(\chi)) \prod_{p|q} \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}(\chi)}(p)}{p^s} \right). \end{aligned} \tag{3.19}$$

4. PROOF OF THEOREM 1.2

Now we consider the sum $S_\pi(x, \alpha) = \sum_{n \leq x} \lambda_\pi(n) e(n\alpha)$. We are just required to work on the interval $\alpha \in [0, 1)$. Let Q be a parameter to be chosen later. By the Dirichlet approximation theorem, for any $\alpha \in [0, 1)$ there exists a rational number a/q with $(a, q) = 1$, $0 \leq a < q \leq Q$ such that

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{qQ}. \quad (4.1)$$

The initial step of the Hardy-Littlewood circle method would be to divide all α into two classes. Let $1 < P < Q$, $P = P(x)$ be also a parameter to be chosen later. For $0 \leq a < q \leq P$, we first define

$$\mathfrak{M}(a, q) = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right].$$

In general, these $\mathfrak{M}(a, q)$ are called the major arcs. Write $\mathfrak{M}$ for the union of all the major arcs

$$\mathfrak{M} = \bigcup_{q \leq P} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \mathfrak{M}(a, q).$$

Next, we denote by $\mathfrak{m}$ be the complement of $\mathfrak{M}$ in $[0, 1)$, that is

$$\mathfrak{m} = [0, 1] \setminus \mathfrak{M}.$$

Our uniform estimate for $S_\pi(x, \alpha)$ is a straightforward consequence of the following two results, which shall be proved in Section 5 and Section 6, respectively.

Proposition 4.1. *If $\alpha \in \mathfrak{M}$, then we have*

$$S_\pi(x, \alpha) \ll_\pi \left(q^{\frac{m+1}{m+2} + \varepsilon} + \frac{x}{Q} \right) x^{\frac{m+1}{m+2} + \varepsilon}.$$

Proposition 4.2. *If $\alpha \in \mathfrak{m}$, then we have*

$$S_\pi(x, \alpha) \ll_\pi \frac{x}{\log x} + \frac{x}{\varphi(q)^{1/2}} + (qx)^{1/2} \left(\log \left(\frac{x}{q} \right) \right)^{3/2}.$$

Proof of Theorem 1.2. With the notation as above, we take $P = x^\delta$, $PQ = x$, where δ is a sufficiently small number providing $0 < \delta < \frac{1}{m+1}$. Thus, we derive from Propositions 4.1 and 4.2 that

$$S_\pi(x, \alpha) \ll_\pi \begin{cases} x^{1-\varepsilon} & \text{if } \alpha \in \mathfrak{M}, \\ \frac{x}{\log x} & \text{if } \alpha \in \mathfrak{m}, \end{cases}$$

which immediately implies Theorem 1.2. □

5. PROOF OF PROPOSITION 4.1

Recall Dirichlet's approximation (4.1), it follows from partial summation that

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll \left(1 + \frac{x}{qQ} \right) \max_{1 \leq y \leq x} \left| \sum_{n \leq y} \lambda_\pi(n) e\left(\frac{an}{q} \right) \right|. \quad (5.1)$$

One obtains from the orthogonality of Dirichlet characters that

$$\begin{aligned}
\sum_{n \leq y} \lambda_\pi(n) e\left(\frac{an}{q}\right) &= \sum_{h=1}^q e\left(\frac{ah}{q}\right) \sum_{\substack{n \leq y \\ n \equiv h \pmod{q}}} \lambda_\pi(n) \\
&= \sum_{d|q} \sum_{\substack{h=1 \\ (h, q/d)=1}}^{q/d} e\left(\frac{adh}{q}\right) \sum_{\substack{l \leq y/d \\ l \equiv h \pmod{q/d}}} \lambda_\pi(dl) \\
&= \sum_{d|q} \frac{1}{\varphi(q/d)} \sum_{\chi \pmod{q/d}} \sum_{\substack{h=1 \\ (h, q/d)=1}}^{q/d} \bar{\chi}(h) e\left(\frac{adh}{q}\right) \sum_{l \leq y/d} \lambda_\pi(dl) \chi(l) \\
&= \sum_{d|q} \frac{1}{\varphi(q/d)} \sum_{\chi \pmod{q/d}} \chi(a) \tau(\bar{\chi}) \sum_{l \leq y/d} \lambda_\pi(dl) \chi(l),
\end{aligned}$$

where φ is the Euler function. Since the Gauss sum

$$\tau(\bar{\chi}) = \sum_{\substack{h=1 \\ (h, q/d)=1}}^{q/d} \bar{\chi}(h) e\left(\frac{dh}{q}\right)$$

has the well-known bound

$$\tau(\bar{\chi}) \ll (q/d)^{\frac{1}{2}},$$

we get

$$\sum_{n \leq y} \lambda_\pi(n) e\left(\frac{an}{q}\right) \ll q^{1/2} \sum_{d|q} \frac{1}{d^{1/2}} \max_{\chi \pmod{q/d}} \left| \sum_{l \leq y/d} \lambda_\pi(dl) \chi(l) \right|.$$

The innermost sum over l requires factoring the arithmetic functions $\lambda_\pi(dl)$. So we write

$$l = l_1 l_2 \quad \text{with } l_1 | d^\infty \text{ and } (l_2, d) = 1,$$

where $l_1 | d^\infty$ means that $p | l_1$ implies $p | d$ for any prime p . By the inequality (3.10) and the asymptotic formula (3.9), we have

$$\sum_{n \leq x} |\lambda_\pi(n)|^2 \ll_\pi x. \quad (5.2)$$

This together with the multiplicative property of $\lambda_\pi(n)$ yields

$$\sum_{n \leq y} \lambda_\pi(n) e\left(\frac{an}{q}\right) \ll q^{1/2} \sum_{d|q} \sum_{\substack{l_1 | d^\infty \\ l_1 \leq y}} l_1^{\frac{1}{2}} \max_{\chi \pmod{q/d}} \left| \sum_{l_2 \leq y/(dl_1)} \lambda_\pi(l_2) \chi(l_2) \right|. \quad (5.3)$$

Now it suffices to estimate the sum of type

$$\sum_{n \leq X} \lambda_\pi(n) \chi(n) \quad (5.4)$$

for any $\chi \pmod{r}$ with $0 < r \leq q$ and $0 < X \leq y$. We choose a function ϕ supported on $[0, X + Y]$, such that $\phi(z) = 1$ if $Y \leq z \leq X$ and $\phi^{(j)}(x) \ll_j Y^{-j}$ for all $j \geq 0$. Here, the

parameter Y will be chosen later subject to $1 \leq Y \leq X$. By partial integration, the Mellin transform of ϕ satisfies

$$\hat{\phi}(s) = \int_0^{X+Y} \phi(z) z^{s-1} dz \ll \frac{Y}{X^{1-\sigma}} \cdot \left(\frac{X}{|s|Y} \right)^j$$

for any $j \geq 1$ and $1/2 \leq \sigma = \Re s \leq 2$. Moreover, we derive from (5.2) and the Cauchy-Schwarz inequality that

$$\sum_{X < n \leq X+Y} |\lambda_\pi(n)| \ll Y^{\frac{1}{2}} \left(\sum_{n \leq X+Y} |\lambda_\pi(n)|^2 \right)^{\frac{1}{2}} \ll (XY)^{\frac{1}{2}}$$

for $0 < Y \leq X$. Thus we can smooth the sum (5.4) by writing

$$\sum_{n \leq X} \lambda_\pi(n) \chi(n) = \sum_n \lambda_\pi(n) \chi(n) \phi(n) + O\left((XY)^{\frac{1}{2}}\right).$$

By Mellin's inverse transform, we can write

$$\sum_n \lambda_\pi(n) \chi(n) \phi(n) = \frac{1}{2\pi i} \int_{(2)} \hat{\phi}(s) \sum_{n=1}^{\infty} \frac{\lambda_\pi(n) \chi(n)}{n^s} ds. \quad (5.5)$$

If χ is induced by a primitive character $\chi_1(\bmod r_1)$, then $r_1 | r$. We further get from (3.18) that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda_\pi(n) \chi(n)}{n^s} &= \left(\prod_p \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi}(p) \chi_1(p)}{p^s} \right)^{-1} \right) \prod_{p | \frac{r}{r_1}} \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi}(p) \chi_1(p)}{p^s} \right) \\ &= L(s, \pi \times \chi_1) \left(\prod_{p | r_1} \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi \times \chi_1}(p)}{p^s} \right) \right) \prod_{p | \frac{r}{r_1}} \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi}(p) \chi_1(p)}{p^s} \right). \end{aligned} \quad (5.6)$$

Due to the estimate (3.2), for any $\varepsilon > 0$,

$$\left(\prod_{p | r_1} \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi \times \chi_1}(p)}{p^s} \right) \right) \prod_{p | \frac{r}{r_1}} \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi}(p) \chi_1(p)}{p^s} \right) \ll r^\varepsilon, \quad (\Re s = 1/2).$$

Lemma 3.2 gives the convexity bound

$$L\left(\frac{1}{2} + it, \pi \times \chi_1\right) \ll_\pi (r_1(|t| + 1))^{\frac{m}{4} + \varepsilon}. \quad (5.7)$$

Moving the vertical line of integration in (5.5) to $\Re s = 1/2$, we obtain by Cauchy's theorem and (5.7) that

$$\begin{aligned} \sum_{n \leq X} \lambda_\pi(n) \chi(n) \phi(n) &\ll r^\varepsilon \int_{(1/2)} |\hat{\phi}(s) L(s, \pi \times \chi_1)| ds \\ &\ll r^\varepsilon \left(\int_0^{X/Y} \frac{X^{\frac{1}{2}}}{t+1} (r_1(t+1))^{\frac{m}{4}+\varepsilon} dt \right. \\ &\quad \left. + \int_{X/Y}^\infty \frac{Y}{X^{1-\sigma}} \cdot \left(\frac{X}{tY} \right)^{\frac{m}{4}+2} (r_1 t)^{\frac{m}{4}+\varepsilon} dt \right) \\ &\ll r^\varepsilon \left(\frac{r_1 X}{Y} \right)^{\frac{m}{4}+\varepsilon} X^{\frac{1}{2}}. \end{aligned}$$

Gathering the above results we arrive at

$$\sum_{n \leq X} \lambda_\pi(n) \chi(n) \ll r^\varepsilon \left(\frac{r_1 X}{Y} \right)^{\frac{m}{4}+\varepsilon} X^{\frac{1}{2}} + (XY)^{\frac{1}{2}}.$$

We choose $Y = (r_1 X)^{\frac{m}{m+2}}$ getting

$$\sum_{n \leq X} \lambda_\pi(n) \chi(n) \ll r^\varepsilon r_1^{\frac{m}{2m+4}} X^{\frac{m+1}{m+2}+\varepsilon}.$$

Inserting this bound into (5.3) yields

$$\sum_{n \leq y} \lambda_\pi(n) e\left(\frac{an}{q}\right) \ll (qy)^{\frac{m+1}{m+2}+\varepsilon}. \quad (5.8)$$

Finally, we plug (5.8) back into (5.1) and then obtain

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll \left(q^{\frac{m+1}{m+2}+\varepsilon} + \frac{x}{Q} \right) x^{\frac{m+1}{m+2}+\varepsilon},$$

which completes the proof of Proposition 4.1.

