Bismut hypoelliptic Laplacians for manifolds with boundaries

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Abstract

Boundary conditions for Bismut's hypoelliptic Laplacian which naturally correspond to Dirichlet and Neumann boundary conditions for Hodge Laplacians are considered. Those are related with specific boundary conditions for the differential and its various adjoints. Once the closed realizations of those operators are well understood, the commutation of the differential with the resolvent of the hypoelliptic Laplacian is checked with other properties like the PT-symmetry, which are important for the spectral analysis.

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Contents

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1 Introduction

This text is devoted to boundary conditions which extend naturally to Bismut's hypoelliptic Laplacians, Dirichlet and Neumann's boundary conditions for Hodge and Witten Laplacians. Actually such boundary conditions were proposed in [Nie] and the functional analysis was carried out, essentially relying on the scalar principal parts while neglecting the complicated lower order terms related with curvature tensors. While considering the commutation of the resolvent of this operator with the differential, with the suitable boundary conditions, a good understanding

of the geometrical content of the whole operators cannot be skipped. This article answers one of the questions asked at the end of [Nie] about the proper boundary conditions for the differential and Bismut's codifferential, which ensure the commutation with the resolvent of Bismut's hypoelliptic Laplacian. A motivation for this comes from the accurate spectral analysis of the low-lying spectrum which is now well understood for Witten Laplacian in the low temperature limits, for possibly non Morse potential functions. An instrumental tool in [LNV1][LNV2], was the introduction of artificial Dirichlet and Neumann realizations of Witten Laplacians, as a localization technique. Although it is a first step, ignoring for the moment the asymptotic analysis issues, the results of this article pave the way to such an analysis for Bismut's hypoelliptic Laplacian.
This introduction rapidly presents the objects, the main results, some notations and conventions. It ends by pointing out the problems to be solved and the followed strategy. The outline of the article is given in this last paragraph.

1.1 Bismut hypoelliptic Laplacian

When Q is a closed (compact) riemannian manifold endowed with the metric $g = g^{TQ} = g_{ij}(\underline{q})d\underline{q}^i d\underline{q}^j$, the hypoelliptic Laplacian introduced by J-M. Bismut in [Bis04][Bis05] is a Hodge type Laplacian defined on the cotangent space $X = T^*Q$, $\pi_X : X = T^*Q \to Q$, of which we briefly recall the construction here. Details will be given in the text and may be found in [Bis04][Bis04-2][Bis05][BiLe][Leb1][Leb2]. The Levi-Civita connection associated with g^{TQ} gives rise to the decomposition into horizontal and vertical space

$$TX = \underbrace{TX^H}_{\simeq \pi_X^*(TQ)} \oplus \underbrace{TX^V}_{\simeq \pi_X^*(T^*Q)} \quad ,$$

and by duality to

$$T^*X = \underbrace{T^*X^H}_{\simeq \pi_X^*(T^*Q)} \oplus \underbrace{T^*X^V}_{\pi_X^*(TQ)} .$$

By using the dual metric $g^{T^*Q} = g^{ij}(\underline{q})\frac{\partial}{\partial \underline{q}^i}\frac{\partial}{\partial \underline{q}^j}$, one may define metrics on TX and T^*X which make the horizontal vertical decomposition orthogonal. By tensorization this provides metrics on the vector bundles $E' = \Lambda TX$ and $E = \Lambda T^*X$. It is convenient to introduce from the begining a p-dependent weight where an element of $X = T^*Q$ is locally written $x = (q,p)$, by setting

$$|p|_q = \sqrt{g_q^{T^*Q}(p,p)} = \sqrt{g^{ij}(q)p_i p_j} \quad , \quad \langle p\rangle_q = \sqrt{1 + g_q^{T^*Q}(p,p)} = \sqrt{1 + g^{ij}(q)p_i p_j} .$$

The metric g^E and $g^{E'}$, dual to each other, are given by

$$g^E = \langle p\rangle_q^{N_V - N_H}\pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ}) \quad , \quad g^{E'} = \langle p\rangle_q^{N_H - N_V}\pi_X^*(g^{\Lambda TQ} \otimes g^{\Lambda T^*Q}),$$

where N_V and N_H are the vertical and horizontal number operators. The volume associated with g^E coincides with the symplectic volume on X and it is simply

written dv_X.
Additionally we may add a hermitian vector bundle structure by starting from $(\mathfrak{f},\nabla^{\mathfrak{f}},g^{\mathfrak{f}})$ where $\pi_{\mathfrak{f}}:\mathfrak{f}\to Q$ is a complex vector bundle, endowed with a flat connection $\nabla^{\mathfrak{f}}$, and $g^{\mathfrak{f}}$ is a hermitian metric. While identifying the anti-dual vector bundle with $\mathfrak{f}$ via the metric $g^{\mathfrak{f}}$, the anti-dual metric remains $g^{\mathfrak{f}}$ but the anti-dual flat connection differ and will be denoted by $\nabla^{\mathfrak{f}'}$. By setting

$$\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})=(g^{\mathfrak{f}})^{-1}\nabla^{\mathfrak{f}}g^{\mathfrak{f}}\in\mathscr{C}^\infty(Q;T^*Q\otimes L(\mathfrak{f}))$$

we get

$$\nabla^{\mathfrak{f}'}=\nabla^{\mathfrak{f}}+\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})$$

and we can introduce the unitary connection on $(\mathfrak{f},g^{\mathfrak{f}})$

$$\nabla^{\mathfrak{f},u}=\nabla^{\mathfrak{f}}+\frac{1}{2}\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}}).$$

Simple examples of such vector bundles on Q are given by

- $\mathfrak{f}=Q\times\mathbb{C}$, $\nabla^{\mathfrak{f}}$ is the trivial connection ∇, $g^{\mathfrak{f}}(z)=e^{-2V(q)}|z|^2$, $\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})=-2dV(q)$, where $V\in\mathscr{C}^\infty(Q;\mathbb{R})$ is a potential function,
- the orientation bundle $\mathfrak{f}=(\mathrm{or}_Q\times\mathbb{C})/\mathbb{Z}_2$ where $\mathrm{or}_Q\to Q$ is the orientation double cover of Q. The trivial connection and trivial metric on $\mathrm{or}_Q\times\mathbb{C}$ induce the connection $\nabla^{\mathfrak{f}}$ and $g^{\mathfrak{f}}$.

The total vector bundles $F=E\otimes\pi_X^*(\mathfrak{f})=\pi_X^*(\Lambda T^*Q\otimes\Lambda TQ\otimes\mathfrak{f})$ (resp. $F'=E'\otimes\pi_X^*(\mathfrak{f})$) endowed with the metrics

$$\begin{aligned} & g^F=g^E\otimes\pi_X^*g^{\mathfrak{f}}=\langle p\rangle_q^{N_V-N_H}\pi_X^*(g^{\Lambda T^*Q}\otimes g^{\Lambda TQ}\otimes g^{\mathfrak{f}})\\ \text{resp.}\qquad & g^{F'}=g^{E'}\otimes\pi_X^*g^{\mathfrak{f}}=\langle p\rangle_q^{N_H-N_V}\pi_X^*(g^{\Lambda TQ}\otimes g^{\Lambda T^*Q}\otimes g^{\mathfrak{f}}).\end{aligned}$$

When $\nabla^{Q,g}$ is connection on $\Lambda T^*Q\otimes\Lambda TQ$ induced by the Levi-Civita connection on TQ, and after the identification of $TX=TX^H\oplus TX^V$ with $\pi_X^*(TQ\oplus T^*Q)$, the connection $\nabla^{E,g}$ on E is nothing but $\pi_X^*(\nabla^{Q,g})$. When $\nabla^{\mathfrak{f}}$, $\nabla^{\mathfrak{f}'}=\nabla^{\mathfrak{f}}+\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})$ and $\nabla^{\mathfrak{f},u}=\nabla^{f}+\frac{1}{2}\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})$ are the three above connections on $\mathfrak{f}$, the connections on F and F' are given by

$$\begin{aligned}\nabla^{F,g}&=\pi_X^*(\nabla^{Q,g}+\nabla^{\mathfrak{f}})\\ \nabla^{F',g}&=\pi_X^*(\nabla^{Q,g}+\nabla^{\mathfrak{f}'})\\ \nabla^{F,g,u}&=\pi_X^*(\nabla^{Q,g}+\nabla^{\mathfrak{f},u}).\end{aligned}$$

When there is no ambiguity with a fixed metric g^{TQ}, the exponent ^{g} will be dropped and the above connections will be simply written ∇^E, ∇^F, $\nabla^{F'}$, $\nabla^{F,u}$.
The $L^2(X;F)$ (resp. $L^2(X;F')$) space is the space of L^2-sections for the Hilbert scalar product

$$\begin{aligned} & \langle s\,,s'\rangle_{L^2}=\int_X\langle s(x)\,,s'(x)\rangle_{g^F}\,dv_X(x)\\ \text{resp.}\qquad & \langle t\,,t'\rangle_{L^2}=\int_X\langle t(x)\,,t'(x)\rangle_{g^{F'}}\,dv_X(x).\end{aligned}$$

From those structures one could define a Hodge Laplacian for sections of F. However this is not the way to introduce Bismut's hypoelliptic Laplacian. Instead one works with the non degenerate bilinear form ϕ_b on TX combining the metric and the symplectic form $\sigma = d\theta$, where $\theta = p_i dq^i$ is the tautological one-form on X, with $X = T^*Q$. For $b \neq 0$, the isomorphism $\phi_b : TX \to T^*X$ extended to $\phi_b : E' = \Lambda TX \to E = \Lambda T^*X$ is the one given by the non degenerate form

$$\eta_{\phi_b}(U,V) = g^{TQ}(\pi_{X,*}(U), \pi_{X,*}(V)) + b\sigma(U,V) = U.\phi_b V\,, \quad U,V \in TX\,.$$

When $(\underline{e}_1,\ldots,\underline{e}_d)$ is a local frame of TQ, with the dual frame $(\underline{e}^1,\ldots,\underline{e}^d)$ in T^*Q, we denote by $e_i \in TX^H$ and $\hat{e}^j \in TX^V$, the corresponding vectors obtained via $TX^H \simeq \pi_X^*(TQ)$ and $TX^V \simeq \pi_X^*(X^V)$ and we notice that $(e_i, \hat{e}^j)$ is a symplectic frame of TX. The dual basis is $e^i \in T^*X^H$ and $\hat{e}_j \in T^*X^V$. Then the matrix of η_{ϕ_b} or $\phi_b : TX \to T^*X$ in those bases is given by

$$\phi_b = \begin{pmatrix} g^{TQ} & -b\mathrm{Id} \\ b\mathrm{Id} & 0 \end{pmatrix}.$$

The dual bilinear form on ΛT^*X is then given by

$$\eta^*_{\phi_b}(\omega,\theta) = (\phi_b^{-1}\omega).\theta \quad , \quad \omega,\theta \in \Lambda T^*X\,.$$

Where the matrix of $\eta^*_{\phi_b}\big|_{T^*X\times T^*X}$ or of ${}^t\phi_b^{-1} : T^*X \to TX$ equals

$${}^t\phi_b^{-1} = \begin{pmatrix} 0 & -b^{-1}\mathrm{Id} \\ b^{-1}\mathrm{Id} & b^{-2}g^{TQ} \end{pmatrix}.$$

The tensorization with $\mathfrak{f}$ is done by writing $\eta^*_{\phi_b,\mathfrak{f}} = \eta^*_{\phi_b} \otimes \pi_X^*(g^{\mathfrak{f}})$ on $F = E \otimes \pi_X^*(\mathfrak{f})$ so that the non-degenerate sesquilinear form on sections of $F = E \otimes \pi_X^*(\mathfrak{f})$ is given by

$$\langle s\,, s'\rangle_{\phi_b} = \int_X \eta^*_{\phi_b,\mathfrak{f}}(s(x), s'(x))\; dv_X(x)\,.$$

By introducing the kinetic energy

$$\mathfrak{h}(q,p) = \frac{|p|_q^2}{2} = \frac{g^{ij}(q)p_i p_j}{2}$$

and the deformed differential

$$d_{\mathfrak{h}} = e^{-\mathfrak{h}} d e^{\mathfrak{h}} = d + d\mathfrak{h}\wedge$$

Bismut's codifferential $d_{\mathfrak{h}}^{\phi_b}$ is the formal adjoint of the differential $d_{\mathfrak{h}}$ for the above duality product $\langle\ ,\ \rangle_{\phi_b}$. Bismut's hypoelliptic Laplacian is nothing but

$$B_{\mathfrak{h}}^{\phi_b} = \frac{1}{4}(d_{\mathfrak{h}}^{\phi_b} + d_{\mathfrak{h}})^2 = \frac{1}{4}(d_{\mathfrak{h}}^{\phi_b} d_{\mathfrak{h}} + d_{\mathfrak{h}} d_{\mathfrak{h}}^{\phi_b})\,.$$

In [Bis05] Bismut proved the following Weitzenbock formula

$$\begin{aligned} B_{\mathfrak{h}}^{\phi_b} \;\; = \;\; & \frac{1}{4b^2}\Big[-\Delta^V + |p|_q^2 - \frac{1}{2}\langle R^{TQ}(e_i,e_j)e_k\,, e_\ell\rangle e^i e^j \mathbf{i}_{\hat{e}^k\hat{e}^\ell} + 2N_V - \dim Q\Big] \\ & -\frac{1}{2b}\Big[\mathscr{L}_{Y_{\mathfrak{h}}} + \frac{1}{2}\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})(Y_{\mathfrak{h}}) + \frac{1}{2}e^i \mathbf{i}_{\hat{e}^j} \nabla^F_{e_i}\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})(e_j) \\ & \qquad + \frac{1}{2}\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})(e_i)\nabla^F_{\hat{e}^i}\Big] \end{aligned}$$

where Δ^V is the vertical Laplacian and $\mathcal{O} = \frac{-\Delta^V + |p|_q^2}{2}$ the scalar harmonic oscillator in the vertical direction while $Y_{\mathfrak{h}} = g^{ij}(q)p_i e_i$ is the Hamilton vector field for the kinetic energy $\mathfrak{h} = \frac{|p|_q^2}{2}$. The other terms, which involve various curvatures, like R^{TQ} the riemann curvature tensor of g^{TQ} pulled-back by π_X^*, and $\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})$, are actually lower order terms controlled in the analysis by the scalar principal part. For the analysis of those operators G. Lebeau introduced in [Leb1] the scale of Sobolev spaces $\mathcal{W}^{\mu}(X;E)$, $\mu \in \mathbb{R}$, modelled on the fact that horizontal derivations $\nabla^{F,u}_{e_i}$, $\frac{\partial}{\partial q^i}$, and weighted vertical derivations $\langle p\rangle_q \nabla^{F,u}_{\hat{e}^j}$, $\langle p\rangle_q \frac{\partial}{\partial p_j}$ are of order 1, while the multilplications $p_j\times$ and $\langle p\rangle_q\times$ are of order $\frac{1}{2}$, with $\mathcal{W}^0(X;F) = L^2(X;F)$. They satisfy $\cap_{\mu\in\mathbb{R}}\mathcal{W}^{\mu}(X;F) = S(X;F)$ the space of $\mathcal{C}^{\infty}$ vertically rapidly decaying sections while $\cup_{\mu\in\mathbb{R}}\mathcal{W}^{\mu}(X;F) = S'(X;F)$ is the space of tempered distributional sections. The maximal subelliptic estimates were proved in [Leb2]. They say the following things:

- Given $b \neq 0$, there exist a constant $C_b > 0$ and for any $\mu \in \mathbb{R}$ a constant $C_{b,\mu} > 0$ such that

$$\|\mathcal{O}s\|_{\mathcal{W}^{\mu}} + \|\nabla^{F,u}_{Y_{\mathfrak{h}}} s\|_{W^{\mu}} + \delta_{0,\mu}\langle\lambda\rangle^{1/2}\|s\|_{\mathcal{W}^{\mu}} + \|s\|_{\mathcal{W}^{\mu+2/3}} \leq C_{b,\mu}\|(C_b + B^{\phi_b}_{\mathfrak{h}} - i\delta_{0,\mu}\lambda)s\|_{\mathcal{W}^{\mu}}$$

  for all $s \in S'(X;F)$ and all $\lambda \in \mathbb{R}$ such that $(C_b + B^{\phi_b}_{\mathfrak{h}} - i\delta_{0,\mu}\lambda)s \in \mathcal{W}^{\mu}(X;F)$.

- The operator $C_b + B^{\phi_b}_{\mathfrak{h}}$ with domain $D(B^{\phi_b}_{\mathfrak{h}}) = \left\{s \in L^2(X;F), B^{\phi_b}_{\mathfrak{h}} s \in L^2(X;F)\right\}$ is maximal accretive in $L^2(X;F)$.

The reason for the factor $\delta_{0,\mu}$ which says that the λ-dependent case holds here only for $\mu = 0$ is due to the fact that the more general λ-dependent estimates of [Leb2] require λ-dependent $\mathcal{W}^{\mu}$-norms for $\mu \neq 0$ which won't be considered in this text.

1.2 Boundary conditions and results

Let $\overline{Q}_- = Q_- \sqcup Q'$, $Q' = \partial Q_-$, be a compact riemannian manifold with boundary. The cotangent bundle is a manifold with boundary: $\overline{X}_- = X_- \sqcup X'$, $X_- = T^*Q_-$, $X' = \partial X_- = T^*Q\big|_{\partial Q_-}$. A collar neighborhood $Q_{(-\varepsilon,0]}$ of Q' can be chosen such that $Q_{(-\varepsilon,0]} \simeq (-\varepsilon,0]\times Q'$ and the metric g^{TQ} equals $(d\underline{q}^1)^2 + m^{TQ'}(\underline{q}^1, \underline{q}')$ with $m^{TQ'}(\underline{q}^1,\underline{q}')$ is a $\underline{q}^1$-dependent metric on Q'. Although the following constructions will be checked to make sense geometrically, we follow here the shortest presentation in terms of coordinates. The decomposition

$$TQ_{(-\varepsilon,0]} = \mathbb{R}\frac{\partial}{\partial \underline{q}^1} \overset{\perp}{\oplus} TQ' \quad T^*Q_{(-\varepsilon,0]} = \mathbb{R}d\underline{q}_1 \overset{\perp}{\oplus} T^*Q',$$

provides coordinates $x = (q^1, q', p_1, p')$ where $(q', p') = (q^{i'}, p_{j'})_{2\leq i',j'\leq d}$ are local coordinates on T^*Q', $q^1 = \underline{q}^1(\pi_X(x))$ and $p_1 = \frac{\partial}{\partial \underline{q}^1}.p$. Take $\underline{e}_i = \frac{\partial}{\partial \underline{q}^i}$ and $\underline{e}^i = d\underline{q}^i$. The construction given in Subsection 1.1, relying on $TX = TX^H \oplus TX^V \simeq$

$\pi_X^*(TQ \oplus T^*Q)$, provides the basis $(e_i, \hat{e}^j)_{1\leq i,j\leq \dim Q}$ of TX and the dual basis $(e^i, \hat{e}_j)_{1\leq i,j\leq \dim Q}$ of T^*X.
A section s of $E = \Lambda T^*X$ (or $F = E \otimes \pi_X^*(\mathfrak{f})$) reads in those coordinates

$$s = s_I^J(q^1, q', p_1, p')e^I \hat{e}_J .$$

When s is a regular enough section to admit the following traces we consider

$$\begin{aligned}
&s\big|_{X'} = s_I^J(0, q', p_1, p')e^I \hat{e}_J ,\\
&\mathbf{i}_{e_1} e^1 \wedge s\big|_{X'} = s_{I'}^J(0, q', p_1, p')e^{I'} \hat{e}_J ,\\
&\hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1} s\big|_{X'} = s_I^{\{1\}\cup J'}(0, q', p_1, p')e^I \hat{e}_{\{1\}\cup J'} .
\end{aligned}$$

Note that $s\big|_{X'}$ is not the classical pull back to X'.
When ν is a flat unitary involution of $\mathfrak{f}\big|_{Q'}$, $\nu \in \mathscr{C}^\infty(Q'; L(\mathfrak{f}\big|_{Q'}))$ such that its covariant derivative along Q' vanishes $\nabla^{L(\mathfrak{f}|_{Q'})}\nu = 0$ (the simplest and essential example being $\nu = \pm \mathrm{Id}_{\mathfrak{f}}$), the transformation $\hat{S}_\nu$ acting on sections of $F\big|_{X'}$ is given by

$$\hat{S}_\nu(s_I^J(0, q', p_1, p')e^I \hat{e}_J) = (-1)^{|\{1\}\cap I| + |\{1\}\cap J|} \nu s_I^J(0, q^1, -p_1, p')e^I \hat{e}_J ,$$

where we use the same notation ν for the pulled-backed unitary map $\pi_{X'}^*(\nu)$. Note that $(\hat{S}_\nu)^2 = \mathrm{Id}$ and $\frac{1-\hat{S}_\nu}{2}$ is a projection.
We shall prove the following results for closed realizations of the differential operators $P = d_{\mathfrak{h}}$, $P = d_{\mathfrak{h}}^{\phi_b}$ and $P = B_{\mathfrak{h}}^{\phi_b}$, where an important step consists in proving trace theorems for sections $s \in L^2(X_-; F)$ such that $Ps \in L^2(X_-; F)$, so that the definition of the domains makes sense.

Theorem 1.1. *The operator $(\overline{d}_{g,\mathfrak{h}}, D(\overline{d}_{g,\mathfrak{h}}))$ in $L^2(X_-; F)$ defined by*

$$\begin{aligned}
&D(\overline{d}_{g,\mathfrak{h}}) = \left\{ s \in L^2(X_-; F), \quad d_{\mathfrak{h}} s \in L^2(X_-; F), \quad \frac{1-\hat{S}_\nu}{2} \mathbf{i}_{e_1} e^1 \wedge s\big|_{X'} = 0 \right\}\\
&\forall s \in D(\overline{d}_{g,\mathfrak{h}}), \quad \overline{d}_{g,\mathfrak{h}} s = d_{\mathfrak{h}} s ,
\end{aligned}$$

is closed and satisfies $\overline{d}_{g,\mathfrak{h}} \circ \overline{d}_{g,\mathfrak{h}} = 0$.
The set $\mathscr{C}_0^\infty(\overline{X}_-; F) \cap D(\overline{d}_{g,\mathfrak{h}})$ is a core for $\overline{d}_{g,\mathfrak{h}}$.

Theorem 1.2. *The operator $(\overline{d}_{g,\mathfrak{h}}^{\phi_b}, D(\overline{d}_{g,\mathfrak{h}}^{\phi_b}))$ in $L^2(X_-; F)$ defined by*

$$\begin{aligned}
&D(\overline{d}_{g,\mathfrak{h}}^{\phi_b}) = \left\{ s \in L^2(X_-; F), \quad d_{\mathfrak{h}}^{\phi_b} s \in L^2(X_-; F), \quad \frac{1-\hat{S}_\nu}{2} \hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1} s\big|_{X'} = 0 \right\}\\
&\forall s \in D(\overline{d}_{g,\mathfrak{h}}^{\phi_b}), \quad \overline{d}_{g,\mathfrak{h}}^{\phi_b} s = d_{\mathfrak{h}}^{\phi_b} s ,
\end{aligned}$$

is closed and satisfies $\overline{d}_{g,\mathfrak{h}}^{\phi_b} \circ \overline{d}_{g,\mathfrak{h}}^{\phi_b} = 0$.
The set $\mathscr{C}_0^\infty(\overline{X}_-; F) \cap D(\overline{d}_{g,\mathfrak{h}}^{\phi_b})$ is a core for $\overline{d}_{g,\mathfrak{h}}^{\phi_b}$.

Theorem 1.3. *The operator* $(\overline{B}^{\phi_b}_{g,\mathfrak{h}}, D(\overline{B}^{\phi_b}_{g,\mathfrak{h}}))$ *defined in* $L^2(X_-;F)$ *by*

$$D(\overline{B}^{\phi_b}_{g,\mathfrak{h}}) = \left\{ s \in L^2(X_-;F), \quad \nabla^F_{\frac{\partial}{\partial p}} s \text{ and } B^{\phi_b}_{\mathfrak{h}} s \in L^2(X_-;F), \quad \frac{1-\hat{S}_v}{2} s\big|_{X'} = 0 \right\}$$
$$\forall s \in D(\overline{B}^{\phi_b}_{g,\mathfrak{h}}), \quad \overline{B}^{\phi_b}_{g,\mathfrak{h}} s = B^{\phi_b}_{\mathfrak{h}} s ,$$

is closed and there exists a constant $C_b \in \mathbb{R}$ *such that* $C_b + \overline{B}^{\phi_b}_{g,\mathfrak{h}}$ *is maximal accretive. The set* $\mathscr{C}^\infty_0(\overline{X}_-;F) \cap D(\overline{B}^{\phi_b}_{g,\mathfrak{h}})$ *is a core for* $\overline{B}^{\phi_b}_{g,\mathfrak{f}}$.
There exists a constant $C'_b > 0$ *such that the estimates*

$$\left. \begin{array}{l} \|(1+\mathscr{O})^{1/2} s\|_{L^2} + \|s\|_{\mathscr{W}^{1/3}} \\ + \langle\lambda\rangle^{1/4} \|s\|_{L^2} + \|\langle p\rangle_q^{-1} s\big|_{X'}\|_{L^2(X',|p_1| dv_{X'})} \end{array} \right\} \leq C'_b \|(1 + C_b + \overline{B}^{\phi_b}_{g,\mathfrak{f}} - i\lambda)s\|_{L^2} ,$$
$$\|(1+\mathscr{O})^{1/2} s\|_{L^2} \leq C'_b \mathbb{R}\mathrm{e}\langle s, (1 + C_b + \overline{B}^{\phi_b}_{g,\mathfrak{f}})s\rangle$$

hold for all $s \in D(\overline{B}^{\phi_b}_{g,\mathfrak{f}})$ *and all* $\lambda \in \mathbb{R}$.
The semigroup $(e^{-t\overline{B}^{\phi_b}_{g,\mathfrak{h}}})_{t\geq 0}$ *preserves* $D(\overline{d}_{g,\mathfrak{h}})$ *and* $D(\overline{d}^{\phi_b}_{g,\mathfrak{h}})$ *with*

$$\forall s \in D(\overline{d}_{g,\mathfrak{h}}), \forall t \geq 0, \quad \overline{d}_{g,\mathfrak{h}} e^{-t\overline{B}^{\phi_b}_{g,\mathfrak{h}}} s = e^{-t\overline{B}^{\phi_b}_{g,\mathfrak{h}}} \overline{d}_{g,\mathfrak{h}} s ,$$
$$\forall s \in D(\overline{d}^{\phi_b}_{g,\mathfrak{h}}), \forall t \geq 0, \quad \overline{d}^{\phi_b}_{g,\mathfrak{h}} e^{-t\overline{B}^{\phi_b}_{g,\mathfrak{h}}} s = e^{-t\overline{B}^{\phi_b}_{g,\mathfrak{h}}} \overline{d}^{\phi_b}_{g,\mathfrak{h}} s .$$

For all $z \in \mathbb{C} \setminus \mathrm{Spec}(\overline{B}^{\phi_b}_{g,\mathfrak{h}})$ *the resolvent* $(z - \overline{B}^{\phi_b}_{g,\mathfrak{h}})^{-1}$ *preserves* $D(\overline{d}_{g,\mathfrak{h}})$ *and* $D(\overline{d}^{\phi_b}_{g,\mathfrak{h}})$ *with*

$$\forall s \in D(\overline{d}_{g,\mathfrak{h}}), \quad \overline{d}_{g,\mathfrak{h}} (z - \overline{B}^{\phi_b}_{g,\mathfrak{h}})^{-1} s = (z - \overline{B}^{\phi_b}_{g,\mathfrak{h}})^{-1} \overline{d}_{g,\mathfrak{h}} s ,$$
$$\forall s \in D(\overline{d}^{\phi_b}_{g,\mathfrak{h}}), \quad \overline{d}^{\phi_b}_{g,\mathfrak{h}} (z - \overline{B}^{\phi_b}_{g,\mathfrak{h}})^{-1} s = (z - \overline{B}^{\phi_b}_{g,\mathfrak{h}})^{-1} \overline{d}^{\phi_b}_{g,\mathfrak{h}} s .$$

1.3 Some notations and conventions

Although we already introduced some notations, let us fix some conventions and notations used throughout the article.

Coordinates: Local coordinates systems on Q or $\overline{Q}_-$ will be underlined and written $(\underline{q}^1, \ldots, \underline{q}^d)$, $d = \dim Q$. While working in a neighborhood of $Q' = \partial Q_-$, they will be chosen such that $g^{TQ} = (d\underline{q}^1)^2 + m^{TQ'}(\underline{q}^1)$.
Primed exponents (or indices) like in $\underline{q}^{i'}$ or $d\underline{q}^{I'} = d\underline{q}^{i'_1} \wedge \ldots \wedge d\underline{q}^{i'_p}$, mean that the value 1 is excluded, $i' \neq 1$ or $1 \notin I'$.
On $X = T^*Q$ local coordinates will be denoted $(q^1, \ldots, q^d, p_1, \ldots, p_d)$ with $q^i = \underline{q}^i(\pi_X(x))$ and $p_i = \frac{\partial}{\partial \underline{q}^i} . p$, with the same convention for primed exponents and indices. Different local coordinate systems on X, which will be specified later, will be used and then they will be written $(\tilde{q}^1, \ldots, \tilde{q}^d, \tilde{p}_1, \ldots, \tilde{p}_d)$.

Local frames: We use $\underline{e}_i = \frac{\partial}{\partial \underline{q}^i}$ and $\underline{e}^i = d\underline{q}^i$ and the notations $(e_i, \hat{e}^j)$ and $(e^i, \hat{e}_j)$ refer to the associated frames of $TX = TX^H \oplus TX^V \simeq \pi_X^*(TQ \oplus T^*Q)$ and $T^*X =$

$T^*X^H \oplus TX^V \simeq \pi_X^*(T^*Q \oplus TQ)$. Similar frames constructed near X' for the metric $g_0^{TQ} = (d\underline{q}^1)^2 + m^{TQ'}(0)$ will be denoted by $(f_i, \hat{f}^j)$ and $(f^i, \hat{f}_j)$. Those frames will be abreviated by $(e,\hat{e})$ or $(f,\hat{f})$. Finally while using a symmetry argument on the double copy $Q = Q_- \sqcup Q' \sqcup Q_+$ we will use the frames $(e_\mp, \hat{e}_\mp)$ on $X_\mp = T^*Q_\mp$. They will be glued in a suitable way along X' and we will use the writing $(e,\hat{e}) = 1_{X_\mp}(x)(e_\mp, \hat{e}_\mp)$ for their global definition on $X = X_+ \sqcup X' \sqcup X_+$.

Fiber bundles, metrics and connections: A general vector bundle being written $\pi_{\mathfrak{F}} : \mathfrak{F} \to M$ with its natural projection $\pi_{\mathfrak{F}}$. The restricted fiber bundle to $M' \subset M$ will be written $\mathfrak{F}\big|_{M'}$.
A metric on $\mathfrak{F}$ will be written $g^{\mathfrak{F}}$. A connection will be written $\nabla^{\mathfrak{F}}$ and $\nabla^{\mathfrak{F}}_U$ for $U \in TM$ will denote the covariant derivative w.r.t U. One exception is the Levi-Civita connection acting on tensors above the riemannian manifold (M, g^{TM}) which will be denoted by ∇^M or $\nabla^{M,g^{TM}}$, the latter being used when it is necessary to specify the metric dependence. Other exponents or indices may be used for specifying the connection and we already made the difference between the flat connection $\nabla^{\mathfrak{f}}$ on $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ and the unitary connection for the metric $g^{\mathfrak{f}}$, $\nabla^{\mathfrak{f},u}$.
When the vector bundle $\pi_{\mathfrak{F}} : \mathfrak{F} \to M$ is endowed with the connection $\nabla^{\mathfrak{F}}$, the exterior covariant derivative acting on $\mathscr{C}^\infty(M; \Lambda T^*M \otimes \mathfrak{F})$, as an exterior derivative, will be denoted $d^{\nabla^{\mathfrak{F}}}$, instead of the sometimes used notation $\nabla^{\mathfrak{F}}$. We recall that $\nabla^{\mathfrak{F}}$ is a flat connection when $d^{\nabla^{\mathfrak{F}}} \circ d^{\nabla^{\mathfrak{F}}} = 0$.
Functional spaces: We shall use the notation $\mathscr{F}(M;\mathfrak{F})$ for sections of a vector bundle $\pi_{\mathfrak{F}} : \mathfrak{F} \to M$ with the regularity specified by $\mathscr{F}$. Example given: We may take $\mathscr{F} = \mathscr{C}^\infty$, $\mathscr{F} = L^2_{loc}$, $\mathscr{F} = L^2$ once a metric $g^{\mathfrak{F}}$ is fixed, $\mathscr{F} = \mathscr{C}^\infty_0$, $\mathscr{F} = \mathscr{D}'$. In particular $\mathscr{C}^\infty(M; \Lambda T^*M)$ stands for $\Omega(M)$ usually denoting the set of smooth differential forms on M.
The local spaces $\mathscr{C}^\infty(M;\mathfrak{F})$, $\mathscr{C}^\infty_0(M;\mathfrak{F})$, $L^2_{loc}(M;\mathfrak{F})$ and $L^2_{comp}(M,\mathfrak{F})$ do not depend on the chosen metric on $\mathfrak{F}$. When $\mathfrak{F}'$ denotes the dual vector bundle with a duality product denoted by $u.v$ (possibly right-antilinear and left-linear for complex vector bundles), remember the duality between $L^2_{loc\ comp}(M;\mathfrak{F})$ and $L^2_{comp\ loc}(M;\mathfrak{F}')$ given by

$$\langle s, s' \rangle = \int_M s(x).s'(x)\, dv_M(x)$$

where dv_M is any given smooth volume measure on M, which can be specified in local charts.
Accordingly the local spaces $W^{\mu,2}_{loc\ comp}(M;\mathfrak{F})$, $\mu \in \mathbb{R}$, saying that there are μ derivatives in $L^2_{loc\ comp}$ when $\mu \in \mathbb{N}$, do not depend on the chosen metric and the dual of $W^{\mu,2}_{comp}(M;\mathfrak{F})$ is $W^{-\mu,2}_{loc}(M;\mathfrak{F})$.
Once metrics are fixed on TM and $\mathfrak{F}$ the global Sobolev space is denoted $W^{\mu,2}(M;\mathfrak{F})$ or $W^{\mu,2}(M;\mathfrak{F}, g^{TM}, g^{\mathfrak{F}})$.
When $M = X$ and $\mathfrak{F} = E$ or $\mathfrak{F} = F$, the Sobolev scale introduced by G. Lebeau in [Leb1] will be denoted by $\mathscr{W}^\mu(X;E)$ or $\mathscr{W}^\mu(X;F)$.

Manifolds with boundaries: All the manifolds with boundaries $\overline{M} = M \sqcup \partial M$,

namely $\overline{Q}_{\mp}$ or $\overline{X}_{\mp}$, will have a $\mathscr{C}^\infty$ boundary. By following the general $\mathscr{C}^\infty$-reflection principle (see [ChPi]) modeled on half-space problems, $\mathscr{C}^\infty$ functions, vector bundle structures, and sections of vector bundles are well defined on $\overline{M}$ as restriction of $\mathscr{C}^\infty$ objects on an extended neighborhood $\tilde{M}$ of $\overline{M}$. Accordingly $\mathscr{C}_0^\infty(\overline{M};\mathfrak{F})$ will denote the space of $\mathscr{C}^\infty$ sections of the vector bundle $\mathfrak{F}$, which have a compact support in $\overline{M}$. The same thing applies to the local spaces $L^2_{loc}(\overline{M};\mathfrak{F})$, $L^2_{comp}(\overline{M};\mathfrak{F})$ which must not be confused with $L^2_{loc}(M;\mathfrak{F})$ and $L^2_{comp}(M;\mathfrak{F})$.
The definition of local Sobolev spaces $W^{\mu,2}_{loc\ comp}(\overline{M};\mathfrak{F})$, $\mu\in\mathbb{R}$, follows the presentation of [ChPi], as the set of restrictions to M of elements of $W^{\mu,2}_{loc\ comp}(\tilde{M};\mathfrak{F})$. When $\mu>1/2$, any element of $W^{\mu,2}_{loc}(\overline{M};\mathfrak{F})$ admits a trace in $W^{\mu-1/2,2}_{loc}(\partial M;\mathfrak{F}\big|_{\partial M})$, while $\mathscr{C}_0^\infty(M;\mathfrak{F})$ is dense in $W^{\mu,2}_{loc\ comp}(\overline{M};\mathfrak{F})$ iff $\mu\leq 1/2$. Remember also that for $\mu\geq 0$, $W^{-\mu,2}_{loc}(\overline{M};\mathfrak{F})$ is the dual of $W^{\mu,2}_{0,comp}(\overline{M};\mathfrak{F})=\overline{\mathscr{C}_0^\infty(M;\mathfrak{F})}^{W^{\mu,2}_{comp}}$.
The definition of the global Sobolev scale introduced by Lebeau on the manifold with boundary $\overline{X}_-$, $\mathscr{W}^\mu(\overline{X}_{\mp};F)$, follows the same scheme and we refer to Subsection 3.3.2 for details.
Operators: On a $\mathscr{C}^\infty$ vector bundle $\pi_{\mathfrak{F}}:\mathfrak{F}\to M$, on a closed manifold M, and when $g^{\mathfrak{F}}$ is a metric on $\mathfrak{F}$, a differential operator P with $\mathscr{C}^\infty(M;L(\mathfrak{F}))$ will not be distinguished by notations from its maximal closed realization with domain

$$D(P)=\left\{s\in L^2(M;\mathfrak{F}),\quad Ps\in L^2(M;\mathfrak{F})\right\}.$$

This will be the case for the the differentials d, $d_{\mathfrak{h}}$, $d^{\phi_b}_{\mathfrak{h}}$ and Bismut's hypoelliptic Laplacian $B^{\phi_b}_{\mathfrak{h}}$ acting on sections of $F=E\otimes\pi_X^*(\mathfrak{f})$.
The situation is different on a manifold with boundary where closed realizations are related with a choice of boundary conditions. Then we will use the notation $\overline{P}_\alpha$ for the closed realization where α is a parameter which specifies the boundary conditions among an admissible family. In our case the parameter α will essentially be $g=g^{TQ}$.

Matched piecewise $\mathscr{C}^\infty$ structures: While using symmetry arguments on the glued double copies $Q_-\sqcup Q'\sqcup Q_+$ or $X_-\sqcup X'\sqcup X_+$, $X_{\mp}=T^*Q_{\mp}$, we are led to use piecewise $\mathscr{C}^\infty$ objects, continuous or not. In order to remember the possible discontinuities we shall use the notation $\widehat{\mathfrak{F}}_g$ for matched fiber bundles, the index $_g$ recalling when it is the case, that the matching along X' depends on the chosen metric $g=g^{TQ}$. Accordingly closed realizations of a differential operators P with piecewise $\mathscr{C}^\infty$ coefficients and interface conditions along X' which may depend on g^{TQ} will be denoted by $\widehat{P}_g$. Redundant $\widehat{}$ notations will be avoided. Example given in $\widehat{B}^{\phi_b}_{g,\mathfrak{h}}$ will be used instead of $\widehat{B}^{\hat{\phi}_b}_{\hat{g},\hat{\mathfrak{h}}}$ despite $\widehat{B}^{\phi_b}_{g,\mathfrak{h}}$ is actually associated with the piecewise $\mathscr{C}^\infty$-versions of ϕ_b, g^{TQ} and $\mathfrak{h}$.

1.4 Issues and strategy

As a first remark, Bismut's hypoelliptic Laplacian, the differential $d_{\mathfrak{h}}$ and Bismut's codifferential $d^{\phi_b}_{\mathfrak{h}}$ are all first order differential operators in the position variable q. Boundary conditions must only involve first, and possibly partial first, traces along the boundary $X'=\partial X_-$. Although the general geometry of $Q'=\partial Q_-$ and $X'=\partial X_-$ depends on curvature terms and in particular the second fundamental form of $Q'\subset(Q,g^{TQ})$, those curvatures should have a limited effect on the analysis of those operators. The analysis carried out in [Nie] worked directly on Geometric Kramers-Fokker-Planck operators as defined by G. Lebeau in [Leb1], which is a larger class of operators including Bismut's hypoelliptic Laplacian and where lower order curvature dependent terms can be neglected. Following the dyadic partition unity in the vertical variable already used by G. Lebeau in [Leb1][Leb2], it was possible to consider terms like $A_j^{ik}(q)p_kp_i\frac{\partial}{\partial p_j}$ as parameter dependent perturbations of $\frac{-\Delta_p+|p|^2}{2}$ and to absorb the large p contribution of the second fundamental form of Q' in $(\overline{Q}_-,g^{TQ})$. This led to subelliptic estimates where the curvature of the boundary nevertheless deteriorates the exponents (compare the maximal hypoellipticity result of G. Lebeau recalled at the end of Subsection 1.1 with Theorem 1.3). It is not known for the moment whether the subelliptic estimates of Theorem 1.3 which are the same as the ones of [Nie] are optimal.

However while considering the exact commutation of $\overline{d}_{g,\mathfrak{h}}$, $\overline{d}_{g,\mathfrak{h}}^{\phi_b}$ with $(z-\overline{B}_{g,\mathfrak{h}}^{\phi_b})^{-1}$ stated in Theorem 1.3, a careful treatment the geometry involved by all the terms of $B_{\mathfrak{h}}^{\phi_b}$, $d_{\mathfrak{h}}$ and $d_{\mathfrak{h}}^{\phi_g}$ cannot be skipped.

The heuristic leading to the boundary conditions of $\overline{d}_{g,\mathfrak{h}}$, $\overline{d}_{g,\mathfrak{h}}^{\phi_b}$ and $\overline{B}_{g,\mathfrak{h}}^{\phi_b}$ given in Theorems 1.1-1.2-1.3, relies on the doubling of the manifold $\overline{Q}_-$ into $Q_-\sqcup Q'\sqcup Q_+$ and then to associate Dirichlet (resp. Neumann) boundary condition to odd (resp. even) sections in the symmetry $(\underline{q}^1,\underline{q}')\mapsto(-\underline{q}^1,\underline{q}')$ between Q_- and Q_+. By working in $X=X_-\sqcup X'\sqcup X_+$, with a totally geodesic boundary Q', namely when $g^{TQ}=(d\underline{q}^1)^2+m^{TQ'}(0)$, this leads naturally to the boundary conditions given in Theorems 1.1-1.2-1.3. The analysis comes from a straightforward translation of G. Lebeau's maximal hypoelliptic results on a closed manifold, because the symmetrization preserves in this case all the $\mathscr{C}^\infty$ structures. When Q' has a non trivial second fundamental form, this is no more possible, e.g. the symmetrized metric $(d\underline{q}^1)^2+m^{TQ'}(|\underline{q}^1|)$ is no more $\mathscr{C}^1$ and only piecewise $\mathscr{C}^\infty$-structures are preserved on X.

Another unusual thing comes from the fact that the boundary conditions for $\overline{d}_{g,\mathfrak{h}}$ in Theorem 1.1 actually depend on the chosen metric g^{TQ} on $\overline{Q}_-$. This is not the case in the elliptic framework of Hodge or Witten Laplacian and this is again a side effect on the cotangent space $X=T^*Q$ of the non trivial second fundamental form of $Q'\subset(Q,g^{TQ})$ which requires a g^{TQ}-dependent matching along X' in order to speak of continuity and traces along X'.

However the followed strategy is reminiscent of what we learned from the carefull analysis of Witten and Hodge Laplacians: Avoid as long as possible the complicated curvature terms, while focussing firstly on the differential $d_{\mathfrak{h}}$ and secondly translate the result on codifferentials $d_{\mathfrak{h}}^{\phi_b}$ by duality. This does not ends the game because it is not possible to write $\overline{B}_{g,\mathfrak{h}}^{\phi_b}$ as a square of $\frac{1}{2}(\overline{d}_{g,\mathfrak{h}}^{\phi_b}+\overline{d}_{g,\mathfrak{h}})$. This has to be combined with the results of [Nie], with specific trace theorems for $B_{\mathfrak{h}}^{\phi_b}$, and with an explicit commutation result for a dense set of smooth sections, where the latter cannot be the same for the commutations with $\overline{d}_{g,\mathfrak{h}}$ or with $\overline{d}_{g,\mathfrak{h}}^{\phi_b}$. A consequence of the pseudospectral subelliptic estimates (with respect to the imaginary spectral parameter $i\lambda$) ensures that $(1+C_b+\overline{B}_{g,\mathfrak{h}}^{\phi_b})^n e^{-t\overline{B}_{g,\mathfrak{h}}^{\phi_b}}$ is bounded as soon as $t>0$ for any $n\in\mathbb{N}$. A bootstrap regularity argument where Lebeau's maximal subelliptic estimates play again a crucial role, shows that taking $n\in\mathbb{N}$ large enough implies $(1+C_b+\overline{B}_{g,\mathfrak{h}}^{\phi_b})^{-n}: L^2(X_-;F)\to\mathscr{W}^1(\overline{X}_-;F)\cap D(\overline{B}_{g,\mathfrak{h}}^{\phi_b})\subset D(\overline{d}_{g,\mathfrak{h}})\cap D(\overline{d}_{g,\mathfrak{h}}^{\phi_b})$. Actually all this analysis, and especially the use of Lebeau's maximal subelliptic estimates for closed manifold, is carried out on the symmetrized phase space $X=X_-\sqcup X'\sqcup X_+$ but for the piecewise $\mathscr{C}^\infty$ and continuous vector bundles $\hat{E}_g$ or $\hat{F}_g$.

Below is the outline of the article:

- Section 2 specifies the geometry of the cotangent bundle $X=T^*Q$ when (Q,g^{TQ}) is a riemannian manifold. Several aspects of the parallel transport for the Levi-Civita connections $\nabla^{Q,g}$ and the pulled-back connection $\pi_X^*(\nabla^{Q,g})$ will be specified. This leads to a natural definition of the piecewise $\mathscr{C}^\infty$ and continuous vector bundles $\hat{E}_g,\hat{E}'_g,\hat{F}_g,\hat{F}'_g$.

- In Section 3, details are given for various functional spaces. In particular the independence of Lebeau's spaces with respect to the chosen metric g^{TQ} is recalled. Functional spaces on the piecewise $\mathscr{C}^\infty$ vector bundles $\hat{E}_g,\hat{E}'_g,\hat{F}_g,\hat{F}'_g$ are specified with the help of parallel transport introduced in Section 2. Isomorphisms and invariances of those functional spaces induced by vector bundle isomorphisms are reviewed.

- Section 4 is devoted to the definition of $\overline{d}_{g,\mathfrak{h}}$ and its symmetrized version $\hat{d}_{g,\mathfrak{h}}$ after proving the suitable trace theorems. A specific paragraph is devoted to checking $\hat{d}_{g,\mathfrak{h}}\circ\hat{d}_{g,\mathfrak{h}}=0$ coming from a $\mathscr{C}^\infty$-interpretation of $\hat{E}_g,\hat{F}_g$.

- After defining the F' adjoint of $\overline{d}_{g,\mathfrak{h}}$, and the $\hat{F}'_g$ adjoint of $\hat{d}_{g,\mathfrak{h}}$, Section 5 specifies the symplectic codifferential $\overline{d}_{g,\mathfrak{h}}^{\sigma}$ for $\phi=\sigma$ and finally Bismut's codifferential $\overline{d}_{g,\mathfrak{h}}^{\phi_b}$ for $\phi=\phi_b$. This follows the scheme of J.M. Bismut in [Bis05]. However for the boundary or interface value problem, the choice of coordinates or $\mathscr{C}^\infty$-structures differ for those three steps and can be put together only at the level of piecewise $\mathscr{C}^\infty$ and continuous vector bundles.

- Section 6 after recalling details about Bismut's hypoelliptic Laplacians and general Geometric Kramers-Fokker-Planck operators, provides a trace theorem for local versions of $B_{g,\mathfrak{h}}^{\phi_b}$. After the definition of $\overline{B}_{g,\mathfrak{h}}^{\phi_b}$, Theorem 1.3 is proved with additional properties concerning bootraped regularity, for powers of the resolvent and the semigroup, and the (formal) PT-symmetry. Note that the commutation $(z-\overline{B}_{\mathfrak{h}}^{\phi_b})^{-1}\overline{d}_{g,\mathfrak{h}}=\overline{d}_{g,\mathfrak{h}}(z-\overline{B}_{\mathfrak{h}}^{\phi_b})^{-1}$ is rather proved in the spirit of [ABG] by making use of the semigroup with $e^{-t\overline{B}_{\mathfrak{h}}^{\phi_b}}\overline{d}_{g,\mathfrak{h}}=\overline{d}_{g,\mathfrak{h}}e^{-t\overline{B}_{\mathfrak{h}}^{\phi_b}}$ for $t\geq 0$.

2 Geometry of the cotangent bundle

This section gathers all the geometric information concerned with : a) the decomposition $TX = TX^H \oplus TX^V$ associated with $g = g^{TQ}$; b) more generally parallel transport for $\nabla^{Q,g}$ and $\pi_X^*(\nabla^{Q,g})$, ∇^E, ∇^F, $\nabla^{E'}$, $\nabla^{F'}$; c) the doubled manifolds $Q = Q_- \sqcup Q' \sqcup Q_+$, and the doubled cotangent $X = X_- \sqcup X' \sqcup X_+$. The piecewise $\mathscr{C}^\infty$ vector bundles $\hat{E}_g, \hat{E}'_g, \hat{F}_g, \hat{F}'_g$ are introduced and some specific vector bundle isomorphisms are studied. All those presentations are done in a coordinate free way and they ensure the independence w.r.t a choice of coordinates. The reader willing to grasp a concrete realization, can first look at the final paragraph where those constructions are expressed in terms of local coordinates.

2.1 The cotangent bundle of a manifold without boundary

Let Q be a smooth manifold (without boundary at the moment). Denote by X the total space of the cotangent bundle T^*Q endowed with the natural projection $\pi_X : X = T^*Q \to Q$.
The vertical subbundle of the tangent vector bundle on X, $\pi_{TX} : TX \to X$, is nothing but

$$TX^V = \pi_X^*(T^*Q). \tag{1}$$

It is a subbundle of TX with the exact sequence of smooth vector bundles on X

$$0 \to TX^V \to TX \to \pi_X^*(TQ) \to 0. \tag{2}$$

By duality $T_x^*X^H = \left\{\alpha \in T_x^*X, \forall t \in T_xX^V, \quad \alpha.t = 0\right\}$ identifies T^*X^H as the subbundle

$$T^*X^H = \pi_X^*(T^*Q), \tag{3}$$

with the exact sequence of smooth vector bundles on X

$$0 \to T^*X^H \to T^*X \to \pi_X^*(TQ) \to 0. \tag{4}$$

Those constructions do not involve any metric.
Now when $g = g^{TQ}$ is a riemannian metric on Q, the Levi-Civita connection $\nabla^{TQ,g}$ induces a connection on tensor algebras written simply $\nabla^{Q,g}$, in particular on $X = T^*Q$. This defines a horizontal vector subbundle of TX

$$TX^H \simeq \pi_X^*(TQ), \tag{5}$$

with the g-dependent direct sum decomposition

$$TX \overset{g}{=} TX^H \oplus TX^V \overset{g}{=} \pi_X^*(TQ \oplus T^*Q). \tag{6}$$

The duality defines

$$T^*X^V \simeq \pi_X^*(TQ) \tag{7}$$
$$T^*X \overset{g}{=} T^*X^H \oplus T^*X^V \overset{g}{=} \pi_X^*(T^*Q \oplus TQ). \tag{8}$$

Let $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ be a smooth vector bundle on Q endowed with a flat connection $\nabla^{\mathfrak{f}}$ and a smooth hermitian metric $g^{\mathfrak{f}}$. It is identified via the metric with the antidual flat connection $\nabla^{\mathfrak{f}'}$. If $\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}}) = (g^{\mathfrak{f}})^{-1}\nabla^{\mathfrak{f}} g^{\mathfrak{f}}$ then $\nabla^{\mathfrak{f}'} = \nabla^{\mathfrak{f}} + \omega(\mathfrak{f}, \nabla^{\mathfrak{f}})$. The vector bundles $\mathfrak{F} = E, F, E', F'$, $\pi_{\mathfrak{F}} : \mathfrak{F} \to Q$, are defined by

$$\begin{aligned} & E = \Lambda T^*X \quad , \quad E' = \Lambda TX \\ \text{and} \qquad & F = E \otimes \pi_X^*(\mathfrak{f}) \quad , \quad F' = E' \otimes \pi_X^*(\mathfrak{f}), \end{aligned}$$

with the g-dependent identifications

$$\begin{aligned} E &\stackrel{g}{=} \pi_X^*(\Lambda T^*Q \otimes \Lambda TQ) \quad , \quad E' \stackrel{g}{=} \pi_X^*(\Lambda TQ \otimes \Lambda T^*Q) \\ F &\stackrel{g}{=} \pi_X^*(\Lambda T^*Q \otimes \Lambda TQ \otimes \mathfrak{f}) \quad , \quad F' \stackrel{g}{=} \pi_X^*(\Lambda TQ \otimes \Lambda T^*Q \otimes \mathfrak{f}) \end{aligned}$$

The metrics on those vector bundles involve the weight

$$\langle p \rangle_q = \sqrt{1 + g_q^{T^*Q}(p,p)} \tag{9}$$

and are defined by

$$g^E = \langle p \rangle_q^{-N_H + N_V} \pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ}), \tag{10}$$
$$g^{E'} = \langle p \rangle_q^{N_H - N_V} \pi_X^*(g^{\Lambda TQ} \otimes g^{\Lambda T^*Q}), \tag{11}$$
$$g^F = \langle p \rangle_q^{-N_H + N_V} \pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ} \otimes g^{\mathfrak{f}}), \tag{12}$$
$$g^{F'} = \langle p \rangle_q^{N_H - N_V} \pi_X^*(g^{\Lambda TQ} \otimes g^{\Lambda T^*Q} \otimes g^{\mathfrak{f}}). \tag{13}$$

The Levi-Civita connection on $\Lambda TQ \otimes \Lambda T^*Q$ associated with $g = g^{TQ}$ being denoted by $\nabla^{Q,g}$ there is a natural connection on $\mathfrak{F} = E, F, E', F'$ simply given by

$$\nabla^{E,g} = \pi_X^*(\nabla^{Q,g}) \quad , \quad \nabla^{E',g} = \pi_X^*(\nabla^{Q,g}), \tag{14}$$
$$\nabla^{F,g} = \pi_X^*(\nabla^{Q,g} + \nabla^{\mathfrak{f}}) \quad , \quad \nabla^{F',g} = \pi_X^*(\nabla^{Q,g} + \nabla^{\mathfrak{f}'}). \tag{15}$$

Remark 2.1. *Discerning what depends on the metric $g = g^{TQ}$ is of outmost importance when boundary value problems are considered in particular because g^{TQ} does not have a product structure near the boundary, in particular when the second fundamental form of the boundary does not vanish.*

2.2 Manifold with boundary

From now on, we will assume that $\overline{Q}_- = Q_- \sqcup Q'$, is a compact manifold with boundary $Q' = \partial Q_-$. Before considering the metric aspects, $\overline{Q}_-$ can be considered as a domain of the doubled manifold $Q = Q_- \sqcup Q' \sqcup Q_+$ where Q_+ (resp. $\overline{Q}_+ = Q' \sqcup Q_+$) is a copy of Q_- (resp. $\overline{Q}_-$) and the $\mathscr{C}^\infty$-structures are matched along Q'. By following the $\mathscr{C}^\infty$-reflection principle (see [ChPi]-I-7), there is a canonical $\mathscr{C}^\infty$ structure on Q which is unique modulo diffeomorphims preserving Q'. However its concrete realization may depend on the choice of a normal bundle with its differential structure, which is equivalent to the choice of a tubular neighborhood of $Q' = \partial Q_-$ in $\overline{Q}_-$ according to [Lan]-Chap IV-6. This may lead to different realizations of the doubled manifold Q which are all diffeomorphic. Actually our analysis is done with a family of metrics for which the normal bundle $N_{Q'}\overline{Q}_-$ is not changed. So both approaches, starting from an abstract definition of Q or from its construction after fixing the normal bundle, are equivalent. The metric $g_- = g^{TQ_-} \in \mathscr{C}^\infty(\overline{Q}_-; T^*Q_- \odot T^*Q_-)$ can thus be thought as the restriction of a $\mathscr{C}^\infty$ metric $g = g^{TQ}$ on Q (another metric will be put on Q in the next paragraph). All the objects, smooth vector bundles and functional spaces (see the $\mathscr{C}^\infty$-reflection principle in [ChPi]-I-7) which are related to the $\mathscr{C}^\infty$ structure of $\overline{Q}_-$ can be thought as restrictions to $\overline{Q}_-$ (or to Q_-) of objects on Q. Those objects will be specified later when necessary.

Hence we can consider the case of a closed hypersurface $Q' \subset Q$, of the compact riemannian manifold (Q, g^{TQ}), which admits a global unit normal vector $\underline{e}_1$:

$$TQ\big|_{Q'} = TQ' \overset{\perp}{\oplus} \mathbb{R}\underline{e}_1, \quad \underline{e}_1 \in \mathscr{C}^\infty(Q'; N_{Q'}Q), \quad g^{TQ}(\underline{e}_1, \underline{e}_1) = 1,$$

where $N_{Q'}Q$ is the normal vector bundle of $Q' \subset (Q, g^{TQ})$. For the manifold $\overline{Q}_-$ with boundary Q', $\underline{e}_1$ is the outward unit normal vector.

For $\underline{q'} \in Q'$, let $(\exp^{Q,g}_{\underline{q'}}(t\underline{e}_1))_{t\in(-\varepsilon,\varepsilon)}$ be the geodesic curve on Q starting from $\underline{q'}$ in the direction $\underline{e}_1$ which is well defined for $t \in (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ can be chosen uniform w.r.t $\underline{q'} \in Q'$ by compactness. This provides diffeomorphisms

$$\begin{array}{rcl} (-\varepsilon,\varepsilon) \times Q' & \to & \left\{q \in Q, d_g(q, Q') < \varepsilon\right\} = Q_{(-\varepsilon,\varepsilon)}, \\ (\underline{q}^1, \underline{q'}) & \mapsto & \exp^{Q,g}_{\underline{q'}}(\underline{q}^1 \underline{e}_1) \end{array} \tag{16}$$

$$\begin{array}{rcl} (-\varepsilon,0] \times Q' & \to & \left\{q \in \overline{Q}_-, d_g(q, Q') < \varepsilon\right\} = Q_{(-\varepsilon,0]}, \\ (\underline{q}^1, \underline{q'}) & \mapsto & \exp^{Q,g}_{\underline{q'}}(\underline{q}^1 \underline{e}_1) \end{array} \tag{17}$$

In the sequel, we will not distinguish the global coordinate $(\underline{q}^1, \underline{q'})$ with the natural projections

$$\underline{q}^1 : Q_I \to I, \qquad \underline{q'} : Q_I \to Q' \quad \text{for } I = (-\varepsilon, \varepsilon) \text{ or } I = (-\varepsilon, 0]. \tag{18}$$

By (16)(17), we have

$$TQ_I \simeq \underline{q}^{1*} TI \oplus \underline{q'}^* TQ'. \tag{19}$$

Gauss Lemma, over Q_I, says

$$g^{TQ} = (d\underline{q}^1)^2 + m^{TQ'}(\underline{q}^1), \tag{20}$$

where $m = m^{TQ'}$ is a $\underline{q}^1$-dependent metric on Q'.

By following Bismut-Lebeau in [BiLe91]-VIII, for $(\underline{q}^1, \underline{q'}) \in Q_{(-\varepsilon,0]}$, we can identify $T_{\underline{q'}}Q'$ with $T_{(\underline{q}^1,\underline{q'})}Q_I$ by the parallel transport with respect to the Levi-Civita connection $\nabla^{Q,g}$, along the geodesic $\exp^{Q,g}_{\underline{q'}}(t\underline{e}_1)$ from $t = 0$ to $t = \underline{q}^1$. This gives over Q_I, a smooth identification of vector bundles,

$$TQ_I = \underline{q'}^*(\mathbb{R}\underline{e}_1 \oplus TQ'). \tag{21}$$

Contrarily to (19) the latter decomposition defined as the pull-back of an abstract vector bundle with $(\underline{q}^1, \underline{q}') \mapsto \underline{q}'$ does not give rise in general to an integrable decomposition of TQ_I. The extrinsic curvature of $Q' \subset (Q_I, g)$ when $\partial_{\underline{q}^1} m(0) \neq 0$ prevents from integrability.
Since the parallel transport is an isometry, via (21), the metric $g = g^{TQ}$ becomes

$$g^{TQ} = (d\underline{q}^1)^2 + \underline{q}'^* m(0). \tag{22}$$

Note that the identification (21) depends on $\underline{q}^1$, while the right hand side of (22) is independent of $\underline{q}^1$.
More generally, if $\pi_{\mathfrak{F}} : \mathfrak{F} \to Q$ is a vector bundle on Q endowed with a connection $\nabla^{\mathfrak{F}}$, the fiber $\mathfrak{F}_{\underline{q}'}$ above $\underline{q}' \in Q'$ can be identified with $\mathfrak{F}_{(\underline{q}^1, \underline{q}')}$, $\underline{q}^1 \in (-\varepsilon, \varepsilon)$, by using the parallel transport along $(\exp^{Q,g}_{\underline{q}'}(t\underline{e}_1))_{t \in (-\varepsilon,\varepsilon)}$ associated with $\nabla^{\mathfrak{F}}$.
Hence over $Q_{(-\varepsilon,\varepsilon)}$,

$$\mathfrak{F} = \underline{q}'^* \mathfrak{F}|_{Q'}. \tag{23}$$

Under this identification, the covariant derivatives equals

$$\nabla^{\mathfrak{F}}_{(U^1 \underline{e}_1 + U')} = U^1 \frac{\partial}{\partial \underline{q}^1} + \nabla^{\mathfrak{F}|_{Q'}, \underline{q}^1}_{U'}, \tag{24}$$

where $\nabla^{\mathfrak{F}|_{Q'}, \underline{q}^1}$ is a $\underline{q}^1$-dependent connection on $\mathfrak{F}|_{Q'}$. The exterior covariant derivative $d^{\nabla^{\mathfrak{F}}}$ is then

$$d^{\nabla^{\mathfrak{F}}} = d\underline{q}^1 \wedge \frac{\partial}{\partial \underline{q}^1} + d^{\nabla^{\mathfrak{F}|_{Q'}, \underline{q}^1}}. \tag{25}$$

More precisely, for $\underline{q}^1 \in (-\varepsilon, \varepsilon)$, there is a section $A_{\underline{q}^1} \in C^\infty(Q', T^*Q' \otimes \mathrm{End}(\mathfrak{F}|_{Q'}))$ which depends smoothly on $\underline{q}^1$, such that

$$\nabla^{\mathfrak{F}}\big|_{Q', \underline{q}^1} = \nabla^{\mathfrak{F}}\big|_{Q', 0} + A_{\underline{q}^1}. \tag{26}$$

In $A_{\underline{q}^1}$ there is no component of $d\underline{q}^1$, because the identification (24) is obtained by parallel transport.
This general construction will be applied with the following vector bundles:

- $\mathfrak{F} = X = T^*Q$, $\nabla^{\mathfrak{F}} = \nabla^{Q,g}$ which is actually obtained by duality from the case $\mathfrak{F} = TQ_{(-\varepsilon,\varepsilon)}$ treated above;
- $\mathfrak{F} = \mathfrak{f}$, where $\mathfrak{f}$ is endowed with the hermitian metric $g^{\mathfrak{f}}$, the flat connection $\nabla^{\mathfrak{f}}$, or its antidual flat connection $\nabla^{\mathfrak{f}'}$, after identifying $\mathfrak{f}$ with its antidual via the metric;
- $\mathfrak{F} = \Lambda TQ \otimes \Lambda T^*Q \otimes \mathfrak{f}$, $\nabla^{\mathfrak{F}} = \nabla^{Q,g} + \nabla^{\mathfrak{f}}$ and $\mathfrak{F} = \Lambda T^*Q \otimes \Lambda TQ \otimes \mathfrak{f}$, $\nabla^{\mathfrak{F}} = \nabla^{Q,g} + \nabla^{\mathfrak{f}'}$.

2.3 The doubled riemannian manifold $(Q, \hat{g})$

Like in the previous paragraph consider $\overline{Q}_- \subset Q_- \sqcup Q' \sqcup Q_+$ and use the identification via the exponential map for a smooth metric $g = g^{TQ}$, $g^{TQ}\big|_{\overline{Q}_-} = g_-$, $Q_I \simeq I \times Q'$ for an interval $I \subset (-\varepsilon, \varepsilon)$.
With this isomorphism the map S_Q

$$S_Q: \begin{array}{ccc} Q_{(-\varepsilon,\varepsilon)} & \to & Q_{(-\varepsilon,\varepsilon)} \\ (\underline{q}^1, \underline{q}') & \mapsto & (-\underline{q}^1, \underline{q}') \end{array} \tag{27}$$

is an involutive diffeomorphism. The push-forward and pull-back maps coincide. They are given by,

$$S_{Q,*} = S_Q^* : TQ_{(-\varepsilon,\varepsilon)} \to TQ_{(-\varepsilon,\varepsilon)} \quad ,$$
$$S_{Q,*}(\alpha \underline{e}_1, t') = (-\alpha \underline{e}_1, t') \in T_{(-\underline{q}^1,\underline{q}')}Q \quad \text{for } (\alpha \underline{e}_1, t') \in T_{(\underline{q}^1,\underline{q}')}Q, \tag{28}$$
$$S_{Q,*} = S_Q^* : T^*Q_{(-\varepsilon,\varepsilon)} \to T^*Q_{(-\varepsilon,\varepsilon)} \quad ,$$
$$S_{Q,*}(\alpha \underline{e}^1, \theta') = (-\alpha \underline{e}^1, \theta') \in T^*_{(-\underline{q}^1,\underline{q}')}Q \quad \text{for } (\alpha \underline{e}^1, \theta') \in T^*_{(\underline{q}^1,\underline{q}')}Q, \tag{29}$$

and they have a natural action on tensors.
In particular we can define the metric $g_+ = S_{Q,*} g_-$ on $TQ_{(-\varepsilon,\varepsilon)}$ with

$$g_+(\underline{q}^1, \underline{q}') = (d\underline{q}^1)^2 + m(-\underline{q}^1, \underline{q}').$$

Because $g_+ = g_-$ on Q', we can define the continuous metric $\hat{g} = 1_{Q_{(-\varepsilon,0)}} g_- + 1_{Q_{[0,\varepsilon)}} g_+$ on $TQ_{(-\varepsilon,\varepsilon)}$ by

$$\hat{g}(\underline{q}^1, \underline{q}') = g_-(-|\underline{q}^1|, \underline{q}') = (d\underline{q}^1)^2 + m(-|\underline{q}^1|, \underline{q}').$$

In general when $\partial_{\underline{q}^1} m(0, \underline{q}') \neq 0$, which corresponds to a non vanishing second fundamental form of $Q' \subset (Q, g_-)$, the metric $\hat{g}$ is only piecewise $\mathscr{C}^\infty$ on $\overline{Q}_-$ and $\overline{Q}_+$ and continuous (with a discontinuous $\partial_{\underline{q}^1}$ derivative along Q'). As noticed before, $g = g_-$ and g_+ induce the same identification (16), the same involutions (27)(28)(29) and the same vector field $\underline{e}_1$ obtained via (21). Nevertheless the identifications of $\underline{q}'^*(TQ')$ in (21) depends on the chosen metric $g = g^{TQ}$.

When $g = g_- = (d\underline{q}^1)^2 + m(0, \underline{q}') = g_0$ the metric $\hat{g}_0 = g_0$ is the initial smooth metric on $Q_{(-\varepsilon,\varepsilon)}$. The identifications $Q_{(-\varepsilon,\varepsilon)} = (-\varepsilon, \varepsilon) \times Q'$ and $TQ_{(-\varepsilon,\varepsilon)} = \underline{q}'^*(\mathbb{R}\underline{e}_1 \oplus TQ')$, made for the metric $\hat{g} = (d\underline{q}^1)^2 + m(-|\underline{q}^1|; \underline{q}')$ and for the metric $g_0 = (d\underline{q}^1)^2 + m(0, \underline{q}')$ provide a piecewise $\mathscr{C}^\infty$ diffeomorphims continously coinciding with $\mathrm{Id}_{Q_{(-\varepsilon,\varepsilon)}}$ and a piecewise $\mathscr{C}^\infty$ and continuous isometry $\hat{\Psi}_Q^{g,g_0}$ from $(TQ_{(-\varepsilon,\varepsilon)}, g_0^{TQ})$ to $(TQ_{(-\varepsilon,\varepsilon)}, \hat{g}^{TQ})$ such that the following diagram commutes

$$\begin{tikzcd}
(TQ_{(-\varepsilon,\varepsilon)}, g_0^{TQ}) \arrow[r, "\hat{\Psi}_Q^{g,g_0}"] \arrow[d, leftrightarrow, "S_{Q,*}"'] & (TQ_{(-\varepsilon,\varepsilon)}, \hat{g}^{TQ}) \arrow[d, leftrightarrow, "S_{Q,*}"] \\
(TQ_{(-\varepsilon,\varepsilon)}, g_0^{TQ}) \arrow[r, "\hat{\Psi}_Q^{g,g_0}"] \arrow[d, "\pi_{TQ}"'] & (TQ_{(-\varepsilon,\varepsilon)}, \hat{g}^{TQ}) \arrow[d, "\pi_{TQ}"] \\
Q_{(-\varepsilon,\varepsilon)} \arrow[r, "\mathrm{Id}_{Q_{(-\varepsilon,\varepsilon)}}"'] & Q_{(-\varepsilon,\varepsilon)}
\end{tikzcd} \tag{30}$$

with $\widehat{\Psi}_Q^{g,g_0}\big|_{Q'} = \mathrm{Id}_{TQ\big|_{Q'}}$. A similar result holds for $T^*Q_{(-\varepsilon,\varepsilon)}$ endowed with the dual metrics $g_0^{T^*Q}$ and $\hat{g}^{T^*Q}$, with natural tensorial extensions.
The situation already encountered with a piecewise $\mathscr{C}^\infty$-identification, above $\overline{Q}_{(-\varepsilon,0]}$ and $\overline{Q}_{[0,\varepsilon)}$, of $\pi_{TQ} : TQ_{(-\varepsilon,\varepsilon)} \to Q_{(-\varepsilon,\varepsilon)}$ can be generalized for a general vector bundle $\pi_{\mathfrak{F}} : \mathfrak{F} \to Q_{(-\varepsilon,\varepsilon)}$ endowed with a connection $\nabla^{\mathfrak{F}}$.
When $\pi_{\mathfrak{F}} : \mathfrak{F} \to Q$ is a vector bundle on Q endowed with a connection $\nabla^{\mathfrak{F}}$, formula (23) remains valid

$$\mathfrak{F} = \underline{q}'^{*}\mathfrak{F}|_{Q'}. \tag{31}$$

In order to complete the picture we spectify the double of the flat vector bundle $\pi_{\mathfrak{f}} : \mathfrak{f} \to \overline{Q}_-$ endowed with the flat connection $\nabla^{\mathfrak{f}}$ and the hermitian metric $g^{\mathfrak{f}}$. Because it is flat the parallel transport along $\exp_{\underline{q}'}^{Q,\hat{g}}$ is trivial but applications, in particular the treatment of Dirichlet and Neumann boundary conditions, requires an additional modification along Q'. As mentionned in the introduction $\nu \in \mathscr{C}^\infty(Q'; L(\mathfrak{f}\big|_{Q'}))$ is an involutive isometry of $(\mathfrak{f}\big|_{Q'}, g^{\mathfrak{f}})$ such that the covariant derivative vanishes $\nabla^{L(\mathfrak{f}|_{Q'})}\nu = 0$.
Actually this is equivalent to $\mathfrak{f}\big|_{Q'} = \mathfrak{f}_+\big|_{Q'} \oplus^\perp \mathfrak{f}_-$ with $\nu\big|_{\mathfrak{f}_\pm} = \pm\mathrm{Id}_{\mathfrak{f}_\pm}$ and all the theory can be done by assuming $\nu = \pm\mathrm{Id}_{\mathfrak{f}\big|_{Q'}}$, which is also our main concern.

Definition 2.2. *The double, with respect to ν, of $\pi_{\mathfrak{f}} : \mathfrak{f} \to \overline{Q}_-$ endowed with the smooth flat connection $\nabla^{\mathfrak{f}} \in \mathscr{C}^\infty(Q; T^*Q \otimes L(\mathfrak{f}))$ and the metric $g^{\mathfrak{f}}$, is the double copy, still denoted by, $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ using $Q = Q_- \sqcup Q' \sqcup Q_+$, $\overline{Q}_+ \simeq \overline{Q}_-$ endowed with the flat connection $\nabla^{\mathfrak{f}}\big|_{\overline{Q}_\mp} = \nabla^{\mathfrak{f}}_{\overline{Q}_-}$, with the metric $\hat{g}^{\mathfrak{f}}(\underline{q}^1, \underline{q}') = g^{\mathfrak{f}}(-|\underline{q}^1|, \underline{q}')$ and the continuity condition*

$$\mathfrak{f}_{(0^+,\underline{q}')} \ni (0^+, \underline{q}', \nu v) = (0^-, \underline{q}', v) \in \mathfrak{f}_{(0^-,\underline{q}')}.$$

Since ν is flat, when $(v_1, \ldots, v_{d_f})$ is a local flat frame of $\pi_{\mathfrak{f}} : \mathfrak{f} \to \overline{Q}_-$ around $q_0 \in Q'$, $[1_{(-\varepsilon,0]}(\underline{q}^1) + 1_{(0,\varepsilon)}(\underline{q}^1)\nu]v_i$, $i = 1, \ldots, d_f$ is a local flat frame of $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$. Thus $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ has a natural $\mathscr{C}^\infty$ structure associated with $\nabla^{\mathfrak{f}}$. The involution S_Q lifts to $\mathfrak{f}$ and therefore to $\Lambda TQ \otimes \Lambda T^*Q \otimes \mathfrak{f}$. This lifting on $\mathfrak{f}$ or $\Lambda TQ \otimes \Lambda T^*Q \otimes \mathfrak{f}$ will be denoted by $S_{Q,\nu}$ in order to recall that the symetric extension of $\mathfrak{f}$ to Q depends on ν. With the symmetric metric $\hat{g}^{\mathfrak{f}}$, $S_{Q,\nu}$ is an isometry of $(\mathfrak{f}, \hat{g}^{\mathfrak{f}})$ and the diagram (30) can be completed by replacing $TQ_{(-\varepsilon,\varepsilon)}$ with

$$\mathfrak{F} = \Lambda TQ_{(-\varepsilon,\varepsilon)} \otimes \Lambda T^*Q_{(-\varepsilon,\varepsilon)} \otimes \mathfrak{f},$$

$$\begin{tikzcd}
(\mathfrak{F}, g_0^{\Lambda TQ} \otimes g_0^{\Lambda T^*Q} \otimes \hat{g}^{\mathfrak{f}}) \arrow[r, "\widehat{\Psi}_Q^{g,g_0}"] \arrow[d, leftrightarrow, "S_{Q,\nu}"'] & (\mathfrak{F}, \hat{g}^{\Lambda TQ} \otimes \hat{g}^{\Lambda T^*Q} \otimes \hat{g}^{\mathfrak{f}}) \arrow[d, leftrightarrow, "S_{Q,\nu}"] \\
(\mathfrak{F}, g_0^{\Lambda TQ} \otimes g_0^{\Lambda T^*Q} \otimes \hat{g}^{\mathfrak{f}}) \arrow[r, "\widehat{\Psi}_Q^{g,g_0}"] \arrow[d, "\pi_{\mathfrak{F}}"'] & (\mathfrak{F}, \hat{g}^{\Lambda TQ} \otimes \hat{g}^{\Lambda T^*Q} \otimes \hat{g}^{\mathfrak{f}}) \arrow[d, "\pi_{\mathfrak{F}}"] \\
Q_{(-\varepsilon,\varepsilon)} \arrow[r, "\mathrm{Id}_{Q_{(-\varepsilon,\varepsilon)}}"'] & Q_{(-\varepsilon,\varepsilon)}
\end{tikzcd} \tag{32}$$

Because the flat connection $\nabla^{\mathfrak{f}}$ differs from the unitary connection $\nabla^{\mathfrak{f},u}$ it is important to keep the $\underline{q}^1$-dependent metric $g^{\mathfrak{f}}$ in $g_0^{\Lambda TQ} \otimes g_0^{\Lambda T^*Q} \otimes g^{\mathfrak{f}}$ while $g_0^{TQ} = (d\underline{q}^1)^2 \oplus m(0,\underline{q}')$.
Since the metric $\hat{g}^{\mathfrak{f}}$ is only piecewise $\mathscr{C}^\infty$ and continuous, attention must be paid to the identification with $\mathfrak{f}$ of the antidual $\mathfrak{f}'$ via the metric. The double $\pi_{\mathfrak{f}'}: \mathfrak{f}' \to Q$ can be thought as $\mathscr{C}^\infty$-vector bundle on Q with the flat connection $\nabla^{\mathfrak{f}'}$ antidual to $\nabla^{\mathfrak{f}}$ but the identification with $\mathfrak{f}$ gives a new $\mathscr{C}^\infty$-structure on $\mathfrak{f}$. This construction yields the following statement.

Proposition 2.3. *The identification of $\pi_{\mathfrak{f}'} : \mathfrak{f}' \to Q$ the antidual to the $\mathscr{C}^\infty$ flat hermitian vector bundle $(\mathfrak{f}, \nabla^{\mathfrak{f}}, \hat{g}^{\mathfrak{f}})$ can be identified via the metric $\hat{g}^{\mathfrak{f}}$ with $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ in the class of piecewise $\mathscr{C}^\infty$ and continous vector bundle. The antidual flat connection $\nabla^{\mathfrak{f}'}$ on $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ differs from $\nabla^{\mathfrak{f}}$ in general and gives rise to a different $\mathscr{C}^\infty$ structure on $\mathfrak{f}$ (remember $\nabla^{\mathfrak{f}'} = \nabla^{\mathfrak{f}} + \omega(\mathfrak{f}, \hat{g}^{\mathfrak{f}}) = \nabla^{\mathfrak{f}} + (\hat{g}^{\mathfrak{f}})^{-1}\nabla^{\mathfrak{f}}\hat{g}^{\mathfrak{f}}$).*

Remark 2.4. *Since Q is smooth or when Q is endowed with the smooth metric g such that $g\big|_{Q_-} = g_-$, all the differential geometric constructions make sense on Q. However, all the constructions which involve the Riemannian structure with the symmetric metric $\hat{g}$, will be only piecewise smooth, and even sometimes not continuous, extending the subtlety already appearing in Proposition 2.3 with the metric $\hat{g}^{\mathfrak{f}}$. In particular on the total space $X = T^*Q$: the tangent and cotangent space TX and T^*X are smooth vector bundles but the horizontal subbundle TX^H and vertical subbundle TX^V, TX^V, which rely on the chosen metric g^{TQ} will lead to piecewise $\mathscr{C}^\infty$ and a priori discontinuous structures for the non smooth metric $\hat{g}^{TQ}$. The continuity issue is discussed in the next Subsection.*

2.4 The doubled cotangent and its vector bundles

The manifold X is the total space of the cotangent T^*Q, $Q = Q_- \sqcup Q' \sqcup Q_+$ and $\overline{X}_- = X_- \sqcup X'$ is the boundary manifold $\overline{X}_- = T^*Q\big|_{\overline{Q}_-}$ with boundary $X' = T^*Q\big|_{Q'}$. So X (resp. $\overline{X}_-$) can be considered as $\mathscr{C}^\infty$-vector bundles on Q (resp. $\overline{Q}_-$) with projections $\pi_X : X \to Q$ (resp. $\pi_{\overline{X}_-} : \overline{X}_- \to \overline{Q}_-$) and as a symplectic manifold (resp. the domain of a symplectic manifold). We follow the two steps approach of Subsections 2.2 and 2.3 by first considering the smooth case with a smooth metric $g_- = g = g^{TQ}$ and then the symmetric non smooth metric $\hat{g}^{TQ}$on $Q_{(-\varepsilon,\varepsilon)}$, with $\hat{g}\big|_{Q_{(-\varepsilon,0]}} = g\big|_{Q_{(-\varepsilon,0]}} = g_-\big|_{Q_{(-\varepsilon,0]}}$ and $\hat{g}\big|_{Q_{[0,\varepsilon)}} = g_+\big|_{Q_{[0,\varepsilon]}}$.

Definition 2.5. *For an interval $I \subset (-\varepsilon, \varepsilon)$, X_I will denote $X\big|_{Q_I} = T^*Q\big|_{Q_I}$.*
The map $S_{Q,*} : T^*Q_{(-\varepsilon,\varepsilon)} \to T^*Q_{(-\varepsilon,\varepsilon)}$ will be denoted $\Sigma : X_{(-\varepsilon,\varepsilon)} \to X_{(-\varepsilon,\varepsilon)}$ as a symplectic smooth involution of $X_{(-\varepsilon,\varepsilon)}$ with push-forward and pull-back $\Sigma_* = \Sigma^*$ acting on $\Lambda TX_{(-\varepsilon,\varepsilon)}$ and $\Lambda T^*X_{(-\varepsilon,\varepsilon)}$.
On $F = \Lambda T^*X \otimes \pi_X^*(\mathfrak{f})$ or $F' = \Lambda TX \otimes \pi_X^*(\mathfrak{f})$, where $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ given of Definition 2.2 depends on the isometric smooth involution ν, the involution $\Sigma_* \otimes \pi_X^*(S_{Q,\nu})$ will be denoted by Σ_ν.

By using the decomposition (19) we can write

$$X_{(-\varepsilon,\varepsilon)} = \underline{q}^{1,*}(\mathbb{R}\underline{e}^1) + \underline{q}^*(T^*Q')$$
$$g^{T^*Q} = \underline{e}_1 \otimes \underline{e}_1 \oplus^{\perp} m^{T^*Q}(\underline{q}^1)$$

so that $X_I = T^*I \oplus T^*Q'$. Hence we can write $x \in X_{(-\varepsilon,\varepsilon)}$ as $x = (q^1, q', p_1, p')$ with $(q^1, p_1) \in I \times \mathbb{R} = T^*I$ and $(q', p') \in T^*Q'$, $(q^1, q') = \pi_X(x)$ and

$$\Sigma(q^1, q', p_1, p') = (-q^1, q', -p_1, p')$$
$$\text{and} \qquad 2\mathfrak{h}(x) = g_q^{T^*Q}(p,p) = p_1^2 + m^{T^*Q'}(q^1, q')(p', p').$$

The domain $X_{(-\varepsilon,0]}$ is a natural collar neighborhood of X' in $\overline{X}_-$.
Additionally, this shows that the kinetic energy $\mathfrak{h}$ is not invariant by Σ, in general. The latter point is solved by introducing the metric $\hat{g}^{TQ}$ of Subsection 2.3 and the kinetic energy

$$2\hat{\mathfrak{h}}(x) = p_1^2 + m^{T^*Q}(-|q^1|, q')(p', p').$$

But this leads to a discontinuous Levi-Civita connection $\nabla^{Q,\hat{g}}$ and therefore to discontinuous horizontal-vertical decomposition. This discontinuity must be handled in the vector bundles $E = \Lambda T^*X$, $E' = \Lambda TX$, $F = E \otimes \pi_X^*(\mathfrak{f})$ and $F' = E' \otimes \pi_X^*(\mathfrak{f})$, where we recall that $(\mathfrak{f}, \nabla^{\mathfrak{f}}, \hat{g}^{\mathfrak{f}})$ used for F and $(\mathfrak{f}, \nabla^{\mathfrak{f}'}, \hat{g}^{\mathfrak{f}'})$ used for F' are two $\mathscr{C}^\infty$-flat vector bundles on Q, with antidual flat connections identified via $\hat{g}^{\mathfrak{f}}$ and possibly different $\mathscr{C}^\infty$-structures (see Definition 2.2 and Proposition 2.3).
However such a discontinuity as well as the isometry with the case when $g_- = g_0$, $\hat{g}_0 = g_0$, and all the constructions are smooth, can be solved by a repeated application of $\mathfrak{F} = \underline{q}'^* \mathfrak{F}\big|_{Q'}$ written in (31).
A) $\boxed{\mathfrak{F} = X_{(-\varepsilon,\varepsilon)} = T^*Q_{(-\varepsilon,\varepsilon)}\ :}$ With $\mathfrak{F} = X_{(-\varepsilon,\varepsilon)} = T^*Q_{(-\varepsilon,\varepsilon)}$ (31) provides the vector bundle isomorphism $\widehat{\Psi}_Q^{g,g_0}$ (see diagrams (30) and (32)) which is piecewise $\mathscr{C}^\infty$ and continuous.

Definition 2.6. *On $X_{(-\varepsilon,\varepsilon)} = T^*Q_{(-\varepsilon,\varepsilon)}$ the piecewise $\mathscr{C}^\infty$ and continuous vector bundle isomorphism $(\widehat{\Psi}_Q^{g,g_0})$ will be denoted $\hat{\varphi}_X^{g,g_0}$ and the coordinates $(\tilde{q}, \tilde{p}) = (\tilde{q}^1, \tilde{q}', \tilde{p}_1, \tilde{p}')$ of $x \in TX_{(-\varepsilon,\varepsilon)}$ will be given by*

$$\tilde{q} = q \quad , \quad \tilde{p}_1 = p_1[(\hat{\varphi}_X^{g,g_0})^{-1}(x)] \quad , \quad \tilde{p}' = p'[(\varphi_X^{g,g_0})^{-1}(x)].$$

With those new coordinates

$$2\mathfrak{h}(x) = \tilde{p}_1^2 + m^{T^*Q'}(0, \tilde{q}')(\tilde{p}', \tilde{p}'),$$

the parallel transport in $X_{(-\varepsilon,\varepsilon)} = T^*Q_{(-\varepsilon,\varepsilon)}$ along the geodesic $(\exp_{\underline{q}'}^{Q,\hat{g}}(te_1))_{t\in(-\varepsilon,\varepsilon)}$ in $Q_{(-\varepsilon,\varepsilon)}$ is nothing but $(t, \tilde{q}', \tilde{p}_1, \tilde{p}')_{t\in(-\varepsilon,\varepsilon)}$ and $e_1 = \frac{\partial}{\partial \tilde{q}^1} \in TX^H\big|_{X_- \cup X_+}$. Finally the diagrams (30) and (32) and Definition 2.5 ensure

$$\Sigma(\tilde{q}^1, \tilde{q}', \tilde{p}_1, \tilde{p}') = (-\tilde{q}^1, \tilde{q}', -\tilde{p}_1, \tilde{p}'). \tag{33}$$

However this change of variables does not preserve the symplectic form σ on $X_{(-\varepsilon,\varepsilon)}$. We won't use the coordinates $(\tilde{q}, \tilde{p})$ when the symplectic structure of X is

required.
B) $\boxed{\mathfrak{F} = \Lambda T^*Q_{(-\varepsilon,\varepsilon)} \otimes \Lambda TQ_{(-\varepsilon,\varepsilon)} \otimes \mathfrak{f}\ :}$ We focus on $F = \Lambda T^*X \otimes \pi_X^*(\mathfrak{f})$ but the constructions have obvious translations in $F' = \Lambda TX \otimes \pi_X^*(\mathfrak{f})$.
Although the horizontal-vertical decomposition $T^*X = T^*X^H \oplus T^*X^V$ made on X_- with $g_- = \hat{g}\big|_{X_-}$ and on X_+ with $g_+ = \hat{g}\big|_{X_+}$ first appears discontinous, with

$$F\big|_{\overline{X}_-} \overset{g_-}{\simeq} \pi_X^*(\mathfrak{F}\big|_{\overline{Q}_-}) \quad , \quad F\big|_{\overline{X}_+} \overset{g_+}{\simeq} \pi_X^*(\mathfrak{F}\big|_{\overline{Q}_+}),$$

we may define a continuous vector bundle after taking a quotient via the map $\pi_{X,*}$.

Definition 2.7. *The vector bundle $\hat{F}_g$ is defined is defined as the quotient vector bundle*

$$\begin{aligned}
&\hat{F}_g = \left(F\big|_{\overline{X}_-} \sqcup F\big|_{\overline{X}_+}\right)/\sim \\
&F\big|_{\overline{X}_-} \overset{g_-}{\simeq} \pi_X^*(\mathfrak{F}\big|_{\overline{Q}_-}) \quad , \quad F\big|_{\overline{X}_+} \overset{g_+}{\simeq} \pi_X^*(\mathfrak{F}\big|_{\overline{Q}_+}), \quad \mathscr{F} = \Lambda T^*Q \otimes \Lambda TQ \otimes \mathfrak{f}, \\
&\left(\begin{array}{c} (x_-, v_-) \sim (x_+, v_+) \\ (x_\mp, v_\mp) \in F\big|_{\overline{X}_\mp} \end{array}\right) \Leftrightarrow \left(\begin{array}{c} x_- = x_+ \in X' = \partial X_\mp \\ \pi_{X,*}(v_-) = \pi_{X,*}(v_+) \in \mathscr{F}_{\pi_X(x)}. \end{array}\right)
\end{aligned}$$

The same definition is used for the continuous vector bundles $\hat{E}_g, \hat{E}'_g, \hat{F}'_g$ by using respectively $\mathfrak{F} = \Lambda T^*Q \otimes \Lambda TQ$, $\Lambda TQ \otimes \Lambda T^*Q$ and $\Lambda TQ \otimes \Lambda T^*Q \otimes \mathfrak{f}$.

By construction $\hat{E}_g$, $\hat{E}'_g$, $\hat{F}_g$ and $\hat{F}'_g$ are piecewise $\mathscr{C}^\infty$ and continuous vector bundles. Additionally because $g_-\big|_{Q'} = g_+\big|_{Q'}$ the horizontal vertical decomposition coincide in $\hat{F}_g\big|_{X'}$ (resp. $\hat{E}_g\big|_{X'}$, $\hat{E}'_g\big|_{X'}$, $\hat{F}'_g\big|_{X'}$) and the metric $g_-^F\big|_{X'}$ (resp. $g_-^E\big|_{X'}$, $g_-^{E'}\big|_{X'}$, $g_-^{F'}\big|_{X'}$) and $g_+^F\big|_{X'}$ (resp. $g_+^E\big|_{X'}$, $g_+^{E'}\big|_{X'}$, $g_+^{F'}\big|_{X'}$) coincide. Therefore we can write

$$\hat{E}_g \overset{\hat{g}}{\simeq} \pi_X^*(\Lambda T^*Q \otimes \Lambda TQ), \tag{34}$$

$$\hat{F}_g \overset{\hat{g}}{\simeq} \pi_X^*(\Lambda T^*Q \otimes \Lambda TQ \otimes \mathfrak{f}), \tag{35}$$

$$\hat{E}'_g \overset{\hat{g}}{\simeq} \pi_X^*(\Lambda TQ \otimes \Lambda T^*Q), \tag{36}$$

$$\hat{F}'_g \overset{\hat{g}}{\simeq} \pi_X^*(\Lambda TQ \otimes \Lambda T^*Q \otimes \mathfrak{f}), \tag{37}$$

where the identification holds in the class of piecewise $\mathscr{C}^\infty$ and continuous vector bundles. Remember that the doubled fiber bundle $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ is the one of Definition 2.2 with the flat connection $\nabla^{\mathfrak{f}}$ for $\hat{F}_g$ while its antidual version of Proposition 2.3 is used for $\hat{F}'_g$.
The complexification of $\mathfrak{F} = E, E'$ is of course a particular case of $\mathfrak{F} = F, F'$ with $\mathfrak{f} = Q \times \mathbb{C}$ endowed with the trivial metric and flat connection and $\nu = 1$. But it is convenient to have a specific notation.
The metric $\hat{g}^F = 1_{\underline{X}_-}(x) g_-^F + 1_{X_+}(x) g_+^F$ (resp. $\hat{g}^{E,E',F'} = 1_{\underline{X}_-}(x) g_-^{E,E',F'} + 1_{X_+}(x) g_+^{E,E',F'}$) is a piecewise $\mathscr{C}^\infty$ and continuous metric on $\hat{F}_g$ (resp. $\hat{E}_g$, $\hat{E}'_g$, $\hat{F}'_g$).
When $g_-^{TQ} = g_0^{TQ}$, the quotient vector bundle $\hat{F}_{g_0}$ is nothing but F.
The diagram (32) is associated with $\mathfrak{F} = \underline{q}'^*(\mathfrak{F}\big|_{Q'})$. Actually this can be lifted to X as follows:

- By A), the exponential map $\exp^{Q,\hat{g}}_{q'}(t\underline{e}_1)$ is lifted to $X_{(-\varepsilon,\varepsilon)}$ as $(t,\tilde{q}',\tilde{p}_1,\tilde{p}')$ and this defines the map $\tilde{x}' : X_{(-\varepsilon,\varepsilon)} \to X'$ with $\tilde{x}'(\tilde{q}^1,\tilde{q}',\tilde{p}_1,\tilde{p}) = (0,\tilde{q}',\tilde{p}_1,\tilde{p}')$. Hence we get $\hat{F}_g = \tilde{x}'^*(\hat{F}_g\big|_{X'})$, where the parallel transport for the connection $\nabla^{Q,\hat{g}} + \nabla^{\mathfrak{f}}$ along $(\exp^{Q,\hat{g}}_{q'}(t\underline{e}_1))_{t\in(-\varepsilon,\varepsilon)}$ is lifted to the parallel transport along $t \mapsto (t,\tilde{x}')$ for $\nabla^{F,\hat{g}}$. When $\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}$ is endowed with the metric $\pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda T^*Q} \otimes g^{\mathfrak{f}})$, pulling back (32) to $X_{(-\varepsilon,\varepsilon)}$ says $\tilde{x}'^*(\hat{F}_g\big|_{X'})$ is isometric to $F\big|_{X_{(-\varepsilon,\varepsilon)}}$ endowed with the metric $\pi_X^*(g_0^{\Lambda T^*Q} \otimes g_0^{\Lambda TQ} \otimes g^{\mathfrak{f}})$.

- The weight $\langle p\rangle_{q,\hat{g}}$ involved in the metrics $\hat{g}^E$ and g^F satisfies
$$\langle p\rangle_{\hat{g},q} = \sqrt{1 + \tilde{p}_1^2 + m^{i'j'}(0,\tilde{q}')\tilde{p}_{i'}\tilde{p}_{j'}}\,,$$
and it is constant along the curve $t \mapsto (t,\tilde{x}')$. With the identification $\hat{F}_g = \tilde{x}'^*(\hat{F}_g\big|_{X'})$ the weight $\langle p\rangle_{\hat{g},q}^{N_H-N_V}$ is thus sent to $\langle p\rangle_{g_0,q}^{N_H-N_V}$.

- It will appear with the explicit coordinate writing (59)(60) or with the $\mathscr{C}^\infty$-structure associated with $\hat{F}_g$ in Subsection 4.2 that the isometric involution on $\hat{F}_g\big|_{X\setminus X'} = F\big|_{X\setminus X'}$ is well defined on the quotient vector bundle $\hat{F}_g$.

Definition 2.8. *The identification $\hat{F}_g = \tilde{x}'^*(F\big|_{X'})$ provides a piecewise $\mathscr{C}^\infty$ and continuous vector bundle isometry from $(F\big|_{X_{(-\varepsilon,\varepsilon)}}, \hat{g}_0^F = g_0^E \otimes \hat{g}^{\mathfrak{f}})$ to $(\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}, \hat{g}^F)$ which is denoted by $\widehat{\Psi}_X^{g,g_0}$. The same notation is used for $\widehat{\Psi}_X^{g,g_0} : (F'\big|_{X_{(-\varepsilon,\varepsilon)}}, \hat{g}_0^{F'} = g_0^{E'} \otimes \hat{g}^{\mathfrak{f}}) \to (\hat{F}'_g\big|_{X_{(-\varepsilon,\varepsilon)}}, \hat{g}^{F'})$ and when F, F' are replaced by E, E'.*

The diagram (32) is now lifted to

$$\begin{tikzcd}
(F\big|_{X_{(-\varepsilon,\varepsilon)}}, g_0^E \otimes \hat{g}^{\mathfrak{f}}) \arrow[r, "\widehat{\Psi}_X^{g,g_0}"] \arrow[d, leftrightarrow, "\Sigma_{0,v}"'] & (\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}, \hat{g}^F) \arrow[d, leftrightarrow, "\Sigma_v"] \\
(F\big|_{X_{(-\varepsilon,\varepsilon)}}, g_0^E \otimes \hat{g}^{\mathfrak{f}}) \arrow[r, "\widehat{\Psi}_X^{g,g_0}"] \arrow[d, "\pi_F"'] & (\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}, \hat{g}^F) \arrow[d, "\pi_F"] \\
X_{(-\varepsilon,\varepsilon)} \arrow[r, "\hat{\varphi}_X^{g,g_0}"'] & X_{(-\varepsilon,\varepsilon)} \\
X' \arrow[u, "i"] \arrow[r, "\mathrm{Id}_{X'}"'] & X' \arrow[u, "i"']
\end{tikzcd} \tag{38}$$

with similar diagrams when F is replaced by E, E', F'.

Remark 2.9. *The $\mathscr{C}^\infty$-structure of $\hat{F}_g$ is based on the non symplectic coordinates $(\tilde{q},\tilde{p})$ with the collar neighborhood $\tilde{q}^1 \in (-\varepsilon, 0]$ and an additional twist presented in Subsection 4.2. However many different structures have to be considered in this analysis: the differential structure, the symplectic structure and the riemannian structure. The presentation of $\hat{F}_g$ as a piecewise $\mathscr{C}^\infty$ and continuous vector bundle, defined as a quotient, is actually the one where all those different aspects are simply formulated.*

2.5 The geometric constructions in local coordinates

The previous sections ensure that the various changes of variables or isomorphisms of vector bundles have a natural geometric meaning independent of a coordinate system. However it is instructive for the analysis to explain them in terms of some specific local coodinates systems. Let us first recall a few facts about the smooth case and then we will shift to the description of piecewise $\mathscr{C}^\infty$ isometries associated with the possibly non smooth metric $\hat{g}$.

2.5.1 The smooth case

Let us start with a smooth riemannian manifold (Q,g) and a local coordinate system $(\underline{q}^1,\ldots,\underline{q}^d)$ in an open neighborhood U of $q_0 \in Q$. The Levi-Civita connection $\nabla^{Q,g}$ associated with $g = g^{TQ} = g_{ij}(\underline{q})d\underline{q}^i d\underline{q}^j$ and the dual metric $g^{T^*Q}(p,p) = g^{ij}(\underline{q})p_i p_j$ can be specified with the Christoffel symbols

$$\Gamma_{ij}^k(\underline{q}) = \frac{1}{2} g^{k\ell}\left[\frac{\partial g_{j\ell}}{\partial \underline{q}^i} + \frac{\partial g_{i\ell}}{\partial \underline{q}^j} - \frac{\partial g_{ij}}{\partial \underline{q}^\ell}\right]. \tag{39}$$

It is given by

$$\nabla^{Q,g}_{\frac{\partial}{\partial \underline{q}^i}} \frac{\partial}{\partial \underline{q}^j} = \Gamma_{ij}^k(\underline{q}) \frac{\partial}{\partial \underline{q}^k} \quad , \quad \nabla^{Q,g}_{\frac{\partial}{\partial \underline{q}^i}} d\underline{q}^j = -\Gamma_{ik}^j(\underline{q}) d\underline{q}^k .$$

Because it is torsion free we have the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$.
An horizontal curve $t \mapsto (q(t),p(t))$ on $X = T^*Q$ being characterized by $\nabla^{Q,g}_{\dot{q}(t)} p(t) = 0$, a basis of TX^H, $e_i \in TX^H$ such that $\pi_{X,*}(e_i) = \frac{\partial}{\partial \underline{q}^i}$, and $\hat{e}^i \in TX^V$, $\pi_{X,*}(\hat{e}^i) = d\underline{q}^i$, is thus given by

$$e_i = \frac{\partial}{\partial q^i} + \Gamma_{ij}^k(q) p_k \frac{\partial}{\partial p_j} \in T_{(q,p)}X^H \quad , \quad \hat{e}^j = \frac{\partial}{\partial p_j} \in T_{(q,p)}X^V . \tag{40}$$

Its dual basis on T^*X is

$$e^i = dq^i \in T^*_{(q,p)}X^H \quad , \quad \hat{e}_j = dp_j - \Gamma_{ji}^k(q) p_k dq^i \in T^*_{(q,p)}X^V . \tag{41}$$

Due to the possible curvature of (Q, g^{TQ})

$$[e_i, e_j] = R_{ijk}^{TQ;\ell}(q) p_\ell \frac{\partial}{\partial p_\ell} \in TX^V \tag{42}$$

where the Riemann curvature tensor $R^{TQ} = R_{ij}^{TQ} d\underline{q}^i \wedge d\underline{q}^j$ is the $\mathrm{End}(TQ)$ valued two-form given

$$R^{TQ}(S,T) = \nabla_S^{TQ,g}\nabla_T^{TQ,g} - \nabla_T^{TQ,g}\nabla_S^{TQ,g} - \nabla_{[S,T]}^{TQ,g}$$

when $S = S^i(\underline{q})\frac{\partial}{\partial \underline{q}^i}$ and $T(\underline{q}) = T^j(\underline{q})\frac{\partial}{\partial \underline{q}^j}$.
However $(e_i, \hat{e}^j)$ is a symplectic basis of $TX = T(T^*Q)$ endowed with its canonical symplectic form $\sigma \overset{loc}{=} dp_i \wedge dq^i$ and

$$\sigma = dp_i \wedge dq^i = \hat{e}_i \wedge e^i .$$

General elements of $E=\Lambda T^*X$ and $E'=\Lambda TX$ will be written locally

$$\omega_I^J e^I \hat{e}_J \quad , \quad u_J^I e_I \hat{e}^J ,$$

with repeated summation convention w.r.t $I,J\subset\{1,\dots,d\}$.
The vertical, total and horizontal number operators, N_V, N_H and N, are given on ΛT^*X and ΛTX by

$$\begin{array}{lllll} N_V(\omega_I^J e^I\hat{e}_J)=|J|\omega_I^J e^I\hat{e}_J & , & N(\omega_I^J e^I\hat{e}_J)=(|I|+|J|)\omega_I^J e^I\hat{e}_J & , & N_H=N-N_V \\ N_H(u_J^I e_I\hat{e}^J)=|I|u_J^I e_I\hat{e}^J & , & N(u_J^I e_I\hat{e}^J)=(|I|+|J|)u_J^I e_I\hat{e}^J & , & N_V=N-N_H . \end{array}$$

Because the Levi-Civita connection preserves the metric on $X=T^*Q$ the horizontal vector fields are tangent to $|p|_q^2=Cte$:

$$e_i f(|p|_q^2)=0 \quad , \quad |p|_q^2=g^{ij}(q)p_ip_j=2\mathfrak{h}(q,p) \quad , \quad f\in\mathscr{C}^1(\mathbb{R};\mathbb{R}).$$

When necessary we will specify the metric g in the notation with an additional index by writing $|p|_q^2=|p|_{g,q}^2$ and $\langle p\rangle_{g,q}=\langle p\rangle_q$.
The metric $g^E=\langle p\rangle_q^{-N_H+N_V}\pi_X^*(g^{\Lambda T^*Q}\otimes g^{\Lambda TQ})$ and $g^{E'}=\langle p\rangle_q^{N_H-N_V}\pi_X(g^{\Lambda TQ}\otimes g^{\Lambda T^*Q})$ on $E=\Lambda T^*X$ and $E'=\Lambda TX$ already introduced in (10) and (11) are such that

$$|e^i|_{(q,p)}=|dq^i|_{(q,p)}=\langle p\rangle_q^{-1/2}\sqrt{g^{ii}(q)}, \tag{43}$$

$$|\hat{e}_j|_{(q,p)}=\left|dp_j-\Gamma_{ji}^k p_k dq^i\right|_{(q,p)}=\langle p\rangle_q^{1/2}\sqrt{g_{jj}(q)}=\mathscr{O}(\langle p\rangle_q)|e^i|_{(q,p)}, \tag{44}$$

$$\langle e^i,\hat{e}_j\rangle_{g^E}=0, \tag{45}$$

$$|\hat{e}^j|_{(q,p)}=\left|\frac{\partial}{\partial p_j}\right|_{(q,p)}=\langle p\rangle_q^{-1/2}\sqrt{g^{jj}(q)}, \tag{46}$$

$$|e_i|_{(q,p)}=\left|\frac{\partial}{\partial q^i}+\Gamma_{ij}^k p_k\frac{\partial}{\partial p_j}\right|_{(q,p)}=\langle p\rangle_q^{1/2}\sqrt{g_{ii}(q)}=\mathscr{O}(\langle p\rangle_q)|\hat{e}^j|_{(q,p)} \tag{47}$$

$$\langle e_i,\hat{e}_j\rangle_{g^{E'}}=0. \tag{48}$$

With this choice the riemannian volume on X, $\mathrm{vol}_{g^{E'}}$, is nothing but the symplectic volume

$$d\mathrm{vol}_{g^{E'}}=|dqdp|=\frac{1}{d!}|\sigma^d| \quad , \quad d=\dim Q ,$$

and coincides with the standard Lebesgue measure in any symplectic coordinates system. Note that X is orientable with the non vanishing volume form $\frac{1}{d!}\sigma^d$.
The connections ∇^E and $\nabla^{E'}$ introduced in (14) satisfy

$$\begin{array}{lllll} \nabla^E_{e_i}e^\ell=-\Gamma^\ell_{ik}(q)e^k & , & \nabla^E_{e_i}\hat{e}_j=\Gamma^k_{ij}(q)\hat{e}_k & , & \nabla^E_{\hat{e}^j}e^\ell=\nabla^E_{\hat{e}_j}\hat{e}_k=0, \\ \nabla^{E'}_{e_i}e_\ell=\Gamma^k_{i\ell}(q)e_k & , & \nabla^{E'}_{e_i}\hat{e}^j=-\Gamma^j_{ik}(q)\hat{e}^k & , & \nabla^{E'}_{\hat{e}^j}e_i=\nabla^{E'}_{\hat{e}_j}\hat{e}^\ell=0. \end{array}$$

Remember that they are defined as pull-backed connections and do not coincide exactly with the Levi-Civita connection associated with $g^{E'}$ due to the weight $\langle p\rangle_q^{N_H-N_V}$.
Let us finish with the flat vector bundle $\mathfrak{f}$ endowed with the flat connection $\nabla^{\mathfrak{f}}$

and the hermitian metric $g^{\mathfrak{f}}$. Locally above $U \ni q_0$, there is a frame $(v^1,\ldots,v^{d_f})$ of $\mathfrak{f}$ such that $\nabla^{\mathfrak{f}}_{\frac{\partial}{\partial \underline{q}^i}} v^k = 0$ and $(\mathfrak{f},\nabla^{\mathfrak{f}}) \simeq (U \times \mathbb{C}^{d_f},\nabla)$ with the trivial connection ∇ and the covariant derivative $\nabla_{\frac{\partial}{\partial \underline{q}_i}} = \frac{\partial}{\partial \underline{q}_i}$. The metric is given by the matrix $A^{ij}(\underline{q}) = g^{\mathfrak{f}}(v^i,v^j)$, $A(\underline{q}) = A(\underline{q})^*$ and $(A(\underline{q})^{-1})_{ij} = A_{ij}(\underline{q})$. The antidual $(\mathfrak{f}',\nabla^{\mathfrak{f}'})$ is also isomorphic to $(U \times \mathbb{C}^d,\nabla)$ but identifying $\mathfrak{f}'$ with $\mathfrak{f}$ via $g^{\mathfrak{f}}(v^*_\ell,v^k) = \delta^k_\ell$ says $v^*_\ell = A^{-1}(\underline{q})v^\ell = A_{\ell,k}(\underline{q})v^k$. With

$$0 = \nabla^{\mathfrak{f}'}_{\frac{\partial}{\partial \underline{q}^i}} v^*_\ell = \nabla^{\mathfrak{f}'}_{\frac{\partial}{\partial \underline{q}^i}}[A^{-1}(\underline{q})v^\ell] = (\partial_{\underline{q}^i}A^{-1})v^\ell + A^{-1}\nabla^{\mathfrak{f}'}_{\frac{\partial}{\partial \underline{q}^i}} v^\ell$$

we deduce

$$\nabla^{\mathfrak{f}'}v^\ell - \underbrace{\nabla^{\mathfrak{f}}v^\ell}_{=0} = -A(dA^{-1})(\underline{q})v^\ell = (dA)A^{-1}(\underline{q})v^\ell .$$

Hence we obtain

$$\omega(g^{\mathfrak{f}},\nabla^{\mathfrak{f}}) = \nabla^{\mathfrak{f}'} - \nabla^{\mathfrak{f}} = (dA)A^{-1}(\underline{q}).$$
$$\nabla^{\mathfrak{f},u} = \nabla^{\mathfrak{f}} + \frac{1}{2}(dA)A^{-1}(\underline{q}) \overset{\mathfrak{f}\simeq I\times\mathbb{C}^{d_f}}{=} \nabla + \frac{1}{2}(dA)A^{-1}(\underline{q}).$$

We are especially interested in the line bundle $\mathfrak{f} = Q \times \mathbb{C}$ with $\nabla^{\mathfrak{f}} = \nabla$ and $g^{\mathfrak{f}}(z) = e^{-2V(q)}|z|^2$. Then $\nabla^{\mathfrak{f}'} = \nabla - 2(dV(\underline{q}))$ and $\nabla^{\mathfrak{f},u} = \nabla - dV(\underline{q})$.
By conjugating with $e^{-V(q)}$, we can actually consider $\mathfrak{f} = Q \times \mathbb{C}$ with $g^{\mathfrak{f}}(z') = |z'|^2$ with the flat connection $\nabla^{\mathfrak{f}} = \nabla + dV(q)$ and its dual flat connection $\nabla^{\mathfrak{f}'} = \nabla - dV(q)$.

2.5.2 The non smooth doubles with the metric $\hat{g}^{TQ}$

With $Q = Q_- \sqcup Q' \sqcup Q_+$ local coordinates in a neighborhood U of $q_0 \in Q'$, ($U \subset Q_{(-\varepsilon,\varepsilon)}$) are chosen such that:

- $g^{TQ}_- = (d\underline{q}^1)^2 \oplus^\perp m^{i'j'}(\underline{q}^1,\underline{q}')d\underline{q}^{i'}d\underline{q}^{j'}$.
- $\hat{g}^{TQ} = (d\underline{q}^1)^2 \oplus^\perp m^{i'j'}(-|\underline{q}^1|,\underline{q}')d\underline{q}^{i'}d\underline{q}^{j'}$, $\hat{m} = m^{TQ'}(-\underline{q}^1,\underline{q}')$ and the corresponding Christoffel symbols are denoted by $\hat{\Gamma}^k_{ij}$. We will keep the notation Γ^k_{ij} for $g^{TQ} = g^{TQ}_-$.
- The associated symplectic coordinates on $\pi_X^{-1}(U) \simeq U \times \mathbb{R}^d$ are written $(q,p) = (q^i,p_j)_{1\leq i,j\leq d}$ and $X' \cap \pi_X^{-1}(U) = \{(q^1,q',p_1,p') \in U \times \mathbb{R}^d, q^1 = 0\}$.

Remember the convention that i' (resp. $I' \subset \{1,\ldots d\}$) denotes an index $i' \neq 1$ (resp. $1 \not\in I'$).
Three things must be noticed with those coordinates

- From (39) a Christoffel symbol $\Gamma^k_{ij}(\underline{q})$ vanishes if 1 appears more than once in i,j,k.

- For the metric $\hat{g}$, the Christoffel symbols $\hat{\Gamma}^{k'}_{i'j'}(\underline{q})$ are continuous but not $\mathscr{C}^1$ on $Q_{(-\varepsilon,\varepsilon)}$ while the possibly non continuous Christoffel symbols $\hat{\Gamma}^{k'}_{1i'} = \hat{\Gamma}^{k'}_{i'1} = \frac{\partial \hat{m}_{i'k'}}{2\partial \underline{q}^1}$ and $\hat{\Gamma}^{1}_{i'j'} = -\frac{\partial \hat{m}_{i'j'}}{2\partial \underline{q}^1}$ satisfy $\hat{\Gamma}^{k'}_{1i'}(0^+,\underline{q}') = -\hat{\Gamma}^{k'}_{1i'}(0^-,\underline{q}')$ and $\hat{\Gamma}^{1}_{i'j'}(0^+,\underline{q}') = -\hat{\Gamma}(0^-,\underline{q}')$.

- When $g^{TQ} = g_0^{TQ} = \hat{g}^{TQ} = (d\underline{q}^1)^2 + m^{TQ'}(0,\underline{q}')$ everything is smooth and $\Gamma^k_{ij} = 0$ when 1 appears in i,j,k.

Definition 2.10. *Work with the local coordinates $(q,p) = (q^1,\ldots,q^d,p_1,\ldots,p_d)$ in $\pi_X^{-1}(U)$, $q_0 \in Q' \cap U \subset Q_{(-\varepsilon,\varepsilon)}$.*
The frame (40) and (41) associated with g_-^{TQ} (resp. g_+^{TQ}) are denoted by $(e_{-,i},\hat{e}^j_-)$ and $(e^i_-,\hat{e}_{-,j})$ (resp. $(e_{-,i},\hat{e}^j_-)$ and $(e^i_-,\hat{e}_{-,j})$).
The abbreviated version is simply $(e_\mp,\hat{e}_\mp)$.
The notations $(e_i,\hat{e}^j)$ and $(e^i,\hat{e}_j)$ now refer to the metric $\hat{g}$ with $(e,\hat{e}) = 1_{Q_-}(q)(e_-,\hat{e}_-) + 1_{Q_+}(q)(e_+,\hat{e}_+)$ on $\pi_X^{-1}(U)\setminus X'$, while working in $\hat{E}_g$, $\hat{E}'_g$, $\hat{F}_g$, $\hat{F}'_g$ means that $(e_-,\hat{e}_-)$ and $(e_+,\hat{e}_+)$ are identified along X' and $(e,\hat{e})$ makes sense on $\pi_X^{-1}(U)$.
When $g^{TQ} = g_0^{TQ}$ those frames are simply denoted $(f_i,\hat{f}^j)$ and $(f^i,\hat{f}_j)$.

Below are the detailed expressions of those frames in the coordinates (q,p):

$$f_1 = \frac{\partial}{\partial q^1} \quad , \quad f_{i'} = \frac{\partial}{\partial q^{i'}} + \Gamma^{k'}_{i',j'}(0,q')p_{k'}\frac{\partial}{\partial p_{j'}} \quad , \quad \hat{f}^j = \frac{\partial}{\partial p_j}, \tag{49}$$

$$f^i = dq^i \quad , \quad \hat{f}_1 = dp_1 \quad , \quad \hat{f}_{j'} = dp_{j'} - \Gamma^{k'}_{j'i'}(0,q')p_{k'}dq^{i'}, \tag{50}$$

$$e_{\mp,1} = \frac{\partial}{\partial q^1} + \hat{\Gamma}^{k'}_{1j'}(q)p_{k'}\frac{\partial}{\partial p_{j'}} \stackrel{\text{on } X'}{=} f_1 \pm \Gamma^{k'}_{1j'}(0,q')p_{k'}\frac{\partial}{\partial p_{j'}} \quad , \tag{51}$$

$$e_{\mp,i'} = \frac{\partial}{\partial q^{i'}} + \hat{\Gamma}^{k'}_{i'j'}(q)p_{k'}\frac{\partial}{\partial p_{j'}} + \hat{\Gamma}^{k'}_{i'1}(q)p_{k'}\frac{\partial}{\partial p_1} + \hat{\Gamma}^{1}_{i'j'}(q)p_1\frac{\partial}{\partial p_{j'}} \tag{52}$$

$$\stackrel{\text{on } X'}{=} f_{i'} \pm \Gamma^{k'}_{i'1}(0,q')p_{k'}\frac{\partial}{\partial p_1} \pm \Gamma^{1}_{i'j'}(0,q')p_1\frac{\partial}{\partial p_{j'}} \quad , \tag{53}$$

$$\hat{e}^j_\mp = \frac{\partial}{\partial p_j} = \hat{f}^j \tag{54}$$

$$e^i_\mp = dq^i = f^i \tag{55}$$

$$\hat{e}_{\mp,1} = dp_1 - \hat{\Gamma}^{k'}_{1i'}(q)p_{k'}dq^{i'} \stackrel{\text{on } X'}{=} \hat{f}_1 \mp \Gamma^{k'}_{1i'}(0,q')p_{k'}dq^{i'} \quad , \tag{56}$$

$$\hat{e}_{\mp,j'} = dp_{j'} - \hat{\Gamma}^{k'}_{j'i'}(q)p_{k'}dq^{i'} - \hat{\Gamma}^{k'}_{j'1}(q)p_{k'}dq^1 - \hat{\Gamma}^{1}_{j'i'}(q)p_1dq^{i'} \tag{57}$$

$$\stackrel{\text{on } X'}{=} f_{j'} \mp \Gamma^{k'}_{j'1}(0,q')p_{k'}dq^1 \mp \Gamma^{1}_{j'i'}(0,q')p_1dq^{i'}. \tag{58}$$

We see in particular on (51)(53)(54) for $E' = \Lambda TX$ and on (55)(56)(58) for $E = \Lambda T^*X$ that

$$\Sigma_*(e_{-,i})\big|_{X'} = (-1)^{\delta_{i1}}e_{+,i}\big|_{X'} \quad , \quad \Sigma_*(\hat{e}^j_-)\big|_{X'} = (-1)^{\delta_{j1}}\hat{e}^j_+\big|_{X'} \tag{59}$$

$$\Sigma_*(e^i_-)\big|_{X'} = (-1)^{\delta_{i1}}e^i_+\big|_{X'} \quad , \quad \Sigma_*(\hat{e}_{-,j})\big|_{X'} = (-1)^{\delta_{j1}}\hat{e}_{+,j}\big|_{X'}. \tag{60}$$

The coordinates $(\tilde{q},\tilde{p})$ introduced in Definition 2.6, and the expression of the frame $(e,\hat{e})$ in those new coordinates can be specified locally. They are characterized by

$$\tilde{q} = q \quad , \quad e_1\tilde{p} = 0 \quad , \quad \tilde{p}\big|_{X'} = p,$$

or

$$\begin{pmatrix} \tilde{q} \\ \tilde{p}_1 \\ \tilde{p}' \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \psi(q^1, q') \end{pmatrix} \begin{pmatrix} q \\ p_1 \\ p' \end{pmatrix}$$

$$\text{with} \qquad \begin{cases} \frac{\partial}{\partial q^1}\psi^{k'}_{j'} = -\hat{\Gamma}^{\ell'}_{1,j'}(q^1,q')\psi^{k'}_{\ell'} \\ \psi^{k'}_{j'}(0,q') = \delta^{k'}_{j'}\,. \end{cases}$$

We deduce

$$\frac{\partial}{\partial \tilde{q}^\ell} = \frac{\partial}{\partial q^\ell} - [\frac{\partial \psi}{\partial q^\ell}\psi^{-1}]^{k'}_{\ell'} p_{k'} \frac{\partial}{\partial p_{\ell'}} \quad , \quad \frac{\partial}{\partial \tilde{p}_1} = \frac{\partial}{\partial p_1} \quad , \quad \frac{\partial}{\partial \tilde{p}_{j'}} = [\psi^{-1}]^{j'}_{\ell'} \frac{\partial}{\partial p_{\ell'}}$$

$$d\tilde{q}^\ell = dq^\ell \quad , \quad d\tilde{p}_1 = dp_1 \quad , \quad d\tilde{p}_{j'} = [\psi(q)]^{k'}_{j'} dp_{k'} + [\frac{\partial \psi}{\partial q^\ell}]^{k'}_{j'} p_{k'} dq^\ell$$

This leads to

$$\frac{\partial}{\partial \tilde{q}^1} = \frac{\partial}{\partial q^1} + \hat{\Gamma}^{k'}_{1j'} p_{k'} \frac{\partial}{\partial p_{j'}} = e_1 \quad , \quad \hat{e}^1 = \frac{\partial}{\partial \tilde{p}_1} \tag{61}$$

$$\hat{e}^{j'} = \frac{\partial}{\partial p_{j'}} = \psi^{j'}_{k'}(q) \frac{\partial}{\partial \tilde{p}_{k'}} \overset{on\_X'}{=} \frac{\partial}{\partial \tilde{p}_{j'}}, \tag{62}$$

$$\begin{aligned} e_{i'} = \frac{\partial}{\partial \tilde{q}^{i'}} + [\psi \frac{\partial \psi}{\partial q^i}]^{k'}_{\ell'} \tilde{p}_{k'} \frac{\partial}{\partial \tilde{p}_{\ell'}} + [\psi \hat{\Gamma}^{\cdot}_{i',\cdot}\psi^{-1}]^{k'}_{j'} \tilde{p}_{k'} \frac{\partial}{\partial \tilde{p}_{j'}} \\ + [\hat{\Gamma}^{\cdot}_{i',1}\psi^{-1}]^{\ell'} \tilde{p}_{\ell'} \frac{\partial}{\partial \tilde{p}_1} + [\psi \hat{\Gamma}^1_{i',\cdot}]_{j'} \tilde{p}_1 \frac{\partial}{\partial \tilde{p}_{j'}} \end{aligned} \tag{63}$$

$$e_{\mp,i'} \overset{on\_X'}{=} \frac{\partial}{\partial \tilde{q}^{i'}} + \Gamma^{k'}_{i'j'}(0,q')\tilde{p}_{k'} \frac{\partial}{\partial \tilde{p}_{j'}} \pm \Gamma^{k'}_{i',1}(0,q')\tilde{p}_k \frac{\partial}{\partial \tilde{p}_1} \pm \Gamma^1_{i'j'}(0,q')\tilde{p}_1 \frac{\partial}{\partial \tilde{p}_{j'}}, \tag{64}$$

and to

$$e^i = dq^i = d\tilde{q}^i \tag{65}$$

$$\hat{e}_1 = d\tilde{p}_1 - [\hat{\Gamma}^{\cdot}_{1j'}\psi^{-1}]^{k'} \tilde{p}_{k'} d\tilde{q}^{j'} \tag{66}$$

$$\hat{e}_1 \overset{on\_X'}{=} d\tilde{p}_1 \mp \Gamma^{k'}_{1j'}(0,q')\tilde{p}_{k'} d\tilde{q}^{i'} \tag{67}$$

$$\begin{aligned} \hat{e}_{j'} = [\psi^{-1}]^{k'}_{j'} d\tilde{p}_{k'} - [\psi^{-1}\frac{\partial \psi}{\partial q^\ell}\psi^{-1}]^{k'}_{j'} \tilde{p}_{k'} d\tilde{q}^\ell - [\hat{\Gamma}^{\cdot}_{j'.}\psi^{-1}]^{k'}_{i'} \tilde{p}_{k'} d\tilde{q}^{i'} \\ - [\hat{\Gamma}^{\cdot}_{j'1}\psi^{-1}]^{k'} \tilde{p}_{k'} d\tilde{q}^1 - \hat{\Gamma}^1_{j'i'} \tilde{p}_1 d\tilde{q}^{i'} \end{aligned} \tag{68}$$

$$\hat{e}_{\mp,j'} \overset{on\_X'}{=} d\tilde{p}_{j'} - \Gamma^k_{j'i'}(0,q')\tilde{p}_{k'} d\tilde{q}^{i'} \mp \Gamma^1_{j'i'}(0,q')\tilde{p}_1 d\tilde{q}^{i'}\,. \tag{69}$$

Remember that the map $\Sigma : (q^1, q', p_1, p') \to (-q^1, q', -p_1, p')$ of Definition 2.5 keeps the same form $\Sigma(\tilde{q}^1, \tilde{q}', \tilde{p}_1, \tilde{p}') = (-\tilde{q}^1, \tilde{q}', -\tilde{p}_1, \tilde{p}')$ in the new coordinates $(\tilde{q}, \tilde{p})$. In particular (59)(60) can be deduced from (61)(62)(64) and (65)(67)(69).
The continuous matching along X' vector bundles $\hat{E}_g$ and $\hat{E}'_g$ simply says that $(e_-, \hat{e}_-)\big|_{\partial X_-}$ and $(e_+, \hat{e}_+)\big|_{\partial X_+}$ are identified.
The lifting to $X \cap U$ of the geodesic curve $\exp^{Q,\hat{g}}_{\underline{q'}}(t\underline{e}_i)$ is nothing but the curve $t \mapsto (t, \tilde{q}', \tilde{p})$ and the map $\tilde{x}' : X \times U \to X'$ is nothing but

$$\tilde{x}'(\tilde{q}^1, \tilde{q}^{i'}, \tilde{p}_j) = (0, \tilde{q}^{i'}, \tilde{p}_j)\,.$$

Using the lifted connection $\nabla^{E,\hat{g}}$ and $\nabla^{E',\hat{g}}$, the frames $(e,\hat{e})\big|_{X'} = (e_-,\hat{e}_-)\big|_{\partial X_-} = (e_+,\hat{e}_+)\big|_{\partial X_+}$ are lifted to the new frames $\tilde{x}'^*\left[(e,\hat{e})\big|_{X'}\right]$. The piecewise $\mathscr{C}^\infty$ and continuous vector bundle isometry $\widehat{\Psi}^{g,g_0}_X$ of diagram (38) is nothing but

$$\Psi^{g,g_0}_X : (q,p,f,\hat{f}) \to (\tilde{q},\tilde{p},\tilde{x}^*[(e,\hat{e})\big|_{X'}]).$$

For the continuity properties in $\hat{E}_g$ and $\hat{E}'_g$ we can work more simply with the frame $(e,\hat{e}) = 1_{Q_\mp}(q)(e_\mp,\hat{e}_\mp)$ than with the frame $\tilde{x}'^*[(e,\hat{e})\big|_{X'}]$.
Let us conclude with the vector bundle $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ of Definition 2.2 endowed with the metric $\hat{g}^{\mathfrak{f}}$ and the two connections $\nabla^{\mathfrak{f}}$ and $\nabla^{\mathfrak{f}'}$ (see also Proposition 2.3) with now:

$$\nabla^{\mathfrak{f}'} - \nabla^{\mathfrak{f}} = \omega(\nabla^{\mathfrak{f}},\hat{g}^{\mathfrak{f}}).$$

With the example of $\mathfrak{f} = \overline{Q}_- \times \mathbb{C}$ and with $\nabla^{\mathfrak{f}} = \nabla$ the trivial connection, and $g^{\mathfrak{f}}(z) = g^{\mathfrak{f}}_-(z) = e^{-2V(\underline{q})}|z|^2$ we get $\hat{g}^{\mathfrak{f}}(z) = e^{-2\hat{V}(\underline{q})}|z|^2$ with $\hat{V}(\underline{q}) = V(-|\underline{q}^1|,\underline{q}')$ and

$$\omega(\nabla^{\mathfrak{f}},\hat{g}^{\mathfrak{f}}) = -2d\hat{V} = -2\left[\mathrm{sign}(-\underline{q}^1)d_{\underline{q}^1} \wedge \frac{\partial V}{\partial \underline{q}^1} + d_{\underline{q}'}V\right](-|\underline{q}^1|,\underline{q}')$$

with a discontinuity along $\underline{q}^1 = 0$ when $\partial_{\underline{q}^1}V(0,\underline{q}') \neq 0$. Then $\nabla^{\mathfrak{f}'}$ and $\nabla^{\mathfrak{f},u}$ become piecewise $\mathscr{C}^\infty$ and not continuous for the initial $\mathscr{C}^\infty$-structure of $Q \times \mathbb{C}$.
Note that if we take $\mathfrak{f} = \overline{Q}_- \times \mathbb{C}$ with the metric $g^{\mathfrak{f}}(z) = |z|^2$ but with the connection $\nabla^{\mathfrak{f}} = \nabla + dV(q)$, the connection on the doubled vector bundle $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ of Definition 2.2 is now $\nabla^{\mathfrak{f}} = \nabla + d\hat{V}(q)$ and while $\nabla^{\mathfrak{f}'} = \nabla - d\hat{V}(q)$ and $\nabla^{\mathfrak{f},u} = \nabla$. Remember also that the continuity in $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ means a change of sign across Q' when $\nu = -1$.

Remark 2.11. *The relations (59)(60) suggest another interpretation of $\hat{E}_g$ and $\hat{E}'_g$ as the exterior algebras of the cotangent and tangent bundle of a smooth manifold, $\overline{X}_-$ and $\overline{X}_+$ being glued by identifying $(0^-,\tilde{q}',\tilde{p}_1,\tilde{p})$ and $(0^+,\tilde{q}',-\tilde{p}_1,\tilde{p}')$. This will be used in Subsection 4.2. However for most of the analysis the above presentation of $\hat{E}_g$ and $\hat{E}'_g$ as piecewise $\mathscr{C}^\infty$ and continuous vector bundles on X is safer and more convenient.*

3 Functional spaces and invariances

We review the functional spaces that we will use. First we start with local spaces in the smooth case, which do not depend on any chosen riemannian metric, then we discuss the case of sections in $\hat{\mathfrak{F}}_g$ for $\mathfrak{F} = E, E', F, F'$, where the metric enters in the game only in the continuity or jump condition along X'. Finally we study how global spaces depend on the chosen metric g^{TQ}. In particular, global spaces of sections of $\hat{\mathfrak{F}}_g$ are characterized after considering the restrictions $s_\mp = s\big|_{\overline{X}_\mp}$ like in the smooth case with a boundary and then possibly adding the local continuity condition $s_-\big|_{X'} = s_+\big|_{X'}$ in $\hat{\mathfrak{F}}_g\big|_{X'}$. Invariances and isomorphisms of those functional spaces via the change of variables or vector bundle isomorphisms introduced in Section 2 are discussed.

3.1 Local spaces for smooth vector bundles

Let M be a $\mathscr{C}^\infty$ manifold and $\pi_{\mathfrak{F}}:\mathfrak{F}\to M$ a $\mathscr{C}^\infty$-vector bundle. A smooth manifold with boundary is denoted $\overline{M}=M\sqcup\partial M$ and accordingly $\pi_{\mathfrak{F}}:\mathfrak{F}\to\overline{M}$ is a $\mathscr{C}^\infty(\overline{M})$-vector bundle endowed with any smooth connection. The cases that we have in mind are $M=Q,X,Q_-,Q_+,X_-,X_+$ and $\overline{M}=\overline{Q}_-,\overline{Q}_+,\overline{X}_-,\overline{X}_+$. By following [ChPi] all the spaces $\mathscr{F}(M;\mathfrak{F})$ or $\mathscr{F}(\overline{M};\mathfrak{F})$ are defined independently of any riemannian structures or metric on $\mathfrak{F}$: $\mathscr{F}=\mathscr{C}^\infty$, $\mathscr{F}=\mathscr{C}^\infty_0$, $\mathscr{F}=L^2_{loc}$, $\mathscr{F}=L^2_{comp}$, $\mathscr{F}=W^{\mu,2}_{loc}$ and $\mathscr{F}=W^{\mu,2}_{comp}$, $\mu\in\mathbb{R}$, where "$W^{\mu,2}$ counts μ-derivatives in L^2" when $\mu\in\mathbb{N}$.

When M is a smooth manifold (no boundary), taking $\mathfrak{F}'$ the dual bundle of $\mathfrak{F}$ and fixing any smooth volume element dv_M on M provide a duality product

$$\langle s\,,s'\rangle=\int_M\langle s\,,s'\rangle_{\mathfrak{F}',\mathfrak{F}}(x)\,dv_M(x)\,,$$

and we will use without distinction real or sesquilinear (left-anti linear and right $\mathbb{C}$-linear) duality products.
The set of distributional sections $\mathscr{D}'(M;\mathfrak{F}')$ is defined as the dual of $\mathscr{C}^\infty_0(M;\mathfrak{F})$ and this duality holds between $W^{\mu,2}_{loc\ comp}(M;\mathfrak{F})$ and $W^{-\mu,2}_{comp\ loc}(M;\mathfrak{F}')$ for $\mu\in\mathbb{R}$.

A smooth manifold with boundary $\overline{M}$, can be considered as a domain of a smooth closed manifold $\tilde{M}$ and $\mathfrak{F}=\tilde{\mathfrak{F}}\big|_{\overline{M}}$ where $\pi_{\tilde{\mathfrak{F}}}:\tilde{\mathfrak{F}}\to\tilde{M}$ is a $\mathscr{C}^\infty$-vector bundle. According to [ChPi], the above functional space $\mathscr{F}(\overline{M};\mathfrak{F})$ is defined as the set of restrictions to M of elements of $\mathscr{F}(\tilde{M};\tilde{\mathfrak{F}})$:

$$\mathscr{F}(\overline{M};\mathfrak{F})=\left\{u\in\mathscr{D}'(M;\mathfrak{F}),\exists\tilde{u}\in\mathscr{F}(\tilde{M};\tilde{\mathfrak{F}}),\quad u=\tilde{u}\big|_M\right\}$$

endowed with the quotient topology. On a manifold with boundary $\overline{M}=M\sqcup\partial M$, compact sets of M and $\overline{M}$ differ and the spaces $\mathscr{F}(M,\mathfrak{F})$ and $\mathscr{F}(\overline{M};\mathfrak{F})$ are distinguished, the later specifying the information up to the boundary ∂M. Finally when dv_M a $\mathscr{C}^\infty(\overline{M})$ volume element, the duality holds between $L^2_{loc\ comp}(\overline{M};\mathfrak{F})$ and $L^2_{comp\ loc}(\overline{M};\mathfrak{F}')$.
A section $s\in\mathscr{F}(M;\mathfrak{F})$ (resp. $s\in\mathscr{F}(\overline{M};\mathfrak{F})$) if for any locally finite partition of unity $\sum_{j\in J}\chi_j\equiv 1$, $\chi_j\in\mathscr{C}^\infty_0(M;\mathbb{R})$ (resp. $\chi_j\in\mathscr{C}^\infty_0(\overline{M};\mathbb{R})$) one has $\chi_j s\in\mathscr{F}(M;\mathfrak{F})$ for all $j\in J$, the latter being checked in any local coordinate system. Those spaces are invariant by $\mathscr{C}^\infty$ diffeomorphisms on M (resp $\overline{M}$) and $\mathscr{C}^\infty$ vector bundle isomorphisms of $\mathfrak{F}$.

When $M=X=T^*Q$ or $\overline{M}=\overline{X}_\mp$, any given riemannian metric g^{TQ} provides the function $|p|^2_q=2\mathfrak{h}(q,p)=g^{ij}(q)p_ip_j$ and $s\in W^{\mu,2}_{loc}(X;\mathfrak{F})$ (resp. $s\in W^{\mu,2}_{loc}(\overline{X}_\mp;\mathfrak{F})$) if and only if for any $\chi\in\mathscr{C}^\infty_0(\mathbb{R};\mathbb{R})$, $\chi(|p|^2_q)s\in W^{\mu,2}_{comp}(M;\mathfrak{F})$ (resp. $\chi(|p|^2_q)s\in W^{\mu,2}_{comp}(\overline{M};\mathfrak{F})$).
Note also that on X and $\overline{X}_\mp$ the symplectic volume $dv_X=|dqdp|$ is fixed independently of any chosen metric.
$W^{(\mu_1,\mu_2),2}$ removed
We now define spaces associated with a continuous operator $P:\mathscr{D}'(M)\to\mathscr{D}'(M)$, which will be in practice a differential operator with smooth coefficients.

Definition 3.1. *Let $\pi_{\mathfrak{F}} : \mathfrak{F} \to M$ be a smooth vector bundle on M (resp $\overline{M} = M \sqcup \partial M$) and let P be a differential operator with smooth coefficients $P : \mathscr{D}'(M;\mathfrak{F}) \to \mathscr{D}'(M;\mathfrak{F})$. The spaces $\mathscr{E}_{loc}(P,\mathfrak{F})$ and $\mathscr{E}_{comp}(P,\mathfrak{F})$ are defined as*

$$\begin{aligned} &\mathscr{E}_{\bullet}(P,\mathfrak{F}) = \left\{\omega \in L^2_{\bullet}(M;\mathfrak{F}),\quad P\omega \in L^2_{\bullet}(M;\mathfrak{F})\right\}, \quad \bullet = loc \text{ or } comp. \\ \text{resp.}\quad &\mathscr{E}_{\bullet}(P,\mathfrak{F}) = \left\{\omega \in L^2_{\bullet}(\overline{M};\mathfrak{F}),\quad P\omega \in L^2_{\bullet}(\overline{M};\mathfrak{F})\right\}, \quad \bullet = loc \text{ or } comp. \end{aligned}$$

3.2 Local spaces for $\hat{E}_g, \hat{E}'_g, \hat{F}_g, \hat{F}'_g$

The vector bundles $\hat{E}_g$, $\hat{E}'_g$, $\hat{F}_g = \hat{E}_g \otimes \pi_X^{-1}(\mathfrak{f})$ and $\hat{F}'_g = \hat{E}_g \otimes \pi_X^{-1}(\mathfrak{f})$ of Definition 2.7 are defined as piecewise $\mathscr{C}^\infty$ vector bundles, with some matching conditions along $X' = \partial X_- = \partial X_+$. Local functional spaces $\mathscr{F}(X;\widehat{\mathfrak{F}}_g)$, with $\widehat{\mathfrak{F}}_g = \hat{E}_g, \hat{E}'_g, \hat{F}_g, \hat{F}'_g$ will be specified accordingly by using $\mathscr{F}(\overline{X}_\mp;\mathfrak{F})$ with the corresponding continuity condition along X' when it makes sense. Remember that $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$ may have different $\mathscr{C}^\infty$ structures according to Proposition 2.3 in $\mathfrak{F} = E \otimes \pi_X^*(\mathfrak{f})$ and $\mathfrak{F} = E' \otimes \pi_X^*(\mathfrak{f})$.
The space $L^2_{loc\ comp}$ is defined piecewise and the distinction between $\widehat{\mathfrak{F}}_g$ and $\mathfrak{F}$ can be forgotten according to

$$L^2_{loc}(X;\widehat{\mathfrak{F}}_g) = L^2_{loc}(\overline{X}_-;\mathfrak{F}) \oplus L^2_{loc}(\overline{X}_+;\mathfrak{F}) = L^2_{loc}(X;\mathfrak{F}).$$

Definition 3.2. *When $\Sigma : X_{(-\varepsilon,\varepsilon)} \to X_{(-\varepsilon,\varepsilon)}$ and Σ_ν are the maps of Definition 2.5 and $\mathfrak{F} = E, E', F, F'$ with $\nu = \pm 1$ when $\mathfrak{F} = E, E'$, the set of even and odd sections of $L^2_{loc}(X_{(-\varepsilon,\varepsilon)};\mathfrak{F})$, is defined by*

$$L^2_{ev\ odd,loc}(X_{(-\varepsilon,\varepsilon)};\mathfrak{F}) = \left\{s \in L^2_{loc}(X_{(-\varepsilon,\varepsilon)};\mathfrak{F}),\quad \Sigma_\nu s = \pm s\right\}. \tag{70}$$

For $s \in L^2_{loc}(X_{(-\varepsilon,0]};\mathfrak{F})$ we define

$$s_{ev} = 1_{X_-}(x)s + 1_{X_+}(x)\Sigma_\nu s). \tag{71}$$

Note that the spaces of even and odd sections are interchanged by a simple change of the unitary flat involution $\nu : \mathfrak{f}\big|_{Q'} \to \mathfrak{f}\big|_{Q'}$ into $-\nu$.
We will make an extensive use of the set of smooth compactly supported sections, $\mathscr{C}_{0,g}(\widehat{\mathfrak{F}}_g)$ defined below.

Definition 3.3. *For $\mathfrak{F} = E, E', F, F'$ possibly restricted to $X_{(-\varepsilon,\varepsilon)}$, the space $\mathscr{C}_{0,g}(\widehat{\mathfrak{F}}_g)$ is defined as*

$$\mathscr{C}_{0,g}(\widehat{\mathfrak{F}}_g) = \left\{s \in \mathscr{C}^0(X;\widehat{\mathfrak{F}}_g),\quad s\big|_{\overline{X}_\mp} \in \mathscr{C}^\infty_0(\overline{X}_\mp;\mathfrak{F})\right\}. \tag{72}$$

The set of even elements is defined by

$$\mathscr{C}_{0,g,ev}(\widehat{\mathfrak{F}}_g) = \mathscr{C}_{0,g}(\widehat{\mathfrak{F}}_g) \cap L^2_{ev,comp}(X_{(-\varepsilon,\varepsilon)};\mathfrak{F}). \tag{73}$$

Finally the space $\mathscr{C}_{0,g}(L(\widehat{\mathfrak{F}}_g))$ is defined like (72) by

$$\mathscr{C}_{0,g}(L(\widehat{\mathfrak{F}}_g)) = \left\{s \in \mathscr{C}^0(X;L(\widehat{\mathfrak{F}}_g)),\quad s\big|_{\overline{X}_\mp} \in \mathscr{C}^\infty(\overline{X}_\mp;L(\mathfrak{F}))\right\}.$$

Definition 2.7 actually provides a $\mathscr{C}^\infty$ structure for $\widehat{\mathfrak{F}}_g$ by taking the one of the right-hand side in the equalities (34)(35)(36)(37), depending on the case. The local Sobolev spaces $W^{\mu,2}_{loc}(X;\widehat{\mathfrak{F}}_g)$, $\mu\in\mathbb{R}$
$W^{(\mu_1,\mu_2),2}$ removed can be defined for this $\mathscr{C}^\infty$-structure. We recall the standard result concerning the existence of traces (see e.g. [ChPi]):

- The trace map $\gamma: W^{\mu,2}_{loc}(\overline{X}_\mp;\widehat{\mathfrak{F}}_g)=W^{\mu,2}_{loc}(\overline{X}_\mp;\mathfrak{F})\to W^{\mu-1/2,2}_{loc}(\partial X_\mp;\mathfrak{F})$, $\gamma s=s\big|_{\partial X_\mp}$, is well defined for $\mu>\frac{1}{2}$.
- For $\mu\in[0,1/2[$, $s\in W^{\mu,2}_{loc}(X;\widehat{\mathfrak{F}}_g)$ if and only if $s_\mp=s\big|_{X_\mp}\in W^{\mu,2}(\overline{X}_\mp;\widehat{\mathfrak{F}}_g)=W^{\mu,2}(\overline{X}_\mp;\mathfrak{F})$.
- For $\mu\in]1/2,3/2[$, $s\in W^{\mu,2}(X;\widehat{\mathfrak{F}}_g)$ if and only if $s_\mp=\big|_{X_\mp}\in W^{\mu,2}_{loc}(\overline{X}_\mp;\widehat{\mathfrak{F}}_g)=W^{\mu,2}_{loc}(\overline{X}_\mp;\mathfrak{F})$ and the traces along $X'=\partial X_-=\partial X_+$ coincide $s_-\big|_{\partial X_-}=s_+\big|_{\partial X_+}$ in $\widehat{\mathfrak{F}}_g\big|_{X'}$.

In all the analysis we will avoid trace issues for half-integer exponents $\mu=\frac{1}{2}+n$, $n\in\mathbb{N}$, which as it is well known (see [LiMa]-Chap 11) is a subtle critical case.
The equality of traces in $\widehat{\mathfrak{F}}_g\big|_{X'}$, for $\mu_1\in]1/2,3/2[$ means that the frames $(e_-,\hat{e}_-)$ and $(e_+,\hat{e}_+)$ are identified along $X'=\partial X_-=\partial X_+$ and this is actually a jump condition with the usual $\mathscr{C}^\infty$-structure of $E=\Lambda T^*X$ and $E'=\Lambda TX$, which also corresponds to the case when $g^{TQ}=g^{TQ}_0=(d\underline{q}^1)^2+m^{TQ}(0,\underline{q}')$. For such a metric g^{TQ}_0, we write simply $W^{\mu,2}_{loc}(X;\widehat{\mathfrak{F}}_{g0})=W^{\mu,2}_{loc}(X;\mathfrak{F})$. For a general metric g^{TQ}, the spaces $W^{\mu,2}_{loc}(X;\widehat{\mathfrak{F}}_g)$ will be considered with $\mu=[0,1]\setminus\{1/2\}$. Since differential operators with possibly discontinuous coefficients along X' and non obvious effects on the jump condition will be studied, it is better to split the analysis on $\overline{X}_-$ and $\overline{X}_+$ and check separately the matching condition along X'.
We keep of course the notation $W^{\mu,2}_{loc}(X;\mathfrak{F})$ when $g^{TQ}=g^{TQ}_0$ for any $\mu\in\mathbb{R}$ such that but for a general g^{TQ}, we take the following definition equivalent to the previous construction.

Definition 3.4. *For $\mathfrak{F}=E,E',F,F'$, $\mu\in[0,1]\setminus\{1/2\}$, the space $W^{\mu,2}_{loc}(X;\widehat{\mathfrak{F}}_g)$ is defined as the space of sections $s\in L^2_{loc}(X;\mathfrak{F})$ such that:*

- $s_\mp=s\big|_{X_\mp}\in W^{\mu,2}_{loc}(\overline{X}_\mp;\mathfrak{F})$*;*
- *if $\mu\in]1/2,1]$, $s_-\big|_{\partial X_-}=s_+\big|_{\partial X_+}$ in $\widehat{\mathfrak{F}}_g\big|_{X'}$.*

The set of even sections of $W^{\mu,2}_{loc}(X;\hat{E}_g)$ is defined as

$$W^{\mu,2}_{ev,loc}(X;\widehat{\mathfrak{F}}_g)=W^{\mu,2}_{loc}(X;\widehat{\mathfrak{F}}_g)\cap L^2_{ev,loc}(X;\mathfrak{F}).$$

Definition 3.5. *In $X'=\partial X_-$ the map $S_1=\Sigma\big|_{X'}$ is given by*

$$S_1(0,q',p_1,p')=(0,q',-p_1,p'). \tag{74}$$

For $\mathfrak{F}=F$ (resp. $\mathfrak{F}=F'$) the map $\hat{S}_\nu:\mathscr{D}'(X';\mathfrak{F}\big|_{X'})\to\mathscr{D}'(\mathfrak{F}\big|_{X'})$ is given by

$$\hat{S}_\nu(\omega^J_I(x')e^I_-\hat{e}_{-,J})=\nu(-1)^{|\{1\}\cap I|+|\{1\}\cap J|}\omega^J_I(S_1(x'))e^I_-\hat{e}_{-,J} \tag{75}$$

resp. $$\hat{S}_\nu(u^I_J(x')e_{-,I}\hat{e}^J_-)=\nu(-1)^{|\{1\}\cap I|+|\{1\}\cap J|}u^I_J(S_1(x'))e_{-,I}\hat{e}^J_-. \tag{76}$$

Proposition 3.6. *For a section $s \in L^2_{loc}(\overline{X}_-;\mathfrak{F})$ and $\mu \in [0,1]\setminus\{1/2\}$, and $\mathfrak{F} = F, F'$, there is an equivalence between:*

a) $s_{ev} \in \mathscr{C}_{0,g}(\widehat{\mathfrak{F}}_g)$ *(resp. $s_{ev} \in W^{\mu,2}_{loc}(X;\widehat{\mathfrak{F}}_g)$);*

b) $s \in \mathscr{C}^\infty_0(\overline{X}_-;\mathfrak{F})$ *(resp. $s \in W^{\mu,2}_{loc}(\overline{X}_-;\mathfrak{F})$) and (resp. when $\mu > 1/2$)*

$$\hat{S}_\nu s\big|_{\partial X_-} = s\big|_{\partial X_-}.$$

Proof. Simply write that $s_{ev} = 1_{X_-}(x)s + 1_{X_+}(\Sigma_\nu s)$ satisfies $s_{ev}\big|_{X_-} = s\big|_{X_-}$ and admits a trace in $\mathscr{D}'(X';\widehat{\mathfrak{F}}_g\big|_{X'})$ when $s_{ev} \in \mathscr{C}_{0,g}(X;\widehat{\mathfrak{F}}_g)$ or $s_{ev} \in W^{\mu,2}_{loc}(X;\widehat{\mathfrak{F}}_g)$, $\mu > 1/2$, with:

- $\Sigma\big|_{X'} = S_1$,
- $\Sigma_*(e^i_-, \hat{e}_{-,j}) = ((-1)^{\delta_{i1}} e^i_+, (-1)^{\delta_{j1}}\hat{e}_{+,j})$ tensorized with ν in $\pi^*_{\overline{X}_-}(\mathfrak{f})$,
- and $(e_-, \hat{e}_-)\big|_{X'} = (e_+, \hat{e}_+)\big|_{X'}$ in $\widehat{\mathfrak{F}}_g\big|_{X'}$.

□

3.3 Global functional spaces

3.3.1 Global L^2-spaces, duality and adjoints

Like for $L^2_{loc}(X;\widehat{\mathfrak{F}}_g) = L^2_{loc}(X;\mathfrak{F}) = L^2_{loc}(\overline{X}_-;\mathfrak{F}) \oplus L^2_{loc}(\overline{X}_+;\mathfrak{F})$ for $\mathfrak{F} = E, E', F, F'$ we can simply work with the vector bundles $E = \Lambda T^*X$, $E' = \Lambda TX$, $F = E \otimes \pi^*_X(\mathfrak{f})$ and $F' = E' \otimes \pi^*_X(\mathfrak{f})$ and forget the distinction between $\hat{\mathfrak{F}}_g$ and $\mathfrak{F}$.
The cotangent space X is endowed with the symplectic volume

$$dv_X = |\frac{1}{d!}\sigma^d| = |\frac{1}{d!}(e^i \wedge \hat{e}_j)^d| = |dqdp|$$

where $|dqdp|$ stands for the Lebesgue measure in the local coordinates $(q,p) = (q^1,\ldots,q^d,p_1,\ldots,p_d)$.
The local coordinates $(\tilde{q},\tilde{p}) = (\tilde{q}^1,\ldots,\tilde{q}^d,\tilde{p}_1,\ldots,\tilde{p}_d)$ of Definition 2.6 are not symplectic coordinates and (65)(66)(68) lead to

$$dv_X = |\frac{1}{d!}\sigma^d| = |\frac{1}{d!}(e^i \wedge \hat{e}_i)^d| = |\det\psi^{-1}(\tilde{q})||d\tilde{q}d\tilde{p}|. \tag{77}$$

Remember that the metrics $g^E, g^{E'}, g^F, g^{F'}$ on E, E', F, F' given by (10)(11)(12)(13) include the weight $\langle p\rangle_q^{\pm N_H \mp N_V}$ and the same is done for $\hat{g}^E, \hat{g}^{E'}, \hat{g}^F, \hat{g}^{F'}$ with g^{TQ} replaced by $\hat{g}^{TQ}$.

Definition 3.7. *Let $\mathfrak{F} = E, E', F, F'$ be endowed with the metric $\tilde{g}^{\mathfrak{F}} = g^{\mathfrak{F}}$ or $\tilde{g}^{\mathfrak{F}} = \hat{g}^{\mathfrak{F}}$. The space $L^2(X;\mathfrak{F})$, or $L^2(X;\mathfrak{F},\tilde{g}^{\mathfrak{F}})$ when we want to specify the metric, is*

$$L^2(X;\mathfrak{F}) = \left\{s \in L^2_{loc}(X;\mathfrak{F}),\quad \int_X |s(q,p)|^2_{\tilde{g}^{\mathfrak{F}}}\ |dqdp| < +\infty\right\}.$$

The scalar product and the duality product between $L^2(X;\mathfrak{F})$ and $L^2(X;\mathfrak{F}')$ are given by

$$\langle s\,,s'\rangle_{L^2}=\langle s\,,s'\rangle_{L^2(\tilde{g}^{\mathfrak{F}})}=\int_X\langle s\,,s'\rangle_{\tilde{g}^{\mathfrak{F}}_{(q,p)}}\,|dqdp|=\overline{\langle s',s\rangle_{L^2}} \tag{78}$$

$$\langle t\,,s\rangle=\int_X\langle t\,,s\rangle_{\mathfrak{F}'_{(q,p)},\mathfrak{F}_{(q,p)}}\,|dqdp|=\overline{\langle s\,,t\rangle} \tag{79}$$

for any $s,s'\in L^2(X;\mathfrak{F})$, $t\in L^2(X;\mathfrak{F}')$.
Finally the set $L^2_{ev\ odd}(X;\mathfrak{F},\hat{g}^{\mathfrak{F}})$ equals

$$L^2_{ev\ odd}(X;\mathfrak{F},\hat{g}^{\mathfrak{F}})=L^2(X;\mathfrak{F},\hat{g}^{\mathfrak{F}})\cap L^2_{ev\ odd,loc}(X;\mathfrak{F}).$$

Proposition 3.8. *Let $\mathfrak{F}=E,E',F,F'$ be endowed with the metric $\tilde{g}^{\mathfrak{F}}=g^{\mathfrak{F}}$ or $\hat{g}^{\mathfrak{F}}$ and let $(\mathfrak{F}',\tilde{g}^{\mathfrak{F}'})$ its antidual, $\mathfrak{F}'=E',E,F',F$ respectively.*

a) With the duality product (79), the dual of $L^2(X;\mathfrak{F},\tilde{g}^{\mathfrak{F}})$ is nothing but $L^2(X;\mathfrak{F}',\tilde{g}^{\mathfrak{F}'})$.

b) When $\varphi:Q\to Q$ is a $\mathscr{C}^\infty$ (resp. piecewise $\mathscr{C}^\infty$ on $\overline{Q}_\mp$) diffeomorphism, the the push-forward $\psi=\varphi_*:X=T^*Q\to X$ viewed as (resp. piecewise) diffeomorphism in X defines a continuous isomorphism $\psi_*:L^2(X;\mathfrak{F},\tilde{g}^{\mathfrak{F}})\to L^2(X;\mathfrak{F},\tilde{g}^{\mathfrak{F}})$.

c) If $\Psi:\Lambda TQ\otimes\Lambda T^*Q\to\Lambda TQ\otimes\Lambda T^*Q$ be a $\mathscr{C}^\infty$ (or piecewise $\mathscr{C}^\infty$ on $\overline{Q}_\mp$) vector bundle isomorphism, then $[\pi_X^*(\Psi)]_*:L^2(X;\mathfrak{F},\tilde{g}^{\mathfrak{F}})\to L^2(X,\mathfrak{F},\tilde{g}^{\mathfrak{F}})$ is a continuous isomorphism.

d) The vector bundle isomorphism $\Sigma_*:\mathfrak{F}\to\mathfrak{F}$ defines a unitary involution of $L^2(X;\mathfrak{F},\hat{g}^{\mathfrak{F}})$ when $\mathfrak{F}=E,E'$. The same holds for Σ_ν when $\mathfrak{F}=F,F'$ and $L^2(X;\mathfrak{F},\hat{g}^{\mathfrak{F}})=L^2_{ev}(X,\mathfrak{F};\hat{g}^{\mathfrak{F}})\oplus^\perp L^2_{odd}(X,\mathfrak{F};\hat{g}^{\mathfrak{F}})$.

e) When $g_0^{TQ}=(d\underline{q}^1)^2+m^{TQ'}(0,\underline{q}')$ and $g^{TQ}=(d\underline{q}^1)^2+m^{TQ'}(\underline{q}^1,\underline{q}')$, the vector bundle isomorphim $\widehat{\Psi}_X^{g,g_0}$ of Definition 2.8 defines a continuous isomorphism $(\widehat{\Psi}_X^{g,g_0})_*$ from $L^2(X_{(-\varepsilon,\varepsilon)};\mathfrak{F},\hat{g}_0^{\mathfrak{F}})$ to $L^2(X_{(-\varepsilon,\varepsilon)};\mathfrak{F},\hat{g}^{\mathfrak{F}})$, with $\hat{g}_0^{\mathfrak{F}}=g_0^E\otimes\hat{g}^{\mathfrak{f}}$ when $\mathfrak{F}=F$ and $\hat{g}_0^{\mathfrak{F}}=g_0^{E'}\otimes\hat{g}^{\mathfrak{f}}$ when $\mathfrak{F}=F'$.

Proof. **a)**It is simply the pointwise duality.
b) When $\varphi:Q\to Q$ is a (piecewise) $\mathscr{C}^\infty$-diffeomorphism

$$(Q,P)=\psi(q,p)=\varphi_*(q,p)=(\varphi(q),{}^tD\varphi_q^{-1}(q)p)=(\varphi(q),A(q)p)$$

the frames $(e,\hat{e})$ are transformed according to

$$dQ^i=[D\varphi_q]^i_jdq^j+[DA_q)^k_{ij}]p_kdq^j\quad,\quad dP_j=[A(q)]^k_jdp_k\,,$$

where all the q-dependent coefficients are uniformly bounded.
The weight $\langle p\rangle_q^{\pm N_H\mp N_V}$ depending on the case for $\mathfrak{F}$, ensures

$$\|\psi^*s\|_{L^2(\tilde{g}^{\mathfrak{F}})}\leq C\|s\|_{L^2(\tilde{g}^{\mathfrak{F}})}$$

and the same can be done for $\psi_* = (\psi^{-1})^* = (\varphi^*)^*$.
c) When $\mathfrak{F} = E, E'$ this comes from the identification of $TX = TX^H \oplus^\perp TX^V = \pi_X^*(TQ \oplus^\perp T^*Q)$ and $T^*X = TX^V \oplus^\perp T^*X^V = \pi_X^*(T^*Q \oplus^\perp TQ)$. When $\mathfrak{F} = F, F'$ we set $\varphi = \pi_{\Lambda TQ \otimes \Lambda T^*Q}$ and we first extend Ψ to $(\Lambda TQ \otimes \Lambda T^*Q) \otimes \mathfrak{f}$ and to $(\Lambda T^*Q \otimes \Lambda TQ) \otimes \mathfrak{f}$ as $\Psi \otimes \varphi_*$ and then pull it back via π_X^*.
d) $\Sigma_\nu : (\mathfrak{F}, \hat{g}^{\mathfrak{F}}) \to (\mathfrak{F}, \hat{g}^{\mathfrak{F}})$ is an isometry and $\Sigma : X \to X$ is a symplectic map.
e) We already know that $\widehat{\Psi}_X^{g,g_0} : (\mathfrak{F}, \hat{g}_0^{\mathfrak{F}}) \to (\widehat{\mathfrak{F}}_g, \hat{g}^{\mathfrak{F}})$ is an isometry projected to $\pi_X(\widehat{\Psi}_X^{g,g_0}) = \hat{\varphi}_X^{g,g_0}$, while the map $\hat{\varphi}_X^{g,g_0}$ is given locally by $\hat{\varphi}_X^{g,g_0}(q,p) = x$ with $\tilde{q}(x) = q$ and $\tilde{p}(x) = p$ according to the Definition 2.6 of the coordinates $(\tilde{q}, \tilde{p})$. With (77) the map $|\det \psi(\tilde{q})|^{1/2} (\hat{\Psi}_X^{g,g_0})_*$ is unitary from $L^2(X_{(-\varepsilon,\varepsilon)}; \mathfrak{F}, \hat{g}_0^{\mathfrak{F}})$ to $L^2(X_{(-\varepsilon,\varepsilon)}; \mathfrak{F}, \hat{g}^{\mathfrak{F}})$ while the multiplication by $|\det \psi(\tilde{q})|^{\pm 1/2}$ is an isomorphism. $\square$

For traces along $X' = \partial X_- = \partial X_+$ we also need global L^2-spaces. It actually suffices to specify the volume element along X'

Definition 3.9. *On X' the volume element denoted $dv_{X'} = |dq' dp|$ equals*

$$dv_{X'} = \left| \mathbf{i}_{e_1} \frac{1}{d!}^d \sigma \right|$$

The volume element $|p_1| dv_{X'} = |p_1||dp_1 dq' dp'|$ equals $|(\mathbf{i}_{Y_\mathfrak{f}} \frac{\sigma^d}{d!})\big|_{TX'}|$ where $Y_\mathfrak{h}$ is the Hamiltonian vector field associated with $\mathfrak{h}$.

The above definition does not rely on a coordinate system. However with the local coordinates (q,p) and $(\tilde{q}, \tilde{p})$ with $(\tilde{q}, \tilde{p})\big|_{X'} = (q,p)\big|_{X'}$, we get

$$dv_{X'} = |dq' dp| = |d\tilde{q}' d\tilde{p}| \quad \text{and} \quad |p_1| dv_{X'} = |p_1||dq' dp_1 dp'| = |\tilde{p}_1||d\tilde{q}' d\tilde{p}_1 d\tilde{p}'|.$$

Because $L^2(X; \mathfrak{F}')$ is the dual of $L^2(X; \mathfrak{F})$ via the duality product (79), the adjoints of operators acting in $L^2(X; \mathfrak{F})$ can be defined in $L^2(X; \mathfrak{F}')$.

Definition 3.10. *The vector bundle $\mathfrak{F} = E, E', F, F'$ with dual $\mathfrak{F}' = E', E, F, F'$ is endowed with the metric $\tilde{g}^{\mathfrak{F}} = g^{\mathfrak{F}}$ or $\tilde{g}^{\mathfrak{F}} = \hat{g}^{\mathfrak{F}}$.*
Let Ω be an open set in X and let $(P, D(P))$ be a densely defined operator in $L^2(\Omega; \mathfrak{F}, \tilde{g}^{\mathfrak{F}})$. The adjoint denoted $(\tilde{P}, D(\tilde{P}))$ in $L^2(\Omega; \mathfrak{F}', \tilde{g}^{\mathfrak{F}'})$ is defined by

$$\begin{aligned}
&\left(t \in D(\tilde{P})\right) \Leftrightarrow \left(\exists C_t \geq 0, \forall s' \in D(P), |\langle t, Ps' \rangle| \leq C_t \|s'\|_{L^2}\right) \\
&\forall t \in D(\tilde{P}), \forall s' \in D(P), \quad \langle \tilde{P} t, s' \rangle = \langle t, Ps \rangle.
\end{aligned}$$

Since (78) and (79) give a unitary mapping $U : L^2(\Omega; \mathfrak{F}', \tilde{g}^{\mathfrak{F}'}) \to L^2(\Omega; \mathfrak{F}, \tilde{g}^{\mathfrak{F}})$, there is a simple relation with the usual adjoint $(P^*, D(P^*))$ for the $L^2(X; \mathfrak{F}, \tilde{g}^{\mathfrak{F}})$ scalar product, $(\tilde{P}, D(\tilde{P})) = (U^{-1} P^* U, U^{-1} D(P^*))$.
However Bismut constructions of adjoint in $L^2(\Omega; \mathfrak{F}, \tilde{g}^{\mathfrak{F}})$ involves a non symmetric (resp. non hermitian) non degenerate bilinear (resp. sesquilinear) on E (resp. F). For an isomorphism $\phi : E' \to E$ with adjoint ${}^t\phi : E' \to E$ we keep the same notation for $\phi = \phi \otimes \mathrm{Id}_{\pi_X^*(\mathfrak{f})} : F' \to F$. Let the vector bundle isomorphism $\phi : E' \to E$ be such that

$$\exists C > 0, \forall x \in X, \forall t \in \mathfrak{F}_x, \quad C^{-1} |\phi t|_{\tilde{g}_x^{\mathfrak{F}}} \leq |t|_{\tilde{g}_x^{\mathfrak{F}'}} \leq C |\phi t|_{\tilde{g}_x^{\mathfrak{F}}}, \tag{80}$$

This gives two sesquilinear forms on $\mathfrak{F}'$ and $\mathfrak{F}$ dual to each other

$$\eta_\phi(U,V) = \langle U\,,\,\phi V\rangle_{\mathfrak{F}',\mathfrak{F}} \quad , \quad \eta^*_\phi(\omega,\theta) = \langle(\phi^{-1}\omega)\,,\,\theta\rangle_{\mathfrak{F}',\mathfrak{F}}\,.$$

For sections $s,s' \in L^2(X,\mathfrak{F})$ we set

$$\langle s\,,\,s'\rangle_\phi = \int_X \eta^*_\phi(s,s')\,dv_X\,. \tag{81}$$

Because ϕ, η_ϕ and η^*_ϕ are not assumed to be neither symmetric (or hermitian) nor anti-symmetric (or anti-hermitian), left and right adjoints must be distinguished.

Definition 3.11. *The vector bundle $\mathfrak{F} = E,E',F,F'$ with dual $\mathfrak{F}' = E',E,F,F'$ is endowed with the metric $\tilde{g}^{\mathfrak{F}} = g^{\mathfrak{F}}$ or $\tilde{g}^{\mathfrak{F}} = \hat{g}^{\mathfrak{F}}$.*
We assume that $\phi : F' \to F$ satisfies (80) and Ω is an open subset of X.
The left ϕ-adjoint of a densely defined operator $(P,D(P))$ in $L^2(\Omega;\mathfrak{F})$, denoted by $(P^\phi,D(P^\phi))$ is defined by

$$\begin{aligned}&\big(s\in D(P^\phi)\big) \Leftrightarrow \big(\exists C_s \geq 0, \forall s' \in D(P), |\langle s\,,\,Ps'\rangle_\phi| \leq C_s\|s'\|_{L^2}\big)\\ &\forall s \in D(P^\phi), \forall s' \in D(P), \quad \langle P^\phi s\,,\,s'\rangle_\phi = \langle s\,,\,Ps\rangle_\phi\,.\end{aligned}$$

The right ϕ-adjoint is defined similarly by considering the continuity of s'-dependent anti-linear form $D(P) \ni s \mapsto \langle Ps\,,\,s'\rangle_\phi$.

Proposition 3.12. *The left ϕ-adjoint $(P^\phi,D(P^\phi))$ equals $(\phi\tilde{P}\phi^{-1},\phi D(\tilde{P}))$ while the right ϕ-adjoint equals $(P^{^t\phi},D(P^{^t\phi})) = ({}^t\phi\tilde{P}{}^t\phi^{-1};{}^t\phi D(\tilde{P}))$.*
As adjoints of densely defined operators, the operators $(P^\phi,D(P^\phi))$ and $(P^{^t\phi},D(P^{^t\phi}))$ are closed in $L^2(\Omega,\mathfrak{F})$. When $(P,D(P))$ is closed and densely defined then the same holds for $(P^\phi,D(P^\phi))$ and $(P^\phi)^{^t\phi} = P$.
When $(P,D(P))$ is densely defined and closable, $(P^\phi)^{^t\phi} = \overline{P}$ the closure of P.

Proof. Just write

$$\langle\phi^{-1}\tilde{P}^\phi s\,,\,s'\rangle = \langle P^\phi s\,,\,s'\rangle_\phi = \langle s\,,\,Ps'\rangle_\phi = \langle\phi^{-1}s\,,\,Ps\rangle = \langle\tilde{P}\phi^{-1}s\,,\,s'\rangle$$

and use the definitions to get $P^\phi = \phi\tilde{P}\phi^{-1}$.
With $\overline{\langle s\,,\,s'\rangle_\phi} = \langle s'\,,\,s\rangle_{^t\phi}$ write for the right ϕ-adjoint $P^{r,\phi}$

$$\langle Ps\,,\,s'\rangle_\phi = \langle s\,,\,P^{r,\phi}s'\rangle_\phi$$

in the form

$$\langle s'\,,\,Ps\rangle_{^t\phi} = \langle P^{r,\phi}s'\,,\,s'\rangle_{^t\phi}\,,$$

so that $P^{r,\phi} = P^{^t\phi} = {}^t\phi\tilde{P}{}^t\phi^{-1}$.
The relation $(P^\phi)^{^t\phi} = \overline{P}$ comes from the standard theory of adjoints in Hilbert (therefore reflexive Banach) spaces, according to

$$(P^\phi)^{^t\phi} = {}^t\phi\widetilde{P^\phi}\;{}^t\phi^{-1} \quad , \quad P^\phi = \phi\tilde{P}\;\phi^{-1} \quad , \quad \widetilde{P^\phi} = {}^t\phi^{-1}\tilde{\tilde{P}}\;{}^t\phi = {}^t\phi^{-1}P\;{}^t\phi\,.$$

$\square$

3.3.2 Global Sobolev scale

In [Leb1], G. Lebeau introduced the Sobolev scale $\mathscr{W}^{\mu}(X;\Lambda T^*X\otimes\mathfrak{f})$, $\mu\in\mathbb{R}$, when (Q,g^{TQ}) is a smooth closed compact riemannian manifold, which is adapted to the geometry of $X=T^*Q$ and to the analysis of Bismut's hypoelliptic Laplacian. We adapt those definitions to our case with our notations.

Definition 3.13. *Let $\mathfrak{F}=E,E',F,F'$ be endowed with the smooth metric $g^{\mathfrak{F}}$.*
For $n\in\mathbb{N}$, the space $\mathscr{W}^n(X;\mathfrak{F})$ is the set of sections $s\in L^2(X;\mathfrak{F},g^{\mathfrak{F}})$ for which there exists $C_s>0$ such that

$$\left\|\langle p\rangle_q^{2n_3}\Big(\prod_{k=1}^{n_1}\nabla^{\mathfrak{F}}_{\tilde{U}_k}\Big)\Big(\prod_{\ell=1}^{n_2}\langle p\rangle_q\nabla^{\mathfrak{F}}_{\tilde{V}^{\ell}}\Big)s\right\|_{L^2(g^{\mathfrak{F}})}\leq C_s\Big(\prod_{k=1}^{n_1}\|U_k\|_{W^{n,\infty}(Q;TQ)}\Big)\Big(\prod_{\ell=1}^{n_2}\|V^{\ell}\|_{W^{n,\infty}(Q;TQ)}\Big)$$

where

$$n_1+n_2+n_3\leq n\,,\quad U_k\in\mathscr{C}^\infty(Q;TQ),V^\ell\in\mathscr{C}^\infty(Q;T^*Q),$$

and where $\tilde{U}_k$ (resp. $\tilde{V}^\ell$) is the horizontal (resp. vertical) lift of U_k (resp. V^ℓ).
The space $\mathscr{W}^n(X;\mathfrak{F})$ can be given a Hilbert space structure (see (84) below) and $\mathscr{W}^{\mu}(X;\mathfrak{F})$ for $\mu\in\mathbb{R}$, is then defined by duality and interpolation.
For $\Omega=X_{\mp}$, $\mathscr{W}^{\mu}(\overline{\Omega};\mathfrak{F})$, $\mu\in\mathbb{R}$, is defined as

$$\mathscr{W}^{\mu}(\overline{\Omega};\mathfrak{F})=\left\{u\in\mathscr{D}'(\Omega;\mathfrak{F}),\exists\tilde{u}\in\mathscr{W}^{\mu}(X;\mathfrak{F}),u=\tilde{u}\big|_{\Omega}\right\}.$$

Here are some explanations and we refer the reader to [Leb1][Leb2] for details. For $\mu=n\in\mathbb{N}$, $s\in\mathscr{W}^n(X;\mathfrak{F})$ can be checked by introducing a partition of unity on Q, $\sum_m\chi_m^2(\underline{q})=1$ subordinate to an atlas and by taking the U_k, V^ℓ in the local frame $(\frac{\partial}{\partial\underline{q}^i},d\underline{q}^j)$ with

$$\tilde{U}_k=e_i=\overbrace{\frac{\partial}{\partial q^i}}^{order\ 1}+\Gamma_{ij}^k(q)p_k\overbrace{\frac{\partial}{\partial p_j}}^{order\ 1}\tag{82}$$

$$\langle p\rangle_q\tilde{V}^\ell=\langle p\rangle_q\hat{e}^j=\overbrace{\langle p\rangle_q\frac{\partial}{\partial p_j}}^{order\ 1}.\tag{83}$$

By setting for $\alpha\in\mathbb{N}^d$, $\nabla_e^{\mathfrak{F},\alpha}=\nabla_{e_1}^{\mathfrak{F},\alpha_1}\ldots\nabla_{e_d}^{\mathfrak{F},\alpha_d}$ and $\nabla_{\hat{e}}^{\mathfrak{F},\alpha}=\nabla_{\hat{e}^1}^{\mathfrak{F},\alpha_1}\ldots\nabla_{\hat{e}^d}^{\mathfrak{F},\alpha_d}$ a Hilbert norm on $\mathscr{W}^n(X,\mathfrak{F})$ is given by

$$\|s\|^2_{\mathscr{W}^n}=\sum_m\sum_{|\alpha|+|\beta|+n_3\leq n}\|\langle p\rangle_q^{2n_3+|\beta|}\nabla_e^{\mathfrak{F},\alpha}\nabla_{\hat{e}}^{\mathfrak{F},\beta}[\chi_m(q)s]\|^2_{L^2(g^{\mathfrak{F}})}.\tag{84}$$

Although the multiplication by $\langle p\rangle^{n_3}$ and the covariant derivatives themselves do not commute, changing the order in the above expression gives an equivalent norm because:

- $\nabla^{\mathfrak{F}}$ is the pull-back of a connection of the fiber bundle $\Lambda TQ\otimes\Lambda T^*Q\otimes\mathfrak{f}$ (resp. $\Lambda TQ\otimes\Lambda T^*Q\otimes\mathfrak{f}$) on Q with $\nabla^{\mathfrak{F}}_{\tilde{U}_1}\nabla^{\mathfrak{F}}_{\tilde{U}_2}-\nabla^{\mathfrak{F}}_{\tilde{U}_2}\nabla^{\mathfrak{F}}_{\tilde{U}_1}=\nabla^{\mathfrak{F}}_{\widetilde{[U_1,U_2]}}+\pi_X^*R(U_1,U_2)$ where $R(U_1,U_2)$ is a smooth endomorphim valued section on Q, therefore independent of p.

- The above additional term $\nabla^{\mathfrak{F}}_{\widehat{[U_1,U_2]}}$ is estimated with (42).
- The covariant vertical derivatives are the trivial ones.
- Changing the position of the weight multiplication brings lower order corrections owing to

$$e_i(f(\mathfrak{h}))=0 \quad , \quad \langle p\rangle_q^t \frac{\partial}{\partial p_j}\langle p\rangle_q^{-t}=\frac{\partial}{\partial p_j}+\mathcal{O}(\langle p\rangle_q^{-1})$$

for any $f\in\mathscr{C}^1(\mathbb{R};\mathbb{R})$, $\langle p\rangle_q=(1+2\mathfrak{h})^{1/2}$, and any $t\in\mathbb{R}$.

The abstract definition of $\mathscr{W}^{\mu}(X;\mathfrak{F})$ by duality and interpolation can be specified as follows (see [Leb1]): Once the localisation in q is made, assume $s=\chi_m(q)s$, take a dyadic partition of unity $\theta_0^2(|p|_q^2)+\sum_{m'=1}^{\infty}\theta_1^2(2^{m'}|p|_q)=\sum_{m'=0}^{\infty}\tilde{\chi}_{m'}^2(|p|_q)\equiv 1$ then $s\in\mathscr{W}^{\mu}(X;\mathfrak{F})$ can be replaced by $2^{m'}$-dependent estimates of $2^{m'd/2}(\tilde{\chi}_{m'}s)(q,2^{m'}p)$ in $W^{\mu,2}_{comp}(X,\mathfrak{F})$, with a fixed compact support in $(q,p)\in\mathbb{R}^{2d}$. And this can be characterized by standard pseudodifferential calculus.

Bismut and Lebeau in [Leb1][Leb2][BiLe] work actually with the metrics

$$\begin{aligned} &\tilde{g}^{E'}=\langle p\rangle_q^{-2N_V}\pi_X^*(g^{\Lambda TQ}\otimes g^{\Lambda T^*Q})=\langle p\rangle_q^{-N}g^{E'} \quad \text{on } E'=\Lambda TX\,,\\ \text{and}\quad &\tilde{g}^{E}=\langle p\rangle_q^{2N_V}\pi_X^*(g^{\Lambda T^*Q}\otimes g^{\Lambda TQ})=\langle p\rangle_q^{N}g^{E} \quad \text{on } E=\Lambda T^*X\,,\end{aligned}$$

with the corresponding metric $\tilde{g}^{F'}=\tilde{g}^{E'}\otimes g^{\mathfrak{f}}$ and $\tilde{g}^{F}=\tilde{g}^{E}\otimes g^{\mathfrak{f}}$. But this is a particular case of the weighted $\mathscr{W}^{\mu}$-spaces which is discussed below.

Proposition 3.14. *Let $\mathfrak{F}=E,E',F,F'$ be endowed with the smooth metric $g^{\mathfrak{F}}$. The spaces $\mathscr{W}^{\mu}(X;\mathfrak{F})$, $\mu\in\mathbb{R}$, have the following properties:*

a) *They do not depend on the chosen metric g^{TQ}.*

b) *When $\varphi:Q\to Q$ is a diffeomorphism and the vector bundle isomorphisms $\psi=\varphi_*:X=T^*Q\to X$ is viewed as as diffeomorphism of X, defines an isomorphism $\psi_*:\mathscr{W}^{\mu}(X;\mathfrak{F})\to\mathscr{W}^{\mu}(X;\mathfrak{F})$.*

c) *If $\Psi:\Lambda TQ\otimes\Lambda T^*Q\to\Lambda TQ\otimes\Lambda T^*Q$ be a $\mathscr{C}^{\infty}$ vector bundle isomorphism, then $[\pi_X^*(\Psi)]_*:\mathscr{W}^{\mu}(X;\mathfrak{F})\to\mathscr{W}^{\mu}(X,\mathfrak{F})$ is a continuous isomorphism.*

d) *The space $\mathscr{W}^{\mu}_{loc}(X;\mathfrak{F})$ is nothing but $\mathrm{W}^{\mu,2}_{loc}(X;\mathfrak{F})$. In particular the trace $s\big|_{X'}$ is well defined as soon as $s\in\mathscr{W}^{\mu}(X;\mathfrak{F})$ with $\mu>1/2$. Additionally for $\mu\in]1/2,1]$, $s\in\mathscr{W}^{\mu}(X;\mathfrak{F})$ is equivalent to $s_{\mp}=s\big|_{X_{\mp}}\in\mathscr{W}^{\mu}(\overline{X}_{\mp};\mathfrak{F})$ and $s_-\big|_{\partial X_-}=s_+\big|_{\partial X_+}$, while the trace condition is dropped when $\mu\in[0,1/2[$.*

e) *Let G be vector bundle isomorphism, $G\in\mathscr{C}^{\infty}(X;L(\mathfrak{F}))$, such that*

$$\begin{aligned}&(\nabla^{L(\mathfrak{F})}_{e_i}G)G^{-1},G^{-1}(\nabla^{L(\mathfrak{F})}_{e_i}G)\in\mathscr{L}(\mathscr{W}^{\mu}(X;\mathfrak{F});\mathscr{W}^{\mu-1}(X;\mathfrak{F}))\\ &(\nabla^{L(\mathfrak{F})}_{\hat{e}^j}G)G^{-1},G^{-1}(\nabla^{L(\mathfrak{F})}_{\hat{e}^j}G)\in\mathscr{L}(\mathscr{W}^{\mu}(X;\mathfrak{F});\mathscr{W}^{\mu-1/2}(X;\mathfrak{F}))\,,\end{aligned}$$

then the norm of $G\mathscr{W}^{n}(X;\mathfrak{F})=\left\{s\in\mathrm{W}^{n,2}_{loc}(X;\mathfrak{F}),G^{-1}s\in\mathscr{W}^{n}(X;\mathfrak{F})\right\}$ can be given by the same expression as (84) where only the metric, $g^{\mathfrak{F}}$, and L^2-norm,

$\| \; \|_{L^2(g^{\mathfrak{F}})}$, *are replaced respectively by* $\tilde{g}^{\mathfrak{F}}(v,v) = g^{\mathfrak{F}}(G^{-1}v, G^{-1}v)$ *and* $\| \; \|_{L^2(\tilde{g}^{\mathfrak{F}})}$. *The weighted spaces* $G\mathscr{W}^{\mu}(X;\mathfrak{F})$, $\mu \in \mathbb{R}$, *can thus be characterized without changing the connection* $\nabla^{\mathfrak{F}}$.

Proof. **a)** It suffices to consider the case $\mathscr{W}^n(X;\mathfrak{F})$ for $n \in \mathbb{N}$, where the result is already known for $n = 0$, and to work locally with the coordinates $(q,p) \in U \times \mathbb{R}^d$, U open set of $\mathbb{R}^d$. We take the euclidean metric $g_e = g_e^{TU} = \sum_{i=1}^d (d\underline{q}^i)^2$ as a reference metric for which the local frame in TX and T^*X are simply $\frac{\partial}{\partial q^i}$, $\frac{\partial}{\partial p_j}$ and dq^i, dp_j, while $\nabla^{\mathfrak{F},g_e}$ can be chosen as the trivial connection. The weights which are powers of $\langle p\rangle_{g,q}^2 = (1 + g^{ij}(q)p_ip_j)$ and $\langle p\rangle_{g_e,q}^2 = 1 + \sum_j^d p_j^2$ are uniformly equivalent with all the derivatives

$$\partial_q^\alpha \partial_p^\beta \left(\frac{\langle p\rangle_{g,q}}{\langle p\rangle_{g_e,q}}\right)^{\pm 1}$$

uniformly bounded. It thus suffices to compare the covariant derivatives:

$$\nabla_{e_i}^{\mathfrak{F},g} - \nabla_{\frac{\partial}{\partial q^i}}^{\mathfrak{F},g_e} = (\nabla^{\mathfrak{F},g} - \nabla^{\mathfrak{F},g_e})_{e_i} + e_i - \frac{\partial}{\partial q^i} = \pi_{X,*}(\nabla^{Q,g} - \nabla^{Q,g_e})_{e_i} + \Gamma_{ij}^k(q)\frac{\partial}{\partial p_j},$$
$$\nabla_{\frac{\partial}{\partial p_j}}^{\mathfrak{F},g} = \nabla_{\frac{\partial}{\partial p_j}}^{\mathfrak{F},g_e} = \frac{\partial}{\partial p_j}.$$

Since $\nabla^{Q,g} - \nabla^{Q,g_e} \in \mathscr{C}^\infty(Q; L(\Lambda TQ \otimes \Lambda T^*Q))$, we deduce that the local expression of the norm of $\mathscr{W}^n(X;\mathfrak{F})$, for $\chi_m s$ with the metric g and g_e a neighborhood of the support of χ_m are uniformly equivalent. This provides the local result for two different metric g_1, g_2 and taking the full finite sum in (84) ends the proof.

b) Again we can work locally and by **a)** we can take the euclidean metric $g^{TQ} = g_e^{TQ}$ on U and $\varphi(U)$. Write $(Q,P) = \psi(q,p) = (\varphi(q), {}^t[D\varphi_q]^{-1}p) = (\varphi(q), A(q)p)$ and

$$\overbrace{\frac{\partial}{\partial q^i}}^{order\ 1} = [D\varphi_q]_i^j \overbrace{\frac{\partial}{\partial Q^j}}^{order\ 1} + [DA_q]_{ij}^k P_k \overbrace{\frac{\partial}{\partial P_j}}^{order\ 1}$$
$$\overbrace{\langle p\rangle \frac{\partial}{\partial p_j}}^{order\ 1} = A_k^j(q)\frac{\langle p\rangle}{\langle A(q)p\rangle} \overbrace{\langle P\rangle_q \frac{\partial}{\partial P_k}}^{order\ 1}.$$

Since locally with $g = g_e$, the connection $\nabla^{\mathfrak{f},g_e}$ becomes a trivial one, the equivalence of the norm (84) of $\chi_m s$ and $\psi_*[\chi_m s]$ follows. By **a)** this equivalence holds for the metric $g^{\mathfrak{F}}$ put on $U \supset\supset \mathrm{supp}\chi_m s$ and $\varphi(U)$, and we conclude by summing with respect to m in (84).

c) By **b)** the problem is reduced to the case when $\varphi = \pi_{\Lambda TQ \otimes \Lambda T^*Q}(\Psi) = \mathrm{Id}_Q$ and $\Psi \in \mathscr{C}^\infty(Q; L(\Lambda TQ \otimes \Lambda T^*Q))$ and the result comes from $\nabla^{\mathfrak{F}} = \pi_X^*(\nabla^{Q,g} \otimes \nabla^{\mathfrak{f}\ or\ \mathfrak{f}'})$ while we already know the result for $n = 0$ by Proposition 3.8-c).

d) Locally, that is while considering χs with $\chi \in \mathscr{C}_0^\infty(X;\mathbb{R})$, the weight $\langle p\rangle_q^{2n_3+|\beta|}$ can be forgotten and $\mathscr{W}_{loc}^n(X;\mathfrak{F})$ is nothing but $W_{loc}^{n,2}(X;\mathfrak{F})$. Choosing χ with a small enough support we can even consider the map $s \mapsto \chi s$ as a continuous map from

$\mathscr{W}^n(X;\mathfrak{F})$ to $W^{n,2}(\mathbb{R}^{2d};\mathbb{C}^{N_{d,\mathfrak{f}}})$ and the continuity from $\mathscr{W}^\mu(X;\mathfrak{F})$ to $W^{\mu,2}(\mathbb{R}^{2d};\mathbb{C}^{N_{d,\mathfrak{f}}})$ for any $\mu\in\mathbb{R}$ holds true by duality and interpolation. This proves $\mathscr{W}^\mu_{loc}(X;\mathfrak{F})=W^{\mu,2}_{loc}(X;\mathfrak{F})$ for all $\mu\in\mathbb{R}$.
e)The norm of $s\in G\mathscr{W}^n(X,\mathfrak{F})$ equals $\|s\|_{G\mathscr{W}^n}=\|G^{-1}s\|_{\mathscr{W}^n}$ while $G^{-1}\langle p\rangle_q^{n_3}=\langle p\rangle_q^{n_3}G^{-1}$. The expression (84) gives

$$\|s\|^2_{G\mathscr{W}^n}=\sum_m\sum_{|\alpha|+|\beta|+n_3\le n}\|\langle p\rangle_q^{2n_3+|\beta|}\tilde{\nabla}_e^{\mathfrak{F},\alpha}\tilde{\nabla}_{\hat{e}}^{\mathfrak{F},\beta}[\chi_m(q)s]\|^2_{L^2(\tilde{g}^{\mathfrak{F}})},$$

with

$$\begin{aligned}
&\tilde{\nabla}^{\mathfrak{F}}=G\nabla^{\mathfrak{F}}G^{-1}=\nabla^{\mathfrak{F}}+G\nabla^{L(\mathfrak{F})}G^{-1}=\nabla^{\mathfrak{F}}-(\nabla^{L(\mathfrak{F})}G)G^{-1}\\
&\nabla^{\mathfrak{F}}=G^{-1}\tilde{\nabla}^{\mathfrak{F}}G=\tilde{\nabla}^{\mathfrak{F}}+G^{-1}(\nabla^{L(\mathfrak{F})}G),\\
\text{and}\quad&\tilde{g}^{\mathfrak{F}}(v,v)=g^{\mathfrak{F}}(G^{-1}v,G^{-1}v).
\end{aligned}$$

The assumptions are exactly the ones which ensure the equivalence with the squared norm

$$\sum_m\sum_{|\alpha|+|\beta|+n_3\le n}\|\langle p\rangle_q^{n_3+|\beta|}\nabla_e^{\mathfrak{F},\alpha}\nabla_{\hat{e}}^{\mathfrak{F},\beta}[\chi_m(q)s]\|^2_{L^2(\tilde{g}^{\mathfrak{F}})},$$

where the initial connection $\nabla^{\mathfrak{F}}$ is used. □

The result **e)** will be used with two types of weights.

$\boxed{G=\langle p\rangle_q^{\pm\frac{N_V+N_H}{2}}}$: The sign depends on the case $\mathfrak{F}=E'$ or $\mathfrak{F}=E$:

- It changes the metric $g^{E'}=\langle p\rangle_q^{N_H-N_V}\pi_X^*(g^{\Lambda TQ}\otimes g^{\Lambda T^*Q})$ into $\tilde{g}^{E'}=\langle p\rangle_q^{-2N_V}\pi_X^*(g^{\Lambda TQ}\otimes g^{\Lambda T^*Q})$
- It changes the metric $g^{E}=\langle p\rangle_q^{-N_H+N_V}\pi_X^*(g^{\Lambda T^*Q}\otimes g^{\Lambda TQ})$ into $\tilde{g}^{E}=\langle p\rangle_q^{2N_V}\pi_X^*(g^{\Lambda T^*Q}\otimes g^{\Lambda TQ})$.

Proposition 3.14-**e)** applies because

$$(\partial_T G)G^{-1},G^{-1}(\partial_T G)\in\mathscr{L}(\mathscr{W}^\mu(X;E))$$

for $T=e_i$ and $T=\hat{e}^j$ and for any $\mu\in\mathbb{N}$ (and therefore for any $\mu\in\mathbb{R}$) because $\partial_{q^i}(\langle p\rangle_q^t)=\mathscr{O}(\langle p\rangle_q^t)$ and $\partial_{p_j}(\langle p\rangle_q^t)=\mathscr{O}(\langle p\rangle_q^{t-1})$ for $t\in\mathbb{R}$.
This choice allows to transfer at once the estimates of [Leb1][Leb2][BiLe] where the metrics $\tilde{g}^E$ and $\tilde{g}^{E'}$ were chosen. The advantage of our choice is that the tensorized map $\phi=\sigma:TX\to T^*X$ sends isometrically $(E',g^{E'})$ to (E,g^E) while it sends isometrically $(E',\tilde{g}^{E'})$ to $(E,\langle p\rangle_q^{-2N}\tilde{g}^E)$.

$\boxed{G=e^{\pm(\mathfrak{h}(q,p)+V(q))}}$: In [BiLe][Bis05] the L^2-norm on $\tilde{F}=E=\Lambda T^*X\otimes\pi_X^*(\tilde{\mathfrak{f}})$, is given by

$$\int_X|s(q,p)|^2_{g^{\tilde{F}}}e^{-2\mathfrak{h}(q,p)}\,|dqdp|=\|s\|^2_{e^{\mathfrak{h}}L^2(g^{\tilde{F}})}.$$

Additionally the metric on $\tilde{\mathfrak{f}}=Q\times\mathbb{C}$ is given by $g^{\tilde{\mathfrak{f}}}(z)=e^{-2V(q)}|z|^2$ while the flat connection is the trivial one $\nabla^{\mathfrak{f}}=\nabla$. Taking $z'=e^{-V(q)}z$ gives $z=e^{V(q)}z'$. It is thus the same as choosing $\mathfrak{f} = Q \times \mathbb{C}$, $\nabla^{\mathfrak{f}} = \nabla + dV(q)$ and $g^{\mathfrak{f}}(z') = |z'|^2$ and the above squared norm equals

$$\|s\|^2_{e^{\mathfrak{h}}L^2(g^{\tilde{F}})} = \|s\|^2_{e^{\mathfrak{h}+V}L^2(g^F)}.$$

while its dual norm satisfies

$$\|t\|^2_{e^{-\mathfrak{h}}L^2(g^{\tilde{F}'})} = \|s\|^2_{e^{-(\mathfrak{h}+V)}L^2(g^{F'})},$$

with $\mathfrak{f}' = Q \times \mathbb{C} = \mathfrak{f}$, $g^{\mathfrak{f}'} = g^{\mathfrak{f}}$, but $\nabla^{\mathfrak{f}'} = \nabla - dV(q)$.
With

$$(\partial_{q^i} G)G^{-1} = G^{-1}(\partial_{q^i} G) = \pm \frac{\partial(\mathfrak{h} + V(q))}{\partial q^i}(q,p) \in \mathcal{L}(\mathcal{W}^{\mu}(X;E);\mathcal{W}^{\mu-1}(X;E))$$

$$\text{and} \quad (\partial_{p_j} G)G^{-1} = G^{-1}(\partial_{p_j} G) = \pm \frac{\partial(\mathfrak{h} + V(q))}{\partial p_j}(q,p) \in \mathcal{L}(\mathcal{W}^{\mu}(X;E);\mathcal{W}^{\mu-1/2}(X;E)).$$

the result of Proposition 3.14-**e)** ensures that the regularity estimates are equivalent after simply applying the weight to the L^2-space.

Definition 3.15. *When $\mu \in [0,1] \setminus \{\frac{1}{2}\}$ we define $\mathcal{W}^{\mu}(X;\hat{\mathfrak{F}}_g)$ as the set of sections $s \in L^2(X;\mathfrak{F})$ such that*

- *$s_{\mp} = s\big|_{X_{\mp}}$ belongs to $\mathcal{W}^{\mu}(\overline{X}_{\mp};\mathfrak{F})$,*
- *if $\mu \in]1/2,1]$, $s_-\big|_{\partial X'} = s_+\big|_{\partial X'}$ in $\hat{\mathfrak{F}}_g\big|_{X'}$.*

Finally for $\mu \in [0,1]$, $\mathcal{W}^{\mu}_{ev}(X;\hat{\mathfrak{F}}_g) = \mathcal{W}^{\mu}(X;\hat{\mathfrak{F}}_g) \cap L^2_{ev}(X;\mathfrak{F})$.

We already noticed that $W^{\mu,2}_{loc}(X;\hat{\mathfrak{F}}_{g_0}) = W^{\mu,2}_{loc}(X;\mathfrak{F})$. Since the definition of $\mathcal{W}^{\mu}(X;\hat{\mathfrak{F}}_{g_0})$ simply adds the global estimates which can be checked separated on both sides we get $\mathcal{W}^{\mu}(X;\hat{\mathfrak{F}}_{g_0}) = \mathcal{W}^{\mu}(X;\mathfrak{F})$.

Proposition 3.16. *The isomorphism $(\hat{\Psi}^{g,g_0}_X)_* : L^2(X_{(-\varepsilon,\varepsilon)};\mathfrak{F},\hat{g}^{\mathfrak{F}}_0) \to L^2(X_{(-\varepsilon,\varepsilon)};\mathfrak{F},\hat{g}^{\mathfrak{F}})$, of Definition 2.8 and Proposition 3.8-**e)**, is actually an continuous isomorphism from $\mathcal{W}^{\mu}(X_{(-\varepsilon,\varepsilon)};\mathfrak{F})$ to $\mathcal{W}^{\mu}(X_{(-\varepsilon,\varepsilon)};\hat{\mathfrak{F}}_g)$ for $\mu \in [0,1]$.*

Proof. For $\mu = 0$, $(\Psi^{g,g_0}_X)_* : L^2(X;\mathfrak{F},g^{\mathfrak{F}}_0) \to L^2(X;\mathfrak{F},g^{\mathfrak{F}})$ is an isomorphism.
Consider now $\mu = 1$. The vector bundle morphism $(\Psi^{g,g_0}_X)_*$ is a piecewise $\mathcal{C}^{\infty}$ vector bundle isomorphism from $\mathfrak{F}$ to $\hat{\mathfrak{F}}_g$ which transforms the continuity condition $s_-\big|_{X'} = s_+\big|_{X'}$ in $\mathfrak{F}\big|_{X'}$ into the same continuity condition in $\hat{\mathfrak{F}}_g\big|_{X'}$. Additionally by Proposition 3.14-**b)c)** applied on both sides (or more exactly for the $\mathcal{C}^{\infty}$-metrics g^{TQ}_- and g^{TQ}_+ and then restricted to $X_{\mp}$), we obtain

$$C^{-1}\|s_{\pm}\|_{\mathcal{W}^1(\overline{X}_{\mp})} \leq \|(\Psi^{g_{\mp},g_0}_X)_* s\|_{\mathcal{W}^1(\overline{X}_{\mp})} \leq C\|s_{\pm}\|_{\mathcal{W}^1(\overline{X}_{\mp})}.$$

This proves that $(\hat{\Psi}^{g,g_0}_X)_* : \mathcal{W}^1(X;\mathfrak{F}) \to \mathcal{W}^1(X;\hat{\mathfrak{F}}_g)$ is an isomorphism. Interpolation yields the result for $\mu \in [0,1]$. □

Finally, Proposition 3.14-**e)** works for $\hat{G}\mathcal{W}^{\mu}(X;\hat{\mathfrak{F}}_g)$, $\mu \in [0,1]$, with the weights $\hat{G}$ described in the three examples, after replacing the smooth metric g^{TQ} by $\hat{g}^{TQ}$, namely

$$\hat{G} = \langle p \rangle_{\hat{g},q}^{\pm\frac{N_H+N_V}{2}} \quad \text{and} \quad \hat{G} = e^{\pm(\hat{\mathfrak{h}}(q,p)+\hat{V}(q))},$$

because those weights are continuous w.r.t q^1.

4 Closed realizations of the differential

In this section trace theorems and boundary conditions for the differential, more generally the exterior covariant derivative, are considered. No riemannian metric is really required here and all the analysis is made by using the proper $\mathscr{C}^\infty$ structure of manifolds made of the two pieces $\overline{X}_-$ and $\overline{X}_+$ glued in the proper way. In particular, the new manifold M_g is introduced in Subsection 4.2. Its construction relies on the coordinates $(\tilde{q},\tilde{p})$ related with the parallel transport in $X = T^*Q$ for the Levi-Civita connection associated with $\hat{g}^{TQ}$. It depends on the metric g_-^{TQ} initially chosen on $\overline{Q}_-$ and accordingly the boundary conditions for the differential finally depend on g_-^{TQ}.

4.1 General partial trace results

Let M (resp. $\overline{M} = M \sqcup M'$) be a smooth oriented manifold (resp. with boundary $\partial M = M'$) and let $\pi_{\mathfrak{F}} : \mathfrak{F} \to M$ (resp. $\pi_{\mathfrak{F}} : \mathfrak{F} \to \overline{M}$) be a $\mathscr{C}^\infty$ vector bundle on M (resp. on $\overline{M}$) endowed with the non necessarily smooth connection

$$\nabla^{\mathfrak{F}} : \mathscr{C}^\infty(M;\mathfrak{F}) \to L^\infty_{loc}(M;T^*M\otimes\mathfrak{F}) \tag{85}$$

$$\text{resp.} \qquad \nabla^{\mathfrak{F}} : \mathscr{C}^\infty(\overline{M};\mathfrak{F}) \to L^\infty_{loc}(\overline{M};T^*M\otimes\mathfrak{F}). \tag{86}$$

The exterior covariant derivatives $d^{\nabla^{\mathfrak{F}}}$ acting on sections of $\Lambda T^*M\otimes\mathfrak{F}$ is written in local coordinates

$$d^{\nabla^{\mathfrak{F}}} = (dx^i\wedge)\frac{\partial}{\partial x^i}\otimes \mathrm{Id}_{\mathfrak{F}} + (dx^i\wedge)\otimes\nabla^{\mathfrak{F}}_{\frac{\partial}{\partial x^i}}\,.$$

Remember the Definition 3.1 of $\mathscr{E}_{loc\ comp}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F})$ in the two cases $M' = \emptyset$ and $M' \neq \emptyset$.

Proposition 4.1. ***a)*** *If $\mathfrak{F}$ is a smooth vector bundle on M (resp. the manifold with boundary $\overline{M} = M\sqcup M'$) and $\nabla^{\mathfrak{F}}_1$, $\nabla^{\mathfrak{F}}_2$ are two connections on $\mathfrak{F}$ which fulfill (85) (resp. (85)), then*

$$\mathscr{E}_\bullet(d^{\nabla^{\mathfrak{F}}_1},\Lambda T^*M\otimes\mathfrak{F}) = \mathscr{E}_\bullet(d^{\nabla^{\mathfrak{F}}_2},\Lambda T^*M\otimes\mathfrak{F}) \quad \bullet = loc \text{ or } comp.$$

b) *When $M' \subset M$ a smooth hypersurface of M (resp. a manifold with boundary $\overline{M} = M\sqcup M'$) with the natural embedding $j_{M'} : M' \to M$, the tangential trace map $s \mapsto j^*_{M'}s$ is well defined an continuous from $\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F})$ to $\mathscr{D}'(M';(\Lambda T^*M'\otimes \mathfrak{F}\big|_{M'})$.*
c) *The space $\mathscr{C}^\infty_0(M;\Lambda T^*M\otimes\mathfrak{F})$ (resp. $\mathscr{C}^\infty_0(\overline{M};\Lambda T^*M\otimes\mathfrak{F})$) is dense in the two spaces $\mathscr{E}_{comp}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F})$ and $\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F})$.*
d) *In the case $\overline{M} = M\sqcup M'$ and $\mathfrak{F} = \overline{M}\times\mathbb{C}$ with the trivial connection, Stokes formula*

$$\int_M d\overline{s}\wedge s' + (-1)^{\deg s}\overline{s}\wedge(ds') = \int_{M'} \overline{j^*_{M'}s}\wedge j^*_{M'}s'\,,$$

holds $s \in \mathscr{E}_{loc}(d,\Lambda T^*M\otimes\mathbb{C})$ and all $s' \in \mathscr{E}_{comp}(d,\Lambda T^*M\otimes\mathbb{C})$, where the right-hand side is the unique sesquilinear continuous extension from $\mathscr{C}^\infty_0(\overline{M};\Lambda T^*M\otimes\mathbb{C})$.

e) When $M = M_-\sqcup M'\sqcup M_+$ and $\overline{M}_\pm$ are smooth domains of M, $s\in\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F})$ iff $s_\mp = s\big|_{M_\mp}$ belongs to $\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{F}}},(\Lambda T^*M\otimes\mathfrak{F})\big|_{\overline{M}_\mp})$ and

$$j_{M'}^* s_- = j_{M'}^* s_+ \quad \text{in } \mathscr{D}'(M';\Lambda T^*M'\otimes\mathfrak{F}\big|_{M'})$$

Proof. We work with a complex vector bundle $\mathfrak{F}$. It does not change anything here.

a) The equality is due to $\nabla_1^{\mathfrak{F}}-\nabla_2^{\mathfrak{F}}\in L^\infty_{loc}(M;T^*M\otimes L(\mathfrak{F}))$ and

$$d^{\nabla_1^{\mathfrak{F}}}-d^{\nabla_2^{\mathfrak{F}}} = dx^i\wedge[\nabla^{\mathfrak{F}}_{1,\frac{\partial}{\partial x^i}}-\nabla^{\mathfrak{F}}_{2,\frac{\partial}{\partial x^i}}],$$

in local coordinates. The result is then a consequence of the equivalences for $s\in L^2_{loc}(M;\Lambda T^*M\otimes\mathfrak{F})$ (resp. $s\in L^2_{loc}(\overline{M};\Lambda T^*M\otimes\mathfrak{F})$):

$$\left(d^{\nabla_1^{\mathfrak{F}}}s\in L^2_{loc}(M;\Lambda T^*M\otimes\mathfrak{F})\right)\Leftrightarrow\left(d^{\nabla_2^{\mathfrak{F}}}s\in L^2_{loc}(M;\Lambda T^*M\otimes\mathfrak{F})\right),$$

$$\text{resp.}\quad\left(d^{\nabla_1^{\mathfrak{F}}}s\in L^2_{loc}(\overline{M};\Lambda T^*M\otimes\mathfrak{F})\right)\Leftrightarrow\left(d^{\nabla_2^{\mathfrak{F}}}s\in L^2_{loc}(\overline{M};\Lambda T^*M\otimes\mathfrak{F})\right).$$

b) For the existence of a trace, the case with a boundary $\pi_{\mathfrak{F}}:\mathfrak{F}\to\overline{M}$ is contained in the case without boundary with $M'\subset M$, by writing $\overline{M}$ as a smooth domain of the smooth manifold $\tilde{M}$ and $\mathfrak{F}=\tilde{\mathfrak{F}}\big|_{\overline{M}}$.
Because $d^{\nabla^{\mathfrak{F}}}\chi=\chi d^{\nabla^{\mathfrak{F}}}+d\chi\wedge$ for $\chi\in\mathscr{C}^\infty_0(M;\mathbb{R})$, $s\in\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F})$ is equivalent to $\chi_j s\in\mathscr{E}_{comp}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F}\big|_{U_j})$ for all j, when $\sum_j\chi_j\equiv 1$ is a locally finite partition of unity subordinate to a trivializing atlas $M=\cup_j U_j$ for $\mathfrak{F}$, $\mathfrak{F}\big|_{U_j}\simeq U_j\times\mathbb{C}^{d_f}$. With **a)**, the connection $\nabla^{\mathfrak{F}}$ can be replaced by the trivial connection on $U_j\times\mathbb{C}^{d_f}$. By possibly refining the atlas we can assume $U_j=(-\varepsilon,\varepsilon)^m$ in a local coordinate system $(x^1,\dots,x^m)$ such that $U_j\cap M'=\{0\}\times(-\varepsilon,\varepsilon)^{m-1}$ for $U_j\cap M'\neq\emptyset$. We now have to verify that

$$\chi_j s\in\mathscr{E}_{comp}(d;\Lambda T^*(-\varepsilon,\varepsilon)^m\otimes\mathbb{C}^{d_f})$$

has a trace along $\{x^1=0\}$. Finally the local L^2-estimates of $\chi_j s\in\mathscr{E}_{comp}(d;\Lambda T^*U_j\otimes\mathbb{C}^{d_f})$ can be expressed with the euclidean metric on $(-\varepsilon,\varepsilon)^m$. Working separtely on components in $\mathbb{C}^{d_f}$ reduces the problem to the scalar case.
In $U_j=(-\varepsilon,\varepsilon)^m$ with the euclidean metric, $\chi_j s=s_I dx^I\in\mathscr{E}_{comp}(d;\Lambda T^*(-\varepsilon,\varepsilon)^m\otimes\mathbb{C})$ means

$$\chi_j s = s_I(x)dx^I\in L^2_{comp}((-\varepsilon,\varepsilon)^m;\Lambda\mathbb{C}^m)\quad\text{and}\quad d\chi_j s\in L^2_{comp}(d;\Lambda\mathbb{C}^m)$$

gives

$$\frac{\partial s_{I'}}{\partial x^1}dx^1\wedge dx^{I'}=ds-\frac{\partial s_I}{\partial x^{i'}}dx^{i'}\wedge dx^I\in L^2((-\varepsilon,\varepsilon);W^{-1,2}((-\varepsilon,\varepsilon)^{m-1};\Lambda\mathbb{C}^m)).$$

Therefore every $s_{I'}$, $1\not\in I'$, belongs to $W^{1,2}((-\varepsilon,\varepsilon);W^{-1,2}((-\varepsilon,\varepsilon)^{m-1};\Lambda\mathbb{C}^{m-1})$ and admits a trace in

$$W^{-1,2}((-\varepsilon,\varepsilon)^{m-1};\Lambda\mathbb{C}^{m-1})\subset\mathscr{D}'((-\varepsilon,\varepsilon)^{m-1};\Lambda\mathbb{C}^{m-1}).$$

Hence, $j^*_{M'}(s_I dx^I) = s_{I'}dx^{I'}$ is well defined in $\mathscr{D}'((-\varepsilon,\varepsilon)^{m-1};\Lambda\mathbb{R}^{m-1}\otimes\mathbb{C})$.
By summing the locally finite different pieces of $s = \sum_j \chi_j s$ where all the $\chi_j s$ belong to $\mathscr{E}_{comp}(d^{\nabla^{\mathfrak{F}}},(\Lambda T^*M\otimes\mathfrak{F})\big|_{U_j})$, we conclude that

$$j^*_{M'}:\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F}))\to\mathscr{D}'(M';\Lambda T^*M' otimes \mathfrak{F}\big|_{M'})$$

is well defined and continuous.
c) For the density and with the local reduction to $U_j = (-\varepsilon,\varepsilon)^m$ used in **b)**, it suffices to approximate $\chi_j s\in\mathscr{E}_{comp}(d;\Lambda T^*(-\varepsilon,\varepsilon)^m\otimes\mathbb{C}^{d_f})$ by $\varphi_\eta*(\chi_j s)$ as $\eta\to 0^+$, with $\varphi_\eta(x) = \eta^{-m}\varphi_1(\eta^{-1}x)$, $\varphi_1\in\mathscr{C}^\infty_0(\mathbb{R}^m)$, $\int_{\mathbb{R}^m}\varphi_1 = 1$. From $d[\varphi_\eta*(\chi_j s)] = \varphi_\eta * d(\chi_j s)$ we deduce

$$\lim_{\eta\to 0^+}\|\chi_j s-\varphi_\eta*(\chi_j s)\|_{L^2}+\|d[\chi_j s-\varphi_\eta*(\chi_j s)]\|_{L^2} = 0,$$

while $\varphi_\eta*(\chi_j s)\in\mathscr{C}^\infty_0((-\varepsilon,\varepsilon)^m;\Lambda\mathbb{R}^m\otimes\mathbb{C}^{d_f})$. On a fixed compact set $K\subset M$, only a finite number of j's in $\sum_j\chi_j s$ have to be considered and this proves the density of $\mathscr{C}^\infty_0(M;\Lambda T^*M\otimes\mathfrak{F})$ in $\mathscr{E}_{loc\ comp}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F})$.
d) Consider the case $\overline{M} = M\sqcup M'$ and $\mathfrak{F} = \overline{M}\times\mathbb{C}$. When $s\in\mathscr{E}_{comp}(d,\Lambda T^*M\otimes\mathfrak{F})$ and $s'\in\mathscr{E}_{loc}(d,\Lambda T^*M\otimes\mathfrak{F})$ there exists $\chi\in\mathscr{C}^\infty_0(\overline{M};\mathbb{R})$ such that $s\wedge s' = \chi s\wedge\chi s'$. We thus assume $s,s'\in\mathscr{E}_{comp}(d;\Lambda T^*M\otimes\mathfrak{F})$.
But the sesquilinear map

$$(s,s')\in\mathscr{E}_{comp}(d,\Lambda T^*M\otimes\mathfrak{F})\to\underbrace{d\overline{s}\wedge s'+(-1)^{\deg s}\overline{s}\wedge ds'}_{=d(\overline{s}\wedge s')}\in L^1_{comp}(M;\Lambda T^*M\otimes\mathfrak{F})$$

is continuous. By **c)**, for any $s,s'\in\mathscr{E}_{comp}(d,\Lambda T^*M\otimes\mathfrak{F})$ there exists two sequences $(\omega_n)_{n\in\mathbb{N}}$ and $(\theta_n)_{n\in\mathbb{N}}$ in $\mathscr{C}^\infty_0(\overline{M};\Lambda T^*M\otimes\mathfrak{F})$ which converge respectively to s and s' in $\mathscr{E}_{comp}(d,\Lambda T^*M\otimes\mathfrak{F})$. For any such sequence $\overline{\omega}_n\wedge\theta_n\in\mathscr{C}^\infty_0(\overline{M};\Lambda T^*M\otimes\mathfrak{F})$ and Stokes formula says

$$\int_M d[\overline{\omega}_n\wedge\theta_n] = \int_{M'} j^*_{M'}(\overline{\omega}_n\wedge\theta_n) = \int_{M'}(j^*_{M'}\overline{\omega}_n)\wedge(j^*_{M'}\theta_n).$$

The left-hand side converges to $\int_M d\overline{s}\wedge s'+(-1)^{\deg s}\overline{s}\wedge ds'$ which is a continuous sesquilinear form on $\mathscr{E}_{comp}(d,\Lambda T^*M\otimes\mathfrak{F})$ and this ends the proof of the extended Stokes formula.
e) One implication is trivial by restriction to $\overline{M}_\mp$.
So assume $s_\mp\in\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{F}}},(\Lambda T^*M\otimes\mathfrak{F})\big|_{\overline{M}_\mp})$ and $j^*_{M'}s_- = j^*_{M'}s_+$ in $\mathscr{D}'(M';(\Lambda T^*M\otimes\mathfrak{F})\big|_{M'})$. With a locally finite partition of unity $\sum_j\chi_j\equiv 1$ in M, we want to prove $\chi_j s\in\mathscr{E}_{comp}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*U_j\otimes\mathfrak{F}\big|_{U_j})$ for all j. By following the scheme of **b)** it suffices to consider $U_j = (-\varepsilon,\varepsilon)^m$ and $\mathfrak{F}\big|_{U_j} = U_j\times\mathbb{C}$ endowed with the trivial connection ∇ and $d^\nabla = d$. The Stokes formula of **d)** is applied with $\chi_j s\big|_{\overline{M}_\mp}$ and $s'\in\mathscr{C}^\infty_0((-\varepsilon,\varepsilon)^m;\Lambda\mathbb{C}^m)$:

$$\left.\begin{array}{l}\int_{(-\varepsilon,0]\times(-\varepsilon,\varepsilon)^{m-1}} d\overline{\chi_j s}\wedge s'+(-1)^{\deg s}\overline{\chi_j s}\wedge(ds')+\\ \int_{[0,\varepsilon)\times(-\varepsilon,\varepsilon)^{m-1}} d\overline{\chi_j s}\wedge s'+(-1)^{\deg s}\overline{\chi_j s}\wedge(ds')\end{array}\right\} = \int_{(-\varepsilon,\varepsilon)^{m-1}} j^*_{M'}\overline{[\chi_j s_+-\chi_j s_-]}\wedge j^*_{M'}s'$$

where the right-hand is 0 for all $s'\in\mathscr{C}_0^\infty((-\varepsilon,\varepsilon)^m;\Lambda\mathbb{C}^m)$. This implies the existence of a constant C_s such that

$$\forall s'\in\mathscr{C}_0^\infty((-\varepsilon,\varepsilon)^m;\Lambda\mathbb{C}^m),\left|\int_{(-\varepsilon,\varepsilon)^m}\overline{\chi_j s}\wedge ds'\right|\leq C_s\|s'\|_{L^2}.$$

But the linear form

$$s'\in\mathscr{C}_0^\infty((-\varepsilon,\varepsilon)^m;\Lambda\mathbb{C}^m)\mapsto -(-1)^{\deg\, s}\int_{(-\varepsilon,\varepsilon)^m}\overline{\chi_j s}\wedge ds'$$

is the definition of $d(\chi_j s)$ as a current, i.e. an element of $\mathscr{D}'((-\varepsilon,\varepsilon)^m,\Lambda\mathbb{C}^m)$, which therefore belongs to $L^2_{comp}((-\varepsilon,\varepsilon)^m;\Lambda T^*(-\varepsilon,\varepsilon)^m\otimes\mathbb{C})$.
Doing this for all components in $\mathbb{C}^{d_f}$, proves $\chi_j s\in\mathscr{E}_{comp}(d^{\nabla^{\mathfrak{F}}},(\Lambda T^*M\otimes\mathfrak{F})\big|_{U_j})$ and therefore $s\in\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{F}}},\Lambda T^*M\otimes\mathfrak{F})$. □

Remark 4.2. *Locally with coordinates such that $M'=\left\{(x^1,x')\in M, x^1=0\right\}$ the partial trace $j^*_{M'}s$ can be replaced by $\mathbf{i}_{\frac{\partial}{\partial x^1}}dx^1\wedge s\big|_{M'}=s_{I'}dx^{I'}\big|_{M'}$ when $s=s_I dx^I$.*

Remark 4.3. *Although the differential d defines an elliptic complex (see [ChPi]), the operator d is not elliptic. In particular the partial trace defined in $\mathscr{D}'(M';\Lambda T^*M)$ does not have neither the $W^{1/2,2}_{loc}$ nor the L^2_{loc} regularity associated with order 1 elliptic differential operators, as shows the example $r^{-\alpha}dr=d\frac{r^{1-\alpha}}{1-\alpha}$, $\alpha\in]1/2,1[$, $M'=\mathbb{R}\times\{0\}$, in $\mathbb{R}^2$ with polar coordinates (r,θ).*

For Proposition 4.1 we used the (local) duality between $\mathscr{C}_0^\infty(M;\Lambda^pT^*M)$ and $\mathscr{D}'(M;\Lambda^{\dim M-p}T^*M)$ and made integration by parts via Stokes theorem. We may instead use the duality between $\mathscr{C}_0^\infty(M;\Lambda T^*M)$ and $\mathscr{D}'(M;\Lambda TM)$ given by the natural duality between ΛTM and ΛT^*M. It is not necessary to assume M oriented here but let us keep this assumption which is fulfilled in our applications. We put a volume element dv_M and by assuming that the hypersurface M' admits a global defining function $x^1\in\mathscr{C}^\infty(M;\mathbb{R})$, $M'=(x^1)^{-1}(\{0\})$, $dx^1\big|_{M'}\neq 0$, this defines a volume element $dv_{M'}$ on M' by writing $dv_M(x)=|dx^1|dv_{M'}(x')$ with local coordinates $x=(x^1,x')$. Let $\mathfrak{F}'$ be the anti-dual $\mathscr{C}^\infty$ vector bundle and let $\nabla^{\mathfrak{F}'}$ be the anti-dual connection of $\nabla^{\mathfrak{F}}$ characterized by

$$\frac{\partial}{\partial x^i}(t.s)=(\nabla^{\mathfrak{F}'}_{\frac{\partial}{\partial x^i}}t).s+t.(\nabla^{\mathfrak{F}}_{\frac{\partial}{\partial x^i}}s)\quad,\quad t\in\mathscr{C}^\infty(M;\mathfrak{F}'),\, s\in\mathscr{C}^\infty(M,\mathfrak{F}),$$

where $t.s(x)$ stands for the natural $\mathfrak{F}'_x-\mathfrak{F}_x$ duality. It satisfies

$$\nabla^{\mathfrak{F}'}:\mathscr{C}^\infty(M;\mathfrak{F}')\to L^\infty_{loc}(M;T^*M\otimes\mathfrak{F}') \tag{87}$$

$$\text{resp.}\quad \nabla^{\mathfrak{F}'}:\mathscr{C}^\infty(\overline{M};\mathfrak{F}')\to L^\infty_{loc}(\overline{M};T^*M\otimes\mathfrak{F}'). \tag{88}$$

The interior covariant derivative $\tilde{d}^{\nabla^{\mathfrak{F}'}}$ acting on sections of $\Lambda TM\otimes\mathfrak{F}'$ is written in local coordinates

$$\tilde{d}^{\nabla^{\mathfrak{F}'}}=-\mathbf{i}_{dx^i}\frac{\partial}{\partial x^i}\otimes\mathrm{Id}_{\mathfrak{F}'}-\mathbf{i}_{dx^i}\otimes\nabla^{\mathfrak{F}'}_{\frac{\partial}{\partial x^i}}.$$

and when $dv_M(x)=\lambda(x)|dx|$

$$\tilde{d}^{\nabla^{\mathfrak{F}'},v_M}=-\mathbf{i}_{dx^i}\frac{\partial}{\partial x^i}\otimes \mathrm{Id}_{\mathfrak{F}'}-\mathbf{i}_{dx^i}\otimes\nabla^{\mathfrak{F}'}_{\frac{\partial}{\partial x^i}}-\mathbf{i}_{dx^i}\frac{\partial\lambda}{\partial x^i}\lambda^{-1}\otimes \mathrm{Id}_{\mathfrak{F}'}$$

The operator $\tilde{d}^{\nabla^{\mathfrak{F}'},v_M}:\mathcal{D}'(M;\Lambda TM\otimes\mathfrak{F}')\to\mathcal{D}'(M;\Lambda TM\otimes\mathfrak{F}')$ is characterized by

$$\forall s\in\mathscr{C}^\infty_0(M;\Lambda T^*M\otimes\mathfrak{F}),\quad \int_M(\tilde{d}^{\nabla^{\mathfrak{F}'},v_M}t).s\;dv_M=\int_M t.(d^{\nabla^{\mathfrak{F}}}s)\;dv_M\,.$$

The following result will be used in Section 5.

Proposition 4.4. ***a)*** *If $\mathfrak{F}$ is a smooth vector bundle on M (resp. the manifold with boundary $\overline{M}=M\sqcup M'$) and $\nabla^{\mathfrak{F}}_1$, $\nabla^{\mathfrak{F}}_2$ are two connections on $\mathfrak{F}$ which fulfill (85) (resp. (85) with antidual versions $\mathfrak{F}'$, $\nabla^{\mathfrak{F}'_1}$ and $\nabla^{\mathfrak{F}'_2}$ and if $dv_{M,1}$ and $dv_{M,2}$ are two Lipschitz continuous volume elements, then*

$$\mathcal{E}_\bullet(d^{\nabla^{\mathfrak{F}'}_1,v_{M,1}},\Lambda TM\otimes\mathfrak{F}')=\mathcal{E}_\bullet(d^{\nabla^{\mathfrak{F}'}_2,v_{M,2}},\Lambda TM\otimes\mathfrak{F}')\quad \bullet=loc\text{ or }comp.$$

b) *When $M'\subset M$ a smooth hypersurface of M (resp. a manifold with boundary $\overline{M}=M\sqcup M'$) with a global defining function x^1, the partial trace map $t\mapsto \mathbf{i}_{dx^1}t\big|_{M'}$ is well defined an continuous from $\mathcal{E}_{loc}(d^{\nabla^{\mathfrak{F}'},v_M},\Lambda TM\otimes\mathfrak{F}')$ to $\mathcal{D}'(M';\Lambda TM'\otimes\mathfrak{F}'\big|_{M'})$.*
c) *The space $\mathscr{C}^\infty_0(M;\Lambda TM\otimes\mathfrak{F}')$ (resp. $\mathscr{C}^\infty_0(\overline{M};\Lambda TM\otimes\mathfrak{F}')$) is dense in the two spaces $\mathcal{E}_{comp}(d^{\nabla^{\mathfrak{F}'},v_M},\Lambda TM\otimes\mathfrak{F}')$ and $\mathcal{E}_{loc}(d^{\nabla^{\mathfrak{F}'},v_M},\Lambda TM\otimes\mathfrak{F}')$.*
d) *In the case $\overline{M}=M\sqcup M'$ and $x^1<0$ in M, the integration by parts*

$$\int_M t.(d^{\nabla^{\mathfrak{F}}}s)\;dv_M-\int_M(\tilde{d}^{\nabla^{\mathfrak{F}'},v_M}t).s\;dv_M=\int_{M'}(\mathbf{i}_{dx^1}t).s\;dv_{M'}$$

holds for all $t\in\mathcal{E}_{loc}(d,\Lambda TM\otimes\mathfrak{F}')$ and all $\in\mathcal{E}_{comp}(d,\Lambda T^*M\otimes\mathfrak{F})$, where the right-hand side is the unique sesquilinear continuous extension from $\mathscr{C}^\infty_0(\overline{M};\Lambda TM\otimes\mathfrak{F}')\times\mathscr{C}^\infty_0(\overline{M};\Lambda T^*M\otimes\mathfrak{F})$.
e) *When $M=M_-\sqcup M'\sqcup M_+$ and $\overline{M}_\pm$ are smooth domains of M, $t\in\mathcal{E}_{loc}(d^{\nabla^{\mathfrak{F}'},v_M},\Lambda TM\otimes\mathfrak{F}')$ iff $t_\mp=t\big|_{M_\mp}$ belongs to $\mathcal{E}_{loc}(d^{\nabla^{\mathfrak{F}'},v_M},(\Lambda TM\otimes\mathfrak{F}')\big|_{\overline{M}_\mp})$ and*

$$\mathbf{i}_{dx^1}t_-\big|_{M'}=\mathbf{i}_{dx^1}t_+\quad in\ \mathcal{D}'(M';\Lambda TM'\otimes\mathfrak{F}'\big|_{M'}).$$

Proof. The proofs of a)b)c) are essentially the same as for a)b)c) in Proposition 4.1. The statement e) is a consequence of d). The proof of d) simply relies on the following computation for $t\in\mathscr{C}^\infty_0(\overline{M};\Lambda TM\otimes\mathfrak{F}')$ and $s\in\mathscr{C}^\infty_0(\overline{M};\Lambda T^*M\otimes\mathfrak{F})$ supported in a chart open domain, $x^{i'}\in(-\varepsilon,\varepsilon)$, $x^1\in(-\varepsilon,0]$:

$$\begin{aligned}\int_M(t.(d^{\nabla^{\mathfrak{F}}}s)-(\tilde{d}^{\nabla^{\mathfrak{F}'}}t).s)\;\lambda(x)|dx| &= \int_{(-\varepsilon,0]\times(-\varepsilon,\varepsilon)^{d-1}}\frac{\partial}{\partial x^i}[((\mathbf{i}_{dx^i}t).s)\times\lambda]\;|dx|\\ &= \int_{(-\varepsilon,\varepsilon)^{d-1}}(\mathbf{i}_{dx^1}t).s\;\lambda(0,x')|dx'|=\int_{M'}(\mathbf{i}_{dx^1}t).s\;dv_{M'}\,.\end{aligned}$$

The sign of the final right-hand side is changed if we assume $x^1>0$ in M. □

4.2 The differential structure of $\hat{E}_g$ and $\hat{F}_g$

The vector bundle $\hat{E}_g$ was introduced in Definition 2.7 as a piecewise $\mathscr{C}^\infty$ and continuous vector bundle above $X = T^*Q$ where $\hat{E}_g\big|_{\overline{X}_\mp} = \Lambda T^*X\big|_{\overline{X}_\mp} \overset{g_\mp}{=} \pi_X^*(\Lambda T^*Q \otimes \Lambda TQ\big|_{\overline{Q}_\mp})$ with the matching condition

$$e_-^i\big|_{\partial X_-} = e_+^i\big|_{\partial X_+} \quad , \quad \hat{e}_{-,j}\big|_{\partial X_-} = \hat{e}_{+,j}\big|_{\partial X_+} \quad \partial X_- = \partial X_+ = X' .$$

We used the frame $(e_\mp, \hat{e}_\mp)$ of Definition 2.10.
It can be given another interpretation. In the manifold $\overline{X}_- = X_- \cup X'$, the coordinates $(\tilde{q}, \tilde{p}) = (\tilde{q}^1, \tilde{x}')$ of Definition 2.6 identify $X_{(-\varepsilon,0]}$ as the tubular neighborhood $(-\varepsilon, 0] \times X'$. Meanwhile $S_1 : X' \to X'$ of Definition 3.5,

$$S_1(0, q', p_1, p') = (0, q', -p_1, p') \quad (0, q', p_1, p') = (0, \tilde{q}', \tilde{p}_1, \tilde{p}'),$$

is a diffeomorphism of X'. By following Milnor in [Mil]-Theorem 1.4 there is a $\mathscr{C}^\infty$-manifold, unique modulo diffeomorphism,

$$M_g = X_- \cup X' \cup X_+ \quad \text{such that} \tag{89}$$
$$\big((0^-, \tilde{q}', \tilde{p}_1, \tilde{p}') = (0^+, \tilde{Q}', \tilde{P}_1, \tilde{P}')\big) \Leftrightarrow ((\tilde{Q}', \tilde{P}_1, \tilde{P}') = S_1(\tilde{q}', \tilde{p}_1, \tilde{p}') = (\tilde{q}', -\tilde{p}_1, \tilde{p}')). \tag{90}$$

The subscripted notation M_g keeps track of the fact that the construction of the coordinates $(\tilde{q}, \tilde{p})$ actually depend on the chosen metric $g_- = g = g^{TQ}$ on $\overline{Q}_-$, symmetrized as $\hat{g}^{TQ}$.
We recall the Definition 2.2 of the double copy $\pi_{\mathfrak{f}} : \mathfrak{f} \to Q$, when $Q = Q_- \sqcup Q' \sqcup Q_+$, given with $(0^+, \underline{q}', vv) = (0^-, \underline{q}', v)$ and $\pi_{M_g}^*(\mathfrak{f})$ is a flat $\mathscr{C}^\infty$ vector bundle on the $\mathscr{C}^\infty$ manifold M_g. The exterior covariant derivative denoted by $d_{M_g}^{\nabla^{\mathfrak{f}}}$ satisfies:

- $d_{M_g}^{\nabla^{\mathfrak{f}}} : \mathscr{F}(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})) \to \mathscr{F}(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$ for $\mathscr{F} = \mathscr{C}_0^\infty$ and for $\mathscr{F} = \mathscr{D}'$;
- $d_{M_g}^{\nabla^{\mathfrak{f}}} \circ d_{M_g}^{\nabla^{\mathfrak{f}}} = 0$ in $\mathscr{C}_0^\infty(M_g; \Lambda T^*M_g \otimes \pi_X^*(\mathfrak{f}))$ and in $\mathscr{D}'(M_g; \Lambda T^*M_g \otimes \pi_X^*(\mathfrak{f}))$;
- in particular $d_{M_g}^{\nabla^{\mathfrak{f}}} : \mathscr{E}_\bullet(d_{M_g}^{\nabla^{\mathfrak{f}}}, \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})) \to \mathscr{E}_\bullet(d_{M_g}^{\nabla^{\mathfrak{f}}}, \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$ respectively for $\bullet = loc$ or $comp$;
- the trace map $s \mapsto j_{X'}s$ or $s \mapsto \mathbf{i}_{e_1} e^1 \wedge s\big|_{X'}$ after Remark 4.2, where we recall $e_1 = \frac{\partial}{\partial \tilde{q}^1}$ and $e^1 = d\tilde{q}^1$, is well defined and continuous from $\mathscr{E}_{loc}(d_{M_g}^{\nabla^{\mathfrak{f}}}, \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$ to $\mathscr{D}'(X'; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})\big|_{X'})$;
- $\mathscr{C}_0^\infty(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$ is dense in $\mathscr{E}_{loc\ comp}(d_{M_g}^{\nabla^{\mathfrak{f}}}, \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$;
- if $\Sigma_{M_g} : M_g \to M_g$ is the natural symmetry specified locally by $\Sigma_{M_g}(\tilde{q}^1, \tilde{q}', \tilde{p}) = (-\tilde{q}_1, \tilde{q}', \tilde{p})$ with its push-forward $\Sigma_{M_g,*}$ acting on $\mathscr{D}'(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$, then $d^{\nabla^{\mathfrak{f}}} \Sigma_{M_g,*} = \Sigma_{M_g,*} d^{\nabla^{\mathfrak{f}}}$ and $d^{\nabla^{\mathfrak{f}}}$ preserves the parity with respect to $\Sigma_{M_g,*}$.

For the smooth manifold M_g it is convenient to introduce the following notations which have already introduced counterparts on $\hat{E}_g$ and $\hat{F}_g$.

Definition 4.5. *Let M_g be the manifold defined by (89)(90). The manifold $M_{g,(-\varepsilon,\varepsilon)}$ is the open domain characterized by $|\tilde{q}^1| < \varepsilon$.*
On $M_{g,(-\varepsilon,\varepsilon)}$ the symmetry Σ_{M_g} is simply given by $\Sigma_{M_g}(\tilde{q}^1,\tilde{x}') = (-\tilde{q}^1,\tilde{x}')$.
The set $L^2_{loc,ev}(M_{g,(-\varepsilon,\varepsilon)};\Lambda T^*M\otimes\pi^*_{M_g}(\mathfrak{f}))$ is defined by

$$L^2_{loc,ev}(M_{g,(-\varepsilon,\varepsilon)};\Lambda T^*M\otimes\pi^*_{M_g}(\mathfrak{f})) = \left\{s\in L^2_{loc}(M_{g,(-\varepsilon,\varepsilon)};\Lambda T^*M\otimes\pi^*_{M_g}(\mathfrak{f})),\quad \Sigma_{M_g,*}s = \nu s\right\},$$

and $L^2_{loc,odd}$ has the same definition with the condition $\Sigma_{M_g,*}s = -\nu s$.
For $\mathfrak{F} = \Lambda T^*M_g\otimes\pi^*_{M_g}(\mathfrak{f})\big|_{M_{g,(-\varepsilon,\varepsilon)}}$, the set $\mathscr{C}_0(\mathfrak{F})$ equals

$$\mathscr{C}^\infty_0(M_{g,(-\varepsilon,0]};\mathfrak{F})\cap\mathscr{C}^\infty_0(M_{g,[0,\varepsilon)};\mathfrak{F})\cap\mathscr{C}^0(M_{g,(-\varepsilon,\varepsilon)};\mathfrak{F})\quad with\quad \mathfrak{F} = \Lambda T^*M_g\otimes\pi^*_{M_g}(\mathfrak{f}),$$

and $\mathscr{C}_{0,ev\ odd}(\mathfrak{F}) = \mathscr{C}_0(\mathfrak{F})\cap L^2_{loc,ev\ odd}(M_{g,(-\varepsilon,\varepsilon)};\mathfrak{F})$.

Lemma 4.6. *Let $M_{g,(-\varepsilon,\varepsilon)}$ be the neighboorhood of X' given by $|\tilde{q}^1| < \varepsilon$ and let the map $\tilde{S}_1 : M_{g,(-\varepsilon,\varepsilon)} \to X_{(-\varepsilon,\varepsilon)}$ given by*

$$\tilde{S}_1(\tilde{q}^1,\tilde{q}',\tilde{p}_1,\tilde{p}') = \begin{cases} (\tilde{q}^1,\tilde{q}',\tilde{p}_1,\tilde{p}') & if\ \tilde{q}^1\le 0,\\ (\tilde{q}^1,\tilde{q}',-\tilde{p}_1,\tilde{p}') & if\ \tilde{q}^1 > 0.\end{cases}$$

When $\Sigma_{M_g}(\tilde{q}^1,\tilde{x}') = (-\tilde{q}^1,x')$ on $M_{g,(-\varepsilon,\varepsilon)}$ we get $\tilde{S}_1\circ\Sigma_{M_g} = \Sigma\circ\tilde{S}_1 : M_{g,(-\varepsilon,\varepsilon)}\to X_{(-\varepsilon,\varepsilon)}$. Moreover $\tilde{S}_{1,*} : \Lambda T^*M_g\otimes\pi^*_{M_g}(\mathfrak{f})\big|_{M_{g,(-\varepsilon,\varepsilon)}} \to \hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}$ is a piecewise $\mathscr{C}^\infty$ and continuous vector bundle isomorphism such that $\tilde{S}_{1,*}$ sends $L^2_{loc}(M_{g,(-\varepsilon,\varepsilon)};\Lambda T^*M\otimes\pi^*_{M_g}(\mathfrak{f}))$ into $L^2_{loc}(X_{(-\varepsilon,\varepsilon)};\hat{F}_g)$, $\mathscr{C}_0(\Lambda T^*M_g\otimes\pi^*_{M_g}(\mathfrak{f})\big|_{M_{g,(-\varepsilon,\varepsilon)}})$ into $\mathscr{C}_0(\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$ of Definition 3.3 and transforms the parity with respect to $\Sigma_{M_g,*}\otimes\nu$ into the parity with respect to Σ_ν.

Proof. We can forget the vector bundle $\pi^*_{M_g}(\mathfrak{f})$. We focus on $\Lambda T^*M_g\big|_{M_{g,(-\varepsilon,\varepsilon)}}$ and $\hat{E}_g\big|_{X_{(-\varepsilon,\varepsilon)}}$. The equality $\tilde{S}_1\circ\Sigma_{M_g} = \Sigma\circ\tilde{S}_1$ is obvious. Meanwhile the push-forward $\tilde{S}_{1,*}$, $\Sigma_{M_g,*}$ and Σ_* are $\mathscr{C}^\infty$ vector bundle isomorphisms when $\tilde{q}^1$ is restricted to $(-\varepsilon,0]$ or $[0,+\infty)$. It thus suffices to check that $\tilde{S}_{1,*} : T^*M_g\big|_{M_{g,(-\varepsilon,\varepsilon)}} \to T^*X_{(-\varepsilon,\varepsilon)}$ is continuous along X'. Restricted to $M_{g,(-\varepsilon,0]} = X_{(-\varepsilon,0]}$, $\tilde{S}_{1,*}$ is the identity and a smooth local frame of $T^*M_{g,(-\varepsilon,0]}$ is given by $(e^i_-,\hat{e}_{j,-})$. On the $\mathscr{C}^\infty$ manifold $M_{g,(-\varepsilon,\varepsilon)}$, it is the restriction of a smooth frame $(\tilde{e}^i,\tilde{e}_j)$ such that $\tilde{e}^i,\tilde{e}_j)\big|_{\tilde{q}^1=0^+} = (e^i_-,\hat{e}_{j,-})\big|_{\tilde{q}^1=0^-}$. By using

$$\tilde{S}_{1,*}\big|_{\tilde{q}^1\le 0} = \mathrm{Id}\quad,\quad \Sigma_*\tilde{S}_{1,*}\Sigma_{M_g,*} = \tilde{S}_{1,*}$$

we deduce from (65)(67)(69)

$$\begin{aligned}\tilde{S}_{1,*}(\tilde{e}^i,\tilde{e}_j)\big|_{\tilde{q}^1=0^-} &= (e^i_-,\hat{e}_{j,-})\big|_{\tilde{q}^1=0^-}\\ \tilde{S}_{1,*}(\tilde{e}^i,\tilde{e}_j)\big|_{\tilde{q}^1=0^+} &= (e^i_+,\hat{e}_{j,+})\big|_{\tilde{q}^1=0^+},\end{aligned}$$

and the continuous local frame of $(\tilde{e}^i,\tilde{e}_j)$ of $T^*M_g\big|_{M_{g,(-\varepsilon,\varepsilon)}}$ is sent to the continuous local frame $(e^i,\hat{e}_j) = 1_{\mathbb{R}_\mp}(\tilde{q}^1)(e^i_\mp,\hat{e}_{j,\mp})$ of $\hat{E}_g\big|_{X_{(-\varepsilon,\varepsilon)}}$.
The parity properties then come from $\Sigma_*\circ\tilde{S}_{1,*} = \tilde{S}_{1,*}\circ\Sigma_{M_g,*}$. $\square$

The trace properties of Proposition 4.1 have been used with the closed manifold $M_g \supset X'$ to review the properties of the exterior covariant derivative $d^{\nabla^{\mathfrak{f}}}_{M_g}$. A separate use in the manifolds with boundaries $\overline{X}_\mp = X_\mp \sqcup X'$ provides a definition of the differential acting on sections of $\hat{E}_g$ and $\hat{F}_g$.

Definition 4.7. *Let $E = \Lambda T^*X$ and $F = E \otimes \pi_X^*(\mathfrak{f})$ and let $d^{\nabla^{\mathfrak{f}}}$ be the covariant exterior derivative for the flat vector bundle $(\pi_X^*(\mathfrak{f}), \pi_X^*(\nabla^{\mathfrak{f}}))$. With the notation $(e,\hat{e}) = 1_{\mathbb{R}_\mp}(q^1)(e_\mp, \hat{e}_\mp)$ of Definition 2.10, a section $\omega = \omega_I^J e^I \hat{e}_J \in L^2_{loc}(X;\hat{F}_g)$ belongs to $\mathcal{E}_{loc}(\hat{d}_g, \hat{F}_g)$ (resp. $\mathcal{E}_{comp}(\hat{d}_g, \hat{F}_g)$) if its restrictions $\omega_\mp = \omega\big|_{X_\mp} = \omega_I^J e_\mp^I \hat{e}_{\mp,J}$ belong to $\mathcal{E}_{loc}(d^{\nabla^{\mathfrak{f}}}, F\big|_{\overline{X}_\mp})$ (resp. $\mathcal{E}_{comp}(d^{\nabla^{\mathfrak{f}}}, F\big|_{\overline{X}_\mp})$) with*

$$\mathbf{i}_{e_{+,1}} e_+^1 \wedge \omega_+\big|_{\partial X_+} = \mathbf{i}_{e_{-,1}} e_{-,1} \wedge \omega_-\big|_{\partial X_-} \quad in\ \mathcal{D}'(X'; \hat{F}_g\big|_{X'}), \tag{91}$$

or

$$\omega_{I'}^J(0^+,.) = \nu \omega_{I'}^J(0^-,.) \quad in\ \ \mathcal{D}'(X', \pi_{X'}^*(\mathfrak{f}\big|_{Q'}))$$

for all $I', J \subset \{1,\ldots,d\}$, $1 \not\in I'$. The differential $\hat{d}_g$ with domain $\mathcal{E}_{loc}(\hat{d}_g; \hat{F}_g)$ is then defined by $\hat{d}\omega\big|_{X_\mp} = d^{\nabla^{\mathfrak{f}}}\omega_\mp$.

The properties of $\hat{d}_g$ and $\mathcal{E}_{loc}(\hat{d}_g, \hat{F}_g)$ are deduced from the one of $d^{\nabla^{\mathfrak{f}}}_{M_g}$ and $\mathcal{E}_{loc}(d^{\nabla^{\mathfrak{f}}}_{M_g}, \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$.

Proposition 4.8. *The differential $\hat{d}_g$ defined on $\mathcal{E}_{loc}(\hat{d}_g, \hat{F}_g)$ satisfies $\hat{d}_g \mathcal{E}_{loc}(\hat{d}_g, \hat{F}_g) \subset \mathcal{E}_{loc}(\hat{d}_g, \hat{F}_g)$ and $\hat{d}_g \circ \hat{d}_g = 0$.*
The map $\Sigma_\nu : L^2_{loc}(X;\hat{F}_g) \to L^2_{loc}(X;\hat{F}_g)$ preserves $\mathcal{E}_{loc}(\hat{d}_g, \hat{F}_g)$ and $\hat{d}_g \Sigma_\nu = \Sigma_\nu \hat{d}_g$ so that $\hat{d}_g$ preserves the parity with respect to Σ_ν.
The space $\mathcal{C}_{0,g}(\hat{F}_g)$ of Definition 3.3 is densely and continuously embedded in $\mathcal{E}_{loc}(\hat{d}_g, \hat{F}_g)$ and $\mathcal{E}_{comp}(\hat{d}_g, \hat{F}_g)$.
Moreover the exists a dense set $\hat{\mathcal{D}}_{g,\nabla^{\mathfrak{f}}}$ of $\mathcal{C}_{0,g}(\hat{F}_g)$, such that $\hat{d}_g \mathcal{D}_{g,\nabla^{\mathfrak{f}}} \subset \mathcal{C}_{0,g}(\hat{F}_g)$.

Proof. With Lemma 4.6 and with $j_{X'}^* s$ written according to Remark 4.2 as $\mathbf{i}_{e_1} e^1 \wedge s\big|_{X'} = \mathbf{i}_{\frac{\partial}{\partial \tilde{q}^1}} d\tilde{q}^1 \wedge s\big|_{X'}$, the map $\tilde{S}_{1,*}$ sends $L^2_{loc}(M_g; \Lambda T^*M \otimes \pi_{M_g}^*(\mathfrak{f}))$ to $L^2_{loc}(X;\hat{F}_g)$ with $d^{\nabla^{\mathfrak{f}}} \tilde{S}_{1,*}\big|_{M_g \setminus X'} = \tilde{S}_{1,*} d^{\nabla^{\mathfrak{f}}}_{M_g}\big|_{M_g \setminus X'}$ while the trace condition $j_{X'} s\big|_{\partial X_-} = j_{X'}^* s\big|_{\partial X_+}$ in $\mathcal{D}'(X'; \Lambda T^*X' \otimes \pi_X^*(\mathfrak{f})\big|_{X'})$ is transformed into (91).
This proves $\tilde{S}_{1,*}\mathcal{E}_{loc}(d^{\nabla^{\mathfrak{f}}}_{M_g}, \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})) = \mathcal{E}_{loc}(d, \hat{F}_g)$ and $\hat{d}_g = \tilde{S}_{1,*} d^{\nabla^{\mathfrak{f}}}_{M_g} \tilde{S}_1^*$.
The property $\hat{d}_g \circ \hat{d}_g = 0$ is thus the consequence of $d^{\nabla^{\mathfrak{f}}}_{M_g} \circ d^{\nabla^{\mathfrak{f}}}_{M_g}$ on the $\mathcal{C}^\infty$ manifold M_g endowed with the flat exterior covariant derivative $d^{\nabla^{\mathfrak{f}}}_{M_g}$.
The dense an continuous embeddings

$$\mathcal{C}_0^\infty(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})) \subset \mathcal{C}_0(\Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})) \subset \mathcal{E}_{loc\ comp}(d^{\nabla^{\mathfrak{f}}}_{M_g}, \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$$

with

$$d^{\nabla^{\mathfrak{f}}}_{M_g} \mathcal{C}_0^\infty(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})) \subset \mathcal{C}_0^\infty(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})) \subset \mathcal{C}_0(\Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})),$$

combined with

$$\tilde{S}_{1,*}\mathcal{C}_0(\Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})) = \mathcal{C}_{0,g}(\hat{F}_g),$$

provides the density results by taking $\hat{\mathcal{D}}_{g,\nabla^{\mathfrak{f}}} = \tilde{S}_{1,*}\mathcal{C}_0^\infty(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$. □

4.3 Boundary conditions for $d_{b'\mathfrak{h}}$ and properties of the associated closed operator

Remember

$$s_{ev}(x) = 1_{X_-}(x)s(x) + 1_{X_+}(x)\Sigma_\nu s(x) \quad \text{when } s \in L^2_{loc}(X_-;F),$$

and

$$\hat{\mathfrak{h}} = \frac{|p|_q^2}{2} = \frac{p_1^2 + m^{i'j'}(-|q^1|,q')p_{i'}p_{j'}}{2} = \frac{\tilde{p}_1^2 + m^{i'j'}(0,\tilde{q}')\tilde{p}_{i'}\tilde{p}_{j'}}{2}.$$

Definition 4.9. *When $\hat{d}_g$ denotes the differential on $\mathscr{E}_{loc}(\hat{d}_g,\hat{F}_g)$ introduced in Definition 4.7 and $b' \geq 0$, the differential $\hat{d}_{g,b'\mathfrak{h}}$ equals*

$$\hat{d}_{g,b'\mathfrak{h}} = e^{-b'\hat{\mathfrak{h}}}\hat{d}_g e^{b'\hat{\mathfrak{h}}} = \hat{d}_g + b'd\hat{\mathfrak{h}} \wedge .$$

We keep the same notation for the operator in $L^2(X;F)$ defined by

$$D(\hat{d}_{g,b'\mathfrak{h}}) = \left\{s \in L^2(X;F) \cap \mathscr{E}_{loc}(\hat{d}_g,\hat{F}_g),\, \hat{d}_{g,b'\mathfrak{h}}s \in L^2(X;F)\right\}$$
$$\forall s \in D(\hat{d}_g), \quad \hat{d}_{g,b'\mathfrak{h}}s = (d^{\nabla^{\mathfrak{f}}} + b'd\hat{\mathfrak{h}}\wedge)(s\big|_{X_-}) + (d^{\nabla^{\mathfrak{f}}} + b'd\hat{\mathfrak{h}}\wedge)(s\big|_{X_+}).$$

The operator $\overline{d}_{g,b'\mathfrak{h}}$ on $\overline{X}_-$ is given by

$$D(\overline{d}_{g,b'\mathfrak{h}}) = \left\{s \in L^2(X_-,F), \quad s_{ev} \in D(\hat{d}_{g,b'\mathfrak{h}})\right\},$$
$$\forall s \in D(\overline{d}_{g,b'\mathfrak{h}}), \quad \overline{d}_{g,b'\mathfrak{h}}s = d^{\nabla^{\mathfrak{f}}}s + b'd\mathfrak{h} \wedge s.$$

Proposition 4.10. *The operator $\hat{d}_{g,b'\mathfrak{h}}$ given in $L^2(X;F)$ by Definition 4.9 is closed, satisfies $\hat{d}_{g,b'\mathfrak{h}} \circ \hat{d}_{g,b'\mathfrak{h}} = 0$, $\hat{d}_{g,b'\mathfrak{h}} \circ \Sigma_\nu = \Sigma_\nu \circ \hat{d}_{g,b'\mathfrak{h}}$. In particular, $\hat{d}_{g,b'\mathfrak{h}}$ preserves the parity:*

$$D(\hat{d}_{g,b'\mathfrak{h}}) = D(\hat{d}_{g,b'\mathfrak{h}}) \cap L^2_{ev}(X;F) \oplus D(\hat{d}_{g,b'\mathfrak{h}}) \cap L^2_{odd}(X;F),$$
$$\text{with} \quad \hat{d}_{g,b'\mathfrak{h}} : D(\hat{d}_{g,b'\mathfrak{h}}) \cap L^2_{ev\ odd}(X;F) \to L^2_{ev\ odd}(X;F).$$

The subset $\mathscr{C}_{0,g}(\hat{F}_g)$ of Definition 3.3 is dense in $D(\hat{d}_{g,b'\mathfrak{h}})$. Additionally the dense subset $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$ of $\mathscr{C}_{0,g}(\hat{F}_g)$ given in Proposition 4.8 satisfies $\hat{d}_{g,b'\mathfrak{h}}\mathscr{D}_{g,\nabla^{\mathfrak{f}}} \subset \mathscr{C}_{0,g}(\hat{F}_g) \subset D(\hat{d}_{g,b'\mathfrak{h}})$.
The domain of the operator $\overline{d}_{g,b'\mathfrak{h}}$ given in $L^2(X_-;F)$ by Definition 4.9 equals

$$D(\overline{d}_{g,b'\mathfrak{h}}) = \left\{s \in L^2(X_-;F), \quad d^{\nabla^{\mathfrak{f}}}_{b'\mathfrak{h}}s \in L^2(X_-;F), \quad \frac{1-\hat{S}_\nu}{2}\mathbf{i}_{e_1}e^1 \wedge s\big|_{X'} = 0\right\}.$$

The operator $\overline{d}_{g,b'\mathfrak{h}}$ with this domain is closed and satisfies $\overline{d}_{g,b'\mathfrak{h}} \circ \overline{d}_{g,b'\mathfrak{h}} = 0$.
The spaces $\mathscr{C}_0^\infty(\overline{X}_-;F) \cap D(\overline{d}_{g,b'\mathfrak{h}})$, $\mathscr{C}_g = \left\{s \in L^2(X_-;F), s_{ev} \in \mathscr{C}_{0,g}(\hat{F}_g)\right\}$ and $\mathscr{D}_{g,\nabla^{\mathfrak{f}}} = \left\{s \in L^2(X_-;F), s_{ev} \in \hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}\right\}$ are dense in $D(\overline{d}_{g,b'\mathfrak{h}})$ with $\overline{d}_{g,b'\mathfrak{h}}\mathscr{D}_{g,\nabla^{\mathfrak{f}}} \subset \mathscr{C}_g$.

Proof. Let us first consider the operator $\hat{d}_{g,b'\mathfrak{h}}$. For a sequence $(u_n)_{n\in\mathbb{N}}$ of $D(\hat{d}_{g,b'\mathfrak{h}})$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty}\hat{d}_{g,b'\mathfrak{h}}u_n = v$ in $L^2(X;F)$, the convergence of $u_n\big|_{X_\mp}$ to $u\big|_{X_\mp}$ holds in $\mathscr{E}_{loc}(d^{\nabla^{\mathfrak{f}}},F\big|_{\overline{X}_\mp})$. The continuity of the trace map $j^*_{X'}$

of Proposition 4.1-b) implies $u \in \mathscr{E}_{loc}(\hat{d}_g,\hat{F}_g)$ and the identification $\hat{d}_{g,b'\mathfrak{h}}u = v$ in $L^2_{loc}(X;F)$. Therefore $(\hat{d}_{g,b'\mathfrak{h}}, D(\hat{d}_{g,b'\mathfrak{h}}))$ is closed. The property $\hat{d}_{g,\mathfrak{h}} \circ \hat{d}_{g,b'\mathfrak{h}}$ and $\hat{d}_{g,b'\mathfrak{h}} \circ \Sigma_\nu = \Sigma_\nu \circ \hat{d}_{g,b'\mathfrak{h}}$ were already proved in Proposition 4.8.
For the density, consider the cut-off function $\chi_n = \chi\left(\frac{\hat{\mathfrak{h}}}{n+1}\right)$ with $\chi \in \mathscr{C}^\infty_0(\mathbb{R};[0,1])$ equal to 1 in a neighborhood of 0. For $s \in D(\hat{d}_{g,b'\mathfrak{h}})$, notice $\chi_n s \in \mathscr{E}_{comp}(\hat{d}_g,\hat{F}_g) \subset D(\hat{d}_{g,b'\mathfrak{h}})$ and write with $s_\mp = s\big|_{X_\mp}$,

$$d^{\nabla^{\mathfrak{f}}}_{b'\mathfrak{h}}[\chi_n s_\mp] = \chi_n(d^{\nabla^{\mathfrak{f}}}_{b'\mathfrak{h}} s_\mp) + [d\chi_n \wedge s_\mp] = \chi_n(d^{\nabla^{\mathfrak{f}}}_{b'\mathfrak{h}} s_\mp) + \frac{1}{n+1}[\chi'\left(\frac{\hat{\mathfrak{h}}}{n+1}\right)(d\hat{\mathfrak{h}} \wedge s_\mp)].$$

With the coordinates $(\tilde{q},\tilde{p})$ and the metric $\hat{g}^E$, we know $|d\tilde{q}^i|_{\hat{g}^E} = \mathscr{O}(\langle\tilde{p}\rangle^{-1/2})$, $|d\tilde{p}_j|_{\hat{g}^E} = \mathscr{O}(\langle\tilde{p}\rangle^{1/2})$ and

$$\begin{aligned} d\hat{\mathfrak{h}} &= \tilde{p}_1 d\tilde{p}_1 + m^{i'j'}(0,\tilde{q}')\tilde{p}_{i'}d\tilde{p}_{j'} + \frac{1}{2}\tilde{p}_{i'}\tilde{p}_{j'}d_{\tilde{q}'}[m^{i'j'}(0,\tilde{q}')] \quad , \\ |d\hat{\mathfrak{h}}|_{\hat{g}^E} &= \mathscr{O}(\langle p\rangle^{3/2}) = \mathscr{O}(\hat{\mathfrak{h}}^{3/4}). \end{aligned}$$

We deduce

$$\|d\chi_n \wedge s_\mp\|_{L^2} \leq \frac{C}{(n+1)^{1/4}}\|s_\mp\|_{L^2}$$

and

$$\lim_{n\to\infty} \|s - \chi_n s\|_{L^2} + \|\hat{d}_{g,b'\mathfrak{h}}(s-\chi_n s)\|_{L^2} = 0.$$

We have just proved that $\mathscr{E}_{comp}(\hat{d}_g,\hat{F}_g)$ is dense (and continuously embedded) in $D(\hat{d}_{g,b'\mathfrak{h}})$. In Proposition 4.8 we proved that $\mathscr{C}_{0,g}(\hat{F}_g)$ and $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$ is dense in $\mathscr{E}_{comp}(\hat{d}_g,\hat{F}_g)$ and that $\hat{d}_g\hat{D}_{g,\nabla^{\mathfrak{f}}} \subset \mathscr{C}_{0,g}(\hat{F}_g)$. By going back to the $\mathscr{C}^\infty$ structure $\Lambda T^*M_g \otimes \pi^*_{M_g}(\mathfrak{f})$ of $\hat{F}_g$, it suffices to notice that $\hat{h} = \frac{\tilde{p}_1^2 + m^{i'j'}(0,\tilde{q}')\tilde{p}_{i'}\tilde{p}_{j'}}{2}$ is actually a $\mathscr{C}^\infty$ function on M_g preserved by $\tilde{S}_1$ to see that the multiplication by $e^{\pm b'\mathfrak{h}}$ preserves $\mathscr{C}_{0,g}(\hat{F}_g) = S_{1,*}\mathscr{C}_0(\Lambda T^*M_g \otimes \pi^*_{M_g}(\mathfrak{f}))$ and $\hat{D}_{g,\nabla^{\mathfrak{f}}} = S_{1,*}\mathscr{C}^\infty_0(M_g;\Lambda T^*M_g \otimes \pi^*_{M_g}(\mathfrak{f}))$.
Finally the properties of $(\overline{d}_{g,b'\mathfrak{h}}, D(\overline{d}_{g,b'\mathfrak{h}}))$ are obvious translations of the properties of $(\hat{d}_{g,b'\mathfrak{h}}, D(\hat{d}_{g,b'\mathfrak{h}}))$ because $\hat{d}_{g,b'\mathfrak{h}}$ preserves the parity with respect to Σ_ν and $s \mapsto \frac{s_{ev}}{\sqrt{2}}$ is a unitary map from $L^2(X_-;F)$ to $L^2_{ev}(X;\hat{F}_g)$. The condition $\frac{1-\hat{S}_\nu}{2}\mathbf{i}_{e_1}e^1 \wedge s\big|_{X'} = 0$ is simply the partial trace version in $\mathscr{D}'(X';F\big|_{X'})$ of Proposition 3.6-b). $\square$

4.4 Comments

The results of this section requires some specifications and explanations.

4.4.1 Dependence of the boundary condition with respect to g^{TQ}

All the analysis was carried out without using any reference to a riemannian metric as it should be when one studies the differential. The weight $\mathfrak{h}$ could be replaced by another function on X with the suitable assumptions. The map $\hat{S}_\nu$ defined on X' actually simply depends only on the identification of a normal vector to

Q'. However the boundary condtions for $\overline{d}_{g,b'\mathfrak{h}}$ depend on the chosen metric g^{TQ} because the tangential trace is written $\mathbf{i}_{e_1}e^1\wedge s\big|_{X'}$. Accordingly the continuity condition in the vector bundles $\hat{E}_g$, $\hat{F}_g$ or the differential $\mathscr{C}^\infty$ manifold M_g introduced in Subsection 4.2 really depend on the chosen metric g^{TQ}. Below is the verification that they must not be confused with more usual boundary conditions for the differential which would correspond to the metric g_0^{TQ} where $\hat{E}_{g_0} = E = \Lambda T^*X$. Let $g^{TQ} = (dq^1)^2 \oplus^\perp m(q^1)$ and $g_0^{TQ} = (dq^1)^2 \oplus^\perp m(0)$ and use the frames $(e^i, \hat{e}_j)$ for the first case and the frame $(f^i, \hat{f}_j)$ for the second case. We forget the vector bundle $\mathfrak{f}$ or simply take $\nu = 1$. Writing a section $\omega \in \mathscr{C}_0^\infty(\overline{X}_-; T^*X)$ in those two local frames gives

$$\begin{aligned}
\omega &= \omega_i e^i + \omega^j \hat{e}_j \\
&= \left[\omega_1 - \Gamma_{j'1}^{k'} p_{k'} \omega^{j'}\right] f^1 + \left[\omega_{i'} - \Gamma_{i'1}^{k'} p_{k'} \omega^1 - \Gamma_{i'j'}^{1} p_1 \omega^{j'}\right] f^{i'} + \omega^j \hat{f}_j .
\end{aligned}$$

The boundary condition for g^{TQ} is

$$\omega_{i'}(0,x') = \omega_{i'}(0, S_1(x')) \quad , \quad \omega^j(0,x') = (-1)^{\delta_{1j}} \omega^j(0, S_1(x'))$$

while for g_0^{TQ} it is

$$\begin{aligned}
&\left[\omega_{i'} - \Gamma_{i'1}^{k'} p_{k'} \omega^1 - \Gamma_{i'j'}^{1} p_1 \omega^{j'}\right] = \left[\omega_{i'} - \Gamma_{i'1}^{k'} p_{k'} \omega^1 - \Gamma_{i'j'}^{1} p_1 \omega^{j'}\right](0, S_1(x')) \\
&\omega^j(0,x') = (-1)^{\delta_{j1}} \omega^j(0, S_1(x')) .
\end{aligned}$$

They coincide iff

$$\forall i' \in \{2, \ldots, d\}\,, \quad \Gamma_{i'1}^{k'} p_{k'} [\omega^1(0,x') + \omega^1(0, S_1(x'))] + 2\Gamma_{i'j'}^{1} p_1 \omega^{j'}(0,x') = 0\,,$$

which is not true in general.

4.4.2 The flat vector bundle $(\mathfrak{f}, \nabla^{\mathfrak{f}})$ and the spaces $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$, $\mathscr{D}_{g,\nabla^{\mathfrak{f}}}$

Once the flat vector bundle $\pi_X^*(\mathfrak{f})$ is made from the two pieces $\mathfrak{f}\big|_{\overline{Q}_-}$ and $\mathfrak{f}\big|_{\overline{Q}_+}$ according to Definition 2.2, the vector bundle $\Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f})$ of Subsection 4.2 can be considered as a $\mathscr{C}^\infty$ flat vector bundle on M_g. However the $\mathscr{C}^\infty$ structure depends on $\nabla^{\mathfrak{f}}$ and this why we keep track of the $\nabla^{\mathfrak{f}}$-dependence in the subset $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$ and $\mathscr{D}_{g,\nabla^{\mathfrak{f}}}$. Let us consider the simple example where a potential is added to the energy $\mathfrak{h}$, with $\nu = 1$. According to the end of Subsection 2.5.2 it can be formulated by starting from $\mathfrak{f} = \overline{Q}_- \otimes \mathbb{C}$ with the trivial connection $\nabla^{\mathfrak{f}} = \nabla$ and the metric $g^{\mathfrak{f}}(z) = e^{-2b'V(q)}|z|^2$ or equivalently with the metric $g^{\mathfrak{f}}(z) = |z|^2$ and the connection $\nabla^{\mathfrak{f}} = \nabla + b'dV(q) = e^{-b'V(q)}\nabla e^{b'V(q)}$. Take the second choice. The corresponding differential $d_{g,b'\mathfrak{h}}^{\nabla^{\mathfrak{f}}}$ on X_- will be

$$d_{g,b'\mathfrak{h}}^{\nabla^{\mathfrak{f}}} = d + b'd(\mathfrak{h} + V)\wedge$$

which is what we expect. Smooth sections $\tilde{s} \in \mathscr{C}^\infty(M_g; \Lambda T^*M_g \otimes \pi_{M_g}^*(\mathfrak{f}))$ are actually sections of ΛT^*M such that $e^{b'\hat{V}(q)}\tilde{s} \in \mathscr{C}^\infty(M_g; \Lambda T^*M_g \otimes \mathbb{C})$ with $\hat{V}(\tilde{q}) =$

$V(-|\tilde{q}^1|,\tilde{q}')$. The set $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$ was defined as the image of $\mathscr{C}_0^\infty(M_g;\Lambda T^*M_g\otimes\pi_{M_g}^*(\mathfrak{f}))$. The set of its even element is the image of the even elements of $\mathscr{C}_0^\infty(M_g;\Lambda T^*M_g\otimes\pi_{M_g}^*(\mathfrak{f}))$ with respect to $\Sigma_{M_g,*}$ that is via the symmetry $(\tilde{q}^1,\tilde{x}')\mapsto(-\tilde{q}^1,\tilde{x}')$. If we simply take a function $\tilde{s}\in\mathscr{C}_0^\infty(M_g;\mathbb{C})$ such that $\tilde{s}(-\tilde{q}^1,\tilde{x}')=\tilde{s}(\tilde{q}^1,\tilde{x}')$ it must satisfy $\frac{\partial}{\partial\tilde{q}^1}(e^{b'\hat{V}}\tilde{s})(0,\tilde{q}')=0$. Written in $\tilde{q}^1=0^-$ it means $\frac{\partial\tilde{s}}{\partial\tilde{q}^1}(0^-,\tilde{x}')+b'\frac{\partial V}{\partial\tilde{q}^1}s(0^-,\tilde{x}')=0$. The corresponding section $s\in\mathscr{D}_{g,\nabla^{\mathfrak{f}}}$ belongs to $\mathscr{C}_0^\infty(\overline{X}_-;\mathbb{C})$ and satisfies

$$s(0,\tilde{q}',\tilde{p}_1,\tilde{p}')=s(0,\tilde{q}',-\tilde{p}_1,\tilde{p}')\quad,\quad\frac{\partial\tilde{s}}{\partial\tilde{q}^1}(0,\tilde{x}')+b'\frac{\partial V}{\partial\tilde{q}^1}s(0,\tilde{x}')=0\quad,\quad\tilde{x}'=(\tilde{q}',\tilde{p}_1,\tilde{p}').$$

It clearly depends on the flat connection $\nabla^{\mathfrak{f}}$.
While considering adjoints, like in Proposition 4.4 we must use the anti-adjoint flat connection $\nabla^{\mathfrak{f}'}$, equal to $\nabla^{\mathfrak{f}'}=\nabla-b'dV\wedge$ here, and the sign in front of $\frac{\partial V}{\partial\tilde{q}^1}$ in the above condition is changed. So the corresponding subset $\mathscr{D}_{g,\nabla^{\mathfrak{f}'}}$ differs from $\mathscr{D}_{g,\nabla^{\mathfrak{f}}}$. A similar subtlety must be watched when we go further in the analysis and play with the extrinsic curvature related with $\frac{\partial m}{\partial\underline{q}^1}(0,\underline{q}')$.
From this point of view working with the vector bundle $\hat{E}_g$ and $\hat{F}_g$ where only the continuity is considered along X', not only simplifies the correspondance $(0^+,x')=(0^-,y')$ into $y'=x'$, but also prevents from mistakes or confusions.

4.4.3 The interface condition of $\hat{d}_{g,b'\mathfrak{h}}$ as a jump condition

Again, we forget the vector bundle $\pi_X^*(\mathfrak{f})$ or take simply $\mathfrak{f}=\mathbb{C}$ with $\nu=1$.
The continuity condition for $s=s_I^J e^I\hat{e}_J\in\mathscr{E}_{loc}(\hat{d}_g,\hat{E}_g)$ written:

$$s_{I'}^J(0^+,x')=s_{I'}^J(0^-,x')$$

can be written

$$j_{X'}^*(A_+s)=j_{X'}(A_-s)$$

where $A_\mp$ is the vector bundle isomorphism sending the frame $(e_\mp^i,\hat{e}_{\mp,j})$ associated with $g_\mp^{TQ}$ to the frame $(f^i,\hat{f}_j)$ associated with g_0^{TQ}, with $j_{X'}^*\tilde{s}$ written as $\mathbf{i}_{f^1}f^1\wedge s\big|_{X'}=\mathbf{i}_{\frac{\partial}{\partial q^1}}(dq^1\wedge s)\big|_{X'}$. With basic linear algebra, it can also be written

$$\mathbf{i}_{\frac{\partial}{\partial q^1}}(dq^1\wedge s)\big|_{\partial X_+}=A\mathbf{i}_{\frac{\partial}{\partial q^1}}(dq^1\wedge s)\big|_{\partial X_-}.$$

By using the $\mathscr{C}^\infty$ natural structure of $E=\Lambda T^*X$ the continuity conditions for $s\in\mathscr{E}_{loc}(\hat{d}_g,\hat{E}_g)$ thus appears as a jump condition in E. In $\mathscr{D}'(X;\Lambda T^*X)$ an element $s\in L^2_{loc}(X;\Lambda T^*X)$ belongs to $\mathscr{E}_{loc}(\hat{d}_g,\hat{E}_g)$ iff

$$ds=(ds_-)+(ds_+)+\delta_0(q^1)dq^1\wedge[(A-\mathrm{Id})j_{X'}*s_-]\quad\text{in }\mathscr{D}'(X;\Lambda T^*X)$$

with $s_\mp=s\big|_{X_\mp}$, $ds_\mp\in L^2_{loc}(\overline{X}_\mp;\Lambda T^*X)$. Checking $\hat{d}_g\mathscr{E}_{loc}(\hat{d}_g,\hat{E}_g)\subset\mathscr{E}_{loc}(\hat{d}_g,\hat{E}_g)$ with $\hat{d}_g\circ\hat{d}_g=0$ means that the right-hand side of

$$d(\hat{d}_gs)=0+0+\delta(q^1)dq^1\wedge[j_{X'}(ds_+)-j_{X'}(ds_-)]=\delta(q^1)dq^1\wedge d'[j_{X'}(A-\mathrm{Id})s_-]$$

can be written $\delta(q^1)\wedge dq^1[(A-\mathrm{Id})(d'j_{X'}s)]$. A sufficient condition is: the vector bundle morphism A satisfies $d'A = Ad'$ in $\mathscr{D}'(X';\Lambda TX')$.
This can be checked by computations in terms of local coordinates by using the expressions (49)(50)(55)(56)(57) of the frames $(e^i_{\mp},\hat{e}_{\mp,j})$ and $(f^i,\hat{f}_j)$. But this is not so simple and may involve the differentiation of the Christoffel symbols $\Gamma^k_{ij}(q)$ which is irrelevant. From this point of view introducing the proper $\mathscr{C}^\infty$ structure of the manifold M_g in Subsection 4.2 is much more effective.
However note that it was introduced with the non symplectic coordinates $(\tilde{q},\tilde{p})$. It is easier to work with the differential structure of X, the coordinates (q,p), and the piecewise $\mathscr{C}^\infty$ and continuous vector bundle $\hat{E}_g$ when the symplectic structure is required.

5 Symplectic and Bismut codifferential

Now that we have a good definition and properties of the closed operator $\overline{d}_{g,\mathfrak{h}}$ in $L^2(X_-;F)$, the codifferential can be defined as the adjoint operator, for various duality products. Bismut's codifferential involves a non-degenerate but non symmetric form, which makes a mixture of symplectic and riemannian Hodge duality. It is therefore simpler to start first from the duality between $F=\Lambda T^*X\otimes\pi_X^*(\mathfrak{f})$ and $F'=\Lambda TX\otimes\pi_X^*(\mathfrak{f})$, $\mathfrak{f}\simeq\mathfrak{f}'$ via the metric $g^{\mathfrak{f}}$ or $\hat{g}^{\mathfrak{f}}$, which does not require any additional information than the symplectic volume measure dv_X on X and then to transfer the information by various isomorphisms from F' to F.

5.1 Adjoints for the (F',F) dual pair

The volume measure on X is the symplectic volume

$$dv_X = \left|\frac{1}{d!}\sigma^d\right| = |dqdp|$$

and coincides with the Lebesgue measure in any symplectic coordinate system (q^i,p_j). However here it is more convenient to work with the non symplectic coordinates $(\tilde{q},\tilde{p})$ of Definition 2.6 where dv_X does not have such a simple form according to (77). However the trace results and the integration by parts of Proposition 4.4 only requires the volume form dv_X and the defining function of the boundary x^1 which is here $\tilde{q}^1=q^1$ in $X_{(-\varepsilon,\varepsilon)}$ with $d\tilde{q}^1=dq^1=e^1$. Therefore the volume form occuring in the boundary integrals is $dv_{X'}=|dq'dp|$.
Although the dual of $F=\Lambda T^*X\otimes\pi_X^*(\mathfrak{f})$ can be identified with $F'=\Lambda T^*X\otimes\pi_X^*(\mathfrak{f})$ via the metric $g^{\mathfrak{f}}$ or $\hat{g}^{\mathfrak{f}}$, which provides the identification at the level of L^2-spaces, the flat $\mathscr{C}^\infty$ connections $\pi_X^*(\nabla^{\mathfrak{f}})$ and $\pi_X^*(\nabla^{\mathfrak{f}'})$ differ and give rise to different $\mathscr{C}^\infty$-structures of vector bundles, especially when the metric $\hat{g}^f$ is used. An example is the case $\nabla^{\mathfrak{f}}=\nabla+b'd\hat{V}$ and $\nabla^{\mathfrak{f}'}=\nabla-b'd\hat{V}$ where $\hat{V}(q^1,q')=V(-|q^1|,q')$ already discussed in Subsection 4.4.2.
Because the operators $\overline{d}_{g,b'\mathfrak{h}}$ and $\hat{d}_{g,b'\mathfrak{h}}$ are densely defined operators respectively

in $L^2(X_-;F)$ and $L^2(X;F)$, we can consider their adjoints via the duality product

$$\langle t, s\rangle = \int_X \langle t, s\rangle_{F'_x,F_x}\, dv_X(x),$$

where the duality between $F' = \Lambda TX \otimes \pi_X^*(\mathfrak{f})$ and $F = \Lambda T^*X \otimes \pi_X^*(\mathfrak{f})$ is made via the metric $g^{\mathfrak{f}}$ (or $\hat{g}^{\mathfrak{f}}$) and for which $L^2(X_-,F)' = L^2(X_-,F')$ (resp. $L^2(X;F)' = L^2(X;F')$).
When we work on X_- or $X_- \cup X_+$ with the symplectic coordinates (q,p), $dv_X = |dqdp|$ and without considering boundary or interface conditions, the formal adjoint of $d_{b'\mathfrak{h}} = d^{\nabla^{\mathfrak{f}}}_{b'\mathfrak{h}} = d^{\nabla^{\mathfrak{f}}} + b'd\mathfrak{h}\wedge$ is nothing but

$$\tilde{d}_{b'\mathfrak{h}} = \tilde{d}^{\nabla^{\mathfrak{f}'}}_{b'\mathfrak{h}} = \tilde{d}^{\nabla^{\mathfrak{f}'}} - b'\mathbf{i}_{d\mathfrak{h}} = -\mathbf{i}_{dq^i}\nabla^{\mathfrak{f}'}_{\frac{\partial}{\partial q^i}} - \mathbf{i}_{dp_j}\frac{\partial}{\partial p_j} - b'\mathbf{i}_{d\mathfrak{h}}. \tag{92}$$

The notation $\tilde{\hat{d}}_{b'\mathfrak{h}}$ on $X_- \cup X_+$ refers to the case when $\mathfrak{h}$ is replaced by the piecewise $\mathscr{C}^\infty$ and continuous function $\hat{\mathfrak{h}}$. According to this, the trace results of Proposition 4.4 lead to the following definition.

Definition 5.1. *The space $\mathscr{E}_{loc}(\tilde{\hat{d}}_{g,b'\mathfrak{h}},\hat{F}'_g) = \mathscr{E}_{loc}(\tilde{\hat{d}}_{g,0} = \tilde{\hat{d}}_g,\hat{F}'_g)$ is the set of sections in $t = L^2_{loc}(X;F')$, $t_\mp = t\big|_{X_\mp}$ such that*

$$\begin{aligned}
& \tilde{d}^{\nabla^{\mathfrak{f}'}} t_\mp \in L^2_{loc}(X_\mp;F'),\\
& \mathbf{i}_{e^1}t_-\big|_{\partial X_-} = \mathbf{i}_{e^1}t_+\big|_{\partial X_+} \quad \text{in } \mathscr{D}'(X';\hat{F}'_g\big|_{X'})\\
\text{or}\quad & e_1\wedge\mathbf{i}_{e^1}t_-\big|_{\partial X_-} = e_1\wedge\mathbf{i}_{e^1}t_+\big|_{\partial X_+} \quad \text{in } \mathscr{D}'(X';\hat{F}'_g\big|_{X'}),\\
\text{and}\quad & \tilde{\hat{d}}_{g,b'\mathfrak{h}}t = (\tilde{d}^{\nabla^{\mathfrak{f}'}}_{b'\hat{\mathfrak{h}}}t_-) + (\tilde{d}^{\nabla^{\mathfrak{f}'}}_{b'\hat{\mathfrak{h}}}t_+).
\end{aligned}$$

Proposition 5.2. *The adjoint $\tilde{\hat{d}}_{g,b'\mathfrak{h}}$ of $(\hat{d}_{g,b'\mathfrak{h}},D(\hat{d}_{g,\mathfrak{h}}))$ of Definition 4.9 is closed and densely defined as*

$$D(\tilde{\hat{d}}_{g,b'\mathfrak{h}}) = \left\{t\in L^2(X;F')\cap\mathscr{E}_{loc}(\tilde{\hat{d}}_g,\hat{F}'_g),\quad \tilde{\hat{d}}_{g,b'\mathfrak{h}}t\in L^2(X;F')\right\}$$
$$\forall t\in D(\tilde{\hat{d}}_{g,b'\mathfrak{h}}),\quad \tilde{\hat{d}}_{g,b'\mathfrak{h}}t = (\tilde{d}^{\nabla^{\mathfrak{f}'}}_{b'\hat{\mathfrak{h}}}t_-) + (\tilde{d}^{\nabla^{\mathfrak{f}'}}_{b'\hat{\mathfrak{h}}}t_+)\quad,\quad t_\mp = t\big|_{X_\mp}.$$

It satisfies $\tilde{\hat{d}}_{g,b'\mathfrak{h}}\circ\tilde{\hat{d}}_{g,b'\mathfrak{h}} = 0$ and $\tilde{\hat{d}}_{g,b'\mathfrak{h}}\circ\Sigma_\nu = \Sigma_\nu\tilde{\hat{d}}_{g,b'\mathfrak{h}}$. In particular, $\tilde{\hat{d}}_{g,b'\mathfrak{h}}$ preserves the parity:

$$\begin{aligned}
& D(\tilde{\hat{d}}_{g,b'\mathfrak{h}}) = D(\tilde{\hat{d}}_{g,b'\mathfrak{h}})\cap L^2_{ev}(X;F')\oplus D(\tilde{\hat{d}}_{g,b'\mathfrak{h}})\cap L^2_{odd}(X;F'),\\
\text{with}\quad & \tilde{\hat{d}}_{g,b'\mathfrak{h}} : D(\tilde{\hat{d}}_{g,b'\mathfrak{h}})\cap L^2_{ev\ odd}(X;F')\to L^2_{ev\ odd}(X;F').
\end{aligned}$$

The subset $\mathscr{C}_{0,g}(\hat{F}'_g)$ of Definition 3.3 is dense in $D(\tilde{\hat{d}}_{g,b'\mathfrak{h}})$. Additionally there exists a dense subset $\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}$ of $\mathscr{C}_{0,g}(\hat{F}'_g)$ such that $\tilde{\hat{d}}_{g,b'\mathfrak{h}}\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}} \subset \mathscr{C}_{0,g}(\hat{F}'_g)\subset D(\tilde{\hat{d}}_{g,b'\mathfrak{h}})$.
The adjoint $\tilde{d}_{g,b'\mathfrak{h}}$ of $(\overline{d}_{g,\mathfrak{h}},D(\overline{d}_{g,\mathfrak{h}}))$ is densely defined and closed with

$$D(\tilde{d}_{g,\mathfrak{h}}) = \left\{t\in L^2(X_-,F'),\quad d^{\nabla^{\mathfrak{f}'}}_{b'\mathfrak{h}}t\in L^2(X_-,F'),\quad \frac{1-\hat{S}_\nu}{2}e_1\wedge\mathbf{i}_{e^1}t\big|_{X'} = 0\right\}.$$

This adjoint operator $(\tilde{d}_{g,b'\mathfrak{h}}, D(\tilde{d}_{g,b'\mathfrak{h}}))$ satisfies $\tilde{d}_{g,b'\mathfrak{h}} \circ \tilde{d}_{g,b'\mathfrak{h}} = 0$.
The spaces $\mathscr{C}_0^\infty(\overline{X}_-;F') \cap D(\tilde{d}_{g,b'\mathfrak{h}})$, $\mathscr{C}'_g = \left\{t \in L^2(X_-,F'), t_{ev} \in \mathscr{C}_{0,g}(\hat{F}'_g)\right\}$ and $\mathscr{D}'_{g,\nabla^{\mathfrak{f}'}} = \left\{t \in L^2(X_-;F), t_{ev} \in \hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}\right\}$ are dense in $D(\tilde{d}_{g,b'\mathfrak{h}})$ with $\tilde{d}_{g,b'\mathfrak{h}} \mathscr{D}'_{g,\nabla^{\mathfrak{f}'}} \subset \mathscr{C}'_g$.

Proof. For $\tilde{\tilde{d}}_{g,b'\mathfrak{h}}$ we use the manifold $M_g = X_- \cup X' \cup X_+$ introduced in subsection 4.2. By Lemma 4.6 the map $\tilde{S}_{1,*} : \Lambda T^*M_g \otimes \pi^*_{M_g}(\mathfrak{f})\big|_{M_{g,(-\varepsilon,\varepsilon)}} \to \hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}$ provides the transpose map $\tilde{S}_1^* : \hat{F}'_g\big|_{X_{(-\varepsilon,\varepsilon)}} \to \Lambda TM_{g,(-\varepsilon,\varepsilon)} \otimes \pi^*_{M_g}(\mathfrak{f})$ and $TM_{g,(-\varepsilon,\varepsilon)} \otimes \pi^*_{M_g}(\mathfrak{f}') = TM_{g,(-\varepsilon,\varepsilon)} \otimes \pi^*_{M_g}(\mathfrak{f})$ is a $\mathscr{C}^\infty$-vector bundle when the flat connection $\pi^*_{M_g}(\nabla^{\mathfrak{f}'})$ is used. The operator $\hat{d}_{g,b'\mathfrak{h}}$ was identified with $e^{-b'\hat{\mathfrak{h}}} d^{\nabla^{\mathfrak{f}}}_{M_g} e^{b'\hat{\mathfrak{h}}}$ acting on the smooth vector bundle $\Lambda T^*M_g \otimes \pi^*_{M_g}(\mathfrak{f})$ and where $\hat{\mathfrak{h}}$ is a smooth function on M_g. We are thus led to consider the adjoint of the differential on the smooth manifold M_g without boundary, $\tilde{\tilde{d}}_{g,b'\hat{\mathfrak{h}}}$ is identified with $e^{b'\hat{\mathfrak{h}}} \tilde{d}^{\nabla^{\mathfrak{f}'}}_{M_g} e^{-b'\hat{\mathfrak{h}}}$ and all the properties follow. In particular $\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}$ is nothing but the image of $\mathscr{C}_0^\infty(M_g; \Lambda TM_g \otimes \pi^*_{M_g}(\mathfrak{f}'))$, where we write $\mathfrak{f}'$ to remind that the $\mathscr{C}^\infty$-structure is the one given by $\nabla^{\mathfrak{f}'}$, by $\tilde{S}_{1,*} : \Lambda M_g \otimes \pi^*_{M_g}(\mathfrak{f}) \to \hat{F}'_g$.
The study of $(\tilde{d}_{g,b'\mathfrak{h}}, D(\tilde{d}_{g,b'\mathfrak{h}}))$ then relies on parity arguments with respect to Σ_ν on $\hat{F}'_g$ (or with respect to $\Sigma_{M_g,*}$ on $\Lambda T^*M_g \otimes \pi^*_{M_g}(\mathfrak{f}')$), as we did for $(\overline{d}_{g,b'\mathfrak{h}}, D(\overline{d}_{g,b'\mathfrak{h}}))$. $\square$

5.2 Symplectic codifferential

As a non degenerate 2-form the symplectic form σ on TX, defines a morphism $\sigma : TX \to T^*X$ by writing $\sigma(S,T) = S.(\sigma T)$ for $S, T \in TX$. By tensorization, this defines a morphism still denoted $\sigma : E' = \Lambda TX \to E = \Lambda T^*X$ and $\sigma : F' = \Lambda TX \otimes \pi^*_X(\mathfrak{f}) \to F = \Lambda T^*X \otimes \pi^*_X(\mathfrak{f})$ which fulfills the condition (80) (see below for details). We can thus consider the σ-adjoint of densely defined operators in $L^2(X;F)$ as closed operators in $L^2(X;F)$, according to Definition 3.11 and deduce their properties from the (F',F)-adjoint according to Proposition 3.12. Because the symplectic form is anti-symmetric the left and right adjoints are equal.

Definition 5.3. *The operators $\hat{d}^\sigma_{g,b'\mathfrak{h}}$ and $d^\sigma_{g,b'\mathfrak{h}}$ are the symplectic adjoints, that is the ϕ-adjoint Definition 3.11 for $\phi = \sigma : TX \to T^*X$, of the operators $\hat{d}_{g,b'\mathfrak{h}}$ and $\overline{d}_{g,b'\mathfrak{h}}$ defined in Definition 4.9 and characterized in Proposition 4.10.*

Before giving the properties of $\hat{d}^\sigma_{g,b'\mathfrak{h}}$ and $d^\sigma_{g,b'\mathfrak{h}}$ let us specify some formulas.

Lemma 5.4. *For $t \in L^\infty_{loc}(X;TX)$ and $\omega \in L^\infty_{loc}(X;T^*X)$, the symplectic adjoint of $\mathbf{i}_t$ (resp. $\omega\wedge$) equals $d(\sigma t)\wedge$ (resp. $-\mathbf{i}_{\sigma^{-1}\omega}$).*

Proof. The general formula of Proposition 3.12 says $P^\sigma = \sigma \tilde{P} \sigma^{-1} = {}^t\sigma \tilde{P} {}^t\sigma$ with ${}^t\sigma = -\sigma : TX \to T^*X$. Therefore

$$(\mathbf{i}_t)^\sigma = \sigma \widetilde{(\mathbf{i}_t)} \sigma^{-1} = \sigma (t\wedge) \sigma^{-1} = (\sigma t)\wedge ,$$
$$(\omega\wedge)^\sigma = \sigma \widetilde{(\omega\wedge)} \sigma^{-1} = \sigma \mathbf{i}_\omega \sigma^{-1} = \mathbf{i}_{{}^t\sigma^{-1}\omega} = \mathbf{i}_{-\sigma^{-1}\omega} = -\mathbf{i}_{\sigma^{-1}\omega} .$$

$\square$

In particular when φ is a locally Lipschitz continuous function the σ-adjoint of $d\varphi\wedge$ is

$$(d\varphi\wedge)^{\sigma} = -\mathbf{i}_{\sigma^{-1}d\varphi} = -\mathbf{i}_{Y_{\varphi}}$$

where Y_{φ} is the Hamiltonian vector field characterized by $\sigma(t, Y_{\varphi}) = t.\sigma Y_{\varphi} = d\varphi(t)$ or $Y_{\varphi} = \sigma^{-1}d\varphi$.
When we use the symplectic coordinates (q,p) with $dv_X = |dqdp|$ the adjoint of the covariant derivative $\nabla^{\mathfrak{f}}_T$ is $-\nabla^{\mathfrak{f}'}_T$ and the formal symplectic adjoint of

$$d_{b'\mathfrak{h}} = dq^i \wedge \nabla^{\mathfrak{f}}_{\frac{\partial}{\partial q^i}} + dp_j \wedge \frac{\partial}{\partial p_j} + b'd\mathfrak{h}\,,$$

with $\sigma(\frac{\partial}{\partial q^i}) = dp_i$, $\sigma(\frac{\partial}{\partial p_j}) = -dq^j$, equals

$$d^{\sigma}_{b'\mathfrak{h}} = -\mathbf{i}_{\frac{\partial}{\partial p_i}}\nabla^{\mathfrak{f}'}_{\frac{\partial}{\partial q^i}} + \mathbf{i}_{\frac{\partial}{\partial q^j}}\frac{\partial}{\partial p_j} - b'\mathbf{i}_{Y_{\mathfrak{h}}} = -\mathbf{i}_{\frac{\partial}{\partial p_i}}\nabla^{\mathfrak{f}}_{\frac{\partial}{\partial q^i}} + \mathbf{i}_{\frac{\partial}{\partial q^j}}\frac{\partial}{\partial p_j} - \mathbf{i}_{\frac{\partial}{\partial p_i}}\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})\left(\frac{\partial}{\partial q^i}\right) - b'\mathbf{i}_{Y_{\mathfrak{h}}}\,. \tag{93}$$

The same formula holds on both sides X_- and X_+, when $g^{\mathfrak{f}}$ and $\mathfrak{h}$ are replaced by $\hat{g}^{\mathfrak{f}}$ and $\hat{\mathfrak{h}}$.
The boundary conditions for $\hat{d}_{g,b'\mathfrak{h}}$, $\overline{d}_{g,b'\mathfrak{h}}$, $\tilde{\hat{d}}_{g,b'\mathfrak{h}}$ and $\tilde{d}_{g,b'\mathfrak{h}}$ were studied with the non symplectic coordinates $(\tilde{q},\tilde{p})$ but were finally formulated with $e^1 = dq^1$ and $e_1 = \frac{\partial}{\partial \tilde{q}^1}$. Because $(e_i, \hat{e}^j)$ is a symplectic basis with dual basis $(e^i, \hat{e}_j)$, we can use

$$\begin{array}{lllll} & \sigma(e_i) = \hat{e}_i \quad , & \sigma(\hat{e}^j) = -e^j \quad , & \sigma^{-1}(\hat{e}_j) = e_j \quad , & \sigma^{-1}(e^i) = -\hat{e}^i\,, \\ \text{and} & \sigma\mathbf{i}_{e^i}\sigma^{-1} = \mathbf{i}_{-\hat{e}^i} & \sigma\mathbf{i}_{\hat{e}_j}\sigma^{-1} = \mathbf{i}_{e_j} & \sigma^{-1}\mathbf{i}_{e_i}\sigma = \mathbf{i}_{\hat{e}_i} & \sigma^{-1}\mathbf{i}_{\hat{e}^j}\sigma = -\mathbf{i}_{e^j}\,, \end{array}$$

without referring to coordinates. It implies

$$|\sigma(e_i)|_{g^E} = |\hat{e}^i|_{g^E} = \langle p\rangle_q^{1/2} = |e_i|_{g^{E'}} \quad , \quad |\sigma(\hat{e}^j)|_{g^E} = |e^j|_{g^E} = \langle p\rangle_q^{-1/2} = |\hat{e}^j|_{g^{E'}}\,,$$

and the condition (80) is satisfied.
The operator $\hat{S}_\nu$ of Definition 3.5 and involved in the boundary conditions for $\overline{d}_{g,b'\mathfrak{h}}$ and $\tilde{d}_{g,b'\mathfrak{h}}$ satisfies

$$\sigma\hat{S}_\nu\sigma^{-1} = \hat{S}_\nu\,.$$

Finally σ belongs to $\mathscr{C}^\infty(\overline{X}_-;L(E',E))\cap\mathscr{C}^\infty(\overline{X}_+;L(E',E))\cap\mathscr{C}^0(X;L(\hat{E}'_g,\hat{E}_g))$, sends $\mathscr{C}^0_g(\hat{F}'_g)$ to $\mathscr{C}^0_g(\hat{F}_g)$ and preserves the parity with respect to Σ_ν.

Proposition 5.5. *The σ-adjoint of $(\hat{d}_{g,b'\mathfrak{h}}, D(\hat{d}_{g,b'\mathfrak{h}}))$ equals*

$$\begin{aligned} &D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}}) = \left\{s\in L^2(X;F)\cap\sigma\mathscr{E}_{loc}(\tilde{d},\hat{F}'_g),\quad \hat{d}^{\sigma}_{g,b'\mathfrak{h}}s\in L^2(X;F)\right\},\\ &\forall s\in D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}}),\quad \hat{d}^{\sigma}_{g,b'\mathfrak{h}}s = (\hat{d}^{\sigma}_{g,b'\mathfrak{h}}s_-) + (\hat{d}^{\sigma}_{b'\mathfrak{h}}s_+) \quad , \quad s_\mp = s\big|_{X_\mp}\,,\\ &\hat{d}^{\sigma}_{g,b'\mathfrak{h}}s_\mp = -\mathbf{i}_{\frac{\partial}{\partial p_i}}\nabla^{\mathfrak{f}}_{\frac{\partial}{\partial q^i}}s_\mp + \mathbf{i}_{\frac{\partial}{\partial q^j}}\frac{\partial s_\mp}{\partial p_j} - \mathbf{i}_{\frac{\partial}{\partial p_i}}\omega(\nabla^{\mathfrak{f}},\hat{g}^{\mathfrak{f}})\left(\frac{\partial}{\partial q^i}\right)s_\mp - b'\mathbf{i}_{Y_{\hat{\mathfrak{h}}}}s_\mp\,. \end{aligned}$$

It satisfies $\hat{d}^{\sigma}_{g,b'\mathfrak{h}}\circ\hat{d}^{\sigma}_{g,b'\mathfrak{h}} = 0$ and $\hat{d}^{\sigma}_{g,b'\mathfrak{h}}\circ\Sigma_\nu = \Sigma_\nu\circ\hat{d}^{\sigma}_{g,b'\mathfrak{h}}$. In particular, $\hat{d}^{\sigma}_{g,b'\mathfrak{h}}$ preserves the parity:

$$\begin{array}{ll} & D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}}) = D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}})\cap L^2_{ev}(X;F)\oplus D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}})\cap L^2_{odd}(X;F),\\ \textit{with} & \hat{d}^{\sigma}_{g,b'\mathfrak{h}} : D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}})\cap L^2_{ev\ odd}(X;F)\to L^2_{ev\ odd}(X;F). \end{array}$$

The subset $\mathscr{C}_{0,g}(\hat{F}_g)$ of Definition 3.3 is dense in $D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}})$. Additionally there exists a dense subset $\sigma\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}$ of $\mathscr{C}_{0,g}(\hat{F}_g)$ such that $\hat{d}^{\sigma}_{g,b'\mathfrak{h}}(\sigma\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}})\subset\mathscr{C}_{0,g}(\hat{F}_g)\subset D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}})$. The adjoint $d^{\sigma}_{g,b'\mathfrak{h}}$ of $(\overline{d}_{g,b'\mathfrak{h}},D(\overline{d}_{g,b'\mathfrak{h}}))$ is densely defined and closed with

$$D(d^{\sigma}_{g,b'\mathfrak{h}})=\left\{s\in L^2(X_-,F),\quad d^{\sigma}_{b'\mathfrak{h}}s\in L^2(X_-,F),\quad \frac{1-\hat{S}_\nu}{2}\hat{e}_1\wedge\mathbf{i}_{\hat{e}^1}s\big|_{X'}=0\right\}.$$

$$\forall s\in D(d^{\sigma}_{g,b'\mathfrak{h}}),\quad d^{\sigma}_{g,b'\mathfrak{h}}s=-\mathbf{i}_{\frac{\partial}{\partial p_i}}\nabla^{\mathfrak{f}}_{\frac{\partial}{\partial q^i}}s+\mathbf{i}_{\frac{\partial}{\partial q^j}}\frac{\partial s}{\partial p_j}-\mathbf{i}_{\frac{\partial}{\partial p_i}}\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})\left(\frac{\partial}{\partial q^i}\right)s-b'\mathbf{i}_{Y_{\mathfrak{h}}}s.$$

This adjoint operator $(d^{\sigma}_{g,b'\mathfrak{h}},D(d^{\sigma}_{g,b'\mathfrak{h}}))$ satisfies $d^{\sigma}_{g,b'\mathfrak{h}}\circ d^{\sigma}_{g,b'\mathfrak{h}}=0$.
The spaces $\mathscr{C}^{\infty}_0(\overline{X}_-;F)\cap D(d^{\sigma}_{g,b'\mathfrak{h}})$, $\mathscr{C}_g=\{s\in L^2(X_-,F),s_{ev}\in\mathscr{C}_{0,g}(\hat{F}_g)\}$ and $\mathscr{D}_{g,\nabla^{\mathfrak{f}'}}=\left\{s\in L^2(X_-;F),s_{ev}\in\sigma\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}\right\}$ are dense in $D(d^{\sigma}_{g,b'\mathfrak{h}})$ with $d^{\sigma}_{g,b'\mathfrak{h}}\mathscr{D}_{g,\nabla^{\mathfrak{f}'}}\subset\mathscr{C}_g$.

Proof. It is a straightforward application of the general formula of Proposition 3.12

$$(P^{\sigma},D(P^{\sigma}))=(\sigma\tilde{P}\sigma^{-1},\sigma D(\tilde{P})).$$

All the properties of P^{σ}, $P=\hat{d}_{g,b'\mathfrak{h}}$ or $P=\overline{d}_{g,b'\mathfrak{h}}$, are obtained by conjugating with σ^{-1} the ones of $\tilde{P}$ and the properties of $D(P^{\sigma})$ are obtained by transporting via σ the ones of $D(\tilde{P})$. Thus, it is just a translation of Proposition 5.2 combined with the previous formulas and observations. □

Remark 5.6. *Below are some detailed explanations of the previous result:*

a) *The sets $\mathscr{C}_{0,g}(\hat{F}_g)$ and $\mathscr{C}_g$, respectively dense in $D(\hat{d}^{\sigma}_{g,b'\mathfrak{h}})$ and $D(d^{\sigma}_{g,b'\mathfrak{h}})$, are the same as in Proposition 4.10 where they are shown to be dense respectively in $D(\hat{d}_{g,b'\mathfrak{h}})$ and $D(\overline{d}_{g,b'\mathfrak{h}})$.*

b) *The term $-\mathbf{i}_{\frac{\partial}{\partial p_i}}\omega(\nabla^{\mathfrak{f}},\hat{g}^{\mathfrak{f}})\left(\frac{\partial}{\partial q^i}\right)$ (resp. $-\mathbf{i}_{\frac{\partial}{\partial p_i}}\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})\left(\frac{\partial}{\partial q^i}\right)$) in the expression of $\hat{d}^{\sigma}_{g,b'\mathfrak{h}}$ (resp. $d^{\sigma}_{g,b'\mathfrak{h}}$) comes from the comparison between $\nabla^{\mathfrak{f}'}$, used for the analysis in $\hat{F}'_g$ (resp. $F'\big|_{\overline{X}_-}$), and the initial connection $\nabla^{\mathfrak{f}}$ on $\hat{F}_g$ (resp. $F\big|_{\overline{X}_-}$).*

c) *The sets $\sigma\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}$ and $\mathscr{D}_{g,\nabla^{\mathfrak{f}'}}$ differ from the sets $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$ and $\mathscr{D}_{g,\nabla^{\mathfrak{f}}}$ of Proposition 4.10, mainly because the two flat connections $\nabla^{\mathfrak{f}}$ and $\nabla^{\mathfrak{f}'}$ are related with different $\mathscr{C}^{\infty}$ structures of $\pi_{\mathfrak{f}}:\mathfrak{f}\to Q$ when $\hat{g}^{\mathfrak{f}}$ is only piecewise $\mathscr{C}^{\infty}$ and continuous.*

d) *While working with the symplectic structure the symplectic coordinates (q,p) are more natural than the coordinates $(\tilde{q},\tilde{p})$ which were used in particular in Subsection 4.2 for the $\mathscr{C}^{\infty}$-structure of $\hat{E}_g$ via the manifold M_g. When one uses the coordinates $(\tilde{q},\tilde{p})$ on X, the symplectic form does not have a better regularity that the continuity at the interface. An example is given by the disc $\overline{Q}_-=\overline{D}(0,r_0)$ in $\mathbb{R}^2$ where the metric can be written $g^{TQ}_-=d(\underline{q}^1)^2+(r_0+\underline{q}^1)^2d\underline{q}^2$, where $\underline{q}^1$ is the radial coordinate and $\underline{q}^2$ the angular coordinate. Then the coordinates $(\tilde{q},\tilde{p})$ are given by $(\tilde{q},\tilde{p}_1)=(\underline{q},p_1)$ and*

$\tilde{p}_2 = \frac{r_0}{r_0+q^1} p_2$ with $d\tilde{p}_2 = -\frac{r_0 p_2}{(r_0+q^1)^2} dq^1 + \frac{r_0}{r_0+q^1} dp_2$. For the metric g_+^{TQ} simply replace (r_0+q^1) by (r_0-q^1). When the coordinates $(\tilde{q},\tilde{p})$ are constructed for $\hat{g}^{TQ} = 1_{\overline{Q}_-}(\underline{q}) g_-^{TQ} + 1_{\overline{Q}_+}(\underline{q}) g_+^{TQ}$, the symplectic volume $dv_X = |dq dp|$ equals $|\frac{r_0-|\tilde{q}^1|}{r_0}||d\tilde{q}d\tilde{p}|$ which is clearly only piecewise $\mathscr{C}^\infty$ and continuous in those coordinates. Introducing the symplectic form breaks the $\mathscr{C}^\infty$ structure inherited from M_g. Again, this is a reason why we prefer to work with piecewise $\mathscr{C}^\infty$ and continuous vector bundles: This is the right framework were all the structures can be put together.

e) *It is possible to express the symplectic codifferential with the frame $(e,\hat{e})$ and in terms of connections. If we work only with E, or equivalently with $\mathfrak{f} = Q \times \mathbb{C}$, $\nu = 1$, and with $b' = 0$, Bismut in [Bis05] wrote*

$$d = e^i \wedge \nabla^E_{e_i} + \hat{e}_j \wedge \nabla^X_{\hat{e}^j} + \mathbf{i}_{R^{TQ}p},$$
$$d^\sigma = -\mathbf{i}_{\hat{e}^i} \nabla^E_{e_i} + \mathbf{i}_{e_j} \nabla^X_{\hat{e}^j} + R^{TQ} p \wedge .$$

When we work with the non smooth metric $\hat{g}^{TQ}$, the zeroth order term related with the curvature tensor R^{TQ} is not continuous along X' and rather complicated. It is not obvious to check $d \circ d = 0$ and $d^\sigma \circ d^\sigma = 0$ or to identify easily dense domains of smooth sections. From this point of view, the coordinates $(\tilde{q},\tilde{p})$ for the $\mathscr{C}^\infty$-structure of the manifold M_g and then the symplectic coordinates (q,p) make things more obvious.

5.3 Bismut codifferential

In the previous section we introduced the parameter $b' \geq 0$ in front of $\mathfrak{h}$ in order to make the comparison of local properties in the case $b' = 0$ and $b' = 1$ self-contained. We now fix $b' = 1$ and will use the scaling of [Bis05] recalled in the introduction where the bilinear form η_{ϕ_b} on TX is defined with a parameter $b \in \mathbb{R}^*$ as

$$\eta_{\phi_b}(U,V) = g^{TQ}(\pi_{X,*}(U), \pi_{X,*}(V)) + b\sigma(U,V) = U.\phi_b V, \quad U,V \in TX. \tag{94}$$

The bilinear form ϕ_b is associated with the map $\phi_b : TX \to T^*X$ written, by taking a symplectic basis compatible with the horizontal-vertical decompositions of TX (like $(e_i, \hat{e}^j)$) and T^*X (like $(e^i, \hat{e}_j)$), as

$$\phi_b = \begin{pmatrix} g^{TQ} & -b\mathrm{Id} \\ b\mathrm{Id} & 0 \end{pmatrix}, \quad b \neq 0. \tag{95}$$

The dual bilinear form on T^*X is denoted η^*_ϕ:

$$\eta^*_{\phi_b}(\omega,\theta) = (\phi_b^{-1}\omega).\theta \quad , \quad \phi_b^{-1} = \frac{1}{b^2}\begin{pmatrix} 0 & b\mathrm{Id} \\ -b\mathrm{Id} & g^{TQ} \end{pmatrix}.$$

With the local bases $(e,\hat{e})$, the map ϕ_b can be be simply related with the operator $\sigma : TX \to T^*X$ associated with the symplectic form. With

$$\lambda_0 = g^{TQ}_{ij}(q) e^i \wedge \mathbf{i}_{\hat{e}^i} : T^*X \to T^*X \tag{96}$$

and by assuming $(\pi_{X,*}e_i = \frac{\partial}{\partial \underline{q}^i})_{i=1,\dots d}$ orthogonal along $X_{\underline{q}_0} = T^*_{\underline{q}_0}Q$, writing

$$\begin{pmatrix} 1 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{b} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \exp\left[\begin{pmatrix} 0 & \frac{1}{b} \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$$

shows that $\phi_b = e^{\frac{\lambda_0}{b}} b\sigma$. In the decomposition

$${}^t\phi_b = b\,{}^t\sigma e^{\frac{{}^t\lambda_0}{b}} = {}^t\sigma e^{\frac{{}^t\lambda_0}{b}}({}^t\sigma^{-1})(b\,{}^t\sigma) : TX \to T^*X$$

the factor ${}^t\sigma e^{\frac{{}^t\lambda_0}{b}}\,{}^t\sigma^{-1} : T^*X \to T^*X$ can also computed with the bases according to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{1}{b} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{b} \\ 0 & 0 \end{pmatrix},$$

which leads to ${}^t\phi_b = e^{-\frac{\lambda_0}{b}}({}^t b\sigma) = -e^{-\frac{\lambda_0}{b}} b\sigma$.
Here attention must be paid on the scaling with respect to $b \in \mathbb{R}^*$ while tensorizing $b\sigma$. Actually the multiplication by b on TX or T^*X is tensorized into the multiplication by b^p on $\Lambda^p TX$ or $\Lambda^p T^*X$. Therefore it is better to use the notation

$$\sigma_b = (\otimes_{p=0}^d b^p)\sigma : \Lambda TX \to \Lambda T^*X$$
$$\phi_b = e^{\frac{\lambda_0}{b}}\sigma_b : \Lambda TX \to \Lambda T^*X\,.$$

Definition 5.7. *The linear maps ϕ_b and λ_0 are respectively given by (95), extended by tensorization as a map $\phi_b = e^{\frac{\lambda_0}{b}}\sigma_b : \Lambda TX \to \Lambda T^*X$, and (96). The same notation is used for $\phi_b = \phi_b \otimes \mathrm{Id}_{\mathfrak{f}}$ and $\lambda_0 = \lambda_0 \otimes \mathrm{Id}_{\mathfrak{f}}$ when $E = \Lambda T^*X$ or $E' = \Lambda TX$ are replaced by $F = E \otimes \pi_X^*(\mathfrak{f})$ and $F' = E' \otimes \pi_X^*(\mathfrak{f})$.*
Accordingly the sesquilinear forms η_{ϕ_b} on F' and $\eta^*_{\phi_b}$ on F are defined by

$$\eta_{\phi_b,\mathfrak{f}}(U,V) = g^{\mathfrak{f}}(U,\phi_b V) \quad , \quad \eta^*_{\phi_b,\mathfrak{f}}(\omega,\theta) = g^{\mathfrak{f}}(\phi_b^{-1}\omega,\theta)\,.$$

Finally the same notations ϕ_b and λ_0 are used with $g = g^{TQ}(\underline{q}^1,\underline{q}')$ replaced by $\hat{g} = g^{TQ}(-|\underline{q}^1|,\underline{q}')$ and $E = \Lambda T^*X, E' = \Lambda TX, F, F'$ replaced by the piecewise $\mathscr{C}^\infty$ and continuous vector bundles $\hat{E}_g, \hat{E}'_g, \hat{F}_g, \hat{F}'_g$ of Definition 2.7

The following lemma gather simple elementary properties of those maps λ_0 and ϕ_b.

Lemma 5.8. *On $\hat{F}_g$ the map λ_0 belongs to $\mathscr{C}^\infty(\overline{X}_-;L(F)) \cap \mathscr{C}^\infty(\overline{X}_+;L(F)) \cap \mathscr{C}^0(X;L(F))$ with $\Sigma_\nu\lambda_0 = \lambda_0\Sigma_\nu$. In particular it is a continuous endomorphism of $\mathscr{C}_{0,g}(\hat{F}_g)$ and $\mathscr{C}_{0,g,ev}(\hat{F}_g)$.*
The maps $\phi_b : F'\big|_{X_\mp} \to F\big|_{X_\mp}$ and $\phi_b : \hat{F}'_g \to \hat{F}_g$ fulfill the condition (80) with $\phi_b = e^{\frac{\lambda_0}{b}}\sigma_b$ and ${}^t\phi_b = -e^{-\frac{\lambda_0}{b}}\sigma_b = \phi_{-b}$.
When $(P, D(P))$ is a densely defined operator in $L^2(X;\hat{F}_g)$ (or $L^2(X_\mp;F)$) with a symplectic adjoint $(P^{\sigma_b}, D(P^{\sigma_b}))$, its left and right ϕ_b-adjoints equal

$$P^{\phi_b} = \phi_b\tilde{P}\phi_b^{-1} = e^{\frac{\lambda_0}{b}}\sigma_b\tilde{P}\sigma_b^{-1}e^{-\frac{\lambda_0}{b}} = e^{\frac{\lambda_0}{b}}P^{\sigma_b}e^{-\frac{\lambda_0}{b}}$$
$$P^{{}^t\phi_b} = P^{\phi_{-b}} = e^{-\frac{\lambda_0}{b}}P^{\sigma_b}e^{\frac{\lambda_0}{b}}$$
$$\textit{with} \quad D(P^{\phi_b}) = e^{-\frac{\lambda_0}{b}}D(P^{\sigma_b}) \quad , \quad D(P^{{}^t\phi_b}) = e^{\frac{\lambda_0}{b}}D(P^{\sigma_b})\,.$$

Proof. The first properties come from the definition of

$$\lambda_0 = e^i \wedge \mathbf{i}_{\hat{e}^i} = e^1 \wedge \mathbf{i}_{\hat{e}^1} + m_{ij}(-|q^1|, q') e^{i'} \wedge \mathbf{i}_{\hat{e}^{j'}}\,,$$

where we use $(e,\hat{e}) = 1_{Q_-}(q)(e_-,\hat{e}_-) + 1_{Q_+}(q)(e_+,\hat{e}_+)$. The regularity with respect to $x=(q,p)$ of λ_0 is inherited from the one of $(e,\hat{e})$. The commutation with Σ_ν comes from

$$\Sigma_\nu\left[s_I^J(q^1,q',p_1,p')e^I\hat{e}_J\right] = (-1)^{|I\cap\{1\}|+|J\cap\{1\}|}\nu s_I^J(-q^1,q',-p_1,p')e^I\hat{e}_J\,.$$

We know that σ, and therefore σ_b when $b\neq 0$, fulfills the condition (80). It thus suffices to check the equivalence

$$\exists C_b>0, \forall x\in X, \forall \omega\in F_x,\quad C_b^{-1}|e^{\frac{\lambda_0}{b}}\omega|_{g_x^F} \leq |\omega|_{g_x^F} \leq C_b|e^{\frac{\lambda_0}{b}}\omega|_{g_x^F}\,. \tag{97}$$

With coordinates such that $(\frac{\partial}{\partial \underline{q}^1},\ldots,\frac{\partial}{\partial \underline{q}^d})$ is orthonormal above a fixed $q_0\in Q$, $g_{ij}^{TQ}(q_0)=\delta_{ij}$, decompose $\omega\in F_{(q_0,p)}$ as $\omega=\omega^H\oplus^\perp\omega^V\in F_x = (T_x^*X^H\otimes\pi_X^*(\mathfrak{f}))\oplus^\perp (T_x^*X^V\otimes\pi_X^*(\mathfrak{f}))$ with $x=(q_0,p)$. Write simply $\omega=\begin{pmatrix}\omega^H\\ \omega^V\end{pmatrix}$ and $e^{\pm\lambda_0}\omega=\begin{pmatrix}\omega^H\pm\frac{1}{b}\omega^V\\ \omega^V\end{pmatrix}$. The g_x^F norm of ω and $e^{\pm\lambda_0}\omega$ satisfy

$$\begin{aligned}
|\omega|_{g_x^F}^2 &= \langle p\rangle^{-1}|\omega^H|^2_{g_{q_0}^{\mathfrak{f}}} + \langle p\rangle|\omega^V|^2_{g_{q_0}^{\mathfrak{f}}}\\
|e^{\pm\frac{\lambda_0}{b}}\omega|_{g_x^F}^2 &= \langle p\rangle^{-1}|\omega^H\pm\frac{1}{b}\omega^V|^2_{g_{q_0}^{\mathfrak{f}}} + \langle p\rangle|\omega^V|^2_{g_{q_0}^{\mathfrak{f}}}\\
|e^{\pm\frac{\lambda_0}{b}}\omega|_{g_x^F}^2 &\leq 2\langle p\rangle^{-1}|\omega^H|^2_{g_{q_0}^{\mathfrak{f}}} + \langle p\rangle(1+\frac{2}{\langle p\rangle_q^2 b^2})|\omega^V|^2_{g_{q_0}^{\mathfrak{f}}} \leq \max(2,1+\frac{2}{b^2})|\omega|^2_{g_x^F}\,,.
\end{aligned}$$

Applying the last inequality with $\omega=e^{\mp\frac{\lambda_0}{b}}\eta$ provides the reverse inequality

$$|\eta|^2_{g_x^F} \leq \max(2,1+\frac{2}{b^2})|e^{\mp\frac{\lambda_0}{b}}\eta|_{g_x^F}\,.$$

The equivalence (97) is thus proved with $C_b=\max(2,1+\frac{2}{b^2})$. □

Proposition 5.9. *Take $b'=1$ and $b\neq 0$.*
The ϕ_b left-adjoint of $(\hat{d}_{g,\mathfrak{h}}, D(\hat{d}_{g,\mathfrak{h}}))$ equals

$$\begin{aligned}
&D(\hat{d}^{\phi_b}_{g,b'\mathfrak{h}}) = \left\{s\in L^2(X;F)\cap e^{\frac{\lambda_0}{b}}\sigma_b\mathscr{E}_{loc}(\tilde{d},\hat{F}'_g),\quad \hat{d}^{\phi_b}_{g,\mathfrak{h}}s\in L^2(X;F)\right\},\\
&\forall s\in D(\hat{d}^{\phi_b}_{g,\mathfrak{h}}),\quad \hat{d}^{\phi_b}_{g,\mathfrak{h}}s = (\hat{d}^{\phi_b}_{g,\mathfrak{h}}s_-) + (\hat{d}^{\phi_b}_{\mathfrak{h}}s_+)\quad,\quad s_\mp = s\big|_{X_\mp}\,,\\
&\hat{d}^{\phi_b}_{g,\mathfrak{h}}s_\mp = e^{-\frac{\lambda_0}{b}}d^{\sigma_b}_{\hat{\mathfrak{h}}}e^{\frac{\lambda_0}{b}}s_\mp = \frac{1}{b}e^{-\frac{\lambda_0}{b}}d^{\sigma_b}_{\hat{\mathfrak{h}}}e^{\frac{\lambda_0}{b}}s_\mp\,.
\end{aligned}$$

It satisfies $\hat{d}^{\phi_b}_{g,\mathfrak{h}}\circ\hat{d}^{\phi_b}_{g,\mathfrak{h}}=0$ and $\hat{d}^{\phi_b}_{g,\mathfrak{h}}\circ\Sigma_\nu=\Sigma_\nu\circ\hat{d}^{\phi_b}_{g,\mathfrak{h}}$. In particular, $\hat{d}^{\phi_b}_{g,\mathfrak{h}}$ preserves the parity:

$$\begin{aligned}
&D(\hat{d}^{\phi_b}_{g,\mathfrak{h}}) = D(\hat{d}^{\phi_b}_{g,\mathfrak{h}})\cap L^2_{ev}(X;F)\oplus D(\hat{d}^{\phi_b}_{g,\mathfrak{h}})\cap L^2_{odd}(X;F),\\
\textit{with}\quad &\hat{d}^{\phi_b}_{g,\mathfrak{h}}: D(\hat{d}^{\phi_b}_{g,\mathfrak{h}})\cap L^2_{ev\ odd}(X;F)\to L^2_{ev\ odd}(X;F).
\end{aligned}$$

The subset $\mathscr{C}_{0,g}(\hat{F}_g)$ of Definition 3.3 is dense in $D(\hat{d}^{\phi_b}_{g,\mathfrak{h}})$. Additionally there exists a dense subset $e^{\frac{\lambda_0}{b}}\sigma_b\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}$ of $\mathscr{C}_{0,g}(\hat{F}_g)$ such that $\hat{d}^{\phi_b}_{g,\mathfrak{h}}(e^{\frac{\lambda_0}{b}}\sigma_b\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}) \subset \mathscr{C}_{0,g}(\hat{F}_g) \subset D(\hat{d}^{\phi_b}_{g,\mathfrak{h}})$.
The adjoint $d^{\phi_b}_{g,\mathfrak{h}}$ of $(\overline{d}_{g,\mathfrak{h}}, D(\overline{d}_{g,\mathfrak{h}}))$ is densely defined and closed with

$$D(d^{\phi_b}_{g,\mathfrak{h}}) = \left\{ s \in L^2(X_-,F), \quad d^{\phi_b}_{\mathfrak{h}} s \in L^2(X_-,F), \quad \frac{1-\hat{S}_\nu}{2}\hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1} s\big|_{X'} = 0 \right\}.$$

$$\forall s \in D(d^{\sigma_b}_{g,b'\mathfrak{h}}), \quad d^{\phi_b}_{g,\mathfrak{h}} s = e^{-\frac{\lambda_0}{b}} d^{\sigma_b}_{g,\mathfrak{h}} e^{\lambda_0} s = \frac{1}{b} e^{-\frac{\lambda_0}{b}} d^{\sigma_b}_{g,\mathfrak{h}} e^{\lambda_0} s.$$

This adjoint operator $(d^{\phi_b}_{g,\mathfrak{h}}, D(d^{\phi_b}_{g,\mathfrak{h}}))$ satisfies $d^{\phi_b}_{g,\mathfrak{h}} \circ d^{\phi_b}_{g,\mathfrak{h}} = 0$.
The spaces $\mathscr{C}^\infty_0(\overline{X}_-;F) \cap D(d^{\phi_b}_{g,\mathfrak{h}})$, $\mathscr{C}_g = \{s \in L^2(X_-,F), s_{ev} \in \mathscr{C}_{0,g}(\hat{F}_g)\}$ and $e^{\frac{\lambda_0}{b}}\mathscr{D}_{g,\nabla^{\mathfrak{f}'}} = \left\{s \in L^2(X_-;F), s_{ev} \in e^{\frac{\lambda_0}{b}}\sigma_b\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}\right\}$ are dense in $D(d^{\phi_b}_{g,\mathfrak{h}})$ with $d^{\phi_b}_{g,\mathfrak{h}} e^{\frac{\lambda_0}{b}}\mathscr{D}_{g,\nabla^{\mathfrak{f}'}} \subset \mathscr{C}_g$.
Finally the ϕ_b right-adjoint is simply the ϕ_{-b} left-adjoint.

Proof. Most of the properties are derived from the properties of the symplectic adjoints by conjugation with $e^{-\frac{\lambda_0}{b}}$, according to Lemma 5.8
One thing to be checked is the simplified writing of the boundary conditions for $s \in D(d^{\phi_b}_{g,\mathfrak{h}})$. Actually $s \in D(d^{\phi_b}_{g,\mathfrak{h}}) = e^{\frac{\lambda_0}{b}} D(d^{\sigma_b}_{g,\mathfrak{h}})$ contains the boundary condition

$$\frac{1-\hat{S}_\nu}{2}\hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1} e^{-\frac{\lambda_0}{b}} s\big|_{X'} = 0. \tag{98}$$

With

$$\lambda_0 = \underbrace{e^1 \wedge \mathbf{i}_{\hat{e}^1}}_{=\lambda_0^1} + \underbrace{m_{i'j'}(-|q^1|,q')e^{i'} \wedge \mathbf{i}_{\hat{e}^{j'}}}_{=\lambda_0'} \quad \text{with} \quad \lambda_0^1\lambda_0' = \lambda_0'\lambda_0^1$$

and $\hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1}\lambda_0' = \lambda_0'\hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1}$ and $\hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1}\lambda_0^1 = 0$ we obtain:

$$\hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1} e^{-\frac{\lambda_0}{b}} s = \hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1} e^{-\frac{\lambda_0^1}{b}} e^{-\frac{\lambda_0'}{b}} s = e^{-\frac{\lambda_0'}{b}} \hat{e}^1 \wedge \mathbf{i}_{\hat{e}^1} s.$$

Since $\hat{S}_\nu$ commutes with $e^{-\frac{\lambda_0'}{b}}$ the boundary condition (98) is equivalent to

$$\frac{1-\hat{S}_\nu}{2}\hat{e}_1 \wedge \mathbf{i}_{\hat{e}^1} s\big|_{X'} = 0.$$

□

Remark 5.10. a) *No explicit expression was given for the differential operator $d^{\phi_g}_{\mathfrak{h}}$. It is not really necessary and actually more confusing when properties along the boundaries are considered. Such an expression may be found in [Bis05]:*

$$\begin{aligned} d^{\phi_b}_{\mathfrak{h}} &= \frac{1}{b}(d^\sigma_0 - \mathbf{i}_{Y_\mathfrak{h}}) - \frac{1}{b^2}[d^\sigma_0 - \mathbf{i}_{Y_\mathfrak{h}}, \lambda_0] \\ &= -\frac{1}{b}\mathbf{i}_{\hat{e}^i}\big(\nabla^{F,g}_{e_i} + \omega(\mathfrak{f}, g^{\mathfrak{f}})(e_i) - \underbrace{\nabla_{e_i}\mathfrak{h}}_{=0}\big) - \frac{1}{b}\mathbf{i}_{e_i}\big(\frac{\partial}{\partial p_i} - g^{ik}(q)p_k\big) - \frac{1}{b}R^{TQ} p \wedge \\ &\qquad -\frac{1}{b^2}\mathbf{i}_{\hat{e}^i}\big(\frac{\partial}{\partial p_i} - g^{ik}(q)p_k\big). \end{aligned}$$

b) *Note that the set* $e^{\frac{\lambda_0}{b}}\mathscr{D}_{g,\nabla\mathfrak{f}'}$ *for the* ϕ_b *left-adjoint and* $e^{-\frac{\lambda_0}{b}}\mathscr{D}_{g,\nabla\mathfrak{f}'}$ *for the* ϕ_b *right-adjoint differ. In general, it is not possible to find a same core of smooth sections, which is sent simultaneously to* $\mathscr{C}_g$ *by* $d^{\phi_b}_{g,\mathfrak{h}}$ *and* $d^{\phi_{-b}}_{g,\mathfrak{h}}$*, especially when the second fundamental form of* $Q' \subset (Q, g^{TQ})$ *does not vanish. This is a curvature problem actually similar to the distinction between* $\mathscr{D}_{g,\nabla\mathfrak{f}}$ *and* $\mathscr{D}_{g,\nabla\mathfrak{f}'}$.

c) *However* $\mathscr{D}_{g,\nabla\mathfrak{f}}$ *is dense in* $\mathscr{C}_g$ *and therefore a core for all the operators* $\overline{d}_{g,\mathfrak{h}}$, $d^{\sigma_b}_{g,\mathfrak{h}}$ *and* $d^{\phi_b}_{g,\mathfrak{h}}$ *with*

$$\overline{d}_{g,\mathfrak{h}}\mathscr{D}_{g,\nabla\mathfrak{f}} \subset \mathscr{C}_g \subset D(\overline{d}_{g,\mathfrak{h}}) \cap D(\overline{d}^{\sigma_b}_{g,\mathfrak{h}}) \cap D(\overline{d}^{\phi_b}_{g,\mathfrak{h}}).$$

Symmetric versions with $(\overline{d}_{g,\mathfrak{h}}, \mathscr{D}_{g,\nabla\mathfrak{f}})$ *replaced by* $(d^{\phi_b}_{g,\mathfrak{h}}, e^{\frac{\lambda_0}{b}}\mathscr{D}_{g,\nabla\mathfrak{f}'})$ *for the* ϕ_b *left-adjoint or* $(d^{\phi_{-b}}_{g,\mathfrak{h}}, e^{-\frac{\lambda_0}{b}}\mathscr{D}_{g,\nabla\mathfrak{f}'})$ *for the* ϕ_b *right-adjoint hold true.*

6 Closed realizations of the hypoelliptic Laplacian

This section is split in several parts which follow the general scheme for the analysis of the differential and its adjoints.

- In Subsection 6.1 we review the known results of [Bis05][BiLe][Leb1][Leb2] for the hypoelliptic Laplacian when (Q, g^{TQ}) is a smooth compact riemannian manifold. In particular we recall the class of Geometric Kramers-Fokker-Planck operators introduced in [Leb1][Leb2].
- The Subsection 6.2 focuses on trace theorem local forms of Geometric Kramers-Fokker-Planck operators.
- The definitions of closed realizations of the hypoelliptic Laplacian acting on sections of $\hat{F}_g$ or of $F\big|_{X_-}$ with boundary conditions are given in Subsection 6.3. Global subelliptic estimates derived from the one of the scalar case in [Nie] are reviewed.
- In Subsection 6.4 improved global estimates are given for powers of the resolvent and the semigroup associated with the maximal accretive closed realizations of the hypoelliptic Laplacian.
- The commutation of the resolvent of the closed maximal accretive realizations of the hypoelliptic Laplacian, with the differential and Bismut's codifferential are proved in Subsection 6.5. Because it concerns commutation of closed unbounded operators it is better to adapt the strategy of [ABG] where instead of a $\mathscr{C}_0$-group the closed realizations of the hypoelliptic Laplacian generate "cuspidal" semigroups.

- Finally Subsection 6.6 is concerned with PT-symmetry which implies that the spectrum of the closed realizations of the hypoelliptic Laplacian is symmetric with respect to the real axis. This property, which actually holds only on dense set of the domain or at the formal level, is crucial when the asymptotic spectral analysis in the specific regimes is considered.

6.1 The hypoelliptic Laplacian in the smooth case

We review here definitions and properties useful for the analysis of Bismut's hypoelliptic Laplacian when $X = T^*Q$ and (Q, g^{TQ}) is a smooth closed compact riemannian manifold. The vector bundle F equals $\Lambda T^*X \otimes \pi_X^*(\mathfrak{f})$ where $(\mathfrak{f}, \nabla^{\mathfrak{f}}, g^{\mathfrak{f}})$ is a smooth vector bundle on Q endowed with the smooth hermitian metric $g^{\mathfrak{f}}$, $\nabla^{\mathfrak{f}}$ is a flat connection, $\mathfrak{f}$ is identified with its antidual via the hermitian metric and $\nabla^{\mathfrak{f}'}$ denotes the antidual flat connection. Bismut's hypoelliptic Laplacian is defined as the differential operator

$$\begin{aligned} B_{\mathfrak{h}}^{\phi_b} &= \frac{1}{4}(d_{\mathfrak{h}}^{\phi_b} + d_{\mathfrak{h}})^2 = \frac{1}{4}(d_{\mathfrak{h}}^{\phi_b} d_{\mathfrak{h}} + d_{\mathfrak{h}} d_{\mathfrak{h}}^{\phi_b}) \\ \text{with} \qquad \mathfrak{h}(q,p) &= \frac{g^{TQ,ij}(q) p_i p_j}{2}, \\ \phi_b &= e^{\frac{\lambda_0}{b}} \sigma_b, \quad b \in \mathbb{R}^*, \end{aligned}$$

where ϕ_b and λ_0 were given in the Introduction and in Definition 5.7.
It is a differential operator with $\mathscr{C}^\infty(X; L(F))$ coefficients, acts naturally on $\mathscr{C}_0^\infty(X;F)$ and $\mathscr{D}'(X;F)$ and satisfies

$$B_{\mathfrak{h}}^{\phi_b} d_{\mathfrak{h}} = d_{\mathfrak{h}} B_{\mathfrak{h}}^{\phi_b} \quad , \quad B_{\mathfrak{h}}^{\phi_b} d_{\mathfrak{h}}^{\phi_b} = d_{\mathfrak{h}}^{\phi_b} B_{\mathfrak{h}}^{\phi_b} d_{\mathfrak{h}} .$$

In [BiLe][Leb1][Leb2] Bismut and Lebeau developed the functional and spectral analysis of this differential operator in the L^2-space associated with the metric $\tilde{g}^F = \langle p \rangle^{N_H + N_V} g^F$ and $L^2(X; F, g^F) = \langle p \rangle_q^{\frac{N_H+N_V}{2}} L^2(X; F, \tilde{g}^F)$. With our choice of metric and the unitary equivalence $\langle p \rangle_q^{\frac{N_H+N_V}{2}} : L^2(X; F, \tilde{g}^F) \to L^2(X; F, g^F)$ their results concerns

$$\langle p \rangle_q^{\frac{N_H+N_V}{2}} B_{\mathfrak{h}}^{\phi_b} \langle p \rangle_q^{-\frac{N_H+N_V}{2}} = B_{\mathfrak{h}}^{\phi_b} + \langle p \rangle_q^{\frac{N_H+N_V}{2}} [B_{\mathfrak{h}}^{\phi_b}, \langle p \rangle_q^{-\frac{N_H+N_V}{2}}].$$

We checked in Proposition 3.14-e) and the following discussion that $\langle p \rangle_q^{\frac{N_H+N_V}{2}}$: $\mathscr{W}^\mu(X; F, \tilde{g}^F) \to \mathscr{W}^\mu(X; F, g^F)$ is an isomorphism. The results of [Leb1][Leb2] allow to absorb error terms due to the conjugation via $\langle p \rangle_q^{\pm\frac{N_V+N_H}{2}}$ because they are concerned with the following general class of operators.

Definition 6.1. *The metric $\gamma = g^F$, $\gamma = \tilde{g}^F = \langle p \rangle_q^{N_H+N_V} g^F$ induces a norm $|\,|_\gamma$ on $L(F,F)$. The associated class of γ-symbols of order $m \in \mathbb{R}$ is defined as the set of functions $M \in \mathscr{C}^\infty(X; L(F,F))$ such that*

$$\begin{aligned} &\forall \alpha, \beta \in \mathbb{N}^d, \exists C_{\alpha,\beta} > 0, \forall x = (q,p) \in X, \quad |(\nabla_e^F)^\alpha (\nabla_{\hat{e}}^F)^\beta M(x)|_\gamma \leq C_{\alpha,\beta} \langle p \rangle_q^{m-|\beta|}, \\ \textit{with} \quad &(\nabla_e^F)^\alpha (\nabla^F)_{\hat{e}}^\beta = (\nabla_{e_1}^F)^{\alpha_1} \dots (\nabla_{e_d}^F)^{\alpha_d} (\nabla_{\hat{e}^1}^F)^{\beta_1} \dots (\nabla_{\hat{e}^d}^F)^{\beta_d} . \end{aligned}$$

A geometric Kramers-Fokker-Planck (GKFP) operator for the metric γ is a differential operator acting on $\mathscr{C}^\infty(X;E)$ and $\mathscr{D}'(X;E)$ of the following form:

$$\begin{aligned}
&\mathscr{A}_{\alpha,\mathscr{M}} = \mathscr{O} + \nabla^F_{\alpha Y_{\mathfrak{h}}} + \mathscr{M}\,,\\
\textit{with}\qquad &\mathscr{O} = \frac{-\Delta_p + |p|_q^2 + 2N_V - d}{2} = \frac{-g_{ij}(q)\nabla^F_{\frac{\partial}{\partial p_i}}\nabla^F_{\frac{\partial}{\partial p_j}} + g^{ij}(q)p_ip_j + 2N_V - d}{2}\\
&\mathfrak{h}(q,p) = \frac{|p|_q^2}{2} = \frac{g^{ij}(q)p_ip_j}{2}\,, \quad \alpha \in \mathbb{R}^*\,,\\
&\nabla^F_{\alpha Y_{\mathfrak{h}}} = \alpha g^{ij}(q)p_j\nabla^F_{e_i}\\
&\mathscr{M} = \mathscr{M}_{0,j}\nabla^F_{\frac{\partial}{\partial p_j}} + \mathscr{M}_0^j p_j + \mathscr{M}_0\,,
\end{aligned}$$

where $\mathscr{M}_{0,j}, \mathscr{M}_0^j, \mathscr{M}_0$ are γ-symbols of order 0.

Let us first consider the case $\mathfrak{f} = Q \times \mathbb{C}$ with the trivial connection which, as we will see below, is not a restriction. Because $\nabla^F_{e_i}\langle p\rangle_q = 0$ and $\nabla^F_{\hat{e}^j}\langle p\rangle_q^s = \mathscr{O}(\langle p\rangle_q^{s-1})$ for $s \in \mathbb{R}$, the same discussion as the one following Proposition 3.14 about the equivariance of $\mathscr{W}^\mu$-spaces, shows that conjugating by $G = \langle p\rangle_q^{\pm\frac{N_H+N_V}{2}}$ transfers GKFP operators for g^F to GKFP operators to $\tilde{g}^F$. Results of [BiLe][Leb1][Leb2] are formulated with the metric $\tilde{g}^F$. By working with those weighted metrics g^F and $\tilde{g}^F$, the definition of GKFP operators is the same when $\nabla^F_{\alpha Y_{\mathfrak{h}}}$ is replaced by the Lie derivative $\mathscr{L}_{Y_{\alpha\mathfrak{h}}}$ and we refer the reader to [Leb1]-formula (20) or [Nie]-(113)(114). The term $\mathscr{M}$ in $\mathscr{A}_{\alpha,\mathscr{M}}$ is actually a perturbation which can be absorbed by the regularity estimates for $\mathscr{A}_{\alpha,0}$. Additionally to the change of metric, error terms due to partitions of unity which allow to localize the analysis also appear as type $\mathscr{M}$ corrections:

- **Partition of unity in the q-variable:** For a partition in unity in $q \in Q$, Q is a closed compact manifold, $\sum_{n=1}^N \chi_n(q) \equiv 1$, the comparison is given by

$$\mathscr{A}_{\alpha,\mathscr{M}} - \sum_{n=1}^N \mathscr{A}_{\alpha,\mathscr{M}}\chi_n = \mathscr{M}_{\alpha,\mathscr{M},\chi}\,.$$

  So the analysis can be localized in a ball $B(q_0,\varrho)$ where the coordinates (q,p) can be used. Additionally changing $\mathfrak{f}\big|_{B(q_0,\varrho)}$ with the connection $\nabla^{\mathfrak{f}}$ by $Q \times \mathbb{C}^{d_{\mathfrak{f}}}$ with the trivial connection and replacing g^{TQ} by the euclidean metric in $B(q_0,\varrho)$, simply adds a term $\mathscr{M}_{g,\mathfrak{f}}$. For L^2 or $\mathscr{W}^\mu$ estimates one rather uses a partition of unity $\sum_{n=1}^N \chi_n^2(q) = 1$ while comparing

$$\|\mathscr{A}_{\alpha,\mathscr{M}}\omega\|^2_{\mathscr{W}^\mu} - \sum_{n=1}^N \|\mathscr{A}_{\alpha,\mathscr{M}}\chi_n\omega\|^2_{\mathscr{W}^\mu}$$

  but the idea is the same.

- **Dyadic partition of unity in the p-variable:** After the localization in the q-variable and the reduction to the scalar case, a dyadic partition of unity

$\sum_{j=0}^{\infty}\theta_j(p)=\chi_0(|p|_q)+\sum_{j=1}^{\infty}\chi_1\left(2^{-j}|p|_q\right)\equiv 1$ is used with

$$\mathscr{A}_{\alpha,\mathscr{M}}-\sum_{j=0}^{\infty}\mathscr{A}_{\alpha,\mathscr{M}}\theta_j(p)=\mathscr{M}'_{\alpha,\mathscr{M},\chi}\,.$$

Meanwhile using $\sum_{j=0}^{\infty}\theta_j^2(p)=\chi_0^2(|p|_q)+\sum_{j=1}^{\infty}\chi_1^2(2^{-j}|p|_q)\equiv 1$, lead to comparable error terms for

$$\|\mathscr{A}_{\alpha,\mathscr{M}}u\|^2_{\mathscr{W}^\mu}-\sum_{j=0}^{\infty}\|\mathscr{A}_{\alpha,\mathscr{M}}\theta_j u\|^2_{\mathscr{W}^\mu}\,.$$

This allowed Lebeau to reduced the global subellipticity estimates to parameter dependent local in a fixed ball or shell, and rather standard, subelliptic estimates uniform with respect to the small parameter $h=2^{-j}\to 0$.

Those use of partition of unity are actually the same as the one used for characterizing the spaces $\mathscr{W}^\mu(X;F)$ in terms of standard parameter dependent usual pseudodifferential calculus.
This led Lebeau in [Leb2] to the following optimal results, that we translate with our metric g^F and our spaces $L^2(X;F)$, $\mathscr{W}^\mu(X;F)$ for a GKFP operator $\hat{A}_{\alpha,\mathscr{M}}$, $\alpha\neq 0$:

- For $\mu\in\mathbb{R}$ there exist two constants $C_{\mu,\alpha,\mathscr{M}}>0$ and $C_{\alpha,\mathscr{M}}>0$, the latter independent of μ, such that the estimate

$$\|\mathscr{O}s\|_{\mathscr{W}^\mu}+\|\nabla^F_{Y_\mathfrak{h}}s\|_{\mathscr{W}^\mu}+\|s\|_{\mathscr{W}^{\mu+2/3}}+\delta_{0,\mu}\langle\lambda\rangle^{1/2}\|s\|_{\mathscr{W}^\mu}$$
$$\leq C_{\mu,\alpha,\mathscr{M}}\|(C_{\alpha,\mathscr{M}}+\mathscr{A}_{\alpha,\mathscr{M}}-i\delta_{0,\mu}\lambda)s\|_{\mathscr{W}^\mu}\,, \quad (99)$$

  holds true for all $\lambda\in\mathbb{R}$ and all $s\in\mathscr{S}'(X;F)$, satifying $(C_{\alpha,\mathscr{M}}+\mathscr{A}_{\alpha,\mathscr{M}}-i\delta_{0,\mu}\lambda)s\in\mathscr{W}^\mu(X;F)$. Remember that we do not use the λ-dependent $\mathscr{W}^\mu$-norms of [Leb2] here.

- The above constant $C_{\alpha,\mathscr{M}}>0$ can be chosen such that $C_{\alpha,\mathscr{M}}\geq C_{0,\alpha,\mathscr{M}}$ and $C_{\alpha,\mathscr{M}}+\mathscr{A}_{\alpha,\mathscr{M}}$ with the domain $D(\mathscr{A}_{\alpha,\mathscr{M}})=\left\{s\in L^2(X;F),\ \mathscr{A}_{\alpha,\mathscr{M}}s\in L^2(X;F)\right\}$ is maximally accretive in $L^2(X;F)$ with

$$\forall s\in D(\mathscr{A}_{\alpha,\mathscr{M}}),\quad \|\nabla^F_p s\|^2_{L^2}+\||p|_q s\|^2_{L^2}+\|u\|^2_{L^2}\leq C_{\alpha,\mathscr{M}}\,\mathbb{R}\mathrm{e}\,\langle s\,,(C_{\alpha,\mathscr{M}}+\mathscr{A}_{\alpha,\mathscr{M}})s\rangle\,.$$

- The subspaces $\mathscr{S}(X;E)$ and $\mathscr{C}_0^\infty(X;E)$ are dense in $D(\mathscr{A}_{\alpha,\mathscr{M}})$ endowed with its graph norm.

- The adjoint $\mathscr{A}^*_{\alpha,\mathscr{M}}$, for the $L^2(X;F)$-scalar product, of $(\mathscr{A}_{\alpha,\mathscr{M}},D(\mathscr{A}_{\alpha,\mathscr{M}}))$ is a GKFP operator of the form $\mathscr{A}_{-\alpha,\mathscr{M}'}$ and has the same properties as $\mathscr{A}_{\alpha,\mathscr{M}}$.

As a consequence of the maximal accretivity of $(C_{\alpha,\mathscr{M}}+\mathscr{A}_{\alpha,\mathscr{M}},D(\mathscr{A}_{\alpha,\mathscr{M}}))$ in $L^2(X;F)$ and the lower bound $C_{\alpha,\mathscr{M}}\|(C_{\alpha,\mathscr{M}}+\mathscr{A}_{\alpha,\mathscr{M}}-i\lambda)s\|_{L^2}\geq\langle\lambda\rangle^r\|s\|_{L^2}$, $0<r<1$ contained in (99) with $\mu=0$ and $r=\frac{1}{2}$ actually implies

$$\mathrm{Spec}(C_{\alpha,\mathscr{M}}+\mathscr{A}_{\alpha,\mathscr{M}})\subset\left\{z\in\mathbb{C},\quad \mathbb{R}\mathrm{e}\,z\geq C'_{\alpha,\mathscr{M}}|\mathrm{Im}\,z|^r\right\}$$

and the representation formula of the semigroup

$$e^{-t\mathscr{A}_{\alpha,\mathscr{M}}}=\frac{1}{2i\pi}\int_{\gamma_{\alpha,\mathscr{M}}}\frac{e^{-tz}}{z-\mathscr{A}_{\alpha,\mathscr{M}}}\,dz\,,\quad t>0\,,$$

with $\gamma_{\alpha,\mathscr{M}}$ oriented from $+i\infty$ to $-i\infty$ and given by

$$\gamma_{\alpha,\mathscr{M}}=\left\{z\in\mathbb{C}\,,\,\mathbb{R}\mathrm{e}\,z+C_{\alpha,\mathscr{M}}\geq C'_{\alpha,\mathscr{M}}|\mathrm{Imag}\,z|^{r}\right\}.$$

This enters the class of "cuspidal semigroups" for which the relationships with subelliptic estimates and various functional analytic characterizations have been explained in [HerNi], [HeNi] and [Nie].

The fact that Bismut's hypoelliptic Laplacian $B_{\mathfrak{h}}^{\phi_b}$, or more exactly $2b^2B_{\mathfrak{h}}^{\phi_b}=\mathscr{A}_{-b,\mathscr{M}}$, is a GKFP operator actually comes from the Weitzenbock type formula given in [Bis05]

$$\begin{aligned}B_{\mathfrak{h}}^{\phi_b}=\frac{1}{4b^2}&\left[-\Delta_p+|p|_q^2-\frac{1}{2}\langle R^{TQ}(e_i,e_j)e_k\,,e_\ell\rangle e^ie^j\mathbf{i}_{\hat{e}^k\hat{e}^\ell}+2N_V-\dim Q\right]\\&-\frac{1}{2b}\left[\mathscr{L}_{Y_{\mathfrak{h}}}+\frac{1}{2}\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})(Y_{\mathfrak{h}})+\frac{1}{2}e^i\mathbf{i}_{\hat{e}^j}\nabla^F_{e_i}\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})(e_j)\right.\\&\qquad\left.+\frac{1}{2}\omega(\nabla^{\mathfrak{f}},g^{\mathfrak{f}})(e_i)\nabla^F_{\hat{e}^i}\right].\end{aligned}\tag{100}$$

We refer the reader to [Leb1] for the detailed verification that it is a GKFP and simply recall that the weighted metric ($\tilde{g}^F$ or g^F) is convenient in the verification that $\mathscr{L}_{Y_{\mathfrak{h}}}-\nabla^F_{Y_{\mathfrak{h}}}$ enters in the perturbation term like $\mathscr{M}$ in Definition 6.1.

Although it is simpler to work with the adjoint associated with the usual $L^2(X;F)$ scalar product, example given when the maximal accretivity is considered, calculations which involve $d_{\mathfrak{h}}$, $d_{\mathfrak{h}}^{\phi_b}$ and $B_{\mathfrak{h}}^{\phi_b}$ are easier by using the ϕ_b left or right-adjoint. The comparison of the standard adjoint P^* and the adjoints $\tilde{P}$, P^{ϕ} and $P^{t\phi}$ of a densely defined operator $(P,D(P))$ in $L^2(X;F)$ was explained in Subsection 3.3.1. Remember also that the ϕ_b right-adjoint is nothing but $P^{t\phi_b}=P^{\phi_{-b}}$, which is the ϕ_{-b} left-adjoint.

In particular the relation

$$\forall s,s'\in\mathscr{S}(X;F),\quad\langle d_{\mathfrak{h}}^{\phi_b}s\,,s'\rangle_{\phi_b}=\langle s\,,d_{\mathfrak{h}}s'\rangle_{\phi_b}\,,$$

leads to

$$\forall s,s'\in\mathscr{S}(X;F),\quad\langle B_{\mathfrak{h}}^{\phi_b}s\,,s'\rangle_{\phi_b}=\langle(d_{\mathfrak{h}}^{\phi_b}d_{\mathfrak{h}}+d_{\mathfrak{h}}d_{\mathfrak{h}}^{\phi_b})s\,,s'\rangle_{\phi_b}=\langle s\,,B_{\mathfrak{h}}^{\phi_{-b}}s'\rangle_{\phi_b}$$

or

$$(B_{\mathfrak{h}}^{\phi_b})^{\phi_{-b}}=(B_{\mathfrak{h}}^{\phi_b})^{t\phi_b}=B_{\mathfrak{h}}^{\phi_{-b}}\quad,\quad(B_{\mathfrak{h}}^{\phi_{-b}})^{\phi_b}=B_{\mathfrak{h}}^{\phi_b}\,.$$

6.2 Trace properties for local geometric Kramers-Fokker-Planck operators

Studying the existence of a trace along X' is a local problem. In order to take advantage of the local flexibility, we introduce a wider class of GKFP-operators which have a good local behaviour. We consider firstly the action of those operators on sections of the smooth vector bundle $\pi_F : F \to X$ and we will in a second step consider their properties when acting on sections of the restricted vector bundles $F\big|_{\overline{X}_\mp}$ and sections of the piecewise $\mathscr{C}^\infty$ and continuous vector bundle $\hat{F}_g$. For the latter, a limited regularity of the coefficients is required.

Definition 6.2. *Consider the case where the metric $g = g^{TQ}$ is a $\mathscr{C}^\infty$ metric while the metric $g^{\mathfrak{f}}$ on the flat vector bundle $(\mathfrak{f}, \nabla^{\mathfrak{f}})$ piecewise $\mathscr{C}^\infty$ and continuous like the metric $\hat{g}^{\mathfrak{f}}$ of Definition 2.2. A local geometric Kramers-Fokker-Planck (shortly LGKFP) operator for the metric $g = g^{TQ}$ is a differential operator which can be written locally above any local chart on Q as:*

$$\begin{aligned}
\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}} &= (g^{-1}\kappa)^{ij}(q)p_j\nabla^F_{e_i} - \frac{\gamma_{ij}(q)\nabla^F_{\frac{\partial}{\partial p_i}}\nabla^F_{\frac{\partial}{\partial p_j}}}{2} + \mathscr{M}\\
&= \mathscr{A}^{g,\mathrm{Id},g}_{\alpha,0} + \mathscr{M} + \frac{\Delta^g_p - \Delta^\gamma_p}{2} + \alpha(g^{-1}(\kappa - \mathrm{Id}))^{ij}(q)p_j\nabla^F_{e_i},\\
\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,0} &= \alpha\nabla^F_{Y_{\mathfrak{h}}} - \frac{\Delta^g_p}{2} = \alpha g^{ij}(q)p_j\nabla^F_{e_i} - \frac{g_{ij}(q)\nabla^F_{\frac{\partial}{\partial p_i}}\nabla^F_{\frac{\partial}{\partial p_j}}}{2},\\
\textit{with} \quad & \alpha \in \mathbb{R}^* \quad , \quad \mathscr{M} = \mathscr{M}_j(q,p)\nabla^F_{\frac{\partial}{\partial p_j}} + \mathscr{M}_0(q,p),\\
\textit{and} \quad & \mathscr{M}_j, \nabla^F_{\frac{\partial}{\partial p_k}}\mathscr{M}_j, \mathscr{M}_0 \in L^\infty_{loc}(X; L(F,F)).
\end{aligned}$$

An element $\mathscr{M}$ like above will be called a locally admissible perturbation. Admissible metrics γ are Lipschitz continuous metric $\gamma \in W^{1,\infty}(Q; T^*Q \odot T^*Q)$ and admissible factors κ belong to $\mathscr{C}^\infty(\overline{Q}_-; L(T^*Q)) \cap \mathscr{C}^\infty(\overline{Q}_+; L(T^*Q)) \cap \mathscr{C}^0(Q; L(T^*Q))$. Both satisfy $\|\gamma\|_{W^{1,\infty}} + \|\kappa\|_{W^{1,\infty}} \le R$ and $\|\gamma - g\|_{L^\infty} + \|\kappa - \mathrm{Id}\|_{L^\infty} < \delta_{R,\alpha,g}$ with $\delta_{R,\alpha,g} > 0$ small enough.

The condition $\nabla^F_{\frac{\partial}{\partial p_k}}\mathscr{M}_j \in L^\infty_{loc}(X; L(F,F))$, implies

$$\mathscr{M} = \mathscr{M}_j(q,p)\nabla^F_{\frac{\partial}{\partial p_j}} + \mathscr{M}_0(q,p) = \nabla^F_{\frac{\partial}{\partial p_j}} \circ (\mathscr{M}_j(q,p)\times) + \underbrace{\mathscr{M}_0(q,p) + (\nabla^F_{\frac{\partial}{\partial p_j}}\mathscr{M}_j)(q,p)}_{\in L^\infty_{loc}(X;L(F,F))}.$$

For any open set $U \subset X$, a LGKFP operator $\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}} : \mathscr{C}^\infty_0(U;F) \to L^2_{comp}(U;F) \subset \mathscr{D}'(U;F)$ has a formal adjoint from $\mathscr{C}^\infty_0(U;F) \to \mathscr{D}'(U;F)$ for the usual $L^2(U;F)$ scalar product, which is itself a LGKFP operator $\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}$ with α changed into $-\alpha$. The map κ' is nothing but ${}^t(g^{-1}\kappa g)$. The difference between $\mathscr{M}'$ and the formal adjoint $\mathscr{M}^*$ is due to:

- $\nabla^{F'}_Y = \nabla^F_Y - \omega(\mathfrak{f}, g^{\mathfrak{f}})(\pi_{X,*}Y)$ where the terms $\omega(\mathfrak{f}, g^{\mathfrak{f}})(\frac{\partial}{\partial q^i})$ belong to $L^\infty(Q;\mathbb{R})$;
- the fact that ∇^E is not exactly the Levi-Civita connection for the metric g^E which includes the weight $\langle p\rangle_q^{N_V - N_H}\pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ})$ (remember $e_i\langle p\rangle_q = 0$ and $\hat{e}^j\langle p\rangle_q^r = \mathscr{O}(\langle p\rangle_q^{r-1})$);

- the derivatives $\nabla^F_{\frac{\partial}{\partial p_j}}\mathcal{M}_j \in L^\infty_{loc}(X;L(F,F))$;
- the derivatives with respecto to q^i of κ which belong to $L^\infty(Q;\mathbb{R})$.

When the connection ∇^E (locally the flat vector bundle $\pi_X^*(\mathfrak{f}|_U)$ can be trivialized as $\pi_X^{-1}(U)\times\mathbb{C}^{d_f}$) is replaced by a smooth connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}_{e_i}e^\ell = -\tilde{\Gamma}^\ell_{ik}(q)e^k \quad , \quad \tilde{\nabla}_{e_i}\hat{e}_j = \tilde{\Gamma}^\ell_{ij}(q)\hat{e}_k \quad \tilde{\nabla}_{\hat{e}^j}e^\ell = \tilde{\nabla}_{\hat{e}^j}\hat{e}_k = 0,$$

with $\tilde{\Gamma}^\ell_{ik}\in L^\infty(Q;\mathbb{R})$, then

$$\alpha(g\kappa)^{ij}(q)p_j\tilde{\nabla}_{e_i} - \frac{\gamma_{ij}(q)\tilde{\nabla}_{\frac{\partial}{\partial p_i}}\tilde{\nabla}_{\frac{\partial}{\partial p_j}}}{2} \tag{101}$$

is again a LGKFP operator for the metric g^{TQ}.

Definition 6.3. *Let $\alpha\in\mathbb{R}^*$ and $\Omega = X$ or $\Omega = X_\mp$. Let $\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}}$ be a LGKFP operator for the metric g^{TQ}. The space $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}},F|_{\overline{\Omega}})$ is defined according to Definition 3.1 by*

$$\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}},F|_{\overline{\Omega}}) = \left\{s\in L^2_{loc}(\overline{\Omega};F), \quad \mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}}s\in L^2_{loc}(\overline{\Omega};F)\right\}.$$

The topology of those spaces $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}};F|_{\overline{\Omega}})$ can be given by the seminorms $p_\chi(s) = \|\chi(\mathfrak{h})s\|_{L^2} + \|\chi(\mathfrak{h})\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}}s\|_{L^2}$.
Those spaces are local on $\pi_X(\overline{\Omega}) = Q$ or $\overline{Q}_\mp$ according to the next lemma. This justifies the writing of $\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}}$ with the local frame of vector fields $(e_i,\hat{e}^j = \frac{\partial}{\partial p_j})$ in Definition 6.2.

Lemma 6.4. *For any finite smooth partition of unity $\sum_{j=1}^J\chi_j(q)\equiv 1$, s belongs to $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}},F|_{\overline{\Omega}})$ if and only if $\chi_j s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}},F|_{\overline{\Omega}})$ for all $j=1,\ldots,I$.*

Proof. When $\sum_{j=1}^J\chi_j(q)\equiv 1$ the difference $\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}} - \sum_{j=1}^J\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}}\chi_j(q)$ written locally as $\sum_{j=1}^J(g\kappa)^{ik}(q)p_k(\partial_{q^i}\chi_j)(q)$ is a continuous endomorphism of $L^2_{loc}(\overline{\Omega};F|_{\overline{\Omega}})$. This yields the equivalence

$$\left(s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}},F|_{\overline{\Omega}})\right)\Leftrightarrow\left(\forall j\in\{1,\ldots,J\}\,,\chi_j s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}},F|_{\overline{\Omega}})\right).$$

□

Other properties of the spaces $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathcal{M}};F|_\Omega)$ will be given in various cases. We first consider the smooth case without boundary $\Omega = X$.
We start with a lemma about perturbed GKFP operators, which are of course specific Local Geometric Kramers-Fokker-Planck operators.

Lemma 6.5. *Consider the operator*

$$\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|_q^2}{2}} = \mathscr{A}_{\alpha,0} - N_V + d/2 + \frac{\Delta^g_p - \Delta^\gamma_p}{2} = \mathscr{A}_{\alpha,0} - N_V + d/2 + \frac{(g^{ij}(q)-\gamma^{ij}(q))\partial_{p_i}\partial_{p_j}}{2}$$

according to Definition 6.2 and Definition 6.1 with $\alpha \neq 0$, $\|\gamma - g\|_{L^\infty} \leq \delta_{g,\alpha}$. For $\delta_{g,\alpha} > 0$ small enough, $\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|_q^2}{2}}$ (resp. its formal adjoint) is a relatively bounded perturbation in $L^2(X;F)$ of $\mathscr{A}_{\alpha,0}$ with domain $D(\mathscr{A}_{\alpha,0}) = \{s \in L^2(X;F),\ \mathscr{A}_{\alpha,0}s \in L^2(X;F)\}$ (resp of the adjoint $\mathscr{A}^*_{\alpha,0} = \mathscr{A}_{-\alpha,\mathscr{M}'}$). This operator with domain

$$D(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|^2}{2}}) = D(\mathscr{A}_{\alpha,0}) = \left\{ s \in L^2(X;F),\ \mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|_q^2}{2}} s \in L^2(X;F) \right\} \subset \mathscr{W}^{2/3}(X;F)$$

is closed. Its adjoint has the domain

$$D(\mathscr{A}_{-\alpha,0}) = \left\{ s \in L^2(X;F),\ (\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|_q^2}{2}})^* s \in L^2(X;F) \right\} \subset \mathscr{W}^{2/3}(X;F).$$

By writing shortly $\mathscr{A} = \mathscr{A}_{\alpha,\frac{|p|_q^2}{2}}$ or $\mathscr{A} = (\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|_q^2}{2}})^*$ and by recalling $\mathscr{O} = \frac{-\Delta_p + |p|_q^2 + 2N_V - d}{2}$, there exists a constant $C_{\alpha,g} > 0$ such that $(C_{\alpha,g} + \mathscr{A}) : D(\mathscr{A}) \to L^2(X;F)$ is invertible and

$$(1+\mathscr{O})^t (C_{\alpha,g} + \mathscr{A})^{-1} (1+\mathscr{O})^{1-t} \in \mathscr{L}(L^2(X;F))$$

is bounded in $L^2(X;F)$ for all for all $t \in [0,1]$.

Proof. Because $\mathscr{A}_{\alpha,0}$ is a GKFP operator, the global subelliptic estimate (99) says

$$\forall s \in D(\mathscr{A}_{\alpha,0}), \quad \|\mathscr{O}s\|_{L^2} + \|s\|_{\mathscr{W}^{2/3}} \leq C_{1,\alpha,g} \|(C_{1,\alpha,g} + \mathscr{A}_{\alpha,0})s\|_{L^2}$$

with

$$D(\mathscr{A}_{\alpha,0}) = \{ s \in L^2(X;F), \quad \mathscr{A}_{\alpha,0}s \in L^2(X;F) \}.$$

The same holds for its adjoint $\mathscr{A}_{-\alpha,\mathscr{M}'}$ with the same constant $C_{1,\alpha,g}$ and $\mathscr{C}_0^\infty(X;F)$ is a core for both closed operators.
Because

$$\left\| \frac{\Delta_p^g - \Delta_p^\gamma}{2} s \right\|_{L^2} \leq \delta_{\alpha,g} \|\mathscr{O}s\|_{L^2},$$

it suffices to choose $\delta_{\alpha,g} > 0$ small enough such that $\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,0}$ and its formal adjoint are respectively bounded perturbations of $(\mathscr{A}_{\alpha,0}, D(\mathscr{A}_{\alpha,0}))$ and its adjoint with bound $\leq \frac{1}{2}$. This ensures that $(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,0}, D(\mathscr{A}_{\alpha,0}))$ and $((\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,0}), D(\mathscr{A}^*_{\alpha,0}))$ are closed with the core $\mathscr{C}_0^\infty(X;F)$. They are ajoint to each other. This yiels the characterization of $D(\mathscr{A})$ as $\{s \in L^2(X;F),\ \mathscr{A}s \in L^2(X;F)\}$ for $\mathscr{A} = \mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,0}$ and $\mathscr{A} = \mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,0}$.
Additionally there exists a constant $C_{\alpha,g} \geq 0$ such that

$$\|\mathscr{O}s\|_{L^2} + \|s\|_{\mathscr{W}^{2/3}} \leq C_{\alpha,g} \|(C_{\alpha,g} + \mathscr{A})s\|_{L^2}.$$

Hence $(1+\mathscr{O})(C_{\alpha,g} + \mathscr{A})^{-1}$ and its adjoint $(1+\mathscr{A}^*)^{-1}(1+\mathscr{O})$ are bounded operators in $L^2(X;F)$. The general result for $t \in [0,1]$ follows by interpolation. □

Lemma 6.6. *When κ is the map of Definition 6.2 and $\sum_{j=1}^J \chi_j(q) \equiv 1$ is a smooth partition of unity on Q subordinate to a chart atlas, the linear map κ_χ defined by*

$$\kappa_\chi\left(\sum_{j=1}^J \chi_j(q) s_I^J(q,p) e^I \hat{e}_J\right) = \sum_{j=1}^J \chi_j(q) s_I^J(q, \kappa(q)p) e^I \hat{e}_J$$

is a continuous automorphism of $W^{\mu,2}_{loc}(X;F)$ *for any* $\mu\in[-1,1]$*, of* $\mathscr{C}^\infty(\overline{X}_-;F)\cap\mathscr{C}^\infty(\overline{X}_+;F)\cap\mathscr{C}^0(X;F)$ *and of* $\mathscr{E}_{loc}(\Delta^g_p;F)=\left\{s\in L^2_{loc}(X;F),\ \Delta^g_p s\in L^2_{loc}(X;F)\right\}$*. It is also a continuous automorphism of the global space* $\mathscr{W}^\mu(X;F)$ *for all* $\mu\in[-1,1]$*.*

Proof. All the considered spaces are local spaces on Q. When U is a chart open set of Q we can assume $\mathfrak{f}\big|_U=U\times\mathbb{C}^{d_f}$, section s of F supported in U can be written $s=s^J_I(q,p)e^I\hat{e}_J$ with $s^J_I(q,p)\in\mathbb{C}^{d_f}$. With the $\mathscr{C}^\infty$ frame $(e^i,\hat{e}_j)$, $s\in\mathscr{W}^\mu_{loc}(\pi_X^{-1}(U);F\big|_U)$ (resp. $s\in\mathscr{C}^\infty(\overline{X}_-\cap\pi_X^{-1}(U);F)\cap\mathscr{C}^\infty(\overline{X}_+\cap\pi_X^{-1}(U);F)\cap\mathscr{C}^0(\pi_X^{-1}(U);F)$, resp. $s\in\mathscr{E}_{loc}(\Delta^g_p;F\big|_{\pi_X^{-1}(U)})$) if and only if for all $I,J\subset\{1,\dots,d\}$

$$\begin{aligned}
&s^J_I\in W^\mu_{loc}(\pi_X^{-1}(U);\mathbb{C}^{d_f})\\
\text{resp.}\quad &s^J_I\in\mathscr{C}^\infty(\overline{X}_-\cap\pi_X^{-1}(U);\mathbb{C}^{d_f})\cap\mathscr{C}^\infty(\overline{X}_+\cap\pi_X^{-1}(U);\mathbb{C}^{d_f})\cap\mathscr{C}^0(\pi_X^{-1}(U);\mathbb{C}^{d_f}),\\
\text{resp.}\quad &s^J_I\in\mathscr{E}_{loc}(\Delta^g_p;\mathbb{C}^{d_f}).
\end{aligned}$$

Because $\kappa\in\mathscr{C}^\infty(\overline{Q}_-;L(T^*Q))\cap\mathscr{C}^\infty(\overline{Q}_+;L(T^*Q))\cap\mathscr{C}^0(Q;L(T^*Q))\subset W^{1,\infty}(Q;L(T^*Q))$, those conditions are preserved by $s^J_I\mapsto s^J_I(q,\kappa(q)p)$ for $\mu\in[-1,1]$ for the first space. The same works for κ_χ^{-1}.
For the global $\mathscr{W}^\mu(X;F)$ space for $\mu\in[-1,1]$, it suffices to consider the cases $\mu=0$ and $\mu=1$ and to conclude by duality and interpolation. For $\mu=0$, we recall that $\|s\|^2_{L^2}$ is equivalent to

$$N_0(s)^2=\sum_{j,I,J}\int_X|\chi_j(q)s^J_I(q,p)|^2\langle p\rangle^{-|I|+|J|}\ |dqdp|$$

The estimate $N_0(\kappa_\chi s)\leq C_{\kappa,\chi}N_0(s)$ is then obvious.
For $\mu=1$ the $\mathscr{W}^1(X;F)$, $\|s\|^2_{\mathscr{W}^1}$ is equivalent to

$$\begin{aligned}
N_1(s)^2=&N_0(\langle p\rangle^2_q s)^2+\\
&\sum_{j,i_1,j_1I,J}\int_X\left[|\chi_j(q)\partial_{q^{i_1}}s^J_I(q,p)|^2+|\chi_j(q)\langle p\rangle_q\partial_{p_{j_1}}s^J_I(q,p)|^2\right]\langle p\rangle_q^{-|I|+|J|}\ |dqdp|
\end{aligned}$$

which leads to $N_1(\kappa_\chi s)\leq C_{\kappa,\chi}N_1(s)$. □

Remark 6.7. *We do not use the natural push forward or pull back on sections of* $F=\Lambda T^*X\otimes\pi_X^*(\mathfrak{f})$ *of* $(q,p)\mapsto(q,\kappa(q)p)$ *because it involves its differential acting e.g on* e^i *and* $\hat{e}_j$ *which is only* L^∞_{loc} *and not Lipschitz continous. Therefore* κ_* *preserves the* W^μ_{loc} *regularity for* $\mu\in[-1,1]$ *only when it acts on functions.*

Lemma 6.8. *Let us work in the framework of Definition 6.2 and Definition 6.3 with* $\Omega=X$ *while* $g=g^{TQ}$ *is a smooth metric on* Q*,* $\|\kappa\|_{W^{1,\infty}}+\|\gamma\|_{W^{1,\infty}}\leq R$ *and* $\|\kappa-\mathrm{Id}\|_{L^\infty}+\|\gamma-g\|_{L^\infty}<\delta_{R,\alpha,g}$ *with* $\delta_{\alpha,g}>0$ *small enough. For a given partition of unity* $\sum_{j=1}^J\chi_j(q)\equiv1$ *subordinate to a chart atlas on* Q*,* κ_χ *is the map defined in Lemma 6.6. The spaces* $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}},F)$ *satisfy the following properties:*

i) $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}},F)=\kappa_\chi\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,0},F)\subset W^{2/3,2}_{loc}(X;F)\cap\mathscr{E}_{loc}(\Delta^g_p;F)$ *with a continuous embedding. In particular the trace map* $s\mapsto s\big|_{X'}$ *is well-defined and continuous from* $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}},F)$ *to* $L^2_{loc}(X';F)$*.*

ii) *A section s belongs to $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}},E)$ iff for any $\chi\in\mathscr{C}^\infty_0(\mathbb{R};\mathbb{R})$, $\chi(\mathfrak{h})\kappa_\chi^{-1}s$ belongs to $D(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}})$. As a consequence $\mathscr{C}^\infty_0(\overline{X}_-;F)\cap\mathscr{C}^\infty_0(\overline{X}_+;F)\cap\mathscr{C}^0(X;F)$ is dense in $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}};F)$.*

iii) *The equality $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}},F)=\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha_1,\mathscr{M}},F)$ holds for any other choice of $\alpha_1\in\mathbb{R}^*$ as soon as*

$$\|\kappa-\mathrm{Id}\|_{L^\infty}+\|\gamma-g\|_{L^\infty}<\min(\delta_{R,\alpha,g},\delta_{R,\alpha_1,g}).$$

iv) *Let $\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}$ be the formal adjoint of $\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}}$, a section $s\in L^2_{loc}(X;F)$ belongs to $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}},F)$ iff, for any compact subset $K\subset X$, there exists a constant $C_K>0$ such that*

$$\forall s'\in\mathscr{C}^\infty_0(\overline{X}_-;F)\cap\mathscr{C}^\infty_0(\overline{X}_+;F)\cap\mathscr{C}^0(X;F),\ \mathrm{supp}\, s'\subset K,\ |\langle\mathscr{A}^{g,\kappa'\gamma}_{-\alpha,\mathscr{M}'}s',s\rangle|\leq C_K\|s'\|_{L^2}.$$

Finally the equality $\langle s',\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}}s'\rangle=\langle\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}s',s\rangle$ holds for any compactly supported $s'\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'};F)$ when $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}})$.

v) *When $\mathrm{supp}\, s\subset\pi_X^{-1}(U)$, where U is a chart open set on Q and $s=s_I^J(q,p)e^I\hat{e}_J$ on which $\mathfrak{f}|_U\simeq U\times\mathbb{C}^{d_\mathfrak{f}}$, s belongs to $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}},F)$ iff*

$$s_I^J(q,\kappa(q)^{-1}p)\quad\text{and}\quad\left(\alpha g^{ij}(q)p_je_i-\frac{g_{ij}(q)\partial_{p_i}\partial_{p_j}}{2}\right)[s_I^J(q,\kappa(q)^{-1}p)]$$

belongs to $L^2_{loc}(\pi_X^{-1}(U);\mathbb{C}^{d_\mathfrak{f}})$ for all $I,J\subset\{1,\dots,d\}$

Proof. The last statement **v)** is a technical point which will not be used afterwards. We will first prove **i)** when $\kappa=\mathrm{Id}$ then **v)** and only after the full result for **i)**. The other statements **ii) iii) iv)** will follow.
i) when $\kappa=\mathrm{Id}$: When $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}};F)$ and $\chi\in\mathscr{C}^\infty_0(\mathbb{R};\mathbb{R})$, the section $\chi(\mathfrak{h})s$ satisfies

$$\left[C_{\alpha,g}+\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|^2}{2}}\right]\chi(\mathfrak{h})s=\chi(\mathfrak{h})\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}}s+\left(C_{\alpha,g}+\frac{|p|^2}{2}-\mathscr{M}\right)\chi(\mathfrak{h})s+\left[\mathscr{M}+\frac{\Delta_p^\gamma}{2},\chi(\mathfrak{h})\right]s \tag{102}$$

where our assumptions ensure that the right-hand side belong to $(1+\mathscr{O})^{1/2}L^2(X;F)$. But Lemma 6.5 tells us that for $\delta_{R,\alpha}>0$ small enough and $C_{\alpha,g}$ large enough, $\chi(\mathfrak{h})s$ belongs to $(1+\mathscr{O})^{1/2}L^2(X;F)$. This holds for any $\chi\in\mathscr{C}^\infty_0(\mathbb{R};\mathbb{R})$ and therefore s and $\nabla^F_{\frac{\partial}{\partial p_j}}s$ belong to $L^2_{loc}(X;F)$. This has two consequences:

- Equation (102) implies that $\chi(\mathfrak{h})s$ belongs to $D(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|^2}{2}})=D(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}})$ for $\delta_{\alpha,R}\leq\delta_{\alpha,g}$, according to Lemma 6.5. With $[\mathscr{A}^{g,\mathrm{Id},g},\chi(\mathfrak{h})]s=-\left[\frac{\Delta_p^g}{2},\chi(\mathfrak{h})\right]s\in L^2(X;F)$, this implies $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}};F)$. Thus $\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}};F)\subset\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}})$ and the reverse inclusion is due to the maximal hypoellipticity of $\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}}$.

- With the maximal hypoellipticity result stated in Lemma 6.5 for $\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\frac{|p|^2}{2}}$ when $\delta_{\alpha,R}\leq\delta_{\alpha,g}$, (102) implies $\chi(\mathfrak{h})s\in W^{2/3,2}(X;F)$ and $\Delta_p^g\chi(\mathfrak{h})s\in L^2(X;F)$. This proves $\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}};F)\subset W^{2/3,2}_{loc}(X;F)\cap\mathscr{E}_{loc}(\Delta_p^g,F)$.

v) Assume $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}})$ with $\mathrm{supp}\, s\subset\pi_X^{-1}(U)$. We assume $\mathfrak{f}\big|_U=U\times\mathbb{C}^{d_f}$ with the trivial connexion, which is not restrictive because $\nabla^{\mathfrak{f}}$ is flat, and we write $s=s_I^J(q,p)e^I\hat{e}_J$ with $s_I^J(q,p)\in L^2_{loc}(\pi_X^{-1}(U);\mathbb{C}^{d_f})$. According to the discussion around (101) we can replace the connexion ∇^F by a trivial connexion simply by taking $\tilde{\Gamma}^k_{ij}(q)=0$. The condition $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}})$ simply means

$$[\alpha(g\kappa)^{ij}(q)p_je_i-\frac{\Delta_p^\gamma}{2}]s_I^J+(\mathscr{M}s)_I^J\in L^2_{loc}(\pi_X^{-1}(U);\mathbb{C}^{d_f}).$$

The change of variable $(q,p)\mapsto(q,\kappa^{-1}(q)p)$ gives $dQ=dq$, $dP=(d\kappa^{-1}(q))dqp+dp$ and

$$\frac{\partial}{\partial Q^i}=\frac{\partial}{\partial q^i}+A^k_{ij}(q)p_k\frac{\partial}{\partial p_j}\quad,\quad\frac{\partial}{\partial P}={}^t\kappa(q)^{-1}\frac{\partial}{\partial p}.$$

By recalling $e_i=\frac{\partial}{\partial q^i}+\Gamma^k_{ij}(q)p_k\frac{\partial}{\partial p_j}$, $\tilde{s}=s_I^J(q,\kappa^{-1}(q)p)e^I\hat{e}_J$ belongs to $\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma'}_{\alpha,\mathscr{M}'};F\big|_{\pi_X^{-1}(U)})$ with $\mathrm{supp}\,\tilde{s}\subset\pi_X^{-1}(U)$ where $\gamma'=\kappa(q)^{-1}\gamma{}^t\kappa(q)^{-1}$ satisfies

$$\|\gamma'\|_{W^{1,\infty}}\leq(1+R)^2R\quad,\quad\|\gamma'-g\|_{L^\infty}\leq 3)(1+R)^2\delta_{\alpha,R}.$$

Taking $\delta_{\alpha,R}>0$ small enough such that $3(1+R)^2\delta_{\alpha,R}\leq\delta_{\alpha,g}$ and the analysis of **i)** with $\kappa=\mathrm{Id}$ implies that $\tilde{s}\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,0};F\big|_{\pi_X^{-1}(U)})$. But this is equivalent to

$$\tilde{s}_I^J\quad\text{and}\quad[\alpha g^{ij}(q)p_je_i-\frac{\Delta_p^g}{2}]\tilde{s}_I^J\in L^2_{loc}(\pi_X^{-1};\mathbb{C}^{d_f})$$

for all $I,J\subset\{1,\ldots,d\}$, with $\tilde{s}_I^J(q,p)=s_I^J(q,\kappa(q)^{-1}p)$. The reverse way comes from the maximal hypoellipticity of $\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}}$.

End of i) Applying **v)** to all $\chi_j(q)s$ which is supported in a chart open set, with Lemma 6.4, says that $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}};F)$ if and only if $\kappa_\chi^{-1}s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,0};F)$. This proves $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}};F)=\kappa_\chi\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,0};F)$ and the embedding in $W^{2/3,2}_{loc}(X;F)\cap\mathscr{E}_{loc}(\eta_p^g;F)$ is a consequence of Lemma 6.6.

ii) When $\kappa=\mathrm{Id}$, the commutation $[\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}},\chi(\mathfrak{h})]=\left[-\frac{\Delta_p^g}{2},\chi(\mathfrak{h})\right]$ with $\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}})=\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}};F)\subset\mathscr{E}_{loc}(\Delta_p^g;F)$ implies that $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}};F)$ if an only if $\chi(\mathfrak{h})s\in L^2(X;F)$ and $\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}}\in L^2(X,F)$, which is $\chi(\mathfrak{h})s\in D(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}})$. It is clear that the topology of $\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}})$ is equivalently given by the family of seminorms $\tilde{p}_\chi(s)=\|\chi(\mathfrak{h})s\|_{L^2}+\|\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}}\chi(\mathfrak{h})s\|_{L^2}$ and the sequence $s_n=\chi(\frac{\mathfrak{h}}{n+1})s$ converges to $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}})$ for this topology. Once the problem is reduced to a compactly supported $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},\gamma}_{\alpha,\mathscr{M}})$ we use the properties of the GKFP operator $\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}}$. We know in this case that $\mathscr{C}_0^\infty(X;F)$ is dense in $D(\mathscr{A})^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}}$. Because

$$\mathscr{C}_0^\infty(X;F)\subset\mathscr{C}_0^\infty(\overline{X}_-;F)\cap\mathscr{C}_0^\infty(X;F)\cap\mathscr{C}^0(X;F)\subset D(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}})$$

where the last embedding is a simple application of the jump formula for the first order derivative $p_1\partial_{q^1}$ transverse to X'. The two results hold for $\kappa=\mathrm{Id}$. For a general κ we use $\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}})=\kappa_\chi\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}})$ and the fact that κ_χ is a isomorphim of $\mathscr{C}_0^\infty(\overline{X}_-;F)\cap\mathscr{C}_0^\infty(\overline{X}_+;F)\cap\mathscr{C}^0(X;F)$.

iii) A section $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}},F)$ iff $\chi(\mathfrak{h})\kappa_\chi s\in D(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}})$ when $\delta_{\alpha,R}>0$ is chosen small enough. But the maximal subelliptic estimate (99) with $\mu=0$ ensures $D(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\frac{|p|^2}{2}})=D(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha_1,\frac{|p|^2}{2}})$ for any $\alpha,\alpha_1\in\mathbb{R}^*$.

iv) As a differential operator $\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}}$ is the formal adjoint of the LGKFP operator $\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}$. So for $s\in L^2_{loc}(X;F)$ the condition of **iv)** with test functions $s'\in\mathscr{C}_0^\infty(X;F)$ is nothing but the weak formulation of $\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}}s\in L^2_{loc}(X;F)$. Therefore the condition of **iv)** with $s'\in\mathscr{C}_0^\infty(X;F)$ is equivalent to $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma}_{\alpha,\mathscr{M}})$. By assuming

$$\left|\langle\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}s'\,,\,s\rangle\right|\leq C_K\|s'\|_{L^2}$$

for all $s'\in\mathscr{C}_0^\infty(X;F)$ with $\operatorname{supp}s'\subset K$, the question is whether it holds for all $s'\in\mathscr{C}_0^\infty(\overline{X}_-;F)\cap\mathscr{C}_0^\infty(\overline{X}_+;F)\cap\mathscr{C}^0(X;F)=:D$. We know $s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa,\gamma'}_{\alpha,\mathscr{M}})\subset\mathscr{E}_{loc}(\Delta_p^g,F)$ while $D\subset W^{1,2}_{comp}(X;F)$. Because $\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}$ contains only first order derivatives in the variable q, any sequence $s'_n\in\mathscr{C}_0^\infty(X;F)$ converging to $s'\in W^{1,2}_{comp}(X;F)$ with a fixed compact support $K'\supset K$ will satisfy

$$\lim_{n\to\infty}\langle\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}s'_n\,,\,s\rangle=\langle\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}s'\,,\,s\rangle$$

By taking the limit in $\left|\langle\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'}s'_n\,,\,s\rangle\right|\leq C_{K'}\|s'_n\|_{L^2}$, this proves

$$\forall s'\in D\,,\operatorname{supp}s'\subset K\,,\quad|\langle\mathscr{A}^{g,\kappa',\gamma'}_{-\alpha,\mathscr{M}'}s'\,,s\rangle|\leq C_{K'}\|s'\|_{L^2}\,.$$

The last equality for a compactly supported $s'\in\mathscr{E}_{loc}(\mathscr{A}^{g,\kappa',\gamma}_{-\alpha,\mathscr{M}'})$ is a consequence of **ii)**. $\square$

Remark 6.9. *Actually we could have used in the proof the local maximal hypoellipticity of the scalar operator $g^{ij}(q)p_i\partial_{q^j}-\frac{\Delta_p}{2}$ instead of Lebeau's global result, which is actually derived from this local result via the dyadic partition of unity in p. Our writing is more straightforward for our purpose. We had however to use a reduction to the scalar case via Lemma 6.8-**v)**.*

The previous result is concerned with the case without boundary with the smooth vector bundle $\pi_F:F\to X$. It relies on the control of terms which contain ∂_{p_j}-derivatives by the main part. We used the maximal hypoellipticity because

the $\mathscr{W}^{2/3}_{loc}(X;E)$-regularity will be required later. It could have been done by using intregration by parts with $2\langle u\,,(C+\mathscr{O})u\rangle \geq \sum_{j=1}^{d}\|\partial_{p_j}u\|^2_{L^2}+\|p_ju\|^2_{L^2}$. None of those techniques are relevant for the case $\Omega=X_{\mp}$ or ($\Omega=X$ and F replaced by $\hat{F}_g$) as long as boundary or interface conditions are not specified.
On one side there is a natural way to define the $\mathscr{E}_{loc}(\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}},\hat{F}_g)$ by using the isometry $\hat{\Psi}^{g,g_0}$ of diagram (38) and this is where the perturbative terms $\kappa-\mathrm{Id}$ and $\gamma-g$ enter in the game. On the other side it is possible to write a trace theorem for elements of $\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}}F\big|_{\overline{X}_{\mp}})\cap\mathscr{E}_{loc}(\nabla^F_{\frac{\partial}{\partial p_1}},F\big|_{\overline{X}_{\mp}})$ by following the approach presented in [Nie]. We check below that the two different approaches are actually coherent and that the additional required regularity for $\nabla^F_{\frac{\partial}{\partial p_1}}$ when $\Omega=X_{\mp}$ is actually provided by the symmetrization technique.

Lemma 6.10. *Consider the case $\Omega=X_{\mp}$, where all the data for the vector bundle $F\big|_{\overline{\Omega}}$ are $\mathscr{C}^\infty(\overline{\Omega})$. Let $\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}}$ be a LGKFP operator with $\kappa=\mathrm{Id}$, $\gamma=g$ and $\alpha\in\mathbb{R}^*$ and let $\mathscr{E}_{loc}(A^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}},F\big|_{\overline{\Omega}})$ be given as in Definition 6.3. We assume additionally that the coefficients $\mathscr{M}_j,\mathscr{M}_0$ of $\mathscr{M}=\mathscr{M}_j(q,p)\nabla^F_{\frac{\partial}{\partial p_j}}+\mathscr{M}_0(q,p)$ belong to $\mathscr{C}^\infty(\overline{\Omega};L(F,F))$.*
Any element of $\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}},F\big|_{\overline{\Omega}})\cap\mathscr{E}_{loc}(\nabla^F_{\frac{\partial}{\partial p_1}},F\big|_{\overline{\Omega}})$ admits a trace along $X'=\partial\Omega$. More precisely for any $\chi\in\mathscr{C}^\infty_0(\mathbb{R};\mathbb{R})$ the map $s\to\chi(\mathfrak{h})s\big|_{X'}$ is well defined an continuous from $\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}},F\big|_{\overline{\Omega}})\cap\mathscr{E}_{loc}(\nabla^F_{\frac{\partial}{\partial p_1}},F\big|_{\overline{\Omega}})$ to $L^2(\mathbb{R},|p_1|dp_1;\mathscr{D}^{-2}_{T^*Q'})$where $\mathscr{D}^{-2}_{T^*Q'}$ is a $W^{-2,2}$-space defined on T^*Q'.
When $\mathscr{A}^{g,\mathrm{Id},g}_{-\alpha,\mathscr{M}'}$ is the formal adjoint of $\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}}$ the integration by parts

$$\langle s\,,\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}}s'\rangle-\langle\mathscr{A}^{g,\mathrm{Id},g}_{-\alpha,\mathscr{M}'}s\,,s'\rangle=\pm\int_{X'}\langle s\,,s'\rangle_{g^F}\;p_1|dp_1dq'dp'|.$$

holds for all $s\in\mathscr{C}^\infty_0(\overline{X_{\mp}};F)$ and $s'\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}},F\big|_{\overline{X}_{\mp}})\cap\mathscr{E}_{loc}(\nabla^F_{\frac{\partial}{\partial p_1}},F\big|_{\overline{X}_{\mp}})$.

Proof. We focus on $\Omega=\overline{X}_-$, $\partial X_-=X'$ while the other case $\Omega=\overline{X}_+$ is symmetric. With a partition of unity $\sum_{j=1}^{J}\chi_j(q)\equiv1$ on $\overline{Q}_-$, Lemma 6.4 gives the equivalence

$$\left(s\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{id},g}_{\alpha,\mathscr{M}},F\big|_{\overline{X}_-})\right)\Leftrightarrow\left(\forall j\in\{1,\ldots,J\}\,,\chi_js\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}},F\big|_{\overline{X}_-})\right),$$

while the same equivalence is obvious when $\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}}$ is replaced by $\nabla^F_{\frac{\partial}{\partial p_1}}$.
Since the existence of trace is a local problem, we may assume that $s=s^J_Ie^I\hat{e}_J\in\mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}},F\big|_{\pi_X^{-1}(U)})$ is supported in $\pi_X^{-1}(U)$, U open chart set in $\overline{X}_-$ surrounding $q_0\in\partial X_-=X'$, and replace the connection ∇^F by a connection $\tilde{\nabla}$ which is trivial in the frame $(e^i,\hat{e}_j)$ in $\pi_X^{-1}(U)$ with $\mathfrak{f}\big|_U\simeq U\times\mathbb{C}^{d_\mathfrak{f}}$. We did not, and we actually cannot, get rid of the term $\mathscr{M}$ and we obtain for all $I,J\subset\{1,\ldots,d\}$

$$s^J_I\in L^2_{loc}(\pi_X^{-1}(U);\mathbb{C}^{d_\mathfrak{f}})\quad\text{and}\quad\left(g^{ij}(q)p_je_i-\frac{g_{ij}(q)\partial_{p_i}\partial_{p_j}}{2}\right)s^J_I\in W^{-1,2}_{loc}(\pi_X^{-1}(U);\mathbb{C}^{d_\mathfrak{f}}),$$

The local coordinates may be chosen so that $g^{TQ}=(dq^1)^2\oplus^\perp m(q^1)$ while we recall:

$$e_i=\frac{\partial}{\partial q^i}-\Gamma^k_{ij}(q)p_k\frac{\partial}{\partial p_j}.$$

Meanwhile the condition $s \in \mathscr{E}_{loc}(\nabla^F_{\frac{\partial}{\partial p_1}}, F)$ says

$$\frac{\partial s^J_I}{\partial p_1} \in L^2_{loc}(\pi_X^{-1}U; \mathbb{C}^d) \quad \text{and} \quad (1+\mathscr{O}_1)^{1/2} s^J_I \in L^2_{loc}(\pi_X^{-1}(U), F),$$

where $\mathscr{O}_1 = \frac{-\Delta_{p_1}+p_1^2-1}{2}$ is the vertical one dimensional harmonic hamiltonian in the variable p_1.
By introducing an arbitrary cut-off $\chi(\mathfrak{h})$ $\chi \in \mathscr{C}_0^\infty(\mathbb{R};\mathbb{R})$, and setting $\tilde{s}^J_I = \chi(\mathfrak{h}) s^J_I$, we end with the essentially scalar problem

$$\tilde{s}^J_I \in (1+\mathscr{O}_1)^{-1/2} L^2(\pi_X^{-1}(U);\mathbb{C}^d) \subset (1+\mathscr{O}_1)^{-1/2} L^2(\mathbb{R}_- \times \mathbb{R}, |dp_1 dq^1|; \mathfrak{F}), \tag{103}$$

$$(1+\mathscr{O}_1)^{-1}\left(p_1 \partial_{q^1} + \frac{-\Delta_{p_1}+p_1^2+1}{2}\right)\tilde{s}^J_I \in (1+\mathscr{O}_1)^{-1/2} L^2(\mathbb{R}_- \times \mathbb{R}, |dp_1 dq^1|; \mathfrak{F}), \tag{104}$$

$$\text{with} \qquad \mathfrak{F} = W^{-2,2}(T^*Q';\mathbb{C}^{d_{\mathfrak{f}}}). \tag{105}$$

Since we work with a compactly supported $\tilde{s}$, any global definition $W^{\mu,2}(T^*Q';\mathbb{C}^{d_{\mathfrak{f}}})$ can be chosen.
This is exactly the situation studied in [Nie]-Chap 2) with the for the $\mathfrak{F}$-valued, $\mathfrak{F}$ a Hilbert space, section on $T^*\mathbb{R}_-$ for which a trace at $q^1 = 0$ is defined as

$$\tilde{s}^J_I\big|_{q^1=0} \in L^2(\mathbb{R}, |p_1||dp_1|; \mathfrak{F}).$$

We deduce

$$\tilde{s}\big|_{q^1=0} \in L^2(\mathbb{R}, |p_1||dp_1|; \mathscr{D}^{-2}_{T^*Q'})$$

which is the trace result if we chose $\|\omega\|^2_{\mathscr{D}^{-2}_{T^*Q'}} = \sum_{j,I,J} |(\chi_j(q')\omega)^J_I|^2_{W^{-2,2}(T^*Q';\mathbb{C}^{d_f})}$.
The integration by part is the standard one for $s, s' \in \mathscr{C}_0^\infty(\overline{X}_-;F)$ where the boundary term comes from $p_1\partial_{q^1}$. For a general $s \in \mathscr{E}_{loc}(\mathscr{A}^{g,\mathrm{Id},g}_{\alpha,\mathscr{M}}, F\big|_{\overline{\Omega}}) \cap \mathscr{E}_{loc}(\nabla^F_{\frac{\partial}{\partial p_1}}, F\big|_{\overline{\Omega}})$, we replace by $\tilde{s} = \chi(\mathfrak{h})s$ with $\tilde{\chi} \equiv 1$ in a neighborhood of $\mathrm{supp}\, s'$ and $\tilde{s}$ satisfies (103)(104)(104) while $s' \in \mathscr{C}_0^\infty(\overline{X}_-;F)$ implies $s'^J_I \in \mathscr{S}((-\infty,0]\times\mathbb{R}_{p_1}; W^{2,2}(T^*Q';\mathbb{C}^{d_{\mathfrak{f}}})$.
It thus suffices to apply [Nie]-Proposition 2.10 while replacing the $W^{-2,2}(T^*Q')$ scalar product by the $W^2 - W^{-2}$ duality product. □

We aim at providing a good domain definition for Bismut's hypoelliptic Laplacian $\hat{B}^{\phi_b}_{\mathfrak{h}}$ acting on sections of the piecewise $\mathscr{C}^\infty$ and continuous vector bundle $\hat{F}_g$ associated with the metric $\hat{g}^{TQ} = 1_{Q_-(q)} g^{TQ}_- + 1_{Q_+}(q) g^{TQ}_+$. This means that as a differential operator $\hat{B}^{\phi_b}_{\mathfrak{h}}$ is defined like $B^{\phi_b}_{\mathfrak{h}}$ on X_- and X_+ with the metric g^F_- on X_- and the metric g^F_+ on X_+ and accordingly the energy $\mathfrak{h}$ replaced by $\hat{\mathfrak{h}} = \frac{\hat{g}^{ij}(q)p_ip_j}{2}$.
The continuity of sections of $\hat{F}_g$ is expressed in the frame $(e,\hat{e}) = 1_{X_\mp}(e_\mp, \hat{e}_\mp)$ with the identification $e^i_+\big|_{\partial X_+} = e^i_-\big|_{\partial X_-}$ and $\hat{e}^j_+\big|\partial X_+ = \hat{e}^j_-\big|_{\partial X_-}$. More generally we may consider LGKFP operators $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}$ defined on $X_- \cup X_+$ associated with the metric $\hat{g}^{TQ}$ with the suitable interface condition along $X' = \partial X_- = \partial X_+$. As a differential operator on $X_- \cup X_+$ it is written

$$\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}} = \alpha \nabla^{F,\hat{g}}_{Y_{\hat{\mathfrak{h}}}} - \frac{\hat{g}^{ij}(q)\partial_{p_i}\partial_{p_j}}{2} + \mathscr{M},$$

where the coefficients $\mathscr{M}_j$, $j \in \{0,1,\dots,d\}$ of the perturbation $\mathscr{M} = \mathscr{M}_j \nabla^{F,\hat{g}}_{\frac{\partial}{\partial p_j}} + \mathscr{M}_0$ belong to $\mathscr{C}^\infty(\overline{X}_\mp;L(F))$ and therefore to $L^\infty_{loc}(X;L(\hat{F}_g))$. With the coordinates $(\tilde{q},\tilde{p})$ of Definition 2.6 the energy $\hat{h}$ satisfies

$$2\hat{h} = \tilde{p}_1^2 + m^{i'j'}(0,\tilde{q}')\tilde{p}_{i'}\tilde{p}_{j'},$$

while formulas, $\tilde{p'} = \psi(q^1,q')p'$, (61)(62)(63) imply

$$Y_{\hat{\mathfrak{h}}} = \tilde{p}_1 e_1 + m^{i'k'}(-|\tilde{q}^1|,\tilde{q}')(\psi^{-1}(\tilde{q}^1,\tilde{q}'))^{j'}_{k'}\tilde{p}_{j'}e_{i'} \tag{106}$$

$$\text{with} \quad e_1 = \frac{\partial}{\partial \tilde{q}^1} \quad , \quad e_{i'} = \frac{\partial}{\partial \tilde{q}^{i'}} + M^k_{i'j}(\tilde{q})\tilde{p}_k \frac{\partial}{\partial \tilde{p}_j} \tag{107}$$

$$\hat{e}^1 = \frac{\partial}{\partial \tilde{p}_1} \quad , \quad \hat{e}^{j'} = \psi^{j'}_{k'}(\tilde{q})\frac{\partial}{\partial \tilde{p}_{k'}} \tag{108}$$

where the coefficients $M^k_{i'j}$, $\psi^{j'}_{k'}$ and $m^{i'j'}(-|\tilde{q}^1|,\tilde{q}) = \hat{g}^{i'j'}(\tilde{q})$ are $\mathscr{C}^\infty$ on $\overline{Q}_\mp$ with the additional property

$$\hat{g}^{ij}(0,\tilde{q}') = g_0^{ij}(0,\tilde{q}') \quad , \quad \psi^{k'}_{j'}(0,\tilde{q}') = \delta^{k'}_{j'}.$$

By using the isometry $\hat{\Psi}^{g,g_0} : F \to \hat{F}_g$ which induces an isomorphism $(\hat{\Psi}^{g,g_0}_X)_* : L^2(X_{(-\varepsilon,\varepsilon)};F\big|_{X_{(-\varepsilon,\varepsilon)}},\hat{g}_0^F) \to L^2(X_{-\varepsilon,\varepsilon};F\big|_{X_{(-\varepsilon,\varepsilon)}},\hat{g}^F)$ according to Proposition 3.8-e), we deduce that

$$(\hat{\Psi}^{g,g_0}_X)^{-1}_* \mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}(\hat{\Psi}^{g,g_0}_{X^*})_* = \mathscr{A}^{g_0,\kappa,\gamma}_{\alpha,\mathscr{M}'}$$

is a LGKFP operator for the metric $g_0^{TQ} = (dq^1)^2 + m_{i'j'}(0,q')dq^{i'}dq^{j'}$, for which we recall that $\hat{F}_{g_0} = F$ is a smooth vector bundle. More precisely the perturbations κ and γ satisfy

- $\kappa \in \mathscr{C}^\infty(\overline{Q}_\mp;L(T^*Q)\big|_{Q_\mp})$, $\gamma \in \mathscr{C}^\infty(\overline{Q}_\mp;T^*Q \odot T^*Q\big|_{Q_\mp})$,
- $\kappa \in \mathscr{C}^0(Q;L(T^*Q))$, $\gamma \in \mathscr{C}^0(Q;T^*Q \odot T^*Q)$,
- $\kappa\big|_{Q'} = \mathrm{Id}$, $\gamma\big|_{Q'} = g_0\big|_{Q'}$.

For trace problems along $X' = \pi_X^{-1}(Q')$, restricting the analysis to $X_{(-\varepsilon,\varepsilon)} = \pi_X^{-1}(Q_{(-\varepsilon,\varepsilon)})$ with $\varepsilon > 0$ small enough is possible by using a finite partition of unity $\sum_{j=1}^J \chi_j(q) \equiv 1$ Lemma 6.8-**v)**. But on $X_{(-\varepsilon,\varepsilon)}$ those coefficients κ and γ satisfy $\|\kappa\|_{W^{1,\infty}} + \|\gamma\|_{W^{1,\infty}} \leq R_g$, independent of $\varepsilon > 0$ and

$$\|\kappa - \mathrm{Id}\|_{L^\infty} + \|\gamma - g_0\|_{L^\infty} = \mathscr{O}(\varepsilon).$$

This leads to the following natural definition, after recalling the definition of κ_χ in Lemma 6.6 and the identification of Lemma 6.8-**i)**.

Definition 6.11. *Let $\alpha \in \mathbb{R}^*$ and let $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}$ a LGKFP operator for the metric $\hat{g}$, the space $\mathscr{E}_{loc}(\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}},\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$ equals*

$$(\hat{\Psi}^{g,g_0}_X)_*\mathscr{E}_{loc}(\mathscr{A}^{g_0,\kappa,\gamma}_{\alpha,\mathscr{M}'},F\big|_{X_{(-\varepsilon,\varepsilon)}}) = (\hat{\Psi}^{g,g_0}_X)_*\mathscr{E}_{loc}\kappa_\chi(\mathscr{A}^{g_0,\mathrm{Id},g_0}_{\alpha,0},F\big|_{X_{(-\varepsilon,\varepsilon)}})$$

for $\varepsilon < \varepsilon_{g,\alpha}$ and $\varepsilon_{g,\alpha} > 0$ small enough.

The following statement specifies the relationship between this definition and the previous trace results.

Proposition 6.12. *Let $\alpha \in \mathbb{R}^*$ and let $\mathscr{A}_{\alpha,\mathscr{M}}^{\hat{g},\mathrm{Id},\hat{g}}$ be a LGKFP operator for the metric $\hat{g}$. A section $s \in L^2_{loc}(X;F)$ belongs to $\mathscr{E}_{loc}(\mathscr{A}_{\alpha,\mathscr{M}}^{\hat{g},\mathrm{Id},\hat{g}},\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$ with $\varepsilon < \varepsilon_{g,\alpha}$, $\varepsilon_{g,\alpha} > 0$ small enough, iff one of the following condition is satisfied:*

i) $(\hat{\Psi}_X^{g,g_0})_*^{-1} s \in \kappa_\chi \mathscr{E}_{loc}(\mathscr{A}_{\alpha,0}^{g_0,\mathrm{Id},g_0}, F\big|_{(-\varepsilon,\varepsilon)})$, *which implies* $s \in W^{2/3,2}_{loc}(X_{(-\varepsilon,\varepsilon)};\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$ *and* $s \in \mathscr{E}_{loc}(\Delta_p^{\hat{g}};\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$;

ii) *the restrictions $s_\mp = s\big|_{X_\mp}$ belong respectively to*

$$\mathscr{E}_{loc}(\mathscr{A}_{\alpha,\mathscr{M}}^{g_\mp,\mathrm{Id},g_\mp}, F\big|_{X_{\mp[0,\varepsilon)}}) \cap \mathscr{E}_{loc}(\nabla^{g_\mp}_{\frac{\partial}{\partial p_1}}, F\big|_{\overline{X}_{\mp[0,\varepsilon)}})$$

and the traces $s_-\big|_{\partial X_-}$ and $s_+\big|_{\partial X_+}$ of Lemma 6.10 coincide after the identification $(e_-,\hat{e}_-) = (e_+,\hat{e}_+)$ along $X' = \partial X_- = \partial X_+$.

iii) *If $\mathscr{A}_{-\alpha,\mathscr{M}'}^{\hat{g},\mathrm{Id},\hat{g}}$ denotes the formal adjoint of $\mathscr{A}_{\alpha,\mathscr{M}}^{\hat{g},\mathrm{Id},\hat{g}}$ (defined on the open set $X_- \cup X_+$), for all $s' \in \mathscr{C}_{0,g}(\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$, the equality*

$$\langle s', \mathscr{A}_{\alpha,\mathscr{M}}^{\hat{g},\mathrm{Id}\hat{g}} s\rangle = \langle \mathscr{A}_{-\alpha,\mathscr{M}'}^{\hat{g};\mathrm{Id},\hat{g}} s', s\rangle$$

holds with for any compact set $K \subset X$, the existence of a constant $C_K > 0$ such that

$$\forall s' \in \mathscr{C}_{0,g}(\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}), supp\, s' \subset K, \quad |\langle \mathscr{A}_{-\alpha,\mathscr{M}'}^{\hat{g},\mathrm{Id},\hat{g}} s', s\rangle| \leq C_K \|s'\|_{L^2}.$$

Proof. The statement **i)** is essentially the definition of $\mathscr{E}_{loc}(\mathscr{A}_{\alpha,\mathscr{M}}^{\hat{g},\mathrm{Id},\hat{g}},\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$ with the additional information $s \in W^{2/3,2}_{loc}(X_{(-\varepsilon,\varepsilon)};\hat{F}^g\big|_{X_{(-\varepsilon,\varepsilon)}}) \cap \mathscr{E}_{loc}(\Delta_p^{\hat{g}};\hat{F}^g\big|_{X_{(-\varepsilon,\varepsilon)}})$.
By Lemma 6.8-**i)**, $(\hat{\Psi}_X^{g,g_0})_*^{-1} s \in \kappa_\chi \mathscr{E}_{loc}(A_{\alpha,0}^{g_0,\mathrm{Id},g_0},)$ implies

$$(\hat{\Psi}_X^{g,g_0})_*^{-1} s \in W^{2/3,2}_{loc}(X_{(-\varepsilon,\varepsilon)};F\big|_{X_{(-\varepsilon,\varepsilon)}}) \cap \mathscr{E}_{loc}(\Delta_p^{g_0};F\big|_{X_{(-\varepsilon,\varepsilon)}}).$$

Since $(\hat{\Psi}_X^{g,g_0})_* : \mathscr{W}^{2/3}_{loc}(X;F) \to \mathscr{W}^{2/3}_{loc}(X;\hat{F}_g)$ is a continuous isomorphism by the local version of Proposition 3.16 with $\mathscr{W}^{2/3}_{loc} = W^{2/3}_{loc}$, this implies $s \in W^{2/3,2}_{loc}(X;\hat{F}_g)$. Remember that $u \in \mathscr{W}^{2/3}_{loc}(X;\hat{F}_g)$ means $u_\mp = u\big|_{X_\mp} \in \mathscr{W}^{2/3}_{loc}(\overline{X}_{\mp[0,\varepsilon)};F\big|_{\mp[0,\varepsilon)})$ with the equality of the traces $u_-\big|_{\partial X_-} = u_+\big|_{\partial X_+}$ (always with $(e_-,\hat{e}_-) = (e_+,\hat{e}_+)$ along X'). The vertical regularity $s \in \mathscr{E}_{loc}(\Delta_p^{\hat{g}};F\big|_{X_{(-\varepsilon,\varepsilon)}})$ is even simpler.
i) implies ii): The previous characterization including the $\mathscr{W}^{2/3}_{loc}(X_{(-\varepsilon,\varepsilon)};\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$ regularity clearly implies $s_\mp = s\big|_{X_{\mp[0,\varepsilon)}} \in \mathscr{E}_{loc}(\mathscr{A}_{\alpha,\mathscr{M}}^{g_\mp,\mathrm{Id},g_\mp}, F\big|_{\overline{X}_{\mp[0,\varepsilon)}}) \cap \mathscr{E}_{loc}(\nabla^{F,g_\mp}_{\frac{\partial}{\partial p_1}}, F\big|_{X_{\mp[0,\varepsilon)}})$ and the equality of the traces $s_-\big|_{\partial X_-} = s_+\big|_{\partial X_+} = s\big|_{\partial X'} \in L^2_{loc}(X';F)$. This ends the proof of **i)** $\Rightarrow$ **ii)**.
By assuming **ii)** the integration by part of Lemma 6.10, where the sum of boundary terms along $\partial X_- = X' = \partial X_+$ vanishes, implies **iii)**.

iii) implies i): Let $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{-\alpha,\mathscr{M}'}$ be the formal adjoint of $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}$ defined on $X_{(-\varepsilon,0)}\cup X_{(0,+\varepsilon)}$. Although $\hat{\Psi}^{g,g_0}_X:(F,\hat{g}^F_0)\to(\hat{F}_g,\hat{g}^F)$ is an isometry, the isomorphism $(\hat{\Psi}^{g,g_0}_X)_*: L^2(X;F,\hat{g}_0)\to L^2(X;F,\hat{g})$ is not unitary because $dv_X=|\det(\psi^{-1}(\tilde{q}))||d\tilde{q}d\tilde{p}|$ according to (77). However it can be made unitary by multiplying by the piecewise $\mathscr{C}^\infty$ and continuous function of q, $|\det(\psi^{-1}(q))|^{1/2}$. When $(\hat{\Psi}^{g,g_0}_X)_*$ is replaced by the unitary map $\tilde{\Psi}^{g,g_0}=|\det(\psi^{-1}(q))|^{1/2}(\hat{\Psi}^{g,g_0}_X)_*$, it simply modifies the admissible perturbation term $\mathscr{M}$ in the action by conjugation on LGKFP-operators. Therefore the operator $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}$ and its formal adjoint $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{-\alpha,\mathscr{M}'}$ can be written

$$\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}=\tilde{\Psi}^{g,g_0}\mathscr{A}^{g_0,\kappa,\gamma}_{\alpha,\mathscr{M}_1}(\tilde{\Psi}^{g,g_0})^{-1}\quad,\quad \mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{-\alpha,\mathscr{M}'}=\tilde{\Psi}^{g,g_0}\mathscr{A}^{g_0,\kappa',\gamma}_{-\alpha,\mathscr{M}'_1}(\tilde{\Psi}^{g,g_0})^{-1},$$

where $\mathscr{A}^{g_0,\kappa',\gamma}_{-\alpha,\mathscr{M}'_1}$ is the formal adjoint of $\mathscr{A}^{g_0,\kappa,\gamma}_{\alpha,\mathscr{M}_1}$ in $X_-\cup X_+$. The result is just a consequence of Lemma 6.8-**iv)** if we notice that $\hat{\Psi}^{g,,g_0}$ sends $\mathscr{C}^\infty_0(X_{(-\varepsilon,0]};F)\cap\mathscr{C}^\infty_0(X_{[0,\varepsilon)};F)\cap\mathscr{C}^0(X_{(-\varepsilon,\varepsilon)};F)$ to $\mathscr{C}^g_0(\hat{F}|_{X_{(-\varepsilon,\varepsilon)}})$. □

We now use $\mathscr{E}_{loc}(\mathscr{A}^{g_0,\kappa,\gamma}_{\alpha,0},F)=\mathscr{E}_{loc}(\mathscr{A}^{g_0,\kappa,\gamma}_{-\alpha,0};F)$ stated in Lemma 6.8-iii) when

$$\|\kappa-\mathrm{Id}\|_{L^\infty}+\|\gamma-g\|_{L^\infty}<\min(\delta_{R,\alpha,g},\delta_{R,-\alpha,g}),$$

and the fact that the formal adjoint in $L^2(X_\mp;F)$ of $\nabla^F_{Y_{\hat{\mathfrak{h}}}}$ is $-\nabla^{F'}_{Y_{\hat{\mathfrak{h}}}}=-\nabla^F_{Y_{\hat{\mathfrak{h}}}}-\omega(\hat{f},g^{\hat{f}})(\pi_{X,*}Y_{\hat{\mathfrak{h}}})$ in order to prove an integration by part. When we work globally this singular framework, this will provide a priori upper bound of $\|\mathscr{O}^{1/2}s\|_{L^2}$ before proving subelliptic estimates.

Proposition 6.13. *For a LGKFP operator $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}$ the space $\mathscr{E}_{loc}(\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}},\hat{F}_g|_{X_{(-\varepsilon,\varepsilon)}})$ equals $\mathscr{E}_{loc}(\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha_1,\mathscr{M}'},\hat{F}_g|_{X_{(-\varepsilon,\varepsilon)}})$ for any other $\alpha_1\in\mathbb{R}^*$, any other admissible local perturbation $\mathscr{M}'$ as soon as $\varepsilon<\varepsilon_{\alpha,\alpha_1,g}$ with $\varepsilon_{\alpha,\alpha_1,g}>0$ small enough. .*
With $\mathscr{M}=\mathscr{M}_j(q,p)\nabla^{X,\hat{g}}_{\frac{\partial}{\partial p_j}}+\mathscr{M}_0(q,p)$, and the adjoints $\mathscr{M}^*_j$ of $\mathscr{M}_j$, the integration part formula

$$\begin{aligned}\mathbb{R}\mathrm{e}\,\langle s\,,\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}s\rangle &= \frac{1}{2}\mathbb{R}\mathrm{e}\,\langle s\,,-\Delta^{\hat{g}}_p s\rangle+\mathbb{R}\mathrm{e}\,\langle s\,,\mathscr{M}_j\nabla^{F,\hat{g}}_{\frac{\partial}{\partial p_j}}s\rangle+\mathbb{R}\mathrm{e}\,\langle s\,,\mathscr{M}_0 s\rangle\\ &\qquad -\frac{\alpha}{2}\mathbb{R}\mathrm{e}\,\langle s\,,\omega(\hat{\mathfrak{f}},\hat{g}^{\mathfrak{f}})(\pi_{X,*}Y_{\hat{\mathfrak{h}}})s\rangle\\ &\geq \sum_{j=1}^d\frac{1}{2}\left[\|\nabla^{F,\hat{g}}_{\frac{\partial}{\partial p_j}}s\|^2_{L^2}-(C_g\|s\|_{L^2}+2\|\mathscr{M}^*_j s\|_{L^2})\|\nabla^{F,\hat{g}}_{\frac{\partial}{\partial p_j}}s\|_{L^2}\right]+\mathbb{R}\mathrm{e}\,\langle s\,,\mathscr{M}_0 s\rangle\\ &\qquad -\frac{\alpha}{2}\mathbb{R}\mathrm{e}\,\langle s\,,\omega(\hat{\mathfrak{f}},\hat{g}^{\mathfrak{f}})(\pi_{X,*}Y_{\hat{\mathfrak{h}}})s\rangle\,,\end{aligned}$$

holds true for any compactly supported section $s\in\mathscr{E}_{loc}(\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}},\hat{F}_g|_{X_{(-\varepsilon,\varepsilon)}})$ and $\varepsilon\in(0,\varepsilon_{\alpha,g})$, $\varepsilon_{\alpha,g}>0$ small enough.

Remark 6.14. *Note that the term $-\frac{\alpha}{2}\mathbb{R}\mathrm{e}\,\langle s\,,\omega(\hat{\mathfrak{f}},\hat{g}^{\mathfrak{f}})(\pi_{X,*}Y_{\hat{\mathfrak{h}}})s\rangle$ is due to the fact that we used the flat and possibly non unitary connection $\nabla^{\mathfrak{f}}$ on $\pi_{\mathfrak{f}}:\mathfrak{f}\to Q$. The term $-C_g\|s\|_{L^2}\|\nabla_{\frac{\partial}{\partial p}}s\|_{L^2}$ comes from the fact that the adjoint of $\nabla^F_{\frac{\partial}{\partial p_j}}$ equals $-\nabla^F_{\frac{\partial}{\partial p_j}}+R_j$ with $R_j\in\mathscr{L}(L^2(X;F))$ when we use the weighted metric $\langle p\rangle_q^{-N_H+N_V}\pi^*_X(g^{\Lambda T^*Q}\otimes\Lambda TQ)$ on E.*

Proof. The first result is simply a consequence of

$$\mathscr{E}_{loc}(\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}},\hat{F}_g) = (\hat{\Psi}^{g,g_0}_X)_*\kappa_\chi\mathscr{E}_{loc}(\mathscr{A}^{g_0,\mathrm{Id},g_0}_{\alpha,0},F) = (\hat{\Psi}^{g,g_0}_X)_*\kappa_\chi\mathscr{E}_{loc}(\mathscr{A}^{g_0,\mathrm{Id},g_0}_{\alpha_1,0},F)$$

for any $\alpha_1 \in \mathbb{R}^*$.
The formal adjoint of $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,0} = \alpha\nabla^F_{Y_{\hat{\mathfrak{h}}}} + \frac{-\Delta^{\hat{g}}_p}{2}$ equals

$$\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{-\alpha,\mathscr{M}'} = -\alpha\nabla^F_{Y_{\hat{\mathfrak{h}}}} + \frac{-\Delta_p}{2} + \mathscr{M}',$$

where $\mathscr{M}'$ gathers

$$-\alpha\omega(\hat{f},\hat{g}^f)(\pi_{X,*}Y_{\hat{\mathfrak{h}}}) = -\alpha g^{ij}(-|q^1|,q')p_j\omega(\hat{\mathfrak{f}},\hat{g}^{\mathfrak{f}})(\frac{\partial}{\partial q^i})\times$$

and other terms coming from the fact that the adjoint of $\nabla^F_{\frac{\partial}{\partial p_j}}$ is $-\nabla^F_{\frac{\partial}{\partial p_j}} + R_j$. By using the map $\hat{\Psi}^{g,g_0} = |\det(\psi^{-1}(q))|^{1/2}\hat{\Psi}^{g,g_0}_X$, introduced in the proof of Proposition 6.12 and which is unitary from $L^2(X;F,g_0)$ to $L^2(X;F,\hat{g})$, we get

$$(\hat{\Psi}^{g,g_0})^{-1}\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,0}\hat{\Psi}^{g,g_0} = \mathscr{A}^{g_0,\kappa,\gamma}_{\alpha,\mathscr{M}_1} \quad \text{and} \quad (\hat{\Psi}^{g,g_0})^{-1}\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{-\alpha,\mathscr{M}'}\hat{\Psi}^{g,g_0} = \mathscr{A}^{g_0,\kappa,\gamma}_{-\alpha,\mathscr{M}_1'},$$

where $= \mathscr{A}^{g_0,\kappa,\gamma}_{-\alpha,\mathscr{M}_1'}$ is the formal adjoint of $\mathscr{A}^{g_0,\kappa,\gamma}_{\alpha,\mathscr{M}_1}$. From Lemma 6.8-**iv)**, we deduce that for any compactly supported $s \in \mathscr{E}_{loc}(\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,0};F)$

$$\langle s\,,\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,0}s\rangle = \langle\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{-\alpha,\mathscr{M}'}s\,,s\rangle$$

and

$$\mathbb{R}\mathrm{e}\,\langle s\,,\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,0}s\rangle = \frac{1}{2}\langle s\,,(\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,0}+\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}s\rangle = \frac{1}{2}\mathbb{R}\mathrm{e}\,\langle s\,,-\Delta^{\hat{g}}_p s\rangle - \frac{\alpha}{2}\mathbb{R}\mathrm{e}\,\langle s\,,\omega(\hat{\mathfrak{f}},\hat{g}^{\mathfrak{f}})(\pi_{X,*}Y_{\hat{\mathfrak{h}}})s\rangle,$$

and this ends the proof. □

6.3 Boundary conditions and closed realizations of the hypoelliptic Laplacian

As a differential operator $\hat{B}^{\phi_b}_{\hat{\mathfrak{h}}}$ is defined as the Bismut hypoelliptic Laplacian on the open set $X_-\cup X_+ = X\setminus X'$ for the metric $\hat{g} = \hat{g}^{TQ} = 1_{Q_-}g^{TQ}_- + 1_{Q_+}g^{TQ}_+$, $b \in \mathbb{R}^*$ and the energy $\hat{\mathfrak{h}}(q,p) = \frac{\hat{g}^{ij}(q)p_ip_j}{2}$. Using the fact that $2b^2B^{\phi_b}_{\mathfrak{h}}$ is a local geometric Kramers-Fokker-Planck operator with $\alpha = -b$, we can define its closed realization in $L^2(X;\hat{F}_g)$. By mimicking the symmetry argument used for $\overline{d}_{g,\mathfrak{h}}$, and $\overline{d}^{\phi_b}_{g,\mathfrak{h}}$, one deduces boundary conditions and a closed realization of $B^{\phi_b}_{\mathfrak{h}}$ in $L^2(X_-;F)$. Additional properties for both operators are specified afterwards.

Proposition 6.15. *Let $\hat{B}^{\phi_b}_{\mathfrak{h}}$ be the Bismut hypoelliptic Laplacian defined as a differential operator on $X_-\cup X_+ = X\setminus X'$ for the metric $\hat{g} = \hat{g}^{TQ} = 1_{Q_-}g^{TQ}_- + 1_{Q_+}g^{TQ}_+$, $b\in\mathbb{R}^*$ and the energy $\hat{\mathfrak{h}}(q,p) = \frac{\hat{g}^{ij}(q)p_ip_j}{2}$. In $L^2(X;F)$ it is defined with the domain*

$$D(\hat{B}^{\phi_b}_{\mathfrak{h}}) = \left\{s \in L^2(X;F)\cap\mathscr{E}_{loc}(\hat{B}^{\phi_b}_{\mathfrak{h}},\hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}),\quad B^{\hat{\phi}}_{\alpha\mathscr{H}}s \in L^2(X;\hat{F}_g)\right\},$$

where $\mathscr{E}_{loc}(\hat{B}^{\phi_b}_{\mathfrak{h}}, \hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$ is given by Definition 6.11 with $\varepsilon < \varepsilon_{g,b}$, $\varepsilon_{g,b} > 0$ small enough.
The operator $(\hat{B}^{\phi_b}_{\mathfrak{h}}, D(\hat{B}^{\phi_b}_{\mathfrak{h}}))$ satisfies the following properties:

a) *With $D(\hat{B}^{\phi_b}_{\mathfrak{h}}) \subset \mathscr{E}_{loc}(\hat{B}^{\phi_b}_{\mathfrak{h}}, \hat{F}_g) \subset \mathscr{W}^{2/3}_{loc}(X;\hat{F}_g) = \mathrm{W}^{2/3,2}_{loc}(X;\hat{F}_g)$, any element $s \in D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ admits a trace in $L^2_{loc}(X';\hat{F}^g\big|_{X'})$.*

b) *The operator $(\hat{B}^{\phi_b}_{\mathfrak{h}}, D(\hat{B}^{\phi_b}_{\mathfrak{h}}))$ is closed with a dense domain.*

c) *There exists a constant $C_{b,g} > 0$ such that the inequality*

$$\mathbb{R}\mathrm{e}\,\langle s\,,(C_{b,g} + \hat{B}^{\phi_b}_{\mathfrak{h}})s\rangle \geq \frac{1}{4b^2}\langle s\,,(1+\mathscr{O})s\rangle$$

holds for all $s \in D(\hat{B}^{\mathfrak{h}}_g)$.

d) *The space $\mathscr{C}_{0,g}(\hat{F}_g)$ is dense in $D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ endowed with its graph norm.*

e) *The ${}^t\phi_b = \phi_{-b}$ left-adjoint of $\hat{B}^{\phi_b}_{\mathfrak{h}}$ is nothing but $(\hat{B}^{\phi_{-b}}_{\mathfrak{h}}, D(\hat{B}^{\phi_{-b}}_{\mathfrak{h}}))$.*

f) *The operator $\hat{B}^{\phi_b}_{\mathfrak{h}}$ commutes with Σ_ν and*

$$D(\hat{B}^{\phi_b}_{\mathfrak{h}}) = [L^2_{ev}(X;E) \cap D(\hat{B}^{\phi_b}_{\mathfrak{h}})] \oplus [L^2_{odd}(X;E) \cap D(\hat{B}^{\phi_b}_{\mathfrak{h}})]$$
$$\hat{B}^{\phi_b}_{\mathfrak{h}} : L^2_{ev\ odd}(X;E) \cap D(\hat{B}^{\phi_b}_{\mathfrak{h}}) \to L^2_{ev\ odd}(X;E).$$

Proof. **a)** The definition of $D(\hat{B}^{\phi_b}_{\mathfrak{h}}) \subset \mathscr{E}_{loc}(\hat{B}^{\phi_b}_{\mathfrak{h}}, \hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$, while the graph norm of $s \in D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ is nothing but $\|s\|_{L^2} + \|\hat{B}^{\phi_b}_{\mathfrak{h}} s\|_{L^2}$, combined with Proposition 6.12 which says $\mathscr{E}_{loc}(\hat{B}^{\phi_b}_{\mathfrak{h}}, \hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}) \subset \mathrm{W}^{2/3,2}_{loc}(X;\hat{F}_g)$ implies that $s \mapsto s\big|_{X'}$ is continuous from $D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ to $L^2_{loc}(X';E)$ (with the identification $(e,\hat{e}) = 1_{\overline{X}_{\mp}}(e_{\mp}, \hat{e}_{\mp})$). Therefore any element $s \in D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ admits a trace in $L^2_{loc}(X';\hat{F}_g\big|_{X'})$.
b) Let us check that $(\hat{B}^{\phi_b}_{\mathfrak{h}}, D(\hat{B}^{\phi_b}_{\mathfrak{h}}))$ is closed. According to Definition 6.11, $2b^2(\hat{B}^{\phi_b}_{\mathfrak{h}})$ is a LGKFP operator $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{\alpha,\mathscr{M}}$ with $\alpha = -b$ with a formal adjoint $\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{b,\mathscr{M}'}$. For a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ such that $\lim_{n\to\infty}\|u_n - u\|_{L^2} = 0$ and $\lim_{n\to\infty}\|\hat{B}^{\phi_b}_{\mathfrak{h}} u_n - v\|_{L^2} = 0$, Proposition 6.12-iii) after a partition of unity in q, leads to

$$\forall s' \in \mathscr{C}_{0,g}(X;\hat{F}_g), \quad \langle s', \hat{B}^{\phi_b}_{\mathfrak{h}} u_n\rangle = \langle (2b^2)^{-1}\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{+b,\mathscr{M}'} s', u_n\rangle.$$

The right-hand side converges to $\langle (2b^2)^{-1}\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{+b,\mathfrak{h}} s', u\rangle$ while the left-hand side converges to $\langle s', v\rangle$. We deduce

$$\forall s' \in \mathscr{C}_{0,g}(\overline{X}_-;\hat{F}_g), \quad \langle s', v\rangle = \langle (2b^2)^{-1}\mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{+b,\mathscr{M}'} s', u\rangle,$$
$$\forall s' \in \mathscr{C}_{0,g}(X;\hat{F}_g), \quad |\langle \mathscr{A}^{\hat{g},\mathrm{Id},\hat{g}}_{+b,\mathscr{M}'} s', u\rangle| \leq \|v\|_{L^2}\|s'\|_{L^2},$$

and Proposition 6.12-iii) implies $u \in \mathscr{E}_{loc}(\hat{B}_{\mathfrak{h}}^{\phi_b}, \hat{F}_g)$ and $\hat{B}_{\mathfrak{h}}^{\phi_g} u = v$ in $L^2_{loc}(X;F)$ while $v \in L^2(X;F)$. This proves that $(\hat{B}_{\mathfrak{h}}^{\phi_b}, D(\hat{B}_{\mathfrak{h}}^{\phi_b}))$ is closed.
c) For a finite partition of unity $\sum_{k=1}^{K} \theta_k^2(q) \equiv 1$ we have:

$$\hat{B}_{\mathfrak{h}}^{\phi_b} = \sum_{k=1}^{K} \theta_k(q) \hat{B}_{\mathfrak{h}}^{\phi_b} \theta_k(q)$$

and we reduce the problem to supp $s \subset X_{(-\varepsilon,\varepsilon)}$.
Consider the dyadic partition of unity $\sum_{k=0}^{\infty} \chi_k^2(t) = \chi_0^2(t) + \sum_{k=1}^{\infty} \chi^2(\frac{t}{2^{2k}}) \equiv 1$ on $\mathbb{R}$ with $\chi_0 \in \mathscr{C}_0^\infty(\mathbb{R})$ $\chi_0 \equiv 1$ in a neighborhood of 0 a $\chi \in \mathscr{C}^\infty(]r_1, r_2[)$. Because $\hat{B}_{\mathfrak{h}}^{\phi_b}$ is a GKFP operator we get for $s \in D(\hat{B}_{\mathfrak{h}}^{\phi_b})$:

$$\hat{B}_{\mathfrak{h}}^{\phi_b} - \sum_{k=0}^{\infty} \chi_k(\hat{\mathfrak{h}}) \hat{B}_{\mathfrak{h}}^{\phi_b} \chi_k(\hat{\mathfrak{h}}) = -\frac{1}{4b^2} \sum_{k=0}^{\infty} \hat{g}_{ij}(q) (\partial_{p_i} \chi_k(\frac{\hat{\mathfrak{h}}}{2^{2k}}))(\partial_{p_j} \chi_k(\frac{\hat{\mathfrak{h}}}{2^{2k}})),$$

$$\text{with} \qquad \partial_{p_\ell}[\chi_k(\hat{\mathfrak{h}})] = \nabla_p [\chi(\frac{\hat{g}^{ij}(q) p_j p_j}{2^{2k}})] = \frac{\hat{g}^{\ell j}(q) p_\ell}{2^{2k}} \chi'(\frac{\hat{\mathfrak{h}}}{2^{2k}}) \quad \text{for } k \geq 1,$$

$$\frac{1}{2} \hat{g}_{ij}(q) (\partial_{p_i} \chi_k(\frac{\hat{\mathfrak{h}}}{2^{2k}}))(\partial_{p_j} \chi_k(\frac{\hat{\mathfrak{h}}}{2^{2k}})) = \frac{\hat{\mathfrak{h}}}{2^{4k}} |\chi'(\frac{\hat{\mathfrak{h}}}{2^{2k}})|^{2k} = \mathscr{O}(2^{-2k}) = \mathscr{O}(\langle p \rangle_q^{-2}).$$

Hence there exists $C_{b,g}^1 > 0$ such that

$$\mathbb{R}\mathrm{e}\, \langle s, B_{\mathfrak{h}}^{\hat{\phi}} s \rangle \geq \left[\sum_{k=0}^{\infty} \mathbb{R}\mathrm{e}\, \langle \chi_k(\hat{\mathfrak{h}}) s, \hat{B}_{\mathfrak{h}}^{\phi_b} \chi_k(\mathfrak{h}) s \rangle \right] - C_{b,g}^1 \|s\|^2.$$

The operator $2b^2 \hat{B}_{\mathfrak{h}}^{\phi_b}$ is a LGKFP operator for the metric $\hat{g}$ according to Definition 6.11, we can use Proposition 6.13 with

$$\mathscr{M}_j(q,p) = \mathscr{M}_{0,j}(q,p) \quad , \quad \mathscr{M}_0(q,p) = \mathscr{M}_0^j(q,p) p_j + \mathscr{M}_{0,0}(q,p) + \frac{\hat{g}^{ij}(q) p_j p_j}{4b^2}$$

and $\mathscr{M}_{0,j}, \mathscr{M}_0^j, \mathscr{M}_{0,0}$ are uniformly bounded. Proposition 6.13 applied to the compactly supported $s_k = \chi_k(\mathfrak{h}) s$ of $\mathscr{E}_{loc}(\hat{B}_{\mathfrak{h}}^{\phi_b}, \hat{F}_g)$ leads to

$$\begin{aligned}
\mathbb{R}\mathrm{e}\, \langle s_k, \hat{B}_{\mathfrak{h}}^{\phi_b} s_k \rangle \quad &\geq \quad \sum_{j=1}^{d} \frac{1}{4b^2} \left[\|\nabla^F_{\frac{\partial}{\partial p_j}} s_k\|_{L^2}^2 + \|p_j s_k\|_{L^2}^2 \right] \\
&\qquad - C_{b,g}^2 \|s_k\|_{L^2} \left[\|\nabla^F_{\frac{\partial}{\partial p_j}} s_k\|_{L^2} + \|p_j s_k\|_{L^2} + \|s_k\|_{L^2} \right] \\
&\geq \quad \frac{1}{6b^2} \sum_{j=1}^{d} \left[\|\nabla^F_{\frac{\partial}{\partial p_j}} s_k\|_{L^2}^2 + \|p_j s_k\|_{L^2}^2 \right] - C_{b,g}^3 \|s_k\|_{L^2}^2,
\end{aligned}$$

for any $\delta > 0$ and some $C_{b,g}^2, C_{b,g}^3 > 0$. By putting all together and absorbing in a similar way the error terms, there is a constant $C_{b,g} > 0$ such that

$$\mathbb{R}\mathrm{e}\, \langle s, (C_{b,g} + \hat{B}_{\mathfrak{h}}^{\phi_b}) s \rangle \geq \frac{1}{8b^2} \left[\sum_{j=1}^{d} \|\nabla^F_{\frac{\partial}{\partial p_j}} s\|_{L^2}^2 + \|p_j s\|_{L^2}^2 + \|s\|_{L^2}^2 \right] \geq \frac{1}{4b^2} \langle s, (1 + \mathscr{O}) s \rangle.$$

d) Let us first consider the effect of a truncation on $s \in D(\hat{B}_{\mathfrak{h}}^{\phi_b})$: Take $\chi \in \mathscr{C}_0^\infty(\mathbb{R};\mathbb{R})$, $\chi \equiv 1$ in a neighborhood of 0 and set $\chi_n(t) = \chi(t/n)$. When $s \in D(\hat{B}_{\mathfrak{h}}^{\phi_b})$, $\chi_n(\hat{\mathfrak{h}})s \in D(\hat{B}_{\mathfrak{h}}^{\phi_b})$ while

$$\hat{B}_{\mathfrak{h}}^{\phi_b}\chi_n(\hat{\mathfrak{h}})s = \chi_n(\hat{\mathfrak{h}})\hat{B}_{\mathfrak{h}}^{\phi_b} - \frac{1}{2}[\Delta_p^{\hat{g}}, \chi_n(\hat{\mathfrak{h}})]s$$

$$\|[\Delta_p^{\hat{g}}, \chi_n(\hat{\mathfrak{h}})]s\| \leq \frac{\|s\|_{L^2}}{n}\left[\sum_{k=1}^{d}\|\nabla^F_{\frac{\partial}{\partial p_k}} s\|_{L^2}\right] \leq C_{b,g}\frac{\mathbb{R}\mathrm{e}\,\langle s\,, (C_{b,g} + \hat{B}_{\mathfrak{h}}^{\phi_b})s\rangle}{n}$$

implies that $\chi_n(\hat{\mathfrak{h}})s$ converges to s in $D(\hat{B}_{\mathfrak{h}}^{\phi_b})$ endowed with its graph norm. Now for $s_N = \chi_N(\mathfrak{h})s$ the problem is reduced to the approximation of the compactly supported element of $\mathscr{E}_{loc}(\hat{B}_{\mathfrak{h}}^{\phi_b}, \hat{F}_g)$ by elements of $\mathscr{C}_{0,g}(\hat{F}_g)$. By using a partition of unity in q, $\sum_{k=1}^{K}\theta_k(q) \equiv 1$ with

$$\left[\hat{B}_{\mathfrak{h}}^{\phi_b}, \theta_k(q)\right] = g^{ij}(q)p_j\frac{\partial\theta_k}{\partial q^i}$$

the problem is reduced to a compactly supported element s_N of

$$\mathscr{E}_{loc}(\hat{B}_{\mathfrak{h}}^{\phi_b}, \hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}}) = (\hat{\Psi}_X^{g,g_0})_*\mathscr{E}_{loc}(\mathscr{A}^{g_0,\kappa,\gamma}_{-b,\mathscr{M}'}, F\big|_{X_{(-\varepsilon,\varepsilon)}}).$$

Then the approximation of a compactly supported element $s_N \in \mathscr{E}_{loc}(\hat{B}_{\mathfrak{h}}^{\phi_b}, \hat{F}_g\big|_{X_{(-\varepsilon,\varepsilon)}})$ results from Lemma 6.8-**ii)**.

e) By construction the isomorphism $J_{b,g} : L^2(X;F) \to L^2(X,F)$ given by

$$\forall s' \in L^2(X;F), \quad \langle J_{b,g}s\,, s'\rangle_{L^2} = \langle s\,, s'\rangle_{\phi_{-b}}$$

and its inverse preserve $\mathscr{C}_{0,g}(\hat{F}_g)$ which is a core for $(\hat{B}_{\mathfrak{h}}^{\phi_b}, D(\hat{B}_{\mathfrak{h}}^{\phi_b})$ and its L^2-adjoint $(B_{\mathfrak{h}}^{\phi_b})^*$. The L^2-adjoint is the closure of the operator defined on $\mathscr{C}_{0,g}(\hat{F}_g)$ by

$$\forall s, s' \in \mathscr{C}_{0,g}(\hat{F}_g), \quad \langle s\,, \hat{B}_{\mathfrak{h}}^{\phi_b}s'\rangle = \langle(\hat{B}_{\mathfrak{h}}^{\phi_b})^* s\,, s'\rangle\,.$$

This gives

$$\forall s, s' \in \mathscr{C}_{0,g}(\hat{F}_g), \quad \langle J_{b,g}^{-1}s\,, \hat{B}_{\mathfrak{h}}^{\phi_b}s'\rangle_{\phi_{-b}} = \langle J_{b,g}^{-1}(\hat{B}_{\mathfrak{h}}^{\phi_b})^* s\,, s'\rangle_{\phi_{-b}}\,.$$

We deduce that the ϕ_{-b} left-adjoint, $(\hat{B}_{\mathfrak{h}}^{\phi_b})^{\phi_{-b}}$ of $\hat{B}_{\mathfrak{h}}^{\phi_b}$ satisfies

$$\forall s \in \mathscr{C}_{0,g}(\hat{F}_g), \quad (\hat{B}_{\mathfrak{h}}^{\phi_b})^{\phi_{-b}}s = J_{b,g}^{-1}(\hat{B}_{\mathfrak{h}}^{\phi_b})^* J_{b,g}s\,.$$

Taking a test function $s' \in \mathscr{C}_0^\infty(X_- \cup X_+;F)$ allows to make the integration by part for $s \in \mathscr{C}_{0,g}(\hat{F}_g)$,

$$\langle s\,, \hat{B}^{\phi_b}s'\rangle_{\phi_{-b}} = \langle s\,, (d_{\hat{\mathfrak{h}}}^{\phi_b}d_{\hat{\mathfrak{h}}} + d_{\hat{\mathfrak{h}}}d_{\hat{\mathfrak{h}}}^{\phi_b})s'\rangle_{\phi_{-b}} = \langle (d_{\hat{\mathfrak{h}}}^{\phi_{-b}})d_{\hat{\mathfrak{h}}} + d_{\hat{\mathfrak{h}}}d_{\hat{\mathfrak{h}}}^{\phi_{-b}})s\,, s'\rangle_{\phi_{-b}}$$

without any boundary term. This gives

$$(\hat{B}_{\mathfrak{h}}^{\phi_b})^{\phi_{-b}}s = J_{b,g}^{-1}(\hat{B}_{\mathfrak{h}}^{\phi_b})^* J_{b,g}s = \hat{B}_{\mathfrak{h}}^{\phi_{-b}}s \quad \text{in } \mathscr{D}'(X_- \cup X_+, F)$$

while the left-hand side belongs to $L^2(X;F)$. With $s \in \mathscr{C}_{0,g}(\hat{F}_g) \subset D(\hat{B}_{\mathfrak{h}}^{\phi_{-b}})$ this gives

$$(\hat{B}_{\mathfrak{h}}^{\phi_b})^{\phi_{-b}} s = \hat{B}_{\mathfrak{h}}^{\phi_{-b}} s\,,$$

and, because $\mathscr{C}_{0,g}(\hat{F}_g)$ is a core for both operators, the ϕ_{-b} left-adjoint of $(\hat{B}_{\mathfrak{h}}^{\phi_b}, D(\hat{B}_{\mathfrak{h}}^{\phi_b}))$ equals $(\hat{B}_{\mathfrak{h}}^{\phi_{-b}}, D(\hat{B}_{\mathfrak{h}}^{\phi_{-b}}))$.

f) By construction $\hat{B}_{\mathfrak{h}}^{\phi_b}$ commutes with Σ_ν as a differential operator on $X_- \cup X_+ = X \setminus X'$, that is in $\mathscr{D}'(X_- \cup X_+;F)$. Meanwhile $\mathscr{C}_{0,g}(\hat{F}_g)$ is left invariant by Σ_ν. This proves

$$\forall s \in \mathscr{C}_{0,g}(\hat{F}_g), \quad \Sigma_\nu \hat{B}_{\mathfrak{h}}^{\phi_b} s = \hat{B}_{\mathfrak{h}}^{\phi_b} \Sigma_\nu s\,.$$

Since $\mathscr{C}_{0,g}(\hat{F}_g)$ is a core for $(\hat{B}_{\mathfrak{h}}^{\phi_b}, D(\hat{B}_{\mathfrak{h}}^{\phi_b}))$, the equality holds for all $s \in D(\hat{B}_{\mathfrak{h}}^{\phi_b})$. $\square$

Definition 6.16. *In $L^2(X_-;F\big|_{X_-})$, the operator $\overline{B}_{g,\alpha\mathscr{H}}^{\phi}$ is defined with the domain*

$$D(\overline{B}_{g,\alpha\mathscr{H}}^{\phi}) = \left\{ s \in L^2(X_-;F\big|_{X_-}), \quad s_{ev} \in D(\hat{B}_{\mathfrak{h}}^{\phi_b}) \right\}.$$

Theorem 6.17. *For $b \in \mathbb{R}^*$ the operator $(\overline{B}_{\mathfrak{h}}^{\phi_b}, D(\overline{B}_{\mathfrak{h}}^{\phi_b}))$ satisfies the following properties*

a) *It is closed and a section $s \in D(\overline{B}_{\mathfrak{h}}^{\phi_b})$ has trace $s\big|_{X'} \in L^2_{loc}(X';F\big|_{X'})$, $X' = \partial X_-$, such that $\hat{S}_\nu s\big|_{X'} = s\big|_{X'}$ where $\hat{S}_\nu$ is defined by (75).*

b) *The space*

$$\mathscr{C}_0^\infty(\overline{X}_-;F) \cap D(\overline{B}_{\mathfrak{h}}^{\phi_b}) = \left\{ s \in \mathscr{C}_0^\infty(\overline{X}_-;F), \quad \hat{S}_\nu s\big|_X = s\big|_{X'} \right\}$$

is dense in $D(\overline{B}_{\mathfrak{h}}^{\phi_b})$ endowed with its graph norm.

c) *The ϕ_{-b} left-adjoint of $(\overline{B}_{\mathfrak{h}}^{\phi_b}, D(\overline{B}_{\mathfrak{h}}^{\phi_b}))$ equals $(\overline{B}_{\mathfrak{h}}^{\phi_{-b}}, D(\overline{B}_{\mathfrak{h}}^{\phi_{-b}}))$.*

d) *There exists a constant $C_{b,g} > 0$ such that*

$$\mathrm{Re}\,\langle s\,,(C_{b,g} + \overline{B}_{\mathfrak{h}}^{\phi_b})s\rangle \geq \frac{1}{4b^2}\langle s\,,(1+\mathscr{O})s\rangle$$

holds for all $s \in D(\overline{B}_{\mathfrak{h}}^{\phi_b})$.

e) *The operator $(C_{b,g} + \overline{B}_{\mathfrak{h}}^{\phi_b}, D(\overline{B}_{\mathfrak{h}}^{\phi_b}))$ is maximal accretive and estimate*

$$\left.\begin{array}{l} \sum_{j=1}^d [\|\nabla^F_{\frac{\partial}{\partial p_j}} s\|_{L^2} + \|p_j s\|_{L^2}] \\ + \|s\|_{\mathscr{W}^{1/3}} + \|\langle p\rangle_q^{-1} s\big|_{X'}\|_{L^2(X',|p_1|dv_{X'})} \\ + \langle\lambda\rangle^{1/4}\|s\| \end{array}\right\} \leq C'_{b,g}\|(\overline{B}_{\mathfrak{h}}^{\phi_b} + C'_{b,g} + i\lambda)s\|_{L^2} \quad ,$$

holds for some $C'_{b,g} > 0$, all $\lambda \in \mathbb{R}$ and all $s \in D(\overline{B}_{\mathfrak{h}}^{\phi_b})$. In particular $(\overline{B}_{\mathfrak{h}}^{\phi_b}, D(\overline{B}_{\mathfrak{h}}^{\phi_b}))$ has a compact resolvent.

f) *Finally the domain* $D(\overline{B}_{\mathfrak{h}}^{\phi_b})$ *equals*

$$D(\overline{B}_{\mathfrak{h}}^{\phi_b}) = \left\{ s \in L^2(X_-;F), \quad \nabla^F_{\frac{\partial}{\partial p_1}} s, B_{\mathfrak{h}}^{\phi_b} s \in L^2(X_-;F) \quad , \quad \hat{S}_\nu s\big|_{X'} = s\big|_{X'} \right\}.$$

Proof. The statements **a),b),c),d)** are straightforward consequences of Proposition 6.15 and Definition 6.16 after recalling that $L^2(X_-;F) \ni s \mapsto 2^{-1/2} s_{ev} \in L^2_{ev}(X;F)$ is unitary.
The results of **e)** are deduced from the results of [Nie] for scalar Kramers-Fokker-Planck operators. After localization in q after partition of unity and a possible change of connection, one can write

$$B_{\mathfrak{h}}^{\phi_b}(s_I^J e^I \hat{e}_J) = \left([g^{ij}(q) p_j e_i + \mathscr{O}] s_I^J \right) e_I \hat{e}^J + \mathscr{M}(s_I^J e^I \hat{e}_J)$$

where $\mathscr{M}$ is a global admissible perturbation (see Definition 6.1), such that

$$\|\mathscr{M} s\|_{L^2} \le C\, \mathbb{R}\mathrm{e}\, \langle s, B_{\mathfrak{h}}^{\phi_b} s \rangle .$$

Meanwhile the boundary conditions written

$$s_I^J(0, q', p_1, p') = \nu(-1)^{|\{1\}\cap I| + |\{1\}\cap J|} s_I^J(0, q', -p_1, p')$$

where ν can be replaced by ± 1 are the ones considered in [Nie]-Theorem 1.1. Finally note that the a priori condition, $\nabla^F_{\frac{\partial}{\partial p_j}} s_I^J, p_j s_I^J \in L^2(X_-;\mathbb{C}^{d_{\mathfrak{f}}})$ is actually provided by the integration by part **d)**. The subelliptic estimates and the maximal accretivity proved in [Nie] for scalar Kramers-Fokker-Planck operators, which is stable under admissible global perturbations $\mathscr{M}$ while adapting the constants in the inequalities, are thus valid for $(\hat{B}_{\mathfrak{h}}^{\phi_b}, D(B_{\mathfrak{h}}^{\phi_b}))$.
f) Clearly

$$D(\overline{B}_{\mathfrak{h}}^{\phi_b}) \subset \left\{ s \in L^2(X_-;F), \quad \nabla^F_{\frac{\partial}{\partial p_1}} s, B_{\mathfrak{h}}^{\phi_b} s \in L^2(X_-;F) \quad , \quad \hat{S}_\nu s\big|_{X'} = s\big|_{X'} \right\}.$$

For the reverse inclusion, the condition $s \in \mathscr{E}_{loc}(B_{\mathfrak{h}}^{\phi_b}, F\big|_{X_{(-\varepsilon,0]}}) \cap \mathscr{E}_{loc}(\nabla^F_{\frac{\partial}{\partial p_1}}, F\big|_{X_{(-\varepsilon,0]}})$ and Lemma 6.10 imply that s admits a trace along $X' = \partial X_-$. The condition $\hat{S}_\nu s\big|_{X'} = s\big|_{X'}$ makes sense and this ensures that s_{ev} admits a trace in $L^2_{loc}(X';\hat{F}_g\big|_{X'})$. By using tha map $\hat{\Psi}_X^{g,g_0}$, we deduce that $\omega_{ev} = (\Psi_X^{g,g_0}) s_{ev}$ admits a trace in $L^2_{loc}(X';F)$ while

$$(\hat{\Psi}_X^{g,g_0})_*^{-1} \hat{B}_{\mathfrak{h}}^{\phi_b} (\hat{\Psi}_X^{g,g_0})_* = \mathscr{A}_{-b,\mathscr{M}'}^{g_0,\kappa,\gamma}.$$

Because $\mathscr{A}_{-b,\mathscr{M}'}^{g_0,\kappa,\gamma}$ is contains only one derivative $\frac{\partial}{\partial q^1}$ with while all the coefficients are $\mathscr{C}^\infty$ above $\overline{X}_-$ and $\overline{X}_+$, the jump formula says

$$\mathscr{A}_{-b,0}^{g_0,\mathrm{Id},g_0} \omega_{ev} = \mathscr{A}_{-b,0}^{g,\mathrm{Id},g_0} \omega_{ev}\big|_{X\setminus X'} \quad \text{in } \mathscr{D}'(X;F),$$

while the right-hand side belongs to $L^2(X_{(-\varepsilon,\varepsilon)}; F\big|_{X_{(-\varepsilon,\varepsilon)}})$. By going back to s_{ev}, this implies $s_{ev} \in D(\hat{B}_{\mathfrak{h}}^{\phi_b})$. $\square$

The result of Theorem 6.17-**e)** can be translated for the operator $\hat{B}_{\mathfrak{h}}^{\phi_b}$ after recalling

$$D(\hat{B}_{\mathfrak{h}}^{\phi_b}) = L^2_{ev}(X;E)\cap D(B^{\hat{\phi}}) \oplus L^2_{odd}(X;E)\cap D(\hat{B}_{\mathfrak{h}}^{\phi_b})$$

and writing $s = s_{ev} + s_{odd}$. Although only s_{ev} has been treated, s_{odd} actually enters in the same framework after replacing the unitary involution ν of $\mathfrak{f}|_{Q'}$ with $-\nu$.

Corollary 6.18. *The results of Theorem 6.17-**e)** hold, mutatis mutandis, for the operator* $(\hat{B}_{\mathfrak{h}}^{\phi_b}, D(\hat{B}_{\mathfrak{h}}^{\phi_b}))$.

The summary about "cuspidal semigroups" (terminology introduced in [Nie]) in Subsection 6.1 applies now to $(\overline{B}_{\mathfrak{h}}^{\phi_b}, D(\overline{B}_{\mathfrak{h}}^{\phi_b}))$ and to $(\hat{B}_{\mathfrak{h}}^{\phi_b}, D(\hat{B}_{\mathfrak{h}}^{\phi_b}))$ with the exponent $r = 1/4$.

Corollary 6.19. *For $b \in \mathbb{R}^*$ and $C_{b,g} > 0$ large enough, the operators $(A = C_{b,g} + \overline{B}_{\mathfrak{h}}^{\phi_b}, D(\overline{B}_{\mathfrak{h}}^{\phi_b}))$ and $(A = C_{b,g} + \hat{B}_{\mathfrak{h}}^{\phi_b}, D(\hat{B}_{\mathfrak{h}}^{\phi_b}))$ are maximal accretive and their resolvent are compact.*
Their spectrum is contained in

$$\left\{ z \in \mathbb{C}, \mathrm{Re}\, z \geq C_{b,g}^{-1} \langle Im\, z\rangle^{1/4} \right\}.$$

If $\gamma_{b,g}$ is the contour $\left\{ z \in \mathbb{C}, \mathrm{Re}\, z = \frac{1}{2C_{b,g}} \langle Im\, z\rangle^{1/4} \right\}$ oriented from $+i\infty$ to $-i\infty$, the semigroups are given by

$$\forall t > 0, e^{-tA} = \frac{1}{2i\pi} \int_{\gamma_{b,g}} \frac{e^{-tz}}{z - A}\, dz.$$

Those semigroups satisfy in particular $e^{-tA} \in D(A^N)$ for any $t > 0$.

6.4 Bootstrapped regularity for the powers of the resolvent and the semigroup

The $\mathscr{W}^{1/3}(X_-;\hat{F})$ (resp. $\mathscr{W}^{1/3}(X;\hat{F}_g)$) global regularity estimate of 6.17-**e)** (resp. Corollary 6.18) for $s \in D(\overline{B}_{\mathfrak{h}})$ (resp. $s \in D(\hat{B}_{\mathfrak{h}}^{\phi_b})$) does not correspond to the maximal hypoellipticity result, $s \in \mathscr{W}^{2/3}(X;F)$ obtained by Lebeau in [Leb2] in the smooth case without boundary. As pointed out in [Nie] it is related to the extrinsic curvature of ∂Q and it is not yet known whether it can be improved. However the estimates of Theorem 6.17 and Corollary 6.18 suffice to get higher regularity estimates for high enough powers of the resolvent and subsequently for the semigroup.
We start with weighted estimates which do not use any other regularity properties than the one stated in Theorem 6.17 and Corollary 6.18.

Lemma 6.20. *Let $b \in \mathbb{R}^*$ and set $A = \overline{B}_{\mathfrak{h}}^{\phi_b}$ or $A = \hat{B}_{\mathfrak{h}}^{\phi_b}$. For any $n \in \mathbb{N}$, there exists a constant $C_{n,b,g} > 0$ such that*

$$\langle p\rangle_q^{m+1}(C+A)^{-1}\langle p\rangle_q^{-m} \quad , \quad (1+\mathscr{O})^{1/2}\langle p\rangle_q^{m}(C+A)^{-1}\langle p\rangle_q^{-m}$$

$$\text{and} \qquad \langle p\rangle_q^{m}(C+A)^{-m} \quad (1+\mathscr{O})^{1/2}\langle p\rangle_q^{m}(C+A)^{-m-1}$$

are bounded for all m, $0 \leq m \leq n$ and all $C \geq C_{n,b,g}$.
Finally for any $t > 0$ and any $n \in \mathbb{N}$, $(1+\mathscr{O})^{1/2}\langle p\rangle_q^n e^{-tA}$ is a bounded operator.

Proof. We focus on the case $A = \overline{B}_{\mathfrak{h}}^{\phi_b}$ and the case $A = \hat{B}_{\mathfrak{h}}^{\phi_b}$ can be recovered by symmetrization like Corollary 6.18.
With $1+\mathscr{O} \geq \frac{\langle p\rangle_q^2}{2}$ and

$$\langle p\rangle_q^n (C+A)^{-n} = \prod_{k=0}^{n-1} \langle p\rangle_q^{n-k}(C+A)^{-1}\langle p\rangle_q^{-n+1+k}$$

all the results are consequences of the boundedness of $(1+\mathscr{O})^{1/2}\langle p\rangle_q^m (C+A)^{-1}\langle p\rangle_q^{-m}$ for $C \geq \tilde{C}_{m,b,g} \geq \tilde{C}_{m-1,b,g}$.
Take $u \in \mathscr{C}_0^\infty(\overline{X}_-;F) \cap D(\overline{B}_{\mathfrak{h}}^{\phi_b})$, and write with $(C+A)u = f$:

$$(C+A)\langle p\rangle_q^{-m} u = \langle p\rangle_q^{-m} f + [-\frac{\Delta_p}{2}, \langle p\rangle_q^{-m}]u\,.$$

It becomes

$$\left(C + A + \mathscr{M}_{m,j}(q,p)\nabla_{\partial p_j} + \mathscr{M}_m\right)\langle p\rangle_q^{-m} = \langle p\rangle_q^{-m} f$$

where $\mathscr{M}_{m,j}$, $\mathscr{M}_m$ are symbols of order 0 according to Definition 6.1. With the integration by part of Theorem 6.17-**d)**, the operator $\mathscr{M}_m = \mathscr{M}_{m,j}(q,p)\nabla_{\partial p_j} + \mathscr{M}_m$ is a relatively bounded perturbation of A with infinitesimal bound. Therefore for $C \geq \tilde{C}_{m,b,g} \leq \tilde{C}_{m-1,b,g}$ with $\tilde{C}_{m,b,g}$large enough, $(C+A+\mathscr{M})$ is maximal accretive and we get the uniform bound

$$\|\langle p\rangle_q^{-m} f\| = \|(C+A+\mathscr{M})\langle p\rangle_q^{-m}u\| \geq \frac{1}{2}\|(C+A)\langle p\rangle_q^{-m}u\| \geq C_m^{-1}\|(1+\mathscr{O})^{1/2}\langle p\rangle_q^{-m}u\|$$

Approaching any $u \in \langle p\rangle_q^m D(A)$ by elements u_n in $\mathscr{C}_0^\infty(\overline{X}_-;F) \cap D(A)$ proves the boundedness of $(1+\mathscr{O})^{1/2}\langle p\rangle^m (C+A)^{-1}\langle p\rangle^{-m}$ for $C \geq \tilde{C}_{m,b,g}$.
The final statement about the semigroup is a consequence of

$$(1+\mathscr{O})^{1/2}\langle p\rangle_q^n e^{-tA} = (1+\mathscr{O})^{1/2}\langle p\rangle_q^n (C_{n,\alpha}+A)^{-n-1}(C_{n,\alpha}+A)^{n+1}e^{-tA}\,.$$

□

We will use Lebeau's maximal hypoellipticity estimate (99) with various values of $\mu \in [0,1]$. We already used the fact that $\hat{B}_{\mathfrak{h}}^{\phi_b} = \mathscr{A}_{-b,\mathscr{M}}^{\hat{g},\mathrm{Id},\hat{g}}$ or more precisely $(\hat{\Psi}_X^{g,g_0})_*^{-1}\hat{B}_{\mathfrak{h}}^{\phi_b}(\hat{\Psi}_X^{g,g_0})_* = \mathscr{A}_{-b,\mathscr{M}'}^{g_0,\kappa,\gamma}$ is a LGKFP operator in order to the existence of traces and local regularity properties in Propositions 6.12 and 6.13. Let us look globally at those transformations. Remember that $(\hat{\Psi}_X^{g,g_0})_*$ provides a continuous isomorphism from $\mathscr{W}^\mu(X_{(-\varepsilon,\varepsilon)};F\big|_{X_{(-\varepsilon,\varepsilon)}},g_0) = \mathscr{W}^\mu(X;F)$ to $\mathscr{W}^\mu(X_{(-\varepsilon,\varepsilon)};\hat{F}_g,\hat{g})$ for $\mu \in [-1,1]$ while Lemma 6.6 provides for a given partition of unity $\sum_{j=1}^J \chi_j(q) \equiv 1$ the map κ_χ which is an automorphism of $\mathscr{W}^\mu(X;F)$ for $\mu \in [-1,1]$.

- Firstly the vector field $Y_{\hat{\mathfrak{h}}}$ remains expressed with the vector fiels $(e_i)_{1\leq i\leq d}$ which differ from the vector fields $(f_i)_{1\leq i\leq}$ associated with the metric g_0^{TQ}. From (107) we can write

$$\begin{aligned} e_{i'} &= \frac{\partial}{\partial \tilde{q}^{i'}} + M_{i'j}^{k}(\tilde{q})\tilde{p}_k \frac{\partial}{\partial \tilde{p}_j} \\ &= \underbrace{\frac{\partial}{\partial q'^{i}} + \Gamma_{i',j'}^{k'}(0,\tilde{q}')\tilde{p}_k \frac{\partial}{\partial \tilde{p}_{j'}}}_{\tilde{f}_{i'}} + \underbrace{[M_{i'j}^{k}(\tilde{q}) - (1-\delta_{1j})(1-\delta_{1k})\Gamma_{i'j}^{k}(0,\tilde{q})]\tilde{p}_k \frac{\partial}{\partial \tilde{p}_j}}_{\tilde{R}_{i'}}, \end{aligned}$$

with $(\hat{\Psi}_X^{(g,g_0)})_*^{-1}(\tilde{f}_i)(\hat{\Psi}_X^{g,g_0})_* = f_i$, $(\hat{\Psi}_X^{(g,g_0)})_*^{-1}(\tilde{R}_{i'})(\hat{\Psi}_X^{g,g_0})_* = R_{i'}$ and R_i behaves like $p \times \frac{\partial}{\partial p}$. Hence we deduce that

$$(\hat{\Psi}_X^{(g,g_0)})_*^{-1}(Y_{\hat{\mathfrak{h}}})(\hat{\Psi}_X^{g,g_0})_* = p_1 \frac{\partial}{\partial q^1} + (g_0\kappa)^{i'j'} p_{j'} f_{i'} + (g_0\kappa)^{i'j'} p_{j'} R_{i'}$$

By conjugating with the map κ_χ of Lemma 6.6 one obtains similarly

$$\kappa_\chi (\hat{\Psi}_X^{(g,g_0)})_*^{-1}(Y_{\hat{\mathfrak{h}}})(\hat{\Psi}_X^{g,g_0})_* \kappa_\chi^{-1} = Y_{\mathfrak{h}_0} + R$$

where $\mathfrak{h}_0 = \frac{g_0^{ij}(q)p_ip_j}{2}$ and R is an operator with terms like $M_i^{jk}(q)p_jp_k\frac{\partial}{\partial p_k}$.

- The Levi-Civita connections for the metrics g^{TQ} and g_0^{TQ} differ but we already noticed that such a change of connection adds a perturbative term $\mathcal{M}^j(q)p_j + \mathcal{M}_0$ when expressed in the basis $(e,\hat{e})$. The change of frame to $(f,\hat{f})$ will lead to a perturbative term $\mathcal{M}^{jk}(q)p_jp_k + \mathcal{M}^j(q)p_j + \mathcal{M}^0(q)$.

- The conjugation by κ_χ changes the term Δ_p^γ into $\Delta_p^{\gamma'}$ with $\gamma' = \kappa^{-1}\gamma^t\kappa^{-1}$.

- All the coefficients belong to $\mathcal{C}^\infty(\overline{X}_\pm;\mathbb{R})$ and may be discontinuous, except the metric γ' which coincides with g_0^{TQ} along Q' on both sides.

Hence we can write

$$\kappa_\chi (\hat{\Psi}_X^{g,g_0})_*^{-1} \hat{B}_{\mathfrak{h}}^{\phi_b} (\hat{\Psi}_X^{g,g_0})_* \kappa_\chi^{-1} = \underbrace{-b\nabla_{Y_{\mathfrak{h}_0}}^{F,g_0} + \mathcal{O}^{g_0}}_{\mathcal{A}_{-b,0}^{g_0}} + \frac{\Delta_p^{g_0} - \Delta_p^{\gamma'}}{2} + \langle p\rangle_{g_0,q}^2 \mathcal{M} \tag{109}$$

with

$$\mathcal{M} = [\mathcal{M}_j^0(q,p) + 1_{\mathbb{R}_+}(q^1)\mathcal{M}_{+,j}^0(q,p)]\frac{\partial}{\partial p_j} + [\mathcal{M}^0(q,p) + 1_{\mathbb{R}_+}(q^1)\mathcal{M}_+^0(q,p)],$$

where $\mathcal{M}_j^0, \mathcal{M}_{+,j}^0, \mathcal{M}^0, \mathcal{M}_+^0$ are symbols of order 0 on $X_{(-\varepsilon,\varepsilon)}$ for the metric g_0^{TQ}, and all the superscript ^{g_0} recall that the GKFP operator $\mathcal{A}_{-b,0}^{g_0}$ and the connection ∇^{F,g_0} are the ones associated with the metric g_0^{TQ}.
If $\mathcal{A}_{-b,0}^{g_0}$ is a GKFP operator in the smooth case, the additional term

$$\frac{\Delta_p^{g_0} - \Delta_p^{\gamma'}}{2} + \langle p\rangle_{g_0,q}^2 \mathcal{M}$$

is not an admissible perturbation according to Definition 6.1. The term $\frac{\Delta_p^{g_0}-\Delta_p^{\gamma'}}{2}$ will be absorbed by a relative boundedness with bound less than 1 argument. For $\langle p\rangle_q^2\mathscr{M}$ we have two difficulties:

- The weight $\langle p\rangle_{g_0,q}^2$ is too high but this will be handled via Lemma 6.20.
- The discontinuity along X' prevent for high regularity estimates. This will be handled by using the one dimensional product rule for Sobolev spaces

$$\left(\begin{array}{lll} \varphi_1\in W^{s_1,2}(\mathbb{R}) & , & \varphi_2\in W^{s_2,2}(\mathbb{R}) \\ s_1,s_2\geq s_3 & & \\ s_1+s_2>s_3+\frac{1}{2} & & \end{array}\right)\Rightarrow(\varphi_1\varphi_2\in W^{s_3,2}(\mathbb{R})), \tag{110}$$

while noticing $1_{\mathbb{R}_+}(q^1)\in W_{loc}^{1/2-\delta,2}(\mathbb{R})$ for all $\delta>0$.

Lemma 6.21. *Consider the operator $P=\mathscr{A}_{\alpha,0}^{g_0}+\frac{\Delta_p^{g_0}-\Delta^{\gamma'}}{2}+\langle p\rangle_{g_0,q}^2\mathscr{M}$ in $X_{(-\varepsilon,\varepsilon)}$ for $\varepsilon>0$ small enough. If $u\in\mathscr{W}^{\mu}(X_{(-\varepsilon,\varepsilon)};F\big|_{X_{(-\varepsilon,\varepsilon)}})$ satisfies $\operatorname{supp} u\subset X_{(-\varepsilon/2,\varepsilon/2)}$ and $(1+\mathscr{O}^{g_0})^{1/2}\langle p\rangle^{m+2}u\in\mathscr{W}^{\mu}(X_{(-\varepsilon,\varepsilon)};F\big|_{X_{(-\varepsilon,\varepsilon)}})$ and $\langle p\rangle_{g_0,q}^m(Pu)\in\mathscr{W}^{\mu}(X;F)$ for $\mu\in[0,1/2)$ and $m\in\mathbb{N}$, then $\langle p\rangle_{g_0,q}^m u\in\mathscr{W}^{\mu'}(X_{(-\varepsilon,\varepsilon)};F\big|_{X_{(-\varepsilon,\varepsilon)}})$ for all $\mu'\in[0,\mu+2/3)$ with*

$$\|\langle p\rangle_{g_0,q}^m u\|_{\mathscr{W}^{\mu'}}\leq C_{b,g_0,\gamma',\mathscr{M},\mu,\mu',m}\left[\|(1+\mathscr{O}^{g_0})^{1/2}\langle p\rangle_{g_0,q}^{m+2}u\|_{\mathscr{W}^{\mu}}+\|\langle p\rangle_{g_0,q}^m(Pu)\|_{\mathscr{W}^{\mu}}\right].$$

Proof. Write simply $\langle p\rangle=\langle p\rangle_{g_0,q}$, $\mathscr{O}=\mathscr{O}^{g_0}$ and compute

$$P(\langle p\rangle^m u)=\langle p\rangle^m(Pu)+\left[-\frac{\Delta_b^{\gamma'}}{2}+\langle p\rangle^2\mathscr{M},\langle p\rangle^m\right]\langle p\rangle^{-m-2}\langle p\rangle^{m+2}u.$$

This gives

$$\left[\mathscr{A}_{-b,0}^{g_0}+\frac{\Delta_g^{g_0}-\Delta_p^{\gamma}}{2}\right](\langle p\rangle^m u)=\langle p\rangle^m(Pu)+\langle p\rangle^2\mathscr{M}'(\langle p\rangle^m u)$$

where $\mathscr{M}'$ has the same structure as $\mathscr{M}$:

$$\begin{aligned}\mathscr{M}' &= [\mathscr{M}_j^{0\prime}(q,p)+1_{\mathbb{R}_+}(q^1)\mathscr{M}_{+,j}^{0\prime}(q,p)]\frac{\partial}{\partial p_j}+[\mathscr{M}^{0\prime}(q,p)+1_{\mathbb{R}_+}(q^1)\mathscr{M}_+^{0\prime}(q,p)]\\ &= \mathscr{M}_-'+1_{\mathbb{R}_+}(q^1)\mathscr{M}_+'.\end{aligned}$$

But we know

$$\|\langle p\rangle^2\mathscr{M}_{\mp}'(\langle p\rangle^m)u\|_{\mathscr{W}^{\mu}}\leq C_{b,g_0,\gamma',\mathscr{M},m}\|(1+\mathscr{O})^{1/2}\langle p\rangle^{m+2}u\|_{\mathscr{W}^{\mu}}.$$

For $\mu=0$ this gives

$$\|\langle p\rangle^2\mathscr{M}'\langle p\rangle^m u\|_{\mathscr{W}^0}\leq C'_{b,g_0,\mathscr{M},m}\|(1+\mathscr{O})^{1/2}\langle p\rangle^{m+2}u\|_{\mathscr{W}^0},$$

while for $\mu\in]0,1/2[$, $\mu''<\mu$, $1_{\mathbb{R}_+}\in W^{1/2-\delta_{\mu,\mu''},2}(\mathbb{R})$, $\delta_{\mu,\mu''}\leq 1/2-\mu$, $\mu-\delta_{\mu,\mu''}>\mu''$, the multiplication rule (110) leads to

$$\|\langle p\rangle^2\mathscr{M}'\langle p\rangle^m u\|_{\mathscr{W}^{\mu''}}\leq C_{b,g_0,\gamma',\mathscr{M},m,\mu,\mu''}\|(1+\mathscr{O})^{1/2}\langle p\rangle^m u\|_{\mathscr{W}^{\mu}}.$$

In all cases $\langle p\rangle^m u \in \mathscr{W}^{\mu}(X_{(-\varepsilon,\varepsilon)}, F\big|_{X_{(-\varepsilon,\varepsilon)}})$ with $\mathrm{supp}\, u \subset X_{(-\varepsilon/2,\varepsilon/2)}$ solves

$$\left[\mathscr{A}^{g_0}_{-b,0} + \frac{\Delta^{g_0}_g - \Delta^{\gamma}_p}{2}\right](\langle p\rangle^m u) = \hat{f}$$

with

$$\|\hat{f}\|_{\mathscr{W}^{\mu''}} \leq C_{b,g,\gamma',\mathscr{M},m,\mu,\mu''}\left[\|(1+\mathscr{O})^{1/2}\langle p\rangle^{m+2}u\|_{\mathscr{W}^{\mu}} + \|\langle p\rangle^m(Pu)\|_{\mathscr{W}^{\mu}}\right]$$

for all $\mu'' < \mu$. But the maximal subelliptic result of Lebeau in [Leb2], recalled in (99) implies

$$\|\mathscr{O}(\langle p\rangle^m u)\|_{\mathscr{W}^{\mu''}} + \|\langle p\rangle^m u\|_{\mathscr{W}^{\mu''+2/3}} \leq C_{b,g_0,s''}\left[\|\mathscr{A}^{g_0}_{-b,0}(\langle p\rangle^m u)\|_{\mathscr{W}^{\mu''}} + \|\langle p\rangle^m u\|_{\mathscr{W}^{\mu''}}\right]$$

while

$$\|\frac{\Delta^{\gamma'}_p - \Delta^{g_0}_p}{2}(\langle p\rangle^m u)\|_{\mathscr{W}^{\mu''}} \leq C_{g_0,\gamma'}\varepsilon\|\mathscr{O}(\langle p\rangle^m u)\|_{\mathscr{W}^{\mu''}}.$$

Taking $\varepsilon > 0$ small enough implies $\langle p\rangle^m u \in \mathscr{W}^{\mu''+2/3}(X_{(-\varepsilon,\varepsilon)}; F\big|_{X_{(-\varepsilon,\varepsilon)}})$ for all $\mu'' < \mu$ with the inequality written for $\mu' = \mu'' + 2/3$. □

By using a partition of unity in q, while the result for u with $\mathrm{supp}\, u \subset X_{\mathbb{R}\setminus(-\varepsilon/2,\varepsilon/2)}$ comes from the maximal subelliptic estimate (99), we deduce the

Lemma 6.22. *Let $(\hat{B}^{\phi_b}_{\mathfrak{h}}, D(\hat{B}^{\phi_b}_{\mathfrak{h}}))$ be the Bismut hypoelliptic Laplacian studied in Proposition 6.15 for $b \in \mathbb{R}^*$. If $u \in \mathscr{W}^{\mu}(X;\hat{F}_g) \cap D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ satisfies $(1+\mathscr{O}^g)^{1/2}\langle p\rangle^{m+2}_q u \in \mathscr{W}^{\mu}(X;F)$ and $\langle p\rangle^m_q(\hat{B}^{\phi_b}_{\mathfrak{h}} u) \in \mathscr{W}^{\mu}(X;\hat{F}_g)$ for $\mu \in [0,1/2)$ and $m \in \mathbb{N}$, then $\langle p\rangle^m u \in \mathscr{W}^{\min(\mu',1)}(X_{(-\varepsilon,\varepsilon)}; F\big|_{X_{(-\varepsilon,\varepsilon)}})$ for all $\mu' \in [0,\mu+2/3)$ with*

$$\|\langle p\rangle^m_q u\|_{\mathscr{W}^{\min(1,\mu')}} \leq C_{b,g,\mu,\mu'}\left[\|(1+\mathscr{O}^g)^{1/2}\langle p\rangle^{m+2}_q u\|_{\mathscr{W}^{\mu}} + \|\langle p\rangle^m_q(\hat{B}^{\phi_b}_{\mathfrak{h}} u)\|_{\mathscr{W}^{\mu}}\right].$$

In particular $(1+\mathscr{O}^g)^{1/2}\langle p\rangle^m_q u \in \mathscr{W}^{\mu''}(X;\hat{F}_g)$ for all $\mu'' \in [0,\mu+\frac{1}{6})$.

Proof. As said before it is a consequence of Lemma 6.21 and the interior maximal subelliptic estimate (99) after using a partition of unity in q.
The only thing to be recalled is the fact that $(\hat{\Psi}^{g,g_0}_X)_*$ and K are isomorphism of $\mathscr{W}^{\mu}$-spaces only for $\mu \in [-1,1]$. This explains the exponent $\min(1,\mu')$. □

Proposition 6.23. *For $b \in \mathbb{R}^*$ and $C \geq C_{b,g} > 0$, $C_{b,g}$ large enough, the operators $(\hat{B}^{\phi_b}_{\mathfrak{h}}, D(\hat{B}^{\phi_b}_{\mathfrak{h}}))$ of Proposition 6.15 and the operator $(\overline{B}^{\phi_b}_{\mathfrak{h}}, D(\overline{B}^{\phi_b}_{\mathfrak{h}}))$ of Definition 6.16 the maps*

$$(C + \hat{B}^{\phi_b}_{\mathfrak{h}})^{-9} : L^2(X;F) \to \mathscr{W}^1(X;\hat{F}_g),$$
$$(C + \hat{B}^{\phi_b}_{\mathfrak{h}})^{-9} : \mathscr{W}^{-1}(X;\hat{F}_g) \to L^2(X;F),$$
$$(C + \overline{B}^{\phi_b}_{\mathfrak{h}})^{-9} : L^2(X_-;F) \to \mathscr{W}^1(\overline{X}_-;F)$$

are all bounded.

Proof. For $m \in \mathbb{N}$ to be fixed later we call $u = (C+\hat{B}_{\mathfrak{h}}^{\phi_b})^{-m-3}u_0$ and $f = (C+\hat{B}_{\mathfrak{h}}^{\phi_b})^{-m-2}$ and $f_1 = (C+\hat{B}_{\mathfrak{h}}^{\phi_b})^{-m-1}u_0$ for $u_0 \in L^2(X;F)$ so that $u, f \in D(\hat{B}_{\mathfrak{h}}^{\phi_b})$ satisfy

$$(C+\hat{B}_{\mathfrak{h}}^{\phi_b})u = f \quad , \quad (C+\hat{B}_{\mathfrak{h}}^{\phi_b})f = f_1 .$$

From Lemma 6.20 we know

$$(1+\mathscr{O}^g)^{1/2}\langle p\rangle_q^m u \in \mathscr{W}^0(X;\hat{F}_g) \quad , \quad \langle p\rangle_q^m f \in \mathscr{W}^0(X;\hat{F}_g).$$

But Lemma 6.22 applied to the pair (u,f) then implies

$$(1+\mathscr{O}^g)^{1/2}\langle p\rangle^{m'+2} u \in \mathscr{W}^{\mu'}(X;\hat{F}_g) \tag{111}$$

for $\mu' \in [0,1/6)$ and $m' = m-2$. Applied to the pair (f, f_1) with m replaced by $m-1$ it says

$$\langle p\rangle_q^{m-1} f \in \mathscr{W}^{\min(1)\mu'')}(X;\hat{F}_g) \text{ for } \mu'' \in [0,2/3),$$

and this implies

$$\langle p\rangle_q^{m'} f \in \mathscr{W}^{\mu'}(X;F). \tag{112}$$

for $m' = m-2$ and $\mu' \in [0,1/6)$. From (111)(112) with $\mu' \in [0,1/6)$, we can apply again Lemma 6.22 with any $\mu \in [0,1/6)$ with m replaced by $m-2$. This leads to (111)(112) with $\mu' \in [0,1/3)$ and $m' = m-4$. By doing it once more we obtain (111)(112) for any $\mu' \in [0,1/2)$ and $m' = m-6$. Applying Lemma 6.22 a last time gives

$$\langle p\rangle_q^{m-6} u \in \mathscr{W}^1(X;\hat{F}_g).$$

Taking $m = 6$ proves that

$$(C+\hat{B}_{\mathfrak{h}}^{\phi_b})^{-6-3} : L^2(X;F) \to \mathscr{W}^1(X;\hat{F}_g)$$

is continuous.

The continuity of $(C+\hat{B}_{\mathfrak{h}}^{\phi_b})^{-9} : \mathscr{W}^{-1}(X;\hat{F}_g) \to L^2(X;F)$ is deduced by duality after recalling that the L^2-adjoint $(\hat{B}_{\mathfrak{h}}^{\phi_b})^*$ is a GKFP operator with the same properties as $\hat{B}_{\mathfrak{h}}^{\phi_b}$.

The property for $\overline{B}_{\mathfrak{h}}^{\phi_b}$ is deduced from the fact that $(C+\hat{B}_{\mathfrak{h}}^{\phi_b})^{-1} : L^2_{ev}(X;F) \to L^2_{ev}(X;F)$. $\square$

Corollary 6.24. *For $b \in \mathbb{R}^*$, $k \in \mathbb{N}$ and $t > 0$ the following operators are well defined and continuous:*

$$\hat{d}_{g,\mathfrak{h}}(\hat{B}_{\mathfrak{h}}^{\phi_b})^k e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}} : L^2(X;F) \to L^2(X;F),$$
$$\hat{d}_{g,\mathfrak{h}}^{\phi_b}(\hat{B}_{\mathfrak{h}}^{\phi_b})^k e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}} : L^2(X;F) \to L^2(X;F),$$
$$\overline{d}_{g,\mathfrak{h}}(\overline{B}_{\mathfrak{h}}^{\phi_b})^k e^{-t\overline{B}_{\mathfrak{h}}^{\phi_b}} : L^2(X_-;F) \to L^2(X_-;F),$$
$$\overline{d}_{g,\mathfrak{h}}^{\phi_b}(\overline{B}_{\mathfrak{h}}^{\phi_b})^k e^{-t\overline{B}_{\mathfrak{h}}^{\phi_b}} : L^2(X_-;F) \to L^2(X_-;F).$$

By taking the $\phi_{\pm b}$ left-adjoints the reverse products, initially defined on $D(\hat{d}_{g,\mathfrak{h}})$, $D(\hat{d}_{g,\mathfrak{h}}^{\phi_b})$, $D(\overline{d}_{g,\mathfrak{h}}^{\phi_b})$ or $D(\overline{d}_{g,\mathfrak{h}}^{\phi_b})$, extend as bounded operators in $L^2(X;F)$ and $L^2(X_-;F)$.

Proof. It suffices to write for $A = \hat{B}^{\phi_b}_{\mathfrak{h}}$ $A^k e^{-tA} = (C+A)^{-9}A^k(C+A)^{-k}(C+A)^{k+9}e^{-tA} = A^k(C+A)^{-k}(C+A)^{k+9}e^{-tA}(C+A)^{-9}$ owing to Corollary 6.19, and to notice that

$$\mathscr{W}^1(X;\hat{F}_g) \subset D(\hat{d}_{\alpha\hat{\mathscr{H}}}) \cap D(\hat{d}^{\hat{\phi}}_{\alpha\hat{\mathscr{H}}}),$$

while $\hat{d}_{\alpha\hat{\mathscr{H}}}$ and $\hat{d}^{\hat{\phi}}_{\alpha\hat{\mathscr{H}}}$ are continuous from $L^2(X;E)$ to $\mathscr{W}^{-1}(X;\hat{F}_g)$.
The result for $\overline{B}^{\phi_b}_{\mathfrak{h}}$, $\overline{d}_{g,\mathfrak{h}}$, $\overline{d}^{\phi_b}_{g,\mathfrak{h}}$ are again deduced from the general construction with the parity with respect to Σ_ν. ☐

6.5 Commutation property

We now prove the commutation of the differential and Bismut codifferential with the resolvent of the hypoelliptic Laplacian.

Proposition 6.25. *Let $(\overline{B}^{\phi_b}_{\mathfrak{h}}, D(\overline{B}^{\phi_b}_{\mathfrak{h}}))$ be the operator of Definition 6.16 and Theorem 6.17 for $b \in \mathbb{R}^*$. Let $(\overline{d}_{g,\mathfrak{h}}, D(\overline{d}_{g,\mathfrak{h}})$ and $(\overline{d}^{\phi_b}_{g,\mathfrak{h}}, D(\overline{d}^{\phi_b}_{g,\mathfrak{h}}))$ be the closed realizations of the differential and Bismut codifferential studied in Proposition 4.10 and Proposition 5.9.*
The semigroup $(e^{-t\overline{B}^{\phi_b}_{\mathfrak{h}}})_{t\geq 0}$ preserves on $D(\overline{d}_{g,\mathfrak{h}})$ (resp. $D(\overline{d}^{\phi_b}_{g,\mathfrak{h}})$) with

$$\begin{aligned} &\forall s \in D(\overline{d}_{g,\mathfrak{h}}), \quad \overline{d}_{g,\mathfrak{h}} e^{-t\overline{B}^{\phi_b}_{\mathfrak{h}}} s = e^{-t\overline{B}^{\phi_b}_{\mathfrak{h}}} \overline{d}_{g,\mathfrak{h}} s \\ \textit{resp.} \quad &\forall s \in D(\overline{d}^{\phi_b}_{\mathfrak{h}}), \quad \overline{d}^{\phi_b}_{g,\mathfrak{h}} e^{-t\overline{B}^{\phi_b}_{\mathfrak{h}}} s = e^{-t\overline{B}^{\phi_b}_{\mathfrak{h}}} \overline{d}^{\phi_b}_{g,\mathfrak{h}} s, \end{aligned}$$

for all $t \geq 0$.
Hence for any $z \in \mathbb{C} \setminus Spec(\overline{B}^{\phi_b}_{\mathfrak{h}})$ the following holds

$$\begin{aligned} &\forall s \in D(\overline{d}_{g,\mathfrak{h}}), \quad (z - B^{\phi_b}_{g,\mathfrak{h}})^{-1} s \in D(\overline{d}_{g,\mathfrak{h}}) \\ &\quad \textit{and} \quad \overline{d}_{g,\mathfrak{h}}(z - \overline{B}^{\phi_b}_{\mathfrak{h}})^{-1} s = (z - \overline{B}^{\phi_b}_{\mathfrak{h}})^{-1} \overline{d}_{g,\mathfrak{h}} s, \\ \textit{resp.} \quad &\forall s \in D(\overline{d}^{\phi_b}_{g,\mathfrak{h}}), \quad (z - \overline{B}^{\phi_b}_{\mathfrak{h}})^{-1} s \in D(\overline{d}^{\phi_b}_{g,\mathfrak{h}}) \\ &\quad \textit{and} \quad \overline{d}^{\phi_b}_{g,\mathfrak{h}}(z - \overline{B}^{\phi_b}_{\mathfrak{h}})^{-1} s = (z - \overline{B}^{\phi_b}_{\mathfrak{h}})^{-1} \overline{d}^{\phi_b}_{g,\mathfrak{h}} s. \end{aligned}$$

Although we seek properties of $\overline{B}^{\phi_b}_{\mathfrak{h}}$, it is again more convenient to work with $\hat{B}^{\phi_b}_{\mathfrak{h}}$ and then to translate the results via the parity w.r.t Σ_ν. We start with a lemma.

Lemma 6.26. *Let $\mathscr{C}_{0,g}(\hat{F}_g)$, $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$ and $e^{\frac{\lambda_0}{b}} \sigma_b \hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}$ be the spaces introduced respectively in Definition 3.3, Proposition 4.10 and Proposition 5.9. For all $\omega \in \hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$ (resp $\omega \in e^{\frac{\lambda_0}{b}} \sigma_b \hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}$) the following properties hold:*

$$\begin{aligned} &\omega \in D(\hat{d}_{g,\mathfrak{h}}) \cap D(\hat{B}^{\phi_b}_{\mathfrak{h}}) \quad \hat{d}_{g,\mathfrak{h}}\omega \in D(\hat{B}^{\phi_b}_{\mathfrak{h}}) \quad \hat{B}^{\phi_b}_{\mathfrak{h}}\omega \in D(\hat{d}_{g,\mathfrak{h}}) \\ \textit{and} \quad &\hat{B}^{\phi_b}_{\mathfrak{h}} \hat{d}_{g,\mathfrak{h}}\omega = \hat{d}_{g,\mathfrak{h}} \hat{B}^{\phi_b}_{\mathfrak{h}}\omega, \\ \textit{resp.} \quad &\omega \in D(\hat{d}^{\phi_b}_{g,\mathfrak{h}}) \cap D(\hat{B}^{\phi_b}_{\mathfrak{h}}) \quad \hat{d}^{\phi_b}_{g,\mathfrak{h}}\omega \in D(\hat{B}^{\phi_b}_{\mathfrak{h}}) \quad \hat{B}^{\phi_b}_{\mathfrak{h}}\omega \in D(\hat{d}^{\phi_b}_{g,\mathfrak{h}}) \\ \textit{and} \quad &\hat{B}^{\phi_b}_{g,\mathfrak{h}} \hat{d}^{\phi_b}_{\mathfrak{h}}\omega = \hat{d}^{\phi_b}_{g,\mathfrak{h}} \hat{B}^{\phi_b}_{g,\mathfrak{h}}\omega. \end{aligned}$$

Proof. By definition $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{h}}} \subset \mathscr{C}_{0,g}(\hat{F}_g) \subset D(\hat{d}_{g,\mathfrak{h}}) \cap D(\hat{B}_{\mathfrak{h}}^{\phi_b})$ while Proposition 4.10 ensures $\hat{d}_{g,\mathfrak{h}}\omega = d_{\hat{\mathfrak{h}}}\omega \in \mathscr{C}_{0,g}(\hat{F}_g) \subset D(\hat{B}_{\mathfrak{h}}^{\phi_b})$ when $\omega \in \hat{D}_{g,\nabla^{\mathfrak{h}}}$. Additionally as differential operators on $X_- \cup X_+ = X \setminus X'$, we know

$$d_{\hat{\mathfrak{h}}}\hat{B}_{\mathfrak{h}}^{\phi_b} = d_{\hat{\mathfrak{h}}}(d_{\hat{\mathfrak{h}}}d_{\hat{\mathfrak{h}}}^{\phi_b} + d_{\hat{\mathfrak{h}}}^{\phi_b}d_{\hat{\mathfrak{h}}}) = \hat{B}_{\mathfrak{h}}^{\phi_b}d_{\hat{\mathfrak{h}}}\,.$$

The question to be answered is $\hat{B}_{\mathfrak{h}}^{\phi_b}\omega \in D(\hat{d}_{g,\mathfrak{h}})$ when $\omega \in \hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$. For $\omega \in \hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$, write

$$\langle t', \underbrace{\hat{B}_{\mathfrak{h}}^{\phi_b}\hat{d}_{g,\mathfrak{h}}\omega}_{\in L^2(X;F)}\rangle = \lim_{\varepsilon\to 0}\langle 1_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(q^1)t', \hat{B}_{\mathfrak{h}}^{\phi_b}d_{\hat{\mathfrak{h}}}\omega\rangle = \langle t', d_{\hat{\mathfrak{h}}}\hat{B}_{\mathfrak{h}}^{\phi_b}\omega\big|_{X\setminus X'}\rangle$$

for $t' \in \mathscr{C}_{0,g}(X;\hat{F}'_g)$, by using the duality product

$$\langle t, s\rangle = \int_X \langle t, s\rangle_{F'_x,F_x}\, dv_X(x)\,.$$

It implies

$$\forall t' \in \mathscr{C}_{0,g}(X;\hat{F}_g),\quad \left|\langle t', d_{\hat{\mathfrak{h}}}(\hat{B}_{\mathfrak{h}}^{\phi_b}\omega)\big|_{X\setminus X'}\rangle\right| \le C\|t'\|_{L^2}\,.$$

In particular when we take $t' \in \mathscr{C}_0^\infty(X_\mp;F')$, it says that $u_\mp = \hat{B}_{\mathfrak{h}}^{\phi_b}\omega\big|_{X_\mp}$ belongs to $\mathscr{E}_{loc}(d_{\hat{\mathfrak{h}}};F)$ and $j_{\partial X_\mp}u_\mp$ is well defined.
By working in M_g as we did in the proof of Proposition and taking test sections in $\mathscr{C}_0^\infty(M_{g,(-\varepsilon,\varepsilon)};\Lambda T^*M_g \otimes \pi_X^*(\mathfrak{f}'))$ which make a subset of $\mathscr{C}_{0,g}(\hat{F}'_g)$, we finally obtain that $u = \hat{B}_{\mathfrak{h}}^{\phi_b}\omega = 1_{X_-}u_- + 1_{X_+}u_+$ belongs to $D(\hat{d}_{g,\mathfrak{h}})$.
When $\omega \in e^{\frac{\lambda_0}{b}}\sigma_b\hat{\mathscr{D}}'_{g,\nabla^{\mathfrak{f}'}}$ we know $\hat{d}_{g,\mathfrak{h}}^{\phi_b}\omega \in \mathscr{C}_{0,g}(\hat{F}_g) \subset D(\hat{B}_{\mathfrak{h}}^{\phi_b})$ and we want to prove $\hat{B}_{\mathfrak{h}}^{\phi_b}\omega \in D(\hat{d}_{g,\mathfrak{h}}^{\phi_b})$. By taking $\omega' \in \hat{D}_{g,\nabla^{\mathfrak{f}}}$ we write

$$\langle \omega', \hat{B}_{\mathfrak{h}}^{\phi_b}\hat{d}_{g,\mathfrak{h}}^{\phi_b}\omega\rangle_{\phi_{-b}} = \langle \hat{d}_{g,\mathfrak{h}}\hat{B}_{\mathfrak{h}}^{\phi_{-b}}\omega', \omega\rangle_{\phi_{-b}} = \langle \hat{B}_g^{\phi_{-b}}\hat{d}_{g,\mathfrak{h}}\omega', \omega\rangle_{\phi_{-b}} = \langle \hat{d}_{g,\mathfrak{h}}\omega', \hat{B}_{\mathfrak{h}}^{\phi_b}\omega\rangle_{\phi_{-b}}$$

where all the identity make sense for the closed operators by the first result. But the left-hand side implies that $v = \hat{B}_{\mathfrak{h}}^{\phi_b}\omega \in L^2(X;\hat{F}_g)$ satisfies

$$\forall \omega' \in \hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}},\quad \left|\langle \hat{d}_{g,\mathfrak{h}}\omega', v\rangle_{\phi_{-b}}\right| \le C\|\omega'\|_{L^2}\,.$$

But since $\hat{\mathscr{D}}_{g,\nabla^{\mathfrak{f}}}$ is a core for $\hat{d}_{g,\mathfrak{h}}$ and the ϕ_{-b} right-adjoint of $\hat{d}_{g,\mathfrak{h}}$ is $\hat{d}_{g,\mathfrak{h}}^{\phi_b}$ we deduce $v = \hat{B}_{\mathfrak{h}}^{\phi_b} \in D(\hat{d}_{g,\mathfrak{h}}^{\phi_b})$ and

$$\hat{B}^{\phi_b}\hat{d}_{g,\mathfrak{h}}^{\phi_b}\omega = \hat{d}_{g,\mathfrak{h}}^{\phi_b}\hat{B}_{\mathfrak{h}}^{\phi_b}\omega$$

□

Proof of Proposition 6.25. We work with $\hat{B}_{\mathfrak{h}}^{\phi_b}$, $\hat{d}_{g,\mathfrak{h}}$ and $\hat{d}_{g,\mathfrak{h}}^{\phi_b}$. For $s_0 \in L^2(X;F)$, Corollary 6.24 says that $\hat{d}_{g,\mathfrak{h}}e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}s_0 = \hat{d}_{g,\mathfrak{h}}s_t$ is a $\mathscr{C}^1(]0,+\infty[;L^2(X;F))$ function with

$$\forall t > 0, \frac{d}{dt}[\hat{d}_{g,\mathfrak{h}}e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}s_0] = \hat{d}_{g,\mathfrak{h}}\hat{B}_{\mathfrak{h}}^{\phi_b}e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}s_0 = \hat{d}_{g,\mathfrak{h}}\hat{B}_{\mathfrak{h}}^{\phi_b}s_t\,.$$

For $\omega \in e^{\frac{\lambda_0}{-b}}\sigma_{-b}\hat{\mathscr{D}}'_{g,\nabla\mathfrak{f}'}$ and $t > 0$, Lemma 6.26 allows to write

$$\langle \omega', \hat{d}_{g,\mathfrak{h}}\hat{B}_{\mathfrak{h}}^{\phi_b}s_t\rangle_{\phi_{-b}} = \langle \hat{B}^{\phi_{-b}}\hat{d}_{g,\mathfrak{h}}^{\phi_{-b}}\omega', s_t\rangle_{\phi_{-b}} = \langle \hat{d}_{g,\mathfrak{h}}^{\phi_{-b}}\hat{B}_{\mathfrak{h}}^{\phi_{-b}}\omega', s_t\rangle_{\phi_b} = \langle \hat{B}_{\mathfrak{h}}^{\phi_{-b}}\omega', \hat{d}_{g,\mathfrak{h}}s_t\rangle_{\phi_{-b}}.$$

This implies in particular

$$\forall \omega' \in e^{\frac{\lambda_0}{-b}}\sigma_{-b}\hat{\mathscr{D}}'_{g,\nabla\mathfrak{f}'}, \quad \left|\langle \hat{B}_{\mathfrak{h}}^{\phi_{-b}}\omega', \hat{d}_{g,\mathfrak{h}}s_t\rangle\right| \leq C_t\|\omega'\|_{L^2}.$$

But $\hat{\mathscr{D}}'_{g,\nabla\mathfrak{f}'}$ is dense in $\mathscr{C}_{0,g}(\hat{F}'_g)$ while $e^{\frac{\lambda_0}{-b}}\sigma_{-b}$ is continuous from $\mathscr{C}_{0,g}(\hat{F}'_g)$ to $\mathscr{C}_{0,g}(\hat{F}_g)$ and $\mathscr{C}_{0,g}(\hat{F}_g)$ is continuously and densely embedded in $D(\hat{B}_{\mathfrak{h}}^{\phi_{-b}})$. Since $\hat{B}_{\mathfrak{h}}^{\phi_b}$ is the ϕ_{-b} right-adjoint of $\hat{B}_{\mathfrak{h}}^{\phi_{-b}}$ we deduce

$$\forall t > 0, \quad \hat{d}_{g,\mathfrak{h}}s_t \in D(\hat{B}_{\mathfrak{h}}^{\phi_b}) \quad \text{and} \quad \frac{d}{dt}\hat{d}_{g,\mathfrak{h}}s_t = \hat{B}_{\mathfrak{h}}^{\phi_b}\hat{d}_{g,\mathfrak{h}}s_t.$$

Since for $t_0 > 0$, $\hat{d}_{g,\mathfrak{h}}s_{t_0} \in L^2(X;F)$ by Corollary 6.24, this implies

$$\forall t > t_0 > 0, \quad \hat{d}_{g,\mathfrak{h}}s_t = e^{-(t-t_0)\hat{B}_{\mathfrak{h}}^{\phi_b}}\hat{d}_{g,\mathfrak{h}}s_{t_0},$$

or

$$\forall t > t_0 > 0, \quad \hat{d}_{g,\mathfrak{h}}e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}s_0 = e^{-(t-t_0)\hat{B}_{\mathfrak{h}}^{\phi_b}}\hat{d}_{g,\mathfrak{h}}(e^{-t_0\hat{B}_{\mathfrak{h}}^{\phi_b}}s_0).$$

Let us assume now $s_0 \in D(\hat{d}_{g,\mathfrak{h}})$ and take the limit as $t_0 \to 0^+$. From $\lim_{t_0\to 0^+} e^{-t_0\hat{B}_{\mathfrak{h}}^{\phi_b}}s_0 = s_0$ in $L^2(X;F)$, Corollary 6.24 yields

$$\lim_{t_0\to 0^+} e^{-(t-t_0)\hat{B}_{\mathfrak{h}}^{\phi_b}}\hat{d}_{g,\mathfrak{h}}(e^{-t_0\hat{B}_{\mathfrak{h}}^{\phi_b}}s_0) = e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}\hat{d}_{g,\mathfrak{h}}s_0$$

We have proved that for $s_0 \in D(\hat{d}_{g,\mathfrak{h}})$

$$\forall t > 0, \quad \hat{d}_{g,\mathfrak{h}}e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}s_0 = e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}\hat{d}_{g,\mathfrak{h}}s_0,$$

while the result is obvious for $t = 0$.
Because $e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}\hat{d}_{g,\mathfrak{h}}^{\phi_b}$ and $\hat{d}_{g,\mathfrak{h}}^{\phi_b}e^{-t\hat{B}_{\mathfrak{h}}^{\phi_b}}$ are the ϕ_b left-adjoints respectively of $\hat{d}_{g,\mathfrak{h}}e^{-t\hat{B}_{\mathfrak{h}}^{\phi_{-b}}}$ and $e^{-t\hat{B}_{\mathfrak{h}}^{\phi_{-b}}}\hat{d}_{g,\mathfrak{h}}$, the same result holds when $\hat{d}_{g,\mathfrak{h}}$ is replaced with $\hat{d}_{g,\mathfrak{h}}^{\phi_b}$.

Finally the commutation with the resolvent are proved after writing for $\mathbb{R}\mathrm{e}\, z \leq -C$ with $C > 0$ large enough

$$(\hat{B}_{\mathfrak{h}}^{\phi_b} - z)^{-1} = \int_0^{+\infty} e^{-t(\hat{B}_{\mathfrak{h}}^{\phi_b} - z)}\, dt,$$

and by analytic continuation for the extension to any $z \in \mathbb{C} \setminus \mathrm{Spec}(\hat{B}_{\mathfrak{h}}^{\phi_b})$. $\square$

6.6 PT-symmetry

While working with the metric $\pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ} \otimes g^{\mathfrak{f}})$ without the weight $\langle p\rangle_q^{N_V - N_H}$, Bismut in [Bis05] establishes an important property for the spectral analysis of the hypoelliptic Laplacian: the PT-symmetry or more precisely a formal version of it. Let us first recall Bismut's construction in the smooth case and then we will show how this can be adapted easily to our case with interface or boundary conditions. Remember

$$\phi_b = \begin{pmatrix} g^{TQ} & -b\mathrm{Id} \\ b\mathrm{Id} & 0 \end{pmatrix} = \begin{pmatrix} g & -b\mathrm{Id} \\ b\mathrm{Id} & 0 \end{pmatrix} \quad , \quad \phi_b^{-1} = \begin{pmatrix} 0 & \frac{\mathrm{Id}}{b} \\ -\frac{\mathrm{Id}}{b} & \frac{1}{b^2} g^{T^*Q} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\mathrm{Id}}{b} \\ -\frac{\mathrm{Id}}{b} & \frac{1}{b^2} g^{-1} \end{pmatrix}.$$

The tangent bundle is endowed with the new metric

$$\mathfrak{g}_b^{TX} = \begin{pmatrix} g^{TQ} & b\mathrm{Id} \\ b\mathrm{Id} & 2b^2 g^{T^*Q} \end{pmatrix} = \begin{pmatrix} g & b\mathrm{Id} \\ b\mathrm{Id} & 2b^2 g^{-1} \end{pmatrix} \tag{113}$$

or by calling $\mathfrak{p}U$ the vertical component of U

$$\langle U, U\rangle_{\mathfrak{g}_b^{TX}} = \langle \pi_{X,*}U, \pi_{X,*}U\rangle_{g^{TQ}} + 2b\langle \pi_{X,*}U\,,\,\mathfrak{p}U\rangle_{TQ,T^*Q} + 2b^2\langle \mathfrak{p}U\,,\,\mathfrak{p}U\rangle_{g^{T^*Q}}.$$

Meanwhile the dual metric is given by

$$\mathfrak{g}_b^{T^*Q} = \begin{pmatrix} 2g^{-1} & -\frac{\mathrm{Id}}{b} \\ -\frac{\mathrm{Id}}{b} & \frac{g}{b^2} \end{pmatrix}. \tag{114}$$

This defines a new metric $\mathfrak{g}_b^{\mathfrak{f}} = \mathfrak{g}_b^{\Lambda T^*X} \otimes \pi_X^*(g^{\mathfrak{f}})$ which is uniformly equivalent to $\pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ} \otimes g^{\mathfrak{f}})$. In particular the sesquilinear form

$$\langle s\ s\rangle_{\mathfrak{g}_b^{\mathfrak{f}}} = \int_X \mathfrak{g}_b^{\mathfrak{f}}(s\,,s')\,dv_X(x) \tag{115}$$

is continuous on $L^2(X;F,\pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ} \otimes g^{\mathfrak{f}})$ and it is neither continuous nor everywhere defined on $L^2(X;F,g^F)$. However it is well defined on $\mathscr{C}_0^\infty(X;F)$ which is dense in all the considered L^2-spaces, in the smooth case. An additional modification is used by introducing the maps $F_b : TX \to TX$ and its transpose $\tilde{F}_b : T^*X \to T^*X$ given by the matrix

$$F_b = \begin{pmatrix} \mathrm{Id}_{TQ} & 2bg^{-1} \\ 0 & -\mathrm{Id}_{T^*Q} \end{pmatrix} \quad , \quad \tilde{F}_b = \begin{pmatrix} \mathrm{Id}_{T^*Q} & 0 \\ 2bg^{-1} & -\mathrm{Id}_{TQ} \end{pmatrix}. \tag{116}$$

We need also the maps $H : TX \to TX$ and its transpose $\tilde{H} : T^*X \to T^*X$ and $r : X \to X$ given by

$$H = \begin{pmatrix} \mathrm{Id}_{TQ} & 0 \\ 0 & -\mathrm{Id}_{T^*Q} \end{pmatrix} \quad , \quad \tilde{H} = \begin{pmatrix} \mathrm{Id}_{T^*Q} & 0 \\ 0 & -\mathrm{Id}_{TQ} \end{pmatrix} \quad , \quad r(q,p) = (q,-p).$$

After tensorization of $\tilde{F}_b$, $\tilde{H}$, the map $u_b : \mathscr{C}_0^\infty(X;F) \to \mathscr{C}_0^\infty(X;F)$ is defined by

$$u_b s(q,p) = (\tilde{F}_b s)(q,-p) = r_* \tilde{H}\tilde{F}_b s. \tag{117}$$

Then the hermitian form $\mathfrak{H}_b$ is defined on $\mathscr{C}_0^\infty(X;F)$ by

$$\langle s, s'\rangle_{\mathfrak{H}_b} = \langle u_b s, s'\rangle_{\mathfrak{g}_b^{\mathfrak{f}}}\,. \tag{118}$$

From $r_*H(\mathfrak{g}_b^{\mathfrak{f}})^{-1} = (\mathfrak{g}_b^{\mathfrak{f}})^{-1} r_*\tilde{H}$, and the relation

$$H(\mathfrak{g}_b^{\mathfrak{f}})^{-1}\tilde{F}_b = \begin{pmatrix}\mathrm{Id} & 0\\ 0 & -\mathrm{Id}\end{pmatrix}\begin{pmatrix}2g^{-1} & -\frac{\mathrm{Id}}{b}\\ -\frac{\mathrm{Id}}{b} & \frac{g}{b^2}\end{pmatrix}\begin{pmatrix}\mathrm{Id} & 0\\ 2bg^{-1} & -\mathrm{Id}\end{pmatrix} = \begin{pmatrix}0 & \frac{\mathrm{Id}}{b}\\ -\frac{\mathrm{Id}}{b} & \frac{g}{b^2}\end{pmatrix} = \phi_b^{-1}$$

we deduce

$$\langle s, s'\rangle_{\mathfrak{H}_b} = \langle s, r_* s'\rangle_{\phi_b}\,.$$

Finally since $d_{\mathfrak{h}} r_* = r_* d_{\mathfrak{h}}$, the $\mathfrak{H}_b$ formal adjoint of $d_{\mathfrak{h}}$ is $d_{\mathfrak{h}}^{\phi_b}$ and since $\mathfrak{H}_b$ is hermitian, $(d_{\mathfrak{h}} + d_{\mathfrak{h}}^{\phi_b})$ and its square $B_{\mathfrak{h}}^{\phi_b}$ are formally self-ajoint for $\mathfrak{H}_b$.
The piecewise $\mathscr{C}^\infty$ and continuous version for the vector bundle $\hat{F}_g$ for the metric $\hat{g}^{TQ}$ is defined as follows.

Definition 6.27. *The map $\hat{u}_b$ and hermitian form $\hat{\mathfrak{H}}_g$ on $\mathscr{C}_{0,g}(\hat{F}_g)$ are given by the same formula as (117)(118) after replacing g^{TQ} by $\hat{g}^{TQ} = 1_{\overline{Q}_-} g_-^{TQ} + 1_{Q_+} g_+^{TQ}$ in (113)(114)(115)(116).*

By construction $\hat{u}_b : \mathscr{C}_{0,b}(\hat{F}_g) \to \mathscr{C}_{0,b}(\hat{F}_b)$ and it preserves the parity with respect to Σ_ν, while $\hat{\mathfrak{H}}_g$ well defined on $\mathscr{C}_{0,g}(\hat{F}_b)$ and the direct sum $\mathscr{C}_{0,g,ev}(\hat{F}_g) \oplus \mathscr{C}_{0,g,odd}(\hat{F}_g)$ is $\hat{\mathfrak{H}}_g$ orthogonal. We deduce the following proposition.

Proposition 6.28. *The hermitian form $\hat{\mathfrak{H}}_g$ is well-defined and continuous on $\langle p\rangle_q^{-d/2} L^2(X;F)$.*
The identity

$$\langle \hat{B}_{\mathfrak{h}}^{\phi_b} s, s'\rangle_{\hat{\mathfrak{H}}_b} = \langle s, \hat{B}_{\mathfrak{h}}^{\phi_b} s'\rangle_{\hat{\mathfrak{H}}_b} \tag{119}$$

holds for all $s, s' \in \langle p\rangle_q^{-d/2} D(\hat{B}_{\mathfrak{h}}^{\phi_b}) \subset D(\hat{B}_{\mathfrak{h}}^{\phi_b}) \cap \langle p\rangle_q^{-d/2} L^2(X;F)$.
As a consequence $\mathrm{Spec}(\hat{B}_{\mathfrak{h}}^{\phi_b})$ *is symmetric with respect to the real axis.*
Finally the same results hold for $(\overline{B}_{\mathfrak{h}}^{\phi_b}, D(\overline{B}_{\mathfrak{h}}^{\phi_b}))$ after using the restriction $\mathfrak{H}_b$ of $\hat{\mathfrak{H}}_b$ to $F|_{X_-}$.

Proof. The map $\langle p\rangle_q^{-d/2}$ is continuous from $L^2(X;F,g^F)$ to $L^2(X;F,\pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ} \otimes g^{\mathfrak{f}}))$ while $\mathfrak{H}_b$ is continuous on $L^2(X;F,\pi_X^*(g^{\Lambda T^*Q} \otimes g^{\Lambda TQ} \otimes g^{\mathfrak{f}})$. All those spaces contain $\mathscr{C}_{0,g}(X;\hat{F}_g)$ as a dense subspace.
The set $\mathscr{C}_{0,g}(X;\hat{F}_g)$ is dense in $\langle p\rangle^{-d/2} D(\hat{B}_{\mathfrak{h}}^{\phi_b})$. The continuous embedding of $\langle p\rangle_q^{-d/2} D(\hat{B}_{\mathfrak{h}}^{\phi_b})$ comes from the fact that $\langle p\rangle^{d/2} \hat{B}_{\mathfrak{h}}^{\phi_b} \langle p\rangle_q^{-d/2}$ is a relatively bounded perturbation of $(\hat{B}_{\mathfrak{h}}^{\phi_b}, D(\hat{B}_{\mathfrak{h}}^{\phi_b}))$ with infinitesimal bound. The identity (119) is valid for $s, s' \in \mathscr{C}_{0,g}(\hat{F}_g)$ and by density extends to $s, s' \in \langle p\rangle_q^{-d/2} D(\hat{B}_{\mathfrak{h}}^{\phi_b})$.
Let us consider the spectral problem. We know that $\mathrm{Spec}(\hat{B}_{\mathfrak{h}}^{\phi_b})$ is discrete. Additionally because $(C_b + \hat{B}_{\mathfrak{h}}^{\phi_b})$ is maximal accretive we know that for all $z \in \mathbb{C}$, $(\hat{B}_{\mathfrak{h}}^{\phi_b} - z)(1 + C_b + \hat{B}_{\mathfrak{h}}^{\phi_b})^{-1}$ is a Fredholm operator with index 0 and $\lambda \in \mathrm{Spec}(\hat{B}_{\mathfrak{h}}^{\phi_b})$ iff

$\ker(\hat{B}^{\phi_b}-\lambda)\neq\{0\}$. When λ is an eigenvalue of $\hat{B}^{\phi_b}_{\mathfrak{h}}$ there exists $s_\lambda\in D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ such that

$$(C+\hat{B}^{\phi_b}_{\mathfrak{h}})s_\lambda=(C+\lambda)s_\lambda\,.$$

By chosing $C>0$ such that $(C+\hat{B}^{\phi_b}_{\mathfrak{h}})$ is invertible, we obtain $s_\lambda\in(C+\hat{B}^{\phi_b}_{\mathfrak{h}})^{-n}L^2(X;F)\subset\langle p\rangle_q^{-n}D(\hat{B}^{\phi_b}_{\mathfrak{h}})$ for any $n\in\mathbb{N}$ by Lemma 6.20. Hence we can use (119) which implies

$$\forall s'\in\mathscr{C}_{0,g}(\hat{F}_g),\quad \langle s_\lambda,(\hat{B}^{\phi_b}-\overline{\lambda})s'\rangle_{\hat{\mathfrak{H}}_b}=0\,.$$

But $\mathfrak{H}_g$ is also continuous on $\langle p\rangle^{-d}L^2(X;F)\times L^2(X;F)$ while $s_\lambda\in\langle p\rangle^{-d}L^2(X;F)$ and the density of $\mathscr{C}_{0,g}(\hat{F}_g)$ in $D(\hat{B}^{\phi_b}_{\mathfrak{h}})$. Finally as a non zero element of $\langle p\rangle^{-d}L^2(X;F)$ there exists $s''_\lambda\in L^2(X;F)$ such that $\langle s_\lambda,s''_\lambda\rangle_{\hat{\mathfrak{H}}_b}\neq 0$. Therefore $(\hat{B}^{\phi_b}_{\mathfrak{h}}-\overline{\lambda})$ is not onto and $\overline{\lambda}\in\mathrm{Spec}(\hat{B}^{\phi_b}_{\mathfrak{h}})$. □

Remark 6.29. *The symmetry with respect to the real axis of* $\mathrm{Spec}(\hat{B}^{\phi_b}_{\mathfrak{h}})$ *and* $\mathrm{Spec}\overline{B}^{\phi_b}_{\mathfrak{h}}$ *are not the only issues of Proposition 6.28. Actually the relation (119) is a crucial point while studying the spectrum in a neighborhood of* 0 *in various asymptotic regimes e.g.* $b\to 0^+$ *(see [She][BiLe][HHS]). In particular those asymptotic regimes correspond to cases where the spectral spaces concentrate asymptotically to the kernel of the scalar vertical harmonic oscillator hamiltonian, multiples of a scaled gaussian function in* p*, on which the restricted hermitian form* $\mathfrak{H}_g$ *is positive definite. This property with (119) helps to reduce the asymptotic spectral analysis of the hypoelliptic Laplacian on* $X=T^*Q$ *to the the more standard asymptotic spectral analysis of the Hodge or Witten Laplacian on* Q*. We refer the reader to [Nie14]-Proposition 15.2 for a short abstract version of those arguments.*

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