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ON THE SMOOTH VALUES OF SHIFTED ALMOST-PRIMES

BIN CHEN, XIAODONG LÜ AND ZHIWEI WANG

ABSTRACT. Denote by $P^+(n)$ (resp. $P^-(n)$) the largest (resp. the smallest) prime factor of the integer $n \geq 1$. In this paper, we prove a lower bound of almost-primes $n \leq x$ with $P^-(n) > x^{1/v}$, $v > 4$ such that $P^+(n-a) \leq x^{1/u}$ for $1 \leq u \ll 1$, $a \in \mathbb{Z}$, $a \neq 0$. As an application, we study two patterns on the largest prime factors of consecutive integers with one of which without small prime factor. This is a corrigendum to the original paper [18].

1. INTRODUCTION

For each integer $n \geq 1$, let $P^+(n)$ denote the largest prime factor of n with the convention that $P^+(1) = 1$. The distribution of the largest prime factors of shifted primes $P^+(p-a)$ with $a \in \mathbb{Z}$, $a \neq 0$ have been studied by many mathematicians, such as Erdős [5], Goldfeld [12], Pomerance [20], Fouvry [9], Baker and Harman [1], Banks and Shparlinski [2]. Define

$$\pi(x, y) := \sum_{\substack{p \leq x \\ P^+(p-a) \leq y}} 1. \quad (1.1)$$

In 2008, Granville [13] stated that the asymptotic formula

$$\pi(x, y) \sim \rho(u)\pi(x), \quad y = x^{1/u} \quad (1.2)$$

follows for all fixed $u \geq 1$ from a weak form of the Elliott-Halberstam conjecture, a well-believed conjecture in analytic number theory. Here $\rho(u)$ is the Dickman-de Bruijn function, and $\pi(x)$ denotes the number of primes not exceeding x . Recently the second named author [25] provided the details for proof of (1.2) under the Elliott-Halberstam conjecture.

In this paper, we consider the analogue problem via replacing prime p by almost-prime (with a limited number of prime factors). For $a \in \mathbb{Z}$, $y \leq x$ and $z \leq x$, we define

$$\Phi(x, y, z) := \sum_{\substack{n \leq x, P^-(n) > z \\ P^+(n-a) \leq y}} 1, \quad (1.3)$$

where $P^-(n)$ denotes the smallest prime factor of integer n with the convention $P^-(1) = \infty$. For the case $a = 0$, we may refer to the work of Friedlander [11] and Saias [21, 22, 23]. Here we only consider the case $a \neq 0$. Considering the following formula for almost-primes (see [24, Chapitre III.6]):

$$\Phi(x, z) := \sum_{\substack{n \leq x \\ P^-(n) > z}} 1 = \frac{x\omega(v) - z}{\log z} + O\left(\frac{x}{(\log z)^2}\right), \quad z = x^{1/v}$$

for $x \geq z \geq 2$, where $\omega(v)$ is the Buchstab function, we may guess that

$$\Phi(x, y, z) \sim \rho(u)\omega(v)\frac{x}{\log z}, \quad y = x^{1/u}, z = x^{1/v} \quad (1.4)$$

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for some $u, v \geq 1$.

It's not difficult to prove (1.4) with $1 \leq v$, $1 \leq u \ll 1$ under the Elliott-Halberstam conjecture for almost-primes (replacing primes by almost-primes) by following the argument of Lemma 4.1 in [25]. We leave the details to the interested reader.

In this paper, we first prove an unconditional result on the estimate of $\Phi(x, y, z)$ for some $u, v \geq 1$.

Theorem 1. *For $x \rightarrow \infty$ and $y = x^{1/u}$, $z = x^{1/v}$, there exists a strictly positive constant $c_1(u, v)$ such that*

$$\Phi(x, y, z) \geq c_1(u, v) \frac{x}{\log x} \quad (1.5)$$

for $v > 4$, $1 \leq u \ll 1$.