6. PROOF OF PROPOSITION 4.2

At the beginning of this section, we will apply sieve theory to estimate

$$\sum_{\substack{p \leq x \\ p+h \text{ is prime}}} |\lambda_\pi(p) \lambda_\pi(p+h)| \quad (6.1)$$

and verify Condition (C.3) for coefficients of automorphic L -functions. For other applications of sieve methods to the setting of automorphic L -functions, see, for example, the works of Holowinsky [16] and Soundararajan-Thorner [39]. The content in this section is actually inspired by their nice arguments. Specifically, Holowinsky studied the shifted sums over all positive integers instead of primes as above which is related to the problem of Quantum Unique Ergodicity, and Soundararajan and Thorner made a very important contribution to the subconvexity problem of general L -functions and gave a weak subconvexity without the Ramanujan hypothesis.

Later, we shall apply Theorem 1.1 to the multiplicative function $f(n) = \lambda_\pi(n)$ and then give the argument of Proposition 4.2.

6.1. Double upper-bound sieve heuristics. Let $\mathcal{A} = (a_n)$ be a sequence of non-negative real numbers and $\mathcal{P}$ be a general set of primes. We would like to estimate the sifting function

$$S(x, z) := \sum_{\substack{n \leq x \\ (n, P(z))=1}} a_n.$$

Here $P(z)$ is the product of primes less than z which belong to the set $\mathcal{P}$, that is

$$P(z) := \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} p.$$

After applying an upper-bound sieve with linear sieve weights ξ_d of level D , one has

$$S(x, z) \leq \sum_{\substack{d|P(z) \\ d \leq D}} \xi_d \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{d}}} a_n.$$

Thus, in order to evaluate $S(x, z)$, we need to have certain information on the behavior of the partial sum

$$\mathcal{A}_d(x) := \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{d}}} a_n$$

for $d|P(z)$, $d \leq D$. If an asymptotic formula can be established for $\mathcal{A}_d(x)$ of the form

$$\mathcal{A}_d(x) = g(d)X + r_d(x), \tag{6.2}$$

where $g(d)X$ is the expected main term with a multiplicative function $g(d)$ and $r_d(x)$ is a sufficiently small error term either individually or in a certain sense of mean, then one would expect to establish an upper bound

$$S(x, z) \ll XV$$

with X approximately equal to

$$\mathcal{A}(x) = \sum_{n \leq x} a_n$$

and

$$V = \prod_{p|P(z)} (1 - g(p)).$$

Thus, establishing asymptotics for $\mathcal{A}_d(x)$ is necessary for square-free numbers d .

For example, we take $g(p) = p^{-1}$ and $P(z) = \prod_{p \leq z} p$ for some suitable size of z , then Mertens' formula gives a saving of $\log z$

$$S(x, z) \ll X(\log z)^{-1}.$$

In general, if a multiplicative function g satisfies $0 \leq g(p) < 1$ and behaves like p^{-1} at primes p , then one should see similar results.

This idea can be extended to estimate the double sum

$$S_2(x, z) := \sum_{\substack{n_1 \leq x \\ (n_1, P_1(z))=1}} \sum_{\substack{n_2 \leq x \\ (n_2, P_1(z))=1}} a_{n_1, n_2}$$

for some sequence of non-negative numbers $\mathcal{A} = (a_{n_1, n_2})$, where $P_i(z)$ is the product of primes less than z which belong to some set $\mathcal{P}_i$ for $i = 1, 2$, respectively. In this case, our goal is to study the partial sum

$$\mathcal{A}_{d_1, d_2}(x) := \sum_{\substack{n_1 \leq x \\ n_1 \equiv 0 \pmod{d_1}}} \sum_{\substack{n_2 \leq x \\ n_2 \equiv 0 \pmod{d_2}}} a_{n_1, n_2}$$

and obtain an asymptotic formula of the form

$$\mathcal{A}_{d_1, d_2}(x) = g'(d_1)g''(d_2)X + r_{d_1, d_2}(x) \quad (6.3)$$

as (6.2). Provided that the appropriate conditions are satisfied, we could carry through in the same manner to obtain the bound

$$S_2(x, z) \ll XV'V'', \quad (6.4)$$

where

$$V' = \prod_{p|P_1(z)} (1 - g'(p)) \quad \text{and} \quad V'' = \prod_{p|P_2(z)} (1 - g''(p)).$$

However, if the density functions g' and g'' are not independent in terms of d_1 and d_2 , then establishing (6.4) would require some more care. For example, suppose that asymptotics in (6.3) hold only for $(d_1, d_2) = 1$. On applying a double upper-bound sieve with sieve weights ξ'_{d_1} and ξ''_{d_2} of level D' and D'' respectively, one would arrive at

$$S_2(x, z) \ll X(G' * G''),$$

where

$$G' * G'' := \sum_{\substack{d_1 | P_1(z) \\ d_1 < D'}} \xi'_{d_1} g'(d_1) \sum_{\substack{d_2 | P_2(z) \\ d_2 < D'' \\ (d_1, d_2) = 1}} \xi''_{d_2} g''(d_2).$$

Friedlander and Iwaniec treat this case in the appendix of [13]. We state their result here with the above definition of $G' * G''$.

Lemma 6.1. *Let g' and g'' be density functions satisfying the linear sieve conditions. Let (ξ') and (ξ'') be the optimal linear sieve weights of level D' and D'' , respectively. Then we have*

$$G' * G'' \ll CV'V'',$$

where

$$V' = \prod_{\substack{p|P_1(z) \\ p < D'}} (1 - g'(p)), \quad V'' = \prod_{\substack{p|P_2(z) \\ p < D''}} (1 - g''(p))$$

and

$$C = \prod_{p|(P_1(z), P_2(z))} (1 + h'h''(p))$$

with

$$h'(p) = g'(p)(1 - g'(p))^{-1}, \quad h''(p) = g''(p)(1 - g''(p))^{-1}.$$

6.2. The distribution in arithmetic progressions. We shall apply the double upper-bound sieve to give an upper bound for (6.1). For this purpose, given two square-free numbers d and q with $(d, q) = 1$, we require an asymptotic formula for

$$\sum_{\substack{n \leq x \\ d|n \\ n \equiv a \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n).$$

We shall detect the congruence $n \equiv a \pmod{q}$ via the orthogonality of Dirichlet characters. Thus, we first show two corresponding results for principal and non-principal characters, respectively. Since it is much simpler to make a dyadic decomposition and deal with the related smoothing sums, we introduce two smooth compactly supported functions $w_{\pm}$ such that

- $w_{-}(x) = 1$ for $x \in [X + Y, 2X - Y]$, $w_{-}(x) = 0$ for $x \geq 2X$ and $x \leq X$;
- $w_{+}(x) = 1$ for $x \in [X, 2X]$, $w_{+}(x) = 0$ for $x \geq 2X + Y$ and $x \leq X - Y$;
- $w_{\pm}^{(j)}(x) \ll_j Y^{-j}$ for all $j \geq 0$;
- the Mellin transform of $w_{\pm}(x)$ is

$$\begin{aligned} \widehat{w_{\pm}}(s) &:= \int_0^{\infty} w_{\pm}(x) x^{s-1} dx \\ &= \frac{1}{s \cdots (s+j-1)} \int_0^{\infty} w_{\pm}^{(j)}(x) x^{s+j-1} dx \\ &\ll_j \frac{Y}{X^{1-\sigma}} \left(\frac{X}{|s|Y} \right)^j \quad \forall j \geq 1; \end{aligned} \tag{6.5}$$

- trivially $\widehat{w_{\pm}}(s) \ll X^{\sigma}$ and

$$\widehat{w_{\pm}}(1) = X + O(Y), \tag{6.6}$$

where X, Y are two positive parameters which will be determined in Lemma 6.4.

Lemma 6.2. *Let $\pi \in \mathcal{A}(m)$, and let $d \geq 1$ and $q \geq 1$ be square-free integers with $(d, q) = 1$. Then we have*

$$\sum_{\substack{d|n \\ (n,q)=1}} \lambda_{\pi \times \tilde{\pi}}(n) w_{\pm}(n) = \kappa G_d(1, \pi \times \tilde{\pi}) G_q(1, \pi \times \tilde{\pi}) X + O_{\pi} \left(q^{\varepsilon} Y + (dq)^{\frac{m^2}{2}} \left(\frac{X}{Y} \right)^{\frac{m^2}{4} + \varepsilon} \sqrt{X} \right),$$

where $\kappa = \text{Res}_{s=1} L(s, \pi \times \tilde{\pi})$, $G_d(1, \pi \times \tilde{\pi})$ and $G_q(1, \pi \times \tilde{\pi})$ are given by

$$G_d(1, \pi \times \tilde{\pi}) = \prod_{p|d} (1 - L_p(1, \pi \times \tilde{\pi})^{-1}) \quad \text{and} \quad G_q(1, \pi \times \tilde{\pi}) = \prod_{p|q} L_p(1, \pi \times \tilde{\pi})^{-1}.$$

Proof. By the Mellin inversion formula

$$w_{\pm}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \widehat{w_{\pm}}(s) x^{-s} ds,$$

we may write

$$\sum_{\substack{d|n \\ (n,q)=1}} \lambda_{\pi \times \tilde{\pi}}(n) w_{\pm}(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \widehat{w_{\pm}}(s) \sum_{\substack{d|n \\ (n,q)=1}} \frac{\lambda_{\pi \times \tilde{\pi}}(n)}{n^s} ds.$$

The Dirichlet series appearing above has non-negative coefficients, converges in the region $\Re s > 1$, and matches the Rankin-Selberg L -function $L(s, \pi \times \tilde{\pi})$ except for the Euler factors at primes p dividing dq . Indeed, by multiplicativity and the condition $(d, q) = 1$, we may write

$$\sum_{\substack{d|n \\ (n, q)=1}} \frac{\lambda_{\pi \times \tilde{\pi}}(n)}{n^s} = L(s, \pi \times \tilde{\pi}) G_d(s, \pi \times \tilde{\pi}) G_q(s, \pi \times \tilde{\pi}), \quad (6.7)$$

where

$$G_d(s, \pi \times \tilde{\pi}) = \prod_{p|d} (1 - L_p(s, \pi \times \tilde{\pi})^{-1}) = \prod_{p|d} \left(1 - \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p)}{p^s} \right) \right)$$

and

$$G_q(s, \pi \times \tilde{\pi}) = \prod_{p|q} L_p(s, \pi \times \tilde{\pi})^{-1} = \prod_{p|q} \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p)}{p^s} \right).$$

Thus the integral above equals

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \widehat{w}_{\pm}(s) L(s, \pi \times \tilde{\pi}) G_d(s, \pi \times \tilde{\pi}) G_q(s, \pi \times \tilde{\pi}) ds. \quad (6.8)$$

We evaluate (6.8) by moving the line of integration to $\Re s = 1/2$. We encounter a simple pole at $s = 1$ and the residue here is

$$\widehat{w}_{\pm}(1) G_d(1, \pi \times \tilde{\pi}) G_q(1, \pi \times \tilde{\pi}) \operatorname{Res}_{s=1} L(s, \pi \times \tilde{\pi}).$$

It follows from (3.4) on the upper bound of $|\alpha_{j, j', \pi \times \pi'}(p)|$ and the definitions of $G_d(s, \pi \times \tilde{\pi})$, $G_q(s, \pi \times \tilde{\pi})$ above that

$$G_d(1, \pi \times \tilde{\pi}) \ll \prod_{p|d} p^{-\frac{2}{d^2+1} + \varepsilon} \ll 1, \quad G_q(1, \pi \times \tilde{\pi}) \ll \prod_{p|q} (1 + p^{-\frac{2}{d^2+1} + \varepsilon}) \ll q^{\varepsilon}. \quad (6.9)$$

Combining these with (6.6), the residue equals to

$$\kappa G_d(1, \pi \times \tilde{\pi}) G_q(1, \pi \times \tilde{\pi}) X + O_{\pi}(q^{\varepsilon} Y),$$

where

$$\kappa = \operatorname{Res}_{s=1} L(s, \pi \times \tilde{\pi}).$$

It remains to bound the integral on the line $\Re s = 1/2$. Using (3.8) and Lemma 3.2, we find

$$L\left(\frac{1}{2} + it, \pi \times \tilde{\pi}\right) \ll_{\pi} (1 + |t|)^{\frac{m^2}{4} + \varepsilon}.$$

Similar to (6.9), we have

$$G_d\left(\frac{1}{2} + it, \pi \times \tilde{\pi}\right) \ll \prod_{p|d} \left(1 + \left(1 + p^{\frac{1}{2} - \frac{2}{m^2+1}} \right)^{m^2} \right) \ll d^{\frac{m^2}{2}}, \quad G_q\left(d + it, \pi \times \tilde{\pi}\right) \ll q^{\frac{m^2}{2}}.$$

Thus the integral on the line $\Re s = 1/2$ is

$$\ll_{\pi} (dq)^{\frac{m^2}{2}} \int_{-\infty}^{\infty} (1 + |t|)^{\frac{m^2}{4} + \varepsilon} \left| \widehat{w}_{\pm}\left(\frac{1}{2} + it\right) \right| dt. \quad (6.10)$$

Using (6.5) with $j = 1$ for $|t| \leq X/Y$, and $k = m^2/4 + 2$ for $|t| > X/Y$, we see that (6.10) is estimated by

$$\ll_{\pi} (dq)^{\frac{m^2}{2}} \left(\frac{X}{Y}\right)^{\frac{m^2}{4} + \varepsilon} \sqrt{X}.$$

□

Lemma 6.3. *Let χ be a non-principal character modulo q . Then we have*

$$\sum_{d|n} \lambda_{\pi \times \tilde{\pi}}(n) \chi(n) w_{\pm}(n) \ll_{\pi} d^{\frac{m^2}{2}} q^{\frac{3m^2}{4} + \varepsilon} \left(\frac{X}{Y}\right)^{\frac{m^2}{4} + \varepsilon} \sqrt{X}.$$

Proof. Consider the Dirichlet series

$$D(s) = \sum_{d|n} \frac{\lambda_{\pi \times \tilde{\pi}}(n) \chi(n)}{n^s}.$$

Assume that χ is induced by a primitive character $\chi_1 \pmod{q_1}$, where $q_1 | q$. Similar to the expression (5.6) and (6.7), we obtain from (3.19) that

$$\begin{aligned} D(s) &= \left(\prod_{p|d} \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p) \chi(p)}{p^s} \right)^{-1} \right) \prod_{p|d} \left(\sum_{j=1}^{\infty} \frac{\lambda_{\pi \times \tilde{\pi}}(p^j) \chi(p)^j}{p^{js}} \right) \\ &= \left(\prod_p \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p) \chi(p)}{p^s} \right)^{-1} \right) \prod_{p|d} \left(1 - \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p) \chi(p)}{p^s} \right) \right) \\ &= L(s, \pi \times \tilde{\pi}(\chi_1)) \left(\prod_{p|q_1} \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p) \chi_1(p)}{p^s} \right) \right) \left(\prod_{p|q_1} \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p) \chi_1(p)}{p^s} \right) \right) \\ &\quad \times \prod_{p|d} \left(1 - \prod_{j_1, j_2=1}^m \left(1 - \frac{\alpha_{j_1, j_2, \pi \times \tilde{\pi}}(p) \chi(p)}{p^s} \right) \right), \end{aligned}$$

where the three products in the last line can be bounded by $O((dq)^{\frac{m^2}{2}})$ after inserting (3.4). Moreover, by (3.8) and Lemma 3.2, the convexity of $L(s, \pi(\chi_1) \times \tilde{\pi})$ is

$$L\left(\frac{1}{2} + it, \pi \times \tilde{\pi}(\chi_1)\right) \ll_{\pi} (q_1(1 + |t|))^{\frac{m^2}{4} + \varepsilon}.$$

Modifying the arguments slightly in the proof of Lemma 6.2, this lemma follows. □

Lemma 6.4. *Let $\pi \in \mathcal{A}(m)$ and let $d \geq 1$ and $q \geq 1$ be square-free integers with $(ad, q) = 1$. For any $x > 0$, we have*

$$\sum_{\substack{n \leq x \\ d|n \\ n \equiv a \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n) = \frac{\kappa}{\varphi(q)} G_d(1, \pi \times \tilde{\pi}) G_q(1, \pi \times \tilde{\pi}) x + O_{\pi} \left(d^{\frac{2m^2}{m^2+4} + \varepsilon} q^{\frac{3m^2}{m^2+4} + \varepsilon} x^{\frac{m^2+2}{m^2+4} + \varepsilon} \right),$$

where $\kappa = \text{Res}_{s=1} L(s, \pi \times \tilde{\pi})$, $G_d(1, \pi \times \tilde{\pi})$ and $G_q(1, \pi \times \tilde{\pi})$ are given by

$$G_d(1, \pi \times \tilde{\pi}) = \prod_{p|d} (1 - L_p(1, \pi \times \tilde{\pi})^{-1}) \quad \text{and} \quad G_q(1, \pi \times \tilde{\pi}) = \prod_{p|q} L_p(1, \pi \times \tilde{\pi})^{-1}.$$

Proof. Obviously we have

$$\sum_{\substack{d|n \\ n \equiv a \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n) w_-(n) \leq \sum_{\substack{X < n \leq 2X \\ d|n \\ n \equiv a \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n) \leq \sum_{\substack{d|n \\ n \equiv a \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n) w_+(n). \quad (6.11)$$

Using the Dirichlet character orthogonality relation

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod{q}, \\ 0 & \text{otherwise} \end{cases}$$

when $(a, q) = 1$, we have

$$\begin{aligned} \sum_{\substack{d|n \\ n \equiv a \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n) w_{\pm}(n) &= \frac{1}{\varphi(q)} \sum_{\substack{d|n \\ (n, q) = 1}} \lambda_{\pi \times \tilde{\pi}}(n) w_{\pm}(n) \\ &+ O\left(\max_{\chi \neq \chi_0} \left| \sum_{d|n} \lambda_{\pi \times \tilde{\pi}}(n) \chi(n) w_{\pm}(n) \right|\right). \end{aligned} \quad (6.12)$$

Now it reduces to estimate the above two terms on the right-hand side. Applying Lemma 6.2 and Lemma 6.3 to (6.12), and taking $Y = d^{\frac{2m^2}{m^2+4}} q^{\frac{3m^2}{m^2+4}} X^{\frac{m^2+2}{m^2+4}}$, it immediately follows from the inequalities (6.11) that

$$\sum_{\substack{X < n \leq 2X \\ d|n \\ n \equiv a \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n) = \frac{\kappa}{\varphi(q)} G_d(1, \pi \times \tilde{\pi}) G_q(1, \pi \times \tilde{\pi}) X + O_{\pi} \left(d^{\frac{2m^2}{m^2+4} + \varepsilon} q^{\frac{3m^2}{m^2+4} + \varepsilon} X^{\frac{m^2+2}{m^2+4} + \varepsilon} \right). \quad (6.13)$$

Finally, Lemma 6.4 follows after subdividing our original interval $[1, x]$ into dyadic segments and applying (6.13) for each segment. $\square$

6.3. Application of sieve. Now we are ready to estimate the sum

$$\sum_{\substack{p \leq x \\ p+h \text{ is prime}}} |\lambda_{\pi}(p) \lambda_{\pi}(p+h)|.$$

We first make use of the inequality of arithmetic and geometric means

$$|\lambda_{\pi}(p) \lambda_{\pi}(p+h)| \leq |\lambda_{\pi}(p)|^2 + |\lambda_{\pi}(p+h)|^2.$$

Thus, by symmetry, it is sufficient to give an upper bound of

$$\sum_{\substack{p \leq x \\ p+h \text{ is prime}}} |\lambda_{\pi}(p)|^2.$$

For the above sum, it is easy to see that for $z \leq x^{1/2}$

$$\sum_{\substack{p \leq x \\ p+h \text{ is prime}}} |\lambda_{\pi}(p)|^2 \leq \sum_{\substack{n \leq x \\ (n, P_{N_{\tilde{\pi}}}(z)) = 1 \\ (n+h, P_h(z)) = 1}} \lambda_{\pi \times \tilde{\pi}}(n) + x^{\frac{1}{2}}, \quad (6.14)$$

where N_π is the arithmetic conductor of π and $P_l(z)$ is given by

$$P_l(z) = \prod_{\substack{p \leq z \\ p \nmid l}} p.$$

for any integer l .

In order to apply the double upper-bound sieve to the sum on the right-hand side of (6.14), we seek asymptotic formulae for the partial sums in arithmetic progression with $(dh, q) = 1$

$$\mathcal{A}_{d,q}(x) = \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{d} \\ n \equiv -h \pmod{q}}} \lambda_{\pi \times \tilde{\pi}}(n)$$

to be of the form

$$\mathcal{A}_{d,q}(x) = g'(d)g''(q)X + r_{d,q}(x).$$

Here X is approximately equal to

$$\mathcal{A}(x) = \sum_{n \leq x} \lambda_{\pi \times \tilde{\pi}}(n).$$

Thanks to Lemma 6.4, we may take $X = \kappa x$, the density functions g' and g'' to be

$$g'(p) = 1 - \frac{1}{L_p(1, \pi \times \tilde{\pi})}, \quad g''(p) = \frac{1}{(p-1)L_p(1, \pi \times \tilde{\pi})}, \quad (6.15)$$

and the error term

$$r_{d,q}(x) = O_\pi \left(d^{\frac{2m^2}{m^2+4} + \varepsilon} q^{\frac{3m^2}{m^2+4} + \varepsilon} x^{\frac{m^2+2}{m^2+4} + \varepsilon} \right).$$

For $L_p(1, \pi \times \tilde{\pi})$ in (6.15), we have the estimate by (3.4)

$$\frac{1}{L_p(1, \pi \times \tilde{\pi})} = 1 + O_m(p^{-\frac{1}{m^2}}). \quad (6.16)$$

After a standard application of the double upper-bound sieve with linear sieve weights ξ'_d and ξ''_q of level $D = z^2$, we find

$$\begin{aligned} \sum_{\substack{n \leq x \\ (n, P_{N_\pi}(z))=1 \\ (n+h, P_h(z))=1}} \lambda_{\pi \times \tilde{\pi}}(n) &\leq \kappa x \sum_{\substack{d | P_{N_\pi}(z) \\ d \leq z^2}} \xi'_d g'(d) \sum_{\substack{q | P_h(z) \\ q \leq z^2 \\ (d,q)=1}} \xi''_q g''(q) \\ &+ O_\pi \left(x^{\frac{m^2+2}{m^2+4} + \varepsilon} \sum_{d \leq z^2} |\xi'_d| d^{\frac{2m^2}{m^2+4} + \varepsilon} \sum_{q \leq z^2} |\xi''_q| q^{\frac{3m^2}{m^2+4} + \varepsilon} \right). \end{aligned} \quad (6.17)$$

Because linear sieve properties give $|\xi'|, |\xi''| \leq \tau_3$, where $\tau_3(n)$ is the number of ways to write n as the product of three integers, we may trivially bound the error term in (6.17) by $\ll_\pi z^{14} x^{\frac{m^2+2}{m^2+4} + \varepsilon}$. Next, for the main term in (6.17), it follows from Lemma 6.1 that

$$\sum_{\substack{d | P_{N_\pi}(z) \\ d \leq z^2}} \xi'_d g'(d) \sum_{\substack{q | P_h(z) \\ q \leq z^2 \\ (d,q)=1}} \xi''_q g''(q) \ll \prod_{\substack{p \leq z \\ p \nmid h}} \left(1 + \frac{g'(p)g''(p)}{(1-g'(p))(1-g''(p))} \right) (1-g''(p)) \prod_{\substack{p \leq z \\ p \nmid N_\pi}} (1-g'(p)). \quad (6.18)$$

For the first two products on the right-hand side of (6.18), by (6.15), (6.16) and Mertens' formula, we have

$$\prod_{\substack{p \leq z \\ p \nmid h}} \left(1 + \frac{g'(p)g''(p)}{(1-g'(p))(1-g''(p))} \right) \ll \prod_{p \leq z} \left(1 + O_m \left(\frac{1}{p^{1+\frac{1}{m^2}}} \right) \right) \ll 1, \quad (6.19)$$

and

$$\prod_{\substack{p \leq z \\ p \nmid h}} (1 - g''(p)) \ll \prod_{\substack{p \leq z \\ p \nmid h}} \left(1 - \frac{1}{p} + O_m \left(\frac{1}{p^{1+\frac{1}{m^2}}} \right) \right) \ll \frac{h}{\varphi(h)} \cdot \frac{1}{\log z}. \quad (6.20)$$

Next, we estimate the third product $\prod_{\substack{p \leq z \\ p \nmid N_\pi}} (1 - g'(p))$ in (6.18). For $p \nmid N_\pi$, we see from (3.3) and (3.5) that

$$\begin{aligned} \log L_p(s, \pi \times \tilde{\pi}) &= \sum_{j=1}^m \sum_{k=1}^m \sum_{\nu=1}^{\infty} \frac{(\alpha_{j,\pi}(p) \overline{\alpha_{k,\pi}(p)})^\nu}{\nu p^{\nu s}} \\ &= \sum_{\nu=1}^{\infty} \frac{|a_\pi(p^\nu)|^2}{\nu p^{\nu s}}, \end{aligned}$$

where $a_\pi(p^\nu)$ is defined by

$$a_\pi(p^\nu) = \sum_{j=1}^m \alpha_{j,\pi}(p)^\nu$$

for $\nu \geq 1$. We know from [35, equation (2.24)] that

$$\sum_{n \leq x} \frac{(\log n) \Lambda(n) |a_\pi(n)|^2}{n} \sim \frac{\log^2 x}{2}.$$

By partial summation, we then get

$$\sum_{n \leq x} \frac{\Lambda(n) |a_\pi(n)|^2}{n \log n} \sim \log \log x.$$

So the third product is estimated by

$$\begin{aligned} \prod_{\substack{p \leq z \\ p \nmid N_\pi}} (1 - g'(p)) &= \exp \left(- \sum_{\substack{p \leq z \\ p \nmid N_\pi}} \log L_p(s, \pi \times \tilde{\pi}) \right) \\ &= \exp \left(- \sum_{\substack{p \leq z \\ p \nmid N_\pi}} \sum_{\nu=1}^{\infty} \frac{|a_\pi(p^\nu)|^2}{\nu p^{\nu s}} \right) \\ &\leq \exp \left(- \sum_{n \leq z} \frac{\Lambda(n) |a_\pi(n)|^2}{n \log n} \right) \\ &\ll \frac{1}{\log z}. \end{aligned} \quad (6.21)$$

By substituting (6.19), (6.20) and (6.21) into (6.18), the first double sum on the right-hand side of (6.17) is bounded by $O\left(\frac{h}{\varphi(h) \log^2 z}\right)$. If we take $z = x^\eta$ with a sufficiently small number

$\eta > 0$, then it leads to

$$\sum_{\substack{n \leq x \\ (n, P_{N\pi}(z))=1 \\ (n+h, P_h(z))=1}} \lambda_{\pi \times \tilde{\pi}}(n) \ll_{\pi} \frac{h}{\varphi(h)} \cdot \frac{x}{\log^2 x}. \quad (6.22)$$

Finally, by (6.14) and (6.22), we conclude the following lemma.

Lemma 6.5. *Let $\pi \in \mathcal{A}(m)$. For any $x > 0$ and any $h > 0$, we have*

$$\sum_{\substack{p \leq x \\ p+h \text{ is prime}}} |\lambda_{\pi}(p)\lambda_{\pi}(p+h)| \ll_{\pi} \frac{h}{\varphi(h)} \cdot \frac{x}{\log^2 x}.$$

Now we turn to prove Proposition 4.2. By (5.2) and Lemma 6.5, we see that Conditions (C.1) and (C.3) are verified for the coefficients $\lambda_{\pi}(n)$. As for Condition (C.2), using Shahidi's non-vanishing result of $L(s, \pi \times \tilde{\pi})$ at $\Re s = 1$ (see [37]), we get

$$\sum_{p \leq x} |\lambda_{\pi}(p)|^2 \log p \ll \sum_{n \leq x} \Lambda(n) a_{\pi \times \tilde{\pi}}(n) \sim x,$$

which implies that Condition (C.2) holds for the coefficients $\lambda_{\pi}(n)$. Finally, Theorem 1.1 applies in the context of automorphic L -functions and yields Proposition 4.2.

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REFERENCES

- [1] Bachman, G.: On exponential sums with multiplicative coefficients. II. Acta Arith. 106, 41–57 (2003)
- [2] Barnet-Lamb, T., Geraghty, D., Harris, M., Taylor, R.: A family of Calabi-Yau varieties and potential automorphy II. Publ. Res. Inst. Math. Sci. 47, 29–98 (2011)
- [3] Blomer, V.: Shifted convolution sums and subconvexity bounds for automorphic L -functions. Int. Math. Res. Not. IMRN 73, 3905–3926 (2004)
- [4] Bourgain, J., Sarnak, P., Ziegler, T.: Disjointness of Möbius from horocycle flows. From Fourier Analysis and Number Theory to Radon Transforms and Geometry. Developments in Mathematics, Springer (2013)
- [5] Brumley, F.: Effective multiplicity one on GL_N and narrow zero-free regions for Rankin-Selberg L -functions. Amer. J. Math. 128, 1455–1474 (2006)
- [6] Bushnell, C. J., Henniart, G.: An upper bound on conductors for pairs. J. Number Theory 65, 183–196 (1997).
- [7] Cafferata, M., Perelli, A., Zaccagnini, A.: An extension of the Bourgain-Sarnak-Ziegler theorem with modular applications. Q. J. Math. 71, 359–377 (2020).
- [8] Daboussi, H.: Fonctions multiplicatives presque périodiques B. Astérisque 24-25, 321–324 (1975)
- [9] Deligne, P.: La conjecture de Weil. I. Publ. Math. Inst. Hautes Études Sci. 43, 273–307 (1974)
- [10] Dupain, Y., Hall, R. R., Tenenbaum, G.: Sur l'équirépartition modulo 1 de certaines fonctions de diviseurs. J. London Math. Soc. (2) 26, 397–411 (1982)
- [11] Elliott, P. D. T. A., Moreno, C. J., Shahidi, F.: On the absolute value of Ramanujan's τ -function. Math. Ann. 266, 507–511 (1984)

- [12] Fouvry, É., Tenenbaum, G.: Entiers sans grand facteur premier en progressions arithmétiques. *Proc. Lond. Math. Soc.* 3, 449–494 (1991)
- [13] Friedlander, J., Iwaniec, H.: Hyperbolic Prime Number Theorem, *Acta Math.* 202, 1–19 (2009)
- [14] Halberstam, H., Richert, H.-E.: *Sieve methods*. Academic Press, London-New York (1974)
- [15] Harcos, G.: An additive problem in the Fourier coefficients of cusp forms. *Math. Ann.* 326, 347–365 (2003)
- [16] Holowinsky, R.: A sieve method for shifted convolution sums. *Duke Math. J.* 146, 401–448 (2009)
- [17] Holowinsky, R.: Sieving for mass equidistribution. *Ann. of Math.* 172, 1499–1516 (2010)
- [18] Humphries, P., Brumley, F.: Standard zero-free regions for Rankin-Selberg L -functions via sieve theory. With an appendix by Farrell Brumley. *Math. Z.* 292 (3–4), 1105–1122 (2019).
- [19] Iwaniec, H.: *Spectral methods of automorphic forms*. Graduate Studies in Mathematics, vol. 53. American Mathematical Society, Providence (2002)
- [20] Iwaniec, H., Kowalski, E.: *Analytic number theory*. American Mathematical Society Colloquium Publications, vol. 53. American Mathematical Society, Providence (2004)
- [21] Jiang, Y. J., Lü, G. S.: The generalized Bourgain-Sarnak-Ziegler criterion and its application to additively twisted sums on GL_m . *Sci. China Math.*, Doi:10.1007/s11425-020-1717-1 (2020).
- [22] Jutila, M.: *A variant of the circle method. Sieve Methods, Exponential Sums and their Applications in Number Theory*. Cambridge University Press, Cambridge (1996)
- [23] Li, X. N.: Upper bounds on L -functions at the edge of the critical strip. *Int. Math. Res. Not. IMRN* 4, 727–755 (2010)
- [24] Luo, W. Z., Rudnick, Z., Sarnak, P.: On Selberg’s eigenvalue conjecture. *Geom. Funct. Anal.* 5, 387–401 (1995)
- [25] Maier, H., Sankaranarayanan, A.: On a certain general exponential sum. *Int. J. Number Theory* 1, 183–192 (2005)
- [26] Michel, P.: *Analytic number theory and families of automorphic L -functions: Automorphic forms and applications*. IAS/Park City Math. Ser., Amer. Math. Soc., Providence (2007)
- [27] Miller, S. D.: Cancellation in additively twisted sums on $GL(n)$. *Amer. J. Math.* 128, 699–729 (2006)
- [28] Miller, S. D., Schmid, W.: Automorphic distributions, L -functions, and Voronoi summation for $GL(3)$. *Ann. of Math.* 164, 423–488 (2006)
- [29] Molteni, G.: Upper and lower bounds at $s = 1$ for certain Dirichlet series with Euler product. *Duke Math. J.* 111(1), 133–158 (2002)
- [30] Montgomery, H. L., Vaughan, R. C.: Exponential sums with multiplicative coefficients. *Invent. Math.* 43, 69–82 (1977)
- [31] Munshi, R.: Shifted convolution sums for $GL(3) \times GL(2)$. *Duke Math. J.* 162, 2345–2362 (2013)
- [32] Munshi, R.: The circle method and bounds for L -functions–I. *Math. Ann.* 358, 389–401 (2014)
- [33] Müller, W., Speh, B.: Absolute convergence of the spectral side of the Arthur trace formula for GL_n , with appendix by E. M. Lapid. *Geom. Funct. Anal.* 14 (1), 58–93 (2004).
- [34] Qu, Y.: Linnik-type problems for automorphic L -functions. *J. Number Theory* 130, 786–802 (2010)
- [35] Rudnick, Z., Sarnak, P.: Zeros of principal L -functions and random matrix theory. *Duke Math. J.* 81, 269–322 (1996)
- [36] Sarnak, P.: *Statistical properties of eigenvalues of the Hecke operators. Analytic number theory and Diophantine problems*. Birkhäuser Boston, Boston, MA (1987)
- [37] Shahidi, F.: On Certain L -functions. *Amer. J. Math.* 103, 297–355 (1981)
- [38] Soundararajan, K.: Weak subconvexity for central values of L -functions. *Ann. of Math.* 172, 1469–1498 (2010)
- [39] Soundararajan, K., Thorner, J.: Weak subconvexity without a Ramanujan hypothesis. With an appendix by Farrell Brumley. *Duke Math. J.* 168, 1231–1268 (2019)
- [40] Wirsing, E.: Das asymptotische Verhalten von Summen über multiplikative Funktionen. II. *Acta Math. Acad. Sci. Hungar.* 18, 411–467 (1967)

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