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► **To cite this version:**

Pierrick Bousseau. ON AN EXAMPLE OF QUIVER DONALDSON-THOMAS/RELATIVE GROMOV-WITTEN CORRESPONDENCE. 2021. hal-03277610

HAL Id: hal-03277610

<https://hal.science/hal-03277610>

Preprint submitted on 4 Jul 2021

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ON AN EXAMPLE OF QUIVER DONALDSON-THOMAS/RELATIVE GROMOV-WITTEN CORRESPONDENCE

PIERRICK BOUSSEAU

ABSTRACT. We explain and generalize a recent result of Reineke-Weist by showing how to reduce it to the Gromov-Witten/Kronecker correspondence by a degeneration and blow-up. We also refine the result by working with all genera on the Gromov-Witten side and with refined Donaldson-Thomas invariants on the quiver side.

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1. INTRODUCTION

1.1. Statement of the result. Let Y be a smooth projective complex surface and let D_1 and D_2 be two smooth non-empty divisors on Y , intersecting transversally, and such that the union $D_1 \cup D_2$ is anticanonical. Elementary theory of surfaces implies that D_1 and D_2 are necessarily rational curves and that the intersection $D_1 \cap D_2$ consists of two points. An example to keep in mind is $Y = \mathbb{P}^2$, D_1 a line and D_2 a smooth conic not tangent to D_1 .

For every $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta \cdot D_1 > 0$ and $\beta \cdot D_2 > 0$, let $N_{0,\beta}^{Y/D_1}$ be the Gromov-Witten count of rational curves in Y of class β , intersecting D_1 in a unique point with maximal tangency and passing through $\beta \cdot D_2$ prescribed points in general position in Y . In the present paper, we explain how to construct a quiver Q_β^{Y/D_1} and a dimension vector

$$\mathbf{d}(\beta) \in \mathbb{Z}^{(Q_\beta^{Y/D_1})_0},$$

where $(Q_\beta^{Y/D_1})_0$ is the set of vertices of Q_β^{Y/D_1} . Donaldson-Thomas theory for quivers defines an integer Ