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ENERGY CONSERVATION IN NEWMARK BASED TIME INTEGRATION ALGORITHMS

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ABSTRACT. Energy balance equations are established for the Newmark time integration algorithm, and for the derived algorithms with algorithmic damping introduced via averaging, the so-called α -methods. The energy balance equations form a sequence applicable to: Newmark integration of the undamped equations of motion, an extended form including structural damping, and finally the generalized form including structural as well as algorithmic damping. In all three cases the expression for energy, appearing in the balance equation, is the mechanical energy plus some additional terms generated by the discretization by the algorithm. The magnitude and character of these terms as well as the associated damping terms are discussed in relation to energy conservation and stability of the algorithms. It is demonstrated that the additional terms in the energy lead to periodic fluctuations of the mechanical energy and are the cause of the phenomenon of response ‘overshoot’, previously observed empirically in the application of Newmark based algorithms to high frequency components. It is also demonstrated that the stability limit of the explicit Newmark algorithm is reached, when the stiffness term in the algorithmic energy vanishes, and that energy fluctuations take place for integration intervals close to the stability limit.

1. INTRODUCTION

The basic equation of structural dynamics is the equation of motion, setting up a balance between internal forces due to inertia, damping and stiffness with the external forces constituting the load. When the displacement vector is denoted $\mathbf{u}(t)$, the linear equation of motion is

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \quad (1)$$

In this equation $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices of the system, and $\mathbf{f}(t)$ is the external load vector, conjugate to the displacements $\mathbf{u}(t)$. A central problem to which much effort has been devoted over the last fifty years is the numerical integration of this equation of motion. The preferred methods are of the single-step type consisting of updating the displacement, velocity and acceleration vectors $\mathbf{u}$, $\dot{\mathbf{u}}$, $\ddot{\mathbf{u}}$ at current time t_n to the time $t_{n+1} = t_n + h$, a small time interval h later. It is desirable that the algorithm has at least second order accuracy, and because the spatial discretization used in structural dynamics often leads to inclusion of high-frequency modes in the model, it is also desirable to have unconditional stability.

There are two basically different approaches to the development of single-step time integration algorithms: collocation of the equation of motion at selected points in time in connection with a set of difference relations between the displacement, velocity and acceleration vectors, or the use of an integrated form of the equations of motion whereby the acceleration is eliminated and

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the displacement and velocity vectors appear as state variables satisfying an integrated form of the equation of motion. The first approach dates back to Newmark [1], who introduced a format in which the displacement and velocity increments are expressed in terms of the acceleration vector, with subsequent reduction of the equation of motion at time t_{n+1} to an equation for the acceleration vector $\ddot{\mathbf{u}}_{n+1}$. This procedure is attractive, because it is of ‘single-step, single-solve’ type, i.e. each step only requires the solution of one set of equations of the size of the corresponding static problem. In practice this means that linear as well as non-linear dynamic analysis can be handled computationally in a way very similar to the corresponding static problem. A review of numerical time integration in structural dynamics with main emphasis on the difference based methods has been given recently in [2].

A limitation and potential problem with the collocation based ‘single-solve’ time integration methods occurs in relation to controlled energy dissipation. As the models used in structural dynamics often contain high-frequency components that are artifacts of the spatial discretization and not representative of the problem to be investigated it is desirable to introduce so-called algorithmic damping of high-frequency components. Essentially, all frequency components in excess of the Nyquist frequency corresponding to the time step h can be considered as high-frequency components. While algorithmic damping can be introduced into the Newmark algorithm it was soon realized that the Newmark algorithm format would then also lead to low-frequency damping, reducing the method to first-order accuracy. This led to a sequence of papers [3, 4, 5] in which the algorithmic damping was modified by satisfying the equation of motion in an average sense between the times t_n and t_{n+1} . These generalized methods enjoy great popularity due to their simplicity of implementation for linear as well as non-linear problems. The properties of these methods have been established by frequency analysis of a modal decomposition of the linear problem. While this gives a detailed description of the response of the modal components, typically in terms of complex variables, the energy conservation properties of the original system is only indirectly represented. It turns out that introduction of algorithmic damping in the collocation based ‘single-solve’ algorithms leads to undesirable side effects in the energy balance equation. One of the consequences is a tendency to ‘overshoot’ of the response of high-frequency components, as discussed in [6, 7]. In the present paper exact energy balance equations are derived for the time integration algorithms of the Newmark family, and it is demonstrated that introduction of algorithmic damping leads to an unintended reinterpretation of the energy. One of the effects of the modified energy balance is the possibility of ‘overshoot’ of transient response.

In recent years there has been extensive research on time integration algorithms in which the equations of motion are considered as two sets of first order differential equations: one set specifying the motion in terms of the time derivative of the momenta, and another defining the momenta in terms of the time derivative of the displacements. Thus, the independent variables to be integrated include displacements as well as momenta - or the associated velocities. This work includes e.g. the extensions of the work by Simo et al. [8, 9] on rigid and flexible bodies to rigid-flexible mechanisms [10], the use of specific non-linear properties of stretches [11] and rotations [12], special potential

fields [13], and the Hamiltonian form of the equations of dynamics [14]. These algorithms make use of an integrated form of the equations of motion or special formation of mean values over the integration interval and typically lead to second order accuracy and exact conservation properties, even for systems with non-linear kinematics represented e.g. in terms of the quadratic Green strain. Within linear structural dynamics the accuracy of this type of algorithm can be increased to fourth order by reintroducing the equations of motion via integration by parts and high-frequency algorithmic damping can be obtained by introducing suitable terms in the energy balance equation, [15]. Satisfactory methods for introducing high-frequency algorithmic damping for non-linear problems have still to be developed.

An essential difference between the two approaches to time integration algorithms is their relation to energy conservation and dissipation. The equation of motion (1) leads to an energy conservation relation by multiplication with the velocity vector $\dot{\mathbf{u}}^T$.

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} \right) = \dot{\mathbf{u}}^T \mathbf{f} - \dot{\mathbf{u}}^T \mathbf{C} \dot{\mathbf{u}} \quad (2)$$

The quadratic forms in the parenthesis constitute the mechanical energy, while the two terms on the right side are the rate of work of the external force and the energy dissipation as described by the damping matrix $\mathbf{C}$. Ideally a single-step time integration algorithm should lead to a similar energy balance equation for finite increments, possibly supplemented with a high-frequency dissipation term. While this can be incorporated directly into the state space formulation, the difference based ‘single-solve’ algorithms typically lead to an undesirable reinterpretation of the ‘energy’. This paper contains a derivation of energy balance equations of difference based time integration algorithms. Due to the prominent place of Newmark and generalized Newmark methods the presentation uses the format associated with these algorithms, although the methods used are generally applicable to difference based methods. First the basic format of the Newmark algorithm is presented to set the scene, and then the energy balance equations are derived in three steps: first for the Newmark algorithm applied to systems without damping, then including the effect of structural damping in this energy balance equation, and finally extending the result to the generalized Newmark methods with algorithmic damping obtained by some form of averaging of the equilibrium equations. The last generalization is accomplished by using the recent result that the generalized Newmark methods can be formulated by introducing additional algorithmic damping via an extra state variable, determined by a first order filter [16].

It is demonstrated, how the parameters of the Newmark algorithm modify the energy balance. This leads directly to three results. It establishes the conditions of stability of the linear Newmark algorithm. It identifies the stability limit of the explicit Newmark algorithm as a loss of positive definiteness of the energy as perceived by the algorithm, and with substantial energy fluctuations for time intervals close to the stability limit. And finally it gives a precise description of the energy fluctuations seen as ‘overshoot’ of high-frequency response components, when started from initial velocity conditions for the explicit algorithm and from initial displacement conditions for the implicit algorithm with algorithmic damping. The results are derived both for the classic

Newmark algorithm and the α -modified forms with high-frequency damping. The conclusion is that the energy balance problems are common to all these algorithms, when used in connection with algorithmic damping. Thus, the paper gives precise results for the properties of Newmark based time integration algorithms for linear problems, and points towards the need for a different basis for further developments, most likely in the form of algorithms based on the integrated equations of motion.

2. THE NEWMARK ALGORITHM

The Newmark algorithm [1] is used as starting point for the following investigation of difference based ‘single-solve’ algorithms. It is based on a set of two relations expressing the forward displacement $\mathbf{u}_{n+1}$ and velocity $\dot{\mathbf{u}}_{n+1}$ in terms of their current values and the forward and current values of the acceleration,

$$\begin{aligned}\dot{\mathbf{u}}_{n+1} &= \dot{\mathbf{u}}_n + (1 - \gamma)h\ddot{\mathbf{u}}_n + \gamma h\ddot{\mathbf{u}}_{n+1} \\ \mathbf{u}_{n+1} &= \mathbf{u}_n + h\dot{\mathbf{u}}_n + \left(\frac{1}{2} - \beta\right)h^2\ddot{\mathbf{u}}_n + \beta h^2\ddot{\mathbf{u}}_{n+1}\end{aligned}\tag{3}$$

These relations are used to eliminate $\mathbf{u}_{n+1}$ and $\dot{\mathbf{u}}_{n+1}$ from the equation of motion (1) at the forward time t_{n+1} giving the following equation for the acceleration vector $\ddot{\mathbf{u}}_{n+1}$.

$$\begin{aligned}\left(\mathbf{M} + \gamma h \mathbf{C} + \beta h^2 \mathbf{K}\right) \ddot{\mathbf{u}}_{n+1} &= \mathbf{f}_{n+1} - \mathbf{C} \left(\dot{\mathbf{u}}_n + (1 - \gamma)h \ddot{\mathbf{u}}_n\right) \\ &\quad - \mathbf{K} \left(\mathbf{u}_n + h \dot{\mathbf{u}}_n + \left(\frac{1}{2} - \beta\right)h^2 \ddot{\mathbf{u}}_n\right)\end{aligned}\tag{4}$$

In linear problems the acceleration $\ddot{\mathbf{u}}_{n+1}$ is found from (4), and the velocity $\dot{\mathbf{u}}_{n+1}$ and displacement $\mathbf{u}_{n+1}$ then follow from (3) by simple vector operations. The algorithm can be generalized to non-linear problems, simply by solving the corresponding non-linear equations of motion at the appropriate time instants, see e.g. [17], although considerations of energy conservation may lead to the introduction of correction terms of higher order in the displacement increments, [9].

In the following sections energy balance equations are derived for the Newmark algorithm and its generalized forms, and it is demonstrated, how the conditions for unconditional stability can be extracted from the coefficients of the terms in the energy balance equation. The energy balance equations also illustrate that algorithmic damping with complex amplification factors for all frequencies leads to the introduction of extra terms in the energy expression appearing in the balance equation.

3. ENERGY BALANCE OF THE NEWMARK ALGORITHM

The discrete form of the energy balance equation involves the increment of the mechanical energy over the time interval from t_n to t_{n+1} . This increment can be expressed in terms of mean values and increments of the displacement and velocity by the following identity

$$\left[\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} \right]_n^{n+1} = \frac{1}{2} (\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T \mathbf{M} (\dot{\mathbf{u}}_{n+1} - \dot{\mathbf{u}}_n) + \frac{1}{2} (\mathbf{u}_{n+1} + \mathbf{u}_n)^T \mathbf{K} (\mathbf{u}_{n+1} - \mathbf{u}_n) \tag{5}$$

The following derivations make extensive use of splitting the expressions into mean values and increments, and it is therefore convenient to introduce the notation $\Delta \mathbf{u} = \mathbf{u}_{n+1} - \mathbf{u}_n$ and similarly for the velocity and the acceleration. It is also convenient to express the Newmark representation formulae (3) in terms of increments and mean values,

$$\begin{aligned}\Delta \dot{\mathbf{u}} &= \frac{1}{2}h(\ddot{\mathbf{u}}_{n+1} + \ddot{\mathbf{u}}_n) + (\gamma - \frac{1}{2})h\Delta \ddot{\mathbf{u}} \\ \Delta \mathbf{u} &= \frac{1}{2}h(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) + (\beta - \frac{1}{2}\gamma)h^2\Delta \ddot{\mathbf{u}}\end{aligned}\tag{6}$$

It is seen that these formulae are fully symmetric for $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{2}\gamma$, while different parameter values introduce a bias.

3.1. Energy balance without structural damping

The first step is to eliminate the velocity and displacement increments from (5) by use of the Newmark representation formulae (6). This gives four terms, of which two can be combined to give the load by using the equation of motion.

$$\begin{aligned}\left[\frac{1}{2}\dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} \right]_n^{n+1} &= \frac{1}{4}h(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T(\mathbf{f}_{n+1} + \mathbf{f}_n) \\ &+ (\gamma - \frac{1}{2})\frac{1}{2}h(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T \mathbf{M} \Delta \ddot{\mathbf{u}} + (\beta - \frac{1}{2}\gamma)\frac{1}{2}h^2(\mathbf{u}_{n+1} + \mathbf{u}_n)^T \mathbf{K} \Delta \ddot{\mathbf{u}}\end{aligned}\tag{7}$$

In order to reduce the terms in this balance equation to quadratic forms the factor $\frac{1}{2}h(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T$ in the first term on the right side is eliminated by use of the displacement increment formula (6b).

$$\begin{aligned}\left[\frac{1}{2}\dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} \right]_n^{n+1} &= \frac{1}{2}\Delta \mathbf{u}^T(\mathbf{f}_{n+1} + \mathbf{f}_n) \\ &+ (\gamma - \frac{1}{2})\frac{1}{2}h\Delta \ddot{\mathbf{u}}^T \mathbf{M}(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) + (\beta - \frac{1}{2}\gamma)\frac{1}{2}h^2\Delta \ddot{\mathbf{u}}^T \left\{ \mathbf{K}(\mathbf{u}_{n+1} + \mathbf{u}_n) - (\mathbf{f}_{n+1} + \mathbf{f}_n) \right\}\end{aligned}\tag{8}$$

The terms in the curly braces can be replaced by a single term containing the mass matrix by use of the equation of motion (1). This term is in increment form and can therefore be included on the left side of the equation.

$$\begin{aligned}\left[\frac{1}{2}\dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} + (\beta - \frac{1}{2}\gamma)\frac{1}{2}h^2\ddot{\mathbf{u}}^T \mathbf{M} \ddot{\mathbf{u}} \right]_n^{n+1} &= \\ \frac{1}{2}\Delta \mathbf{u}^T(\mathbf{f}_{n+1} + \mathbf{f}_n) + (\gamma - \frac{1}{2})\frac{1}{2}h\Delta \ddot{\mathbf{u}}^T \mathbf{M}(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)\end{aligned}\tag{9}$$

In the last term the factor $(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)$ is substituted from (6b), whereby

$$\begin{aligned}\left[\frac{1}{2}\dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} + (\beta - \frac{1}{2}\gamma)\frac{1}{2}h^2\ddot{\mathbf{u}}^T \mathbf{M} \ddot{\mathbf{u}} \right]_n^{n+1} &= \\ \frac{1}{2}\Delta \mathbf{u}^T(\mathbf{f}_{n+1} + \mathbf{f}_n) + (\gamma - \frac{1}{2})\Delta \mathbf{u}^T \mathbf{M} \Delta \ddot{\mathbf{u}} - (\gamma - \frac{1}{2})(\beta - \frac{1}{2}\gamma)h^2\Delta \ddot{\mathbf{u}}^T \mathbf{M} \Delta \ddot{\mathbf{u}}\end{aligned}\tag{10}$$

The final step of the reduction consists in reformulating the middle term on the right side by use of the equation of motion (1). This generates terms containing the load, absorbed in the first term, plus a quadratic term containing the stiffness matrix. The final energy equation then is

$$\begin{aligned}\left[\frac{1}{2}\dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T \mathbf{K} \mathbf{u} + (\beta - \frac{1}{2}\gamma)\frac{1}{2}h^2\ddot{\mathbf{u}}^T \mathbf{M} \ddot{\mathbf{u}} \right]_n^{n+1} &= \\ \Delta \mathbf{u}^T \left\{ \frac{1}{2}(\mathbf{f}_{n+1} + \mathbf{f}_n) + (\gamma - \frac{1}{2})\Delta \mathbf{f} \right\} - (\gamma - \frac{1}{2}) \left\{ \Delta \mathbf{u}^T \mathbf{K} \Delta \mathbf{u} + (\beta - \frac{1}{2}\gamma)h^2\Delta \ddot{\mathbf{u}}^T \mathbf{M} \Delta \ddot{\mathbf{u}} \right\}\end{aligned}\tag{11}$$

This relation has the form of an evolution equation for the quantity within the square brackets on the left side of the equation. A special instance of this relation with $\gamma = \frac{1}{2}$ was reported in [18].

For free vibration response the acceleration increment can be expressed in terms of the displacement increment by use of the homogeneous equation of motion, whereby the energy evolution equation takes the form

$$\begin{aligned} & \left[\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^T \left(\mathbf{K} + (\beta - \frac{1}{2}\gamma) h^2 \mathbf{K} \mathbf{M}^{-1} \mathbf{K} \right) \mathbf{u} \right]_n^{n+1} \\ & = -(\gamma - \frac{1}{2}) \Delta \mathbf{u}^T \left(\mathbf{K} + (\beta - \frac{1}{2}\gamma) h^2 \mathbf{K} \mathbf{M}^{-1} \mathbf{K} \right) \Delta \mathbf{u} \end{aligned} \quad (12)$$

From this relation it is seen that the algorithm replaces the original stiffness matrix $\mathbf{K}$ by an equivalent stiffness given by

$$\mathbf{K}_{eq} = \mathbf{K} + (\beta - \frac{1}{2}\gamma) h^2 \mathbf{K} \mathbf{M}^{-1} \mathbf{K} \quad (13)$$

This equivalent stiffness appears in the definition of the equivalent energy inside the brackets on the left side *and* in the quadratic damping term on the right side.

3.2. Stability and modal response

For the algorithm to be stable the equivalent energy inside the brackets on the left side must be positive definite and the right side must be zero or provide positive dissipation. It is seen immediately that these conditions are satisfied and full correspondence with the continuous energy equation (2) obtained for the parameter values $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$. In general stability requires the right side to be zero or negative, implying that $\gamma \geq \frac{1}{2}$. Taking $\gamma > \frac{1}{2}$ introduces so-called algorithmic damping into the time integration. Stability also requires that the equivalent stiffness matrix $\mathbf{K}_{eq}$ is non-negative definite. The stiffness matrix $\mathbf{K}$ itself is assumed to be positive definite, and for $\beta \geq \frac{1}{2}\gamma$ a positive definite equivalent stiffness is guaranteed irrespective of the magnitude of the time increment h . Thus the conditions for unconditional stability are

$$\gamma \geq \frac{1}{2} \quad \text{and} \quad \beta \geq \frac{1}{2}\gamma \quad (14)$$

These conditions are illustrated in Fig. 1. A spectral analysis reveals that for $\gamma > \frac{1}{2}$ the amplification factor associated with a single step of the algorithm may become real-valued for large frequency components, [17, 19]. This is undesirable and may lead to loss of high-frequency damping. The following stronger criterion on β is therefore desirable to obtain complex-valued amplification factors for all frequencies,

$$\beta \geq \frac{1}{4}(\gamma + \frac{1}{2})^2 \quad (15)$$

This criterion is shown in Fig. 1 as a dashed curve.

The condition (14b) on the parameter β can be relaxed, provided the time increment h is bounded by a limit h_{max} to be calculated from the condition that $\mathbf{K}_{eq}$ is non-negative definite. The time limit h_{max} depends on the behavior of the structure described by the stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$. For linear structures the response is conveniently represented in terms of the mode

FIGURE 1. Stability domains of the Newmark algorithm.

shapes $\mathbf{u}_1, \dots, \mathbf{u}_m$ in the form

$$\mathbf{u}(t) = \sum_{j=1}^m u_j(t) \mathbf{u}_j \quad (16)$$

In the representation (16) the modal coordinates $u_j(t)$ describe the time dependence of components described by the mode shapes $\mathbf{u}_j$. The mode shapes are free vibration solutions of the form

$$\mathbf{u}(t) = \mathbf{u}_j e^{i\omega_j t} \quad (17)$$

where i is the imaginary unit, ω_j is the angular frequency of the modal vibration, and the result is interpreted as the real part of the equation. Substitution of this representation into the undamped equation of free vibrations gives the generalized eigenvalue problem

$$(\mathbf{K} - \omega_j^2 \mathbf{M}) \mathbf{u}_j = \mathbf{0} \quad (18)$$

The mode shapes satisfy orthogonality conditions with respect to both the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$. These orthogonality relations are conveniently normalized to the form

$$\mathbf{u}_j^T \mathbf{M} \mathbf{u}_k = \delta_{jk} \quad , \quad \mathbf{u}_j^T \mathbf{K} \mathbf{u}_k = \omega_j^2 \delta_{jk} \quad (19)$$

These orthogonality relations imply that each mode can be treated separately, and substitution of the modal representation (16) into the energy equation (9) gives the following energy equation for mode j ,

$$\begin{aligned} \left[\frac{1}{2} \dot{u}_j^2 + \frac{1}{2} \left(1 + \left(\beta - \frac{1}{2} \gamma \right) (\omega_j h)^2 \right) \omega_j^2 u_j^2 \right]_n^{n+1} = \\ - \left(\gamma - \frac{1}{2} \right) \left(1 + \left(\beta - \frac{1}{2} \gamma \right) (\omega_j h)^2 \right) \omega_j^2 (\Delta u_j)^2 \end{aligned} \quad (20)$$

It is seen from this equation, that the modal stiffness is changed from its proper value ω_j^2 to an equivalent modal stiffness $[1 + (\beta - \frac{1}{2} \gamma) (\omega_j h)^2] \omega_j^2$ on both sides of the equation. This equivalent modal stiffness must be positive for all modes. The condition of positive modal stiffness is conveniently expressed in the form

$$\beta_* = \beta + (\omega_j h)^{-2} \geq \frac{1}{2} \gamma \quad (21)$$

For a given value of β this condition determines an upper limit on h and the smallest of these is the limit h_{max} for conditional stability. Conversely, for a given time increment size h the condition

determines a lower limit on β . The limits for conditional stability are included in the diagram in Fig. 1 simply by replacing β with β_* . The case $\beta = 0$ is of particular interest because the algorithm can then be arranged in explicit form if the mass matrix $\mathbf{M}$ is diagonal, or can be represented as a diagonal matrix, [17]. This leads to a conditionally stable algorithm, and if no algorithmic damping is wanted $\gamma = \frac{1}{2}$, and the upper time increment limit is determined from (21) as $h \leq h_{max} = 2/\omega_{max}$.

FIGURE 2. Phase-plane plots. Initial velocity $\dot{u}_j(0) = 1$ (—), initial displacement $u_j(0) = 1$ (- · -). a) Reduced stiffness $\beta < \frac{1}{2}\gamma$, b) Increased stiffness $\beta > \frac{1}{2}\gamma$.

The effect of changing the modal stiffness is illustrated in Fig. 2. The figure shows the trace of a harmonic oscillation in the normalized phase plane with coordinates $(u_j, \dot{u}_j/\omega_j)$. In this plot the mechanical energy is represented by half the square of the distance from the origin and in the original problem the harmonic response is represented by a circle. The circular contour is retained for $\beta = \frac{1}{2}\gamma$, but for $\beta \neq \frac{1}{2}\gamma$ the elastic stiffness is changed by the algorithm leading to an elliptic trace. For $\beta < \frac{1}{2}\gamma$, e.g. for the explicit algorithm with $\beta = 0$, the major axis of the ellipse is in the u_j -direction. Two instances of this are shown in Fig. 2a. The dashed curve shows the response to initial velocity conditions $u_j(0) = 0, \dot{u}_j(0) = 1$ marked with a circle symbol, while the dash-dot curve shows the response to initial displacement conditions $u_j(0) = 1, \dot{u}_j(0) = 0$ marked with a cross. It is seen that for $\beta < \frac{1}{2}\gamma$ initial velocity conditions lead to a periodic energy fluctuation, in which the peaks of the displacement become too large, the so-called ‘overshoot’. Conversely, displacement initial conditions leads to underestimation of the peaks of the velocity. Figure 2b shows the opposite situation in which $\beta > \frac{1}{2}\gamma$, typical of algorithmic damping in the implicit algorithm. In this case the elliptic contour has major axis along the $\dot{u}_j$ -direction. Therefore the roles of the initial conditions are interchanged, and now ‘overshoot’ is associated with the velocity response $\dot{u}_j(t)$ and initial displacement conditions. These effects are illustrated by specific numerical examples in Section 5.

3.3. Energy balance with structural damping

The effect of viscous structural damping can be included by substituting the structural damping force

$$\mathbf{f}_C = -\mathbf{C}\dot{\mathbf{u}} \quad (22)$$

into the balance equation (11). This gives an additional dissipation contribution

$$-\Delta \mathbf{u}^T \mathbf{C} \left\{ \frac{1}{2}(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) + (\gamma - \frac{1}{2})\Delta \dot{\mathbf{u}} \right\} \quad (23)$$

to the right side of the equation. These two terms are now reduced to quadratic forms and increments of quadratic forms.

It is convenient to start with the second structural damping term $-(\gamma - \frac{1}{2})\Delta \mathbf{u}^T \mathbf{C} \Delta \dot{\mathbf{u}}$. This term is reduced by use of the Newmark representation relations (6). The numerical factor $(\gamma - \frac{1}{2})$ is left out during the reduction for typographical reasons. First the displacement increment is expressed by use of (6b).

$$\begin{aligned} -\Delta \mathbf{u}^T \mathbf{C} \Delta \dot{\mathbf{u}} &= -\left\{ \frac{1}{2}h(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T + (\beta - \frac{1}{2}\gamma)h^2\Delta \ddot{\mathbf{u}}^T \right\} \mathbf{C} \Delta \dot{\mathbf{u}} \\ &= -\frac{1}{2}h \left[\dot{\mathbf{u}}^T \mathbf{C} \dot{\mathbf{u}} \right]_n^{n+1} - (\beta - \frac{1}{2}\gamma)h^2\Delta \ddot{\mathbf{u}}^T \mathbf{C} \Delta \dot{\mathbf{u}} \end{aligned} \quad (24)$$

In the last term the velocity increment is now expressed in terms of the acceleration mean value and the acceleration increment by use of (6a). The term with the acceleration mean value can be absorbed within the square brackets, while the acceleration difference contributes a quadratic term. The full expression is

$$-\Delta \mathbf{u}^T \mathbf{C} \Delta \dot{\mathbf{u}} = -\frac{1}{2}h \left[\dot{\mathbf{u}}^T \mathbf{C} \dot{\mathbf{u}} + (\beta - \frac{1}{2}\gamma)h^2\ddot{\mathbf{u}}^T \mathbf{C} \ddot{\mathbf{u}} \right]_n^{n+1} - (\gamma - \frac{1}{2})(\beta - \frac{1}{2}\gamma)h^3\Delta \ddot{\mathbf{u}}^T \mathbf{C} \Delta \ddot{\mathbf{u}} \quad (25)$$

It is interesting to compare this contribution to the energy balance equation (11) with the terms containing the mass matrix. It is seen that there is a complete match, such that when the factor $(\gamma - \frac{1}{2})$ is applied to the damping term this part of the structural damping is accounted for, if the mass matrix $\mathbf{M}$ is replaced with an equivalent mass matrix $\mathbf{M}_*$ defined by

$$\mathbf{M}_* = \mathbf{M} + (\gamma - \frac{1}{2})h\mathbf{C} \quad (26)$$

For $\gamma > \frac{1}{2}$ the modification of the mass matrix leads to increase of mass and stiffness in the equivalent energy, and to a corresponding increase of the algorithmic dissipation term. Alternatively, it may be argued that $\mathbf{M}_*$ should represent the actual mass matrix, and the matrix $\mathbf{M}$ should be modified accordingly.

The first structural damping term $-\Delta \mathbf{u}^T \mathbf{C} \frac{1}{2}(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)$ would be quadratic, if both the displacement increment and the mean value of the velocity represented the same value. This is not the case in general, and the difference is expressed by writing the Newmark representation formula (6b) in the form

$$\Delta \mathbf{u} - \frac{1}{2}h(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) = (\beta - \frac{1}{2}\gamma)h^2\Delta \ddot{\mathbf{u}} \quad (27)$$

According to this relation the difference between representing the velocity via the displacement increment or via the mean value of the displacements at the interval ends is proportional to the acceleration increment. Multiplication of both sides of this equation from the left and the right on the structural damping matrix $\mathbf{C}$ gives the following expression for the first part of the damping contribution.

$$\begin{aligned} -\Delta \mathbf{u}^T \mathbf{C} \frac{1}{2}(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) &= \\ -\frac{1}{2}h \left\{ h^{-2}\Delta \mathbf{u}^T \mathbf{C} \Delta \mathbf{u} + \frac{1}{4}(\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T \mathbf{C} (\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) \right\} &+ \frac{1}{2}(\beta - \frac{1}{2}\gamma)^2 h^3 \Delta \ddot{\mathbf{u}}^T \mathbf{C} \Delta \ddot{\mathbf{u}} \end{aligned} \quad (28)$$

All terms are now in quadratic form.

When the structural damping terms are incorporated, the energy balance equation (11) takes the form

$$\begin{aligned} & \left[\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M}_* \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} + (\beta - \frac{1}{2}\gamma) \frac{1}{2} h^2 \ddot{\mathbf{u}}^T \mathbf{M}_* \ddot{\mathbf{u}} \right]_n^{n+1} = \\ & \Delta \mathbf{u}^T \left\{ \frac{1}{2} (\mathbf{f}_{n+1} + \mathbf{f}_n) + (\gamma - \frac{1}{2}) \Delta \mathbf{f} \right\} - (\gamma - \frac{1}{2}) \left\{ \Delta \mathbf{u}^T \mathbf{K} \Delta \mathbf{u} + (\beta - \frac{1}{2}\gamma) h^2 \Delta \ddot{\mathbf{u}}^T \mathbf{M}_* \Delta \ddot{\mathbf{u}} \right\} \\ & - \frac{1}{2} h \left\{ h^{-2} \Delta \mathbf{u}^T \mathbf{C} \Delta \mathbf{u} + \frac{1}{4} (\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T \mathbf{C} (\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) \right\} + \frac{1}{2} (\beta - \frac{1}{2}\gamma)^2 h^3 \Delta \ddot{\mathbf{u}}^T \mathbf{C} \Delta \ddot{\mathbf{u}} \end{aligned} \quad (29)$$

In this relation all the terms on the right side, except those representing the effect of the external loads, are of quadratic form. With the exception of the last term they all represent positive damping. The last term is positive and quadratic in the acceleration increments. A similar negative term is present in the modified mass matrix. The total contribution from the viscous structural damping matrix in the quadratic acceleration increment terms is

$$-(\beta - \frac{1}{2}\gamma) \left\{ (\gamma - \frac{1}{2})^2 - \frac{1}{2} (\beta - \frac{1}{2}\gamma) \right\} h^3 \Delta \ddot{\mathbf{u}}^T \mathbf{C} \Delta \ddot{\mathbf{u}} \quad (30)$$

For unconditional stability without restrictions on the structural damping matrix $\mathbf{C}$ both scalar factors must be non-negative. Thus, unconditional stability of systems with viscous structural damping can be secured by $\gamma \geq \frac{1}{2}$ and a parameter β satisfying

$$0 \leq \beta - \frac{1}{2}\gamma \leq 2(\gamma - \frac{1}{2})^2 \quad (31)$$

The lower limit corresponds to the requirement for unconditional stability of the undamped system. The upper limit is similar in form to that of unconditional stability with complex amplification factor, given in (15). This condition may be written as $\beta - \frac{1}{2}\gamma = \frac{1}{4}(\gamma - \frac{1}{2})^2$, and it is seen that the upper limit (31) is obtained by replacing the factor $\frac{1}{4}$ with 2. The upper limit is shown in Fig. 1 as a dotted curve. In practice the upper limit on β for systems with viscous damping is of little use as the lower limit gives better phase behavior.

4. ENERGY BALANCE FOR GENERALIZED NEWMARK ALGORITHMS

It is well known that introduction of algorithmic damping in the Newmark integration method by selecting a value of γ larger than the stability limit $\frac{1}{2}$ introduces low frequency damping, equivalent to reduction of the accuracy of the integration scheme from second to first order. This problem was remedied by averaging procedures, in which either the stiffness and damping terms, the inertial terms, or all of these were introduced as weighted averages of their values at t_n and t_{n+1} , [3, 4, 5]. These generalized Newmark methods often go by the name of ‘ α -methods’ because the symbol α is used for the weights in the averages. It has recently been demonstrated that these methods working with one or two weighted averages of the terms in the equation of motion can be considered as special cases of a general procedure, in which the algorithmic damping of the Newmark algorithm is supplemented by a damping term generated from the response velocity vector via a first order filter [16]. This approach is particularly appealing in relation to energy conservation properties, because it permits a fairly straightforward extension of the analysis given above to include the effects of the generalized Newmark methods by considering the new damping

vector as an additional state variable. A variant of the α -modification developed for Newmark based methods has been introduced into the kinematically non-linear energy conserving algorithm by changing the interval mean values to α -weighting [20].

4.1. Algorithmic damping by an additional state variable

The idea is to introduce a damping contribution in which the velocity vector $\dot{\mathbf{u}}(t)$ is replaced by a vector $\mathbf{w}(t)$ that is in phase with $\dot{\mathbf{u}}(t)$ for low frequencies, but gradually changes to be in phase with the displacement $\mathbf{u}(t)$ for high frequencies. Such a variable can be generated by the linear filter equation

$$\nu h \dot{\mathbf{w}}(t) + \mathbf{w}(t) = \nu h \dot{\mathbf{u}}(t) \quad (32)$$

In this filter equation the time scale νh is expressed in terms of the integration time step h and the non-dimensional parameter ν . The state variable $\mathbf{w}$ has the same dimension as the displacement $\mathbf{u}$, and the phase relation appearing from the filter equation suggests a generalized equation of motion

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) - (\nu h)^{-1}\mathbf{D}\mathbf{w}(t) = \mathbf{f}(t) \quad (33)$$

where a new negative damping term $-(\nu h)^{-1}\mathbf{D}\mathbf{w}(t)$ has been introduced. The factor $(\nu h)^{-1}$ is included to give the matrix $\mathbf{D}$ the dimension of a damping matrix. The term is negative, because it is intended to eliminate the undesirable low-frequency algorithmic damping that is a side effect in the basic Newmark algorithm. This requires that the algorithmic damping matrix $\mathbf{D}$ is proportional to the stiffness matrix,

$$\mathbf{D} = \alpha h \mathbf{K} \quad (34)$$

where α is a non-dimensional parameter controlling the magnitude of the damping term, and the factor h appears for dimensional reasons.

In the algorithm the filter equation (32) is discretized in terms of central differences and mean values as

$$\nu \Delta \mathbf{w} + \frac{1}{2}(\mathbf{w}_{n+1} + \mathbf{w}_n) = \nu \Delta \mathbf{u} \quad (35)$$

The terms on the left can be rearranged as a weighted sum of $\mathbf{w}_{n+1}$ and $\mathbf{w}_n$,

$$\left(\frac{1}{2} + \nu\right)\mathbf{w}_{n+1} + \left(\frac{1}{2} - \nu\right)\mathbf{w}_n = \nu(\mathbf{u}_{n+1} - \mathbf{u}_n) \quad (36)$$

Formation of a linear combination of the equation of motion with the weight $\frac{1}{2} + \nu$ on the forward time t_{n+1} and the weight $\frac{1}{2} - \nu$ on the current time t_n will eliminate explicit dependence on the variable $\mathbf{w}$, leaving the averaged equation of motion

$$\begin{aligned} & \left(\frac{1}{2} + \nu\right)\mathbf{M}\ddot{\mathbf{u}}_{n+1} + \left(\frac{1}{2} + \nu\right)\mathbf{C}\dot{\mathbf{u}}_{n+1} + \left(\frac{1}{2} + \nu - \alpha\right)\mathbf{K}\mathbf{u}_{n+1} = \\ & - \left(\frac{1}{2} - \nu\right)\mathbf{M}\ddot{\mathbf{u}}_n - \left(\frac{1}{2} - \nu\right)\mathbf{C}\dot{\mathbf{u}}_n - \left(\frac{1}{2} - \nu + \alpha\right)\mathbf{K}\mathbf{u}_n + \left(\frac{1}{2} + \nu\right)\mathbf{f}_{n+1} + \left(\frac{1}{2} - \nu\right)\mathbf{f}_n \end{aligned} \quad (37)$$

It is seen that the effect of the algorithmic damping via the first order filter with displacement representation of the discretization is to form a weighted average of the equation of motion, and it was demonstrated in [16] that the improved low frequency damping is obtained by selecting

$$\alpha = \gamma - \frac{1}{2} \quad (38)$$

In the weighted average the inertial and damping forces and the loads are given the weights $(\frac{1}{2} \pm \nu)$ while the stiffness terms are given the weights $[\frac{1}{2} \pm (\nu - \alpha)]$. This form of the equation of motion, in which the terms are given two sets of relative forward weights is similar to that of the generalized α -method of Chung and Hulbert [5]. However, in the present format the damping term and the load have the same weights as the inertial terms, while in the generalized α -method the weights on the loads were chosen to follow those of the stiffness terms. The α -methods by Hilber, Hughes and Taylor [3] and by Wood, Bossak and Zienkiewicz [4] are contained as special cases with $\nu = \frac{1}{2}$ and $\nu = \frac{1}{2} + \alpha$, respectively. It is a feature of the filter approach that the external loads have the same weights as the inertial terms, and it has been demonstrated in [16] that this weighting is more accurate. It can be demonstrated that a marginal improvement of the viscous damping can be obtained by use the weights of the stiffness terms $[\frac{1}{2} \pm (\nu - \alpha)]$ on the viscous damping terms, [16], but this will not be discussed further here.

4.2. Energy balance

An energy balance equation for generalized Newmark methods can be obtained by observing that the equation of motion (33) is similar to that treated previously, but now with an additional damping force

$$\mathbf{f}_D = \frac{\alpha}{\nu} \mathbf{K} \mathbf{w} \quad (39)$$

When this additional force is included in the balance equation (11) it generates an additional term

$$\frac{\alpha}{\nu} \Delta \mathbf{u}^T \mathbf{K} \left\{ \frac{1}{2} (\mathbf{w}_{n+1} + \mathbf{w}_n) + \alpha \Delta \mathbf{w} \right\} \quad (40)$$

on the right side of the equation. The value $\alpha = \gamma - \frac{1}{2}$, necessary for optimal low-frequency damping, has been used. This term will now be reduced to quadratic forms and increments of quadratic forms.

First the mean value of $\mathbf{w}$ is eliminated by use of the discretized filter equation (35). This gives

$$\frac{\alpha}{\nu} \Delta \mathbf{u}^T \mathbf{K} \left\{ \nu \Delta \mathbf{u} - (\nu - \alpha) \Delta \mathbf{w} \right\} = \alpha \Delta \mathbf{u}^T \mathbf{K} \Delta \mathbf{u} - (\nu - \alpha) \frac{\alpha}{\nu} \Delta \mathbf{u}^T \mathbf{K} \Delta \mathbf{w} \quad (41)$$

The quadratic term in $\Delta \mathbf{u}$ is positive and cancels a similar negative term in the energy balance equation (43). In fact, this is the elimination of the undesirable low-frequency algorithmic damping of the Newmark algorithm. In the last term the displacement increment $\Delta \mathbf{u}$ is expressed in terms of mean value and increment of $\mathbf{w}$ by the discretized filter equation (35). When leaving out the numerical factor this gives for the last term,

$$-\Delta \mathbf{u}^T \mathbf{K} \Delta \mathbf{w} = - \left\{ \Delta \mathbf{w}^T + \frac{1}{2\nu} (\mathbf{w}_n + \mathbf{w}_{n+1})^T \right\} \mathbf{K} \Delta \mathbf{w} = -\Delta \mathbf{w}^T \mathbf{K} \Delta \mathbf{w} - \frac{1}{2\nu} \left[\mathbf{w}^T \mathbf{K} \mathbf{w} \right]_n^{n+1} \quad (42)$$

When the terms generated by the damping force $\mathbf{f}_D$ are added into the energy balance equation (43) it takes the form

$$\begin{aligned} & \left[\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M}_* \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} + (\beta - \frac{1}{2}\gamma) \frac{1}{2} h^2 \ddot{\mathbf{u}}^T \mathbf{M}_* \ddot{\mathbf{u}} + \frac{1}{2} (1 - \frac{\alpha}{\nu}) \frac{\alpha}{\nu} \mathbf{w}^T \mathbf{K} \mathbf{w} \right]_n^{n+1} = \\ & \Delta \mathbf{u}^T \left\{ \frac{1}{2} (\mathbf{f}_{n+1} + \mathbf{f}_n) + (\gamma - \frac{1}{2}) \Delta \mathbf{f} \right\} - \alpha \left\{ (1 - \frac{\alpha}{\nu}) \Delta \mathbf{w}^T \mathbf{K} \Delta \mathbf{w} + (\beta - \frac{1}{2}\gamma) h^2 \Delta \ddot{\mathbf{u}}^T \mathbf{M}_* \Delta \ddot{\mathbf{u}} \right\} \\ & - \frac{1}{2} h \left\{ h^{-2} \Delta \mathbf{u}^T \mathbf{C} \Delta \mathbf{u} + \frac{1}{4} (\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T \mathbf{C} (\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) \right\} + \frac{1}{2} (\beta - \frac{1}{2}\gamma)^2 h^3 \Delta \ddot{\mathbf{u}}^T \mathbf{C} \Delta \ddot{\mathbf{u}} \end{aligned} \quad (43)$$

where the parameters γ and α are related by (38). It is seen that the effect of the filter damping term is to replace the low frequency damping term, originally quadratic in $\Delta \mathbf{u}$ with a similar term in $\Delta \mathbf{w}$, and to add a quadratic term in $\mathbf{w}$ to the equivalent energy. The phase of $\mathbf{w}$ varies between that of the velocity $\dot{\mathbf{u}}$ and the displacement $\mathbf{u}$ with increasing frequency, but is bounded relative to these vectors as a consequence of the filter equation (36).

4.3. Parameters of the generalized Newmark method

In the generalized Newmark methods there are three amplification factors associated with the natural modes of vibration, two from the original algorithm and one extra from the new state variable $\mathbf{w}$. An optimal set of parameters may be defined by requiring the pair of amplification factors to be complex conjugate for all frequencies, and to let all three amplification factors coalesce at the same spectral radius $|\lambda_\infty|$ in the limit of infinite frequency. These are the criteria used by Chung and Hulbert [5] for optimal parameters of the generalized α -method. In the present notation these criteria imply that all parameters can be prescribed in terms of α as

$$\gamma = \frac{1}{2} + \alpha \quad , \quad \beta = \frac{1}{4}(1 + \alpha)^2 \quad , \quad \nu = \frac{3}{2}\alpha \quad (44)$$

The parameter α controls the magnitude of the algorithmic damping, and is related to the spectral radius at infinite frequency as

$$|\lambda_\infty| = \frac{1 - \alpha}{1 + \alpha} \quad , \quad \alpha = \frac{1 - |\lambda_\infty|}{1 + |\lambda_\infty|} \quad (45)$$

A detailed study of the influence of the parameter α on damping and phase error has been presented in [16]. The parameter combination (44) represents the most common choice, and it is therefore of interest to specialize the general energy balance equation (43) to this case.

$$\begin{aligned} & \left[\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M}_* \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} + \frac{1}{2} (\frac{1}{2} \alpha h)^2 \dot{\mathbf{u}}^T \mathbf{M}_* \dot{\mathbf{u}} + \frac{1}{9} \mathbf{w}^T \mathbf{K} \mathbf{w} \right]_n^{n+1} = \\ & \Delta \mathbf{u}^T \left\{ \frac{1}{2} (\mathbf{f}_{n+1} + \mathbf{f}_n) + \alpha \Delta \mathbf{f} \right\} - \alpha \left\{ \frac{1}{3} \Delta \mathbf{w}^T \mathbf{K} \Delta \mathbf{w} + (\frac{1}{2} \alpha h)^2 \Delta \dot{\mathbf{u}}^T \mathbf{M}_* \Delta \dot{\mathbf{u}} \right\} \\ & - \frac{1}{2} h^{-1} \left\{ \Delta \mathbf{u}^T \mathbf{C} \Delta \mathbf{u} + \frac{1}{4} h^2 (\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n)^T \mathbf{C} (\dot{\mathbf{u}}_{n+1} + \dot{\mathbf{u}}_n) - (\frac{1}{2} \alpha h)^4 \Delta \dot{\mathbf{u}}^T \mathbf{C} \Delta \dot{\mathbf{u}} \right\} \end{aligned} \quad (46)$$

As already mentioned the last viscous damping term on the right side contributes negative dissipation but constitutes only a fraction of the acceleration term associated with $\Delta \dot{\mathbf{u}}$ and $\mathbf{M}_*$. Thus, all damping terms are non-negative quadratic forms, and the generalized Newmark algorithm with parameter combination (44) is unconditionally stable. However, the energy has been reinterpreted as shown in the square brackets on the left side of the balance equation. The consequence of this is illustrated in the following section.

5. NUMERICAL ILLUSTRATIONS

The original Newmark algorithm is energy conserving in the special case $\beta = \frac{1}{2}\gamma = \frac{1}{4}$. There are two typical cases in which different parameters are chosen: the case of explicit integration $\beta = 0$, $\gamma = \frac{1}{2}$, and the case in which algorithmic damping is introduced by selecting $\gamma > \frac{1}{2}$. In these cases the energy balance equation contains additional terms that lead to undesirable fluctuations of the

proper mechanical energy as illustrated in Fig. 2. This effect is demonstrated in the following by three examples: energy conservation for explicit Newmark integration of a single mode, energy fluctuations in explicit time integration of multi-component response, and finally the attenuation of equivalent and mechanical energy in the presence of algorithmic damping, where the initial part of the response may contain an increase of the energy of undersampled high frequency modes.

5.1. Energy conservation in explicit Newmark integration.

In the case of a single mode the energy balance of the Newmark algorithm is given by (20). For explicit integration with $\beta = 0$, $\gamma = \frac{1}{2}$ this equation takes the form of the conservation equation

$$\left[\frac{1}{2}\dot{u}^2 + \frac{1}{2}\left(1 - \frac{1}{4}(\omega h)^2\right)\omega^2 u^2 \right]_n^{n+1} = 0 \quad (47)$$

It is seen that the equivalent energy, that is conserved, contains a reduced stiffness. The requirement of non-negative stiffness leads to the stability condition $\omega h \leq 2$. The development of the mechanical energy and the equivalent energy are shown in Fig. 3 with time step $\omega h = 0.8$ for the two different initial conditions $v(0) = 1$ and $u(0) = 1$, respectively. In both cases the equivalent energy is conserved, as predicted by the theory, while the mechanical energy shows considerable fluctuations. In the case of initial velocity $v(0) = 1$, shown in Fig. 3a, the initial value of the equivalent energy is determined by the kinetic part alone, and therefore is equal to the theoretical value of the mechanical energy. The equivalent energy is conserved and this implies that when $v(t) = 0$ the normalized mechanical energy is equal to $[1 - \frac{1}{4}(\omega h)^2]^{-1} = 1.19$. In the case of an initial displacement $u(0) = 1$ the initial equivalent energy has the magnitude $1 - \frac{1}{4}(\omega h)^2 = 0.84$, and the mechanical energy fluctuates between the correct initial value and this lower value.

FIGURE 3. Energy (—) and equivalent energy (- -). a) Initial velocity $v(0) = 1$, b) Initial displacement $u(0) = 1$.

It is noted that for the explicit algorithm the equivalent energy always constitutes a lower bound on the mechanical energy. This is a consequence of the parameter combination by which the equivalent stiffness is reduced. The implications on the calculated response are illustrated in Figs. 4 and 5. For the initial velocity condition $v(0) = 1$ it can be seen from Fig. 4 that the velocity has the correct magnitude, while the displacement is *overestimated* by about $\frac{1}{2}(\omega h)^2 = 0.08$. Conversely, for the initial displacement condition $u(0) = 1$ it can be seen from Fig. 5 that the displacement has the correct magnitude, while the velocity is *underestimated* by about $\frac{1}{2}(\omega h)^2 = 0.08$. For a

time step close to the stability limit these errors can become very large, and it is advisable to use $\omega h \lesssim 1$ for modes containing appreciable energy.

FIGURE 4. Initial condition $v(0) = 1$. a) Displacement $u(t)$, b) Velocity $v(t)$.

FIGURE 5. Initial condition $u(0) = 1$. a) Displacement $u(t)$, b) Velocity $v(t)$.

5.2. Explicit Newmark integration of multi-component response

For $\beta = 0$ the Newmark algorithm can be arranged in explicit form, if the mass matrix is diagonal and there is no structural damping, see e.g. [17]. For $\gamma = 0$ the energy balance equation (12) takes the following conservation form for free vibration response.

$$\left[\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \mathbf{u}^T (\mathbf{K} - \frac{1}{4} h^2 \mathbf{K} \mathbf{M}^{-1} \mathbf{K}) \mathbf{u} \right]_n^{n+1} = 0 \quad (48)$$

With these parameters the equivalent stiffness is less than the mechanical stiffness, and the stability limit follows from the condition that the equivalent stiffness term must be positive definite, corresponding to $\omega_{max} h \leq 2$. The implications of the conservation equation (48) for the equivalent energy is illustrated by analysis of the simple portal frame shown in Fig. 6 with $m = 8$ storeys. The mass of all floors are identical as is the inter-floor stiffness. They are normalized to $M = 1$ and $K = 1$, respectively. This gives a reference frequency of $\omega_0 = (K/M)^{1/2} = 1$ corresponding to a similar one-storey building. An upper bound on the maximum modal frequency ω_{max} follows by considering a division into elements, each containing half the floor mass. This results in the upper bound $\omega_{max} < 2\omega_0 = 2$, [19], and the stability limit gives a safe time step limit $h \leq 1$. The natural frequencies and periods of the vibration modes are given in Table 1. It is seen that the upper bound is within 1.7% of the calculated value. The largest modal frequency is about ten

FIGURE 6. Portal frame with m storeys.

TABLE 1. Modal parameters in 8-storey portal building.

	1	2	3	4	5	6	7	8
ω_i	0.185	0.547	0.892	1.205	1.478	1.700	1.865	1.966
T_i	34.048	11.480	7.048	5.213	4.251	3.695	3.369	3.196
$1 - \frac{1}{4}(\omega h)^2$	0.998	0.981	0.950	0.909	0.864	0.819	0.783	0.758
$E_j/E_0 _{v_8}$	0.233	0.218	0.189	0.150	0.107	0.065	0.031	0.008
$E_j/E_0 _{x_8}$	0.008	0.065	0.150	0.218	0.233	0.189	0.107	0.031

times the smallest, and thus even a crude integration of the higher modes leads to a fairly detailed representation of the lower modes.

In this example the time step $h = 0.5$ is selected, corresponding to about half the stability limit with six points per period for the highest mode and about 70 points per period in the lowest mode. The quality of the integrated results depends on the distribution of the energy on the individual modes. Two cases are analyzed. In the first case an initial velocity $v_8(0) = 1$ is imposed on the top floor. The distribution of energy on the individual modes is shown in the fourth row of Table 1. It is seen that the highest three modes account for less than 10 pct. of the energy. The displacement $u_8(t)$ and the velocity $v_8(t)$ of the top floor are shown in Fig. 7. The full curves show the result of explicit integration with $h = 0.5$, while the dotted curves were obtained with $h = 0.05$. As expected the curves illustrate a certain drift due to the phase error developing a noticeable difference after $t = 15 - 20$.

In the second case the initial velocity condition is replaced by the similar initial displacement condition $u_8(0) = 1$. This leads to the same distribution of the amplitudes of the modes, but the elastic energy of each mode is proportional with ω_j^2 , and therefore initial displacement condition results in more energy in the higher modes. In fact the highest three modes now contain about 30 pct. of the energy. The displacement and velocity of the top floor are illustrated in Fig. 8. It is seen that the results are of lower quality due to the higher energy content in the higher modes.

The development of the energy in the two cases is shown in Figs. 9a and 9b, respectively. It is seen that in spite of the seeming acceptable quality of the initial velocity analysis, the mechanical

FIGURE 7. Initial condition $v_g(0) = 1$. a) Displacement $u_g(t)$, b) Velocity $v_g(t)$.

FIGURE 8. Initial condition $u_g(0) = 1$. a) Displacement $u_g(t)$, b) Velocity $v_g(t)$.

energy shows considerable fluctuations, while the equivalent energy remains constant as expected. The initial velocity condition translates into a state of initial kinetic energy. As the kinetic energy is correctly represented in the conservation equation, the mechanical and equivalent energy are identical at the initial time. At later times some of the energy is elastic, and because the equivalent stiffness is too small, this leads to larger amplitudes, and thereby to larger mechanical energy. Thus, for the initial velocity condition the equivalent energy takes the value of the theoretically correct mechanical energy, while the mechanical energy produced by the algorithm fluctuates at larger values. The fluctuations represent the contributions from the various modes. The relative error in stiffness is shown in row three of Table 1.

In the case of imposed initial displacements the roles of kinetic and elastic energy are reversed. In this case the initial displacement state corresponds to an initial state of elastic energy. This state is misrepresented in the equivalent energy due to the misrepresentation of the modal stiffness, and therefore the equivalent energy in this case is lower than the correct energy level. In this case the mechanical energy fluctuates between the correct theoretical value and the level of the equivalent energy. The fluctuations are considerably larger than in the initial velocity case due to

FIGURE 9. Energy (—) and equivalent energy (- -). a) Initial velocity $v_8(0) = 1$, b) Initial displacement $u_8(0) = 1$.

the higher energy content of the higher modes. In both cases the level of the energy fluctuations is proportional with the square of the time step.

5.3. Modal response with algorithmic damping

In the case of algorithmic damping of a single mode without structural damping the energy balance equation (46) takes the following form for free vibrations,

$$\left[\frac{1}{2} \dot{u}^2 + \frac{1}{2} \left(1 + \frac{1}{4} (\alpha \omega h)^2 \right) \omega^2 u^2 + \frac{1}{9} \omega^2 w^2 \right]_n^{n+1} = -\alpha \omega^2 \left(\frac{1}{4} (\alpha \omega h)^2 (\Delta u)^2 + \frac{1}{3} (\Delta w)^2 \right) \quad (49)$$

This equation covers the original Newmark algorithm with $\alpha = \gamma - \frac{1}{2}$, when terms involving the auxiliary state variable w are omitted, and the generalized Newmark algorithm when all terms are included. This energy equation is different from (47) for the explicit Newmark algorithm in several ways. First of all it contains dissipation terms on the right side. The reinterpretation of the energy is also different, depending on $\alpha \omega h$ instead of ωh , with *increased* equivalent stiffness as well as an additional quadratic term in w .

FIGURE 10. Energy (—) and equivalent energy (- -) for $\alpha = 0.1$ and $h/T = 10$. a) Newmark algorithm, b) Generalized α -algorithm.

The parameter α is related to the spectral radius $|\lambda_\infty|$ as given by (45). In order to illustrate the effect of algorithmic damping the damping parameter is taken to be $\alpha = 0.1$ corresponding to the spectral radius $\lambda_\infty = 0.818$. At the Nyquist frequency $\omega h = 1$, and the change of the stiffness is about 0.3 pct. However, algorithmic damping is introduced to reduce or eliminate response components with frequencies higher than those that can be properly handled within the discretization with the chosen time step h , and thus control of response components with ωh considerably larger than unity is also a concern. Here the parameter value $h/T = 10$ is chosen, representing a heavily undersampled response component. The development of the mechanical energy and the equivalent energy for free vibration response to the two sets of initial conditions $u(0)=0, v(0)=1$ and $u(0)=1, v(0)=0$ are illustrated in Fig. 10a and 10b for the Newmark algorithm and the generalized α -algorithms, respectively. The energy development is largely similar for the two algorithms. In the case of initial velocity $v(0) = 1$ the initial value of the equivalent and mechanical energies are identical, and they both decrease monotonically. In the case of initial displacement $u(0) = 1$ the higher equivalent stiffness leads to a higher initial value of the equivalent energy. The equivalent energy decreases monotonically, as predicted by the energy relation (49). In contrast, the mechanical energy increases to nearly double its initial value before starting to decrease. Clearly this excess energy leads to overestimation of the response. The larger equivalent stiffness limits the displacement response, while the velocity response exhibits ‘overshoot’ behavior. This is illustrated in Fig. 11 for the generalized α -algorithm. The behavior for the original Newmark algorithm is similar. This type of ‘overshoot’ behavior was illustrated by Hilber and Hughes [6] and investigated more recently by Zhou and Tamma [7]. It finds a theoretical explanation by the energy relations for the algorithms.

FIGURE 11. Initial condition $u(0) = 1$, $\alpha = 0.1$ and $h/T = 10$. a) Displacement $u(t)$, b) Velocity $v(t)$.

6. DISCUSSION

It has been demonstrated that the response calculated by the Newmark time integration algorithm satisfies an energy balance equation. In the special case of $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$ the algorithm is energy conserving. For different values of the parameters, extra terms appear in the energy balance equation. For $\gamma > \frac{1}{2}$ algorithmic damping in terms of the stiffness matrix appears. For $\beta \neq \frac{1}{2}\gamma$ the quadratic stiffness term is supplemented by a term that is quadratic in the acceleration.

The stiffness term appears in the definition of the mechanical energy and also contributes to the algorithmic damping. For free vibrations the acceleration term can be expressed as an extra stiffness term for each of the vibration modes with a relative magnitude proportional with the square of the modal frequencies. The frequency dependence implies that this modification of the stiffness may become important for the higher modes. Two cases are of particular interest: the explicit integration with $\beta = 0$, and the case when unconditional stability is used in connection with algorithmic damping to suppress undesirable high-frequency components. In both cases the free vibration path in the phase plane of the high-frequency modes traces an ellipse instead of a circle, due to the changed stiffness. In the response this shows up as stable and regular fluctuations of the mechanical energy. In fact, the criterion for conditional stability can be interpreted as the requirement that even the highest mode retains positive equivalent stiffness.

It has recently been demonstrated that the so-called α -methods, in which the algorithmic damping of the original Newmark method is modified by taking a weighted average of the equation of motion at two consecutive times, can be considered as special cases of a procedure in which damping is modified by a first order filter. By casting the α -methods into a format with this new damping variable as a new state variable, the energy balance equations can be generalized to the class of α -methods. The basic features of the energy balance equation are the same as for the original Newmark method, because the α -methods only introduce a low-frequency modification to the algorithmic damping of the Newmark algorithm obtained by selecting $\gamma > \frac{1}{2}$. Thus the α -methods leave the high frequency regime with its proneness to energy modification virtually unchanged from the Newmark algorithm. An effect of the energy fluctuations introduced by the algorithm for $\beta \neq \frac{1}{2}\gamma$ is the phenomenon of ‘overshoot’ by which high-frequency components can increase beyond their initial amplitude, before they are attenuated by the algorithmic damping. The energy balance equations explains this phenomenon quantitatively as well as qualitatively, and suggests that algorithmic damping should only be introduced to the extent necessary, as the stiffness modification is proportional with α^2 .

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