Remark. With the same method to (1.5), we can obtain an upper bound

$$\Phi(x, y, z) \leq c_2(u, v) \frac{x}{\log x} \quad (1.6)$$

for $1 \leq v \ll 1$, $1 \leq u \ll 1$, where $c_2(u, v)$ is a strictly positive constant. In fact, (1.6) is the case “ $q = 1$ ” of Theorem 1 in [17]. An application of the upper bound (1.6) is to study the density of shifted primes with large prime factors. For $0 < \theta < 1$, we define

$$T_\theta(x) := |\{p \leq x : P^+(p-1) \geq p^\theta\}|,$$

and we want to estimate $T_\theta(x)$ with a good lower bound (see [19], [3] and [6]). As in the very recent work of Liu, Wu and Xi [17], with the help of (1.6), for $x \rightarrow \infty$ we have the following lower bound

$$T_\theta(x) \geq \begin{cases} \left(1 - 4\rho(1/\theta) + o(1)\right) \frac{x}{\log x} & \text{for } 0 < \theta \leq \theta_0, \\ (1 - \theta + o(1)) \frac{x}{\log x} & \text{for } \theta_0 < \theta \leq 1/2, \end{cases} \quad (1.7)$$

where $\theta_0 \approx 0.3734$.

As an application of Theorem 1, we treat a problem on the largest prime factors of consecutive integers with one of which almost-prime. In [25], the second named author first exploited this problem and proved that there exists a strictly positive constant $C(\beta)$ such that

$$\sum_{\substack{n \leq x, P^-(n) > x^\beta \\ P^+(n) < P^+(n+2)}} 1 \geq \{C(\beta) + o(1)\} \frac{x}{\log x} \quad (1.8)$$

for $0 < \beta < \frac{1}{3}$. Taking $\beta = \frac{1}{3} - \delta$ with $0 < \delta \leq \frac{1}{12}$ in (1.8), we obtain the lower bound

$$\sum_{\substack{P_3 \leq x, P^-(P_3) > x^{1/3-\delta} \\ P^+(P_3) < P^+(P_3+2)}} 1 \geq \left\{C\left(\frac{1}{3} - \delta\right) + o(1)\right\} \frac{x}{\log x} \quad (1.9)$$

for $0 < \delta \leq \frac{1}{12}$. Here P_3 denotes an integer having at most three prime factors counted with multiplicity, i.e. P_3 is an almost-prime. The most interesting case in this problem is undoubtedly on primes, i.e. replacing P_3 in (1.9) by a prime number p , or equivalently,

extend the value of β in (1.8) to $0 < \beta \leq \frac{1}{2}$, considering the well-known twin prime conjecture is equivalent to

$$|\{p \leq x : P^+(p) < P^+(p+2)\}| \rightarrow \infty \quad \text{for } x \rightarrow \infty.$$

In addition, it should be noted that we can obtain a result similar to (1.8) directly for the small values of β by a result of Fouvry [18, Théorème 2], a theorem of Bombieri-Vinogradov type on sifted integers (without small prime factor) with an exponent of distribution $> \frac{1}{2}$, or applying a weighted sieve combined with a theorem of Bombieri-Vinogradov type (see Fouvry [8, Théorème 2] and [7, Corollaire 3]).

In this paper, we treat the dual problem of (1.8), i.e. we replace the condition $P^+(n) < P^+(n+2)$ by $P^+(n) > P^+(n+2)$ with n free of small prime factor. In fact, this problem was posed by Fouvry in the thesis report that he wrote for the second named author. With the help of (1.5), and the lower bound of $\Phi(x, y, z)$ in Theorem 1, we prove that there exists a positive density of n such that $P^+(n) > P^+(n+2)$, $P^-(n) > x^\alpha$ with $0 < \alpha < \frac{1}{4}$:

Corollary 1. *For $x \rightarrow \infty$ and $0 < \alpha < \frac{1}{4}$, there exists a strictly positive constant $C_1(\alpha)$ such that*

$$\sum_{\substack{n \leq x, P^-(n) > x^\alpha \\ P^+(n) > P^+(n+2)}} 1 \geq \{C_1(\alpha) + o(1)\} \frac{x}{\log x}. \quad (1.10)$$

It would be interesting to extend the domain of α to $0 < \alpha < \frac{1}{2}$.

2. LEMMAS

The first lemma is the Rosser–Iwaniec sieve [16]. Here we only employ the upper-bound sieve.

Lemma 2.1. *Let $D \geq 2$ and $L > 1$. Let $\mathcal{P}$ denote a set of primes, let $z \geq 2$ and write $P(z) := \prod_{p \leq z, p \in \mathcal{P}} p$. There exists two sequences $\{\lambda_d^\pm\}_{d=1}^\infty$ of real numbers, vanishing for $d > D$ or $\mu(d) = 0$, satisfying $\lambda_1^\pm = 1$, $|\lambda_d^\pm| \leq 1$, $\lambda^- * \mathbf{1} \leq \mu * \mathbf{1} \leq \lambda^+ * \mathbf{1}$, and such that*

$$\sum_{d|P(z)} \lambda_d^+ \frac{w(d)}{d} \leq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right) \left\{ F(s) + O\left(\frac{e^{L^{1/2}-s}}{(\log D)^{1/3}}\right) \right\},$$

$$\sum_{d|P(z)} \lambda_d^- \frac{w(d)}{d} \geq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right) \left\{ f(s) - O\left(\frac{e^{L^{1/2}-s}}{(\log D)^{1/3}}\right) \right\}$$

uniformly for all multiplicative functions w satisfying

- (i) $0 < w(p) < p \quad (p \in \mathcal{P})$,
- (ii) $\prod_{\substack{u < p \leq v \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right)^{-1} \leq \frac{\log v}{\log u} \left(1 + \frac{L}{\log u}\right) \quad (2 \leq u \leq v \leq z).$

where $s = \log D / \log z$ and $F(s), f(s)$ are defined by the continuous solutions to the system

$$\begin{cases} sF(s) = 2e^\gamma & 1 \leq s \leq 2, \\ sf(s) = 0 & 0 < s \leq 2, \\ (sF(s))' = f(s-1) & s > 2, \\ (sf(s))' = F(s-1) & s > 2. \end{cases}$$

Here γ is the Euler constant.

Define

$$S(x, y) := \{n \leq x : P^+(n) \leq y\}, \quad \Psi(x, y) := |S(x, y)| \quad (2.1)$$

and

$$\Psi(x, y; a, q) := \sum_{\substack{n \in S(x, y) \\ n \equiv a \pmod{q}}} 1, \quad \Psi_q(x, y) := \sum_{\substack{n \in S(x, y) \\ (n, q) = 1}} 1. \quad (2.2)$$

The following two lemmas are on the distribution of friable numbers, obtained by Hildebrand [15, Theorem 1] and Fouvry-Tenenbaum [10, Théorème 1] respectively.

Lemma 2.2. *For $\varepsilon > 0$, we have*

$$\Psi(x, y) = x\rho(u) \left\{ 1 + O_\varepsilon \left(\frac{\log(u+1)}{\log y} \right) \right\}$$

uniformly for

$$x \geq x_0(\varepsilon), \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \leq y \leq x,$$

where $u = \log x / \log y$ and $\rho(u)$ is the Dickman-de Bruijn function.

Lemma 2.3. *For $\varepsilon > 0$, we have*

$$\Psi_q(x, y) = \frac{\varphi(q)}{q} \Psi(x, y) \left\{ 1 + O \left(\frac{\log_2(qy) \log_2 x}{\log y} \right) \right\}$$

uniformly for

$$x \geq x_0(\varepsilon), \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \leq y \leq x$$

and

$$\log_2(q+2) \leq \left(\frac{\log y}{\log(u+1)} \right)^{1-\varepsilon},$$

where $u = \log x / \log y$ and $\varphi(n)$ is the Euler function.

The fourth lemma is the Bombieri-Vinogradov theorem for friable numbers, concerning the distribution of friable numbers in arithmetic progressions, averaged over a range of moduli. For $\Psi(x, y; a, q)$ and $\Psi_q(x, y)$ defined by (2.2), we have

Lemma 2.4. *For any $A > 0$, there exists a constant $B = B(A) > 0$ such that*

$$\sum_{q \leq x^{1/2}/(\log x)^B} \max_{x' \leq x} \max_{(a, q) = 1} \left| \Psi(x', y; a, q) - \frac{\Psi_q(x', y)}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A} \quad (2.3)$$

uniformly for $x \geq y \geq 2$.

In [26], Wolke proved a theorem of Bombieri-Vinogradov for sifted numbers, and announced Lemma 2.4. In 1991, Fouvry and Tenenbaum [10] proved Lemma 2.4 with a different method. In addition, for Lemma 2.4, we can see the work of Fouvry and Tenenbaum (also the work of Drappeau [4]) with the level of the distribution of friable numbers $> 1/2$ for some small y , or the work of Harper [14].

3. PROOF OF THEOREM 1

For $\Phi(x, y, z)$ defined by (1.3) with $y = x^{1/u}$, $z = x^{1/v}$ and any $a \in \mathbb{Z}^*$ fixed, we apply Lemma 2.1 by taking

$$\mathcal{P} = \{p : p \text{ is prime, } p \nmid a\}.$$

So we have

$$\begin{aligned} \Phi(x, y, z) &= \sum_{\substack{n \leq x+a, (n, a)=1 \\ P^+(n-a) \leq y \\ (n, P(z))=1}} 1 + O(1) \\ &= \sum_{\substack{n \leq x+a, (n, a)=1 \\ P^+(n-a) \leq y}} \mu * \mathbf{1}((n, P(z))) + O(1) \\ &\geq \sum_{d|P(z)} \lambda_d^- \sum_{\substack{n \leq x+a, (n, a)=1 \\ P^+(n-a) \leq y \\ n \equiv 0 \pmod{d}}} 1 + O(1) \\ &= \sum_{d|P(z)} \lambda_d^- \sum_{\substack{n \leq x, (n, a)=1 \\ P^+(n) \leq y \\ n \equiv -a \pmod{d}}} 1 + O(1). \end{aligned} \tag{3.1}$$

Then we write this in the following form

$$\Phi(x, y, z) \geq M(x, y, z) + R(x, y, z) + O(1) \tag{3.2}$$

where

$$M(x, y, z) := \sum_{d \leq D, d|P(z)} \frac{\lambda_d^-}{\varphi(d)} \sum_{\substack{n \leq x, (n, a)=1 \\ P^+(n) \leq y \\ (n, d)=1}} 1$$

and

$$R(x, y, z) := \sum_{d \leq D, d|P(z)} \lambda_d^- \left(\sum_{\substack{n \leq x, (n, a)=1 \\ P^+(n) \leq y \\ n \equiv -a \pmod{d}}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n \leq x, (n, a)=1 \\ P^+(n) \leq y \\ (n, d)=1}} 1 \right).$$

Taking $D = x^{1/2}/(\log x)^B$ for convenient $B > 0$, we see that the error term $R(x, y, z)$ is admissible with the help of Lemma 2.4

$$\begin{aligned}
R(x, y, z) &= \sum_{d \leq D, d|P(z)} \lambda_d^- \left(\sum_{\substack{n \leq x, P^+(n) \leq y \\ n \equiv -a \pmod{d}}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{n \leq x, P^+(n) \leq y \\ (n, d)=1}} 1 \right) \sum_{\ell|(n, a)} \mu(\ell) \\
&= \sum_{d \leq D, d|P(z)} \lambda_d^- \sum_{\ell|a} \mu(\ell) \left(\sum_{\substack{m \leq x/\ell, P^+(m) \leq y \\ m \equiv -a\bar{\ell} \pmod{d}}} 1 - \frac{1}{\varphi(d)} \sum_{\substack{m \leq x/\ell, P^+(m) \leq y \\ (m, d)=1}} 1 \right) \\
&\ll \sum_{d \leq D} \max_{x' \leq x} \max_{(a', d)=1} \left| \Psi(x', y; a', d) - \frac{\Psi_d(x', y)}{\varphi(d)} \right| \\
&\ll \frac{x}{(\log x)^A}
\end{aligned} \tag{3.3}$$

for any $A > 0$ and any fixed integer $a \neq 0$.

Now it remains to estimate $M(x, y, z)$. We first change the order of summation and write

$$M(x, y, z) = \sum_{\substack{n \leq x, (n, a)=1 \\ P^+(n) \leq y}} \sum_{\substack{d \leq D \\ d|P'(z)}} \frac{\lambda_d^-}{\varphi(d)}$$

where $P'(z) = \prod_{p < z, p \in \mathcal{P}'} p$ with $\mathcal{P}' := \{p : p \nmid an\}$. Note that by (3.1) we may deduce that n and a have opposite parity which implies $2|an$. We apply Lemma 2.1 with $w(p) = p/\varphi(p)$ and some large constant L getting

$$\begin{aligned}
M(x, y, z) &\geq \left\{ f\left(\frac{v}{2}\right) - o(1) \right\} \sum_{\substack{n \leq x, (n, a)=1 \\ P^+(n) \leq y}} \prod_{\substack{2 < p \leq z \\ p \nmid an}} \left(1 - \frac{1}{p-1} \right) \\
&= \left\{ f\left(\frac{v}{2}\right) - o(1) \right\} \prod_{\substack{2 < p \leq z \\ p \nmid a}} \left(\frac{p-2}{p-1} \right) \sum_{\substack{n \leq x, (n, a)=1 \\ P^+(n) \leq y}} H(n)
\end{aligned}$$

where $H(n)$ is a multiplicative function defined by

$$H(n) = \prod_{2 < p \leq z, p|n} \left(\frac{p-1}{p-2} \right).$$

By the following formula due to Liu-Wu-Xi (see [17, (3.5)])

$$\sum_{\substack{n \leq x, (n, a)=1 \\ P^+(n) \leq y}} H(n) = \frac{\varphi(a)}{a} \prod_{\substack{2 < p \leq z \\ p \nmid a}} \left(1 + \frac{1}{p(p-2)} \right) \Psi(x, y) \left\{ 1 + O\left(\frac{(\log \log x)^2}{\log y} \right) \right\},$$

we arrive at by Mertens' formula and Lemma 2.2

$$\begin{aligned}
M(x, y, z) &\geq \left\{ f\left(\frac{v}{2}\right) - o(1) \right\} \frac{\varphi(a)}{a} \prod_{\substack{2 < p \leq z \\ p \nmid a}} \left(1 - \frac{1}{p}\right) \Psi(x, y) \\
&\geq \left\{ f\left(\frac{v}{2}\right) - o(1) \right\} \frac{e^{-\gamma}}{\log z} \Psi(x, y) \\
&= \left\{ f\left(\frac{v}{2}\right) - o(1) \right\} \frac{ve^{-\gamma}}{\log x} \rho(u)x.
\end{aligned} \tag{3.4}$$

We remark that if a is odd, we may obtain a better lower bound with an extra constant factor 2.

Finally, combining (3.2), (3.3) and (3.4), we obtain

$$\Phi(x, y, z) \geq v\rho(u)f\left(\frac{v}{2}\right)\frac{e^{-\gamma}x}{\log x}\{1 - o(1)\}$$

for $y = x^{1/u}$, $z = x^{1/v}$ with $v > 4$, $1 \leq u \ll 1$. This completes the proof of Theorem 1.

4. PROOF OF COROLLARY 1

For $0 < \alpha < \frac{1}{4}$, we have

$$\sum_{\substack{n \leq x, P^-(n) > x^\alpha \\ P^+(n) > P^+(n+2)}} 1 \geq \sum_{\substack{n \leq x, P^-(n) > x^\alpha \\ P^+(n+2) < x^\alpha}} 1.$$

By taking $z = x^\alpha$ and $a = -2$ in (1.5) of Theorem 1, we obtain Corollary 1.

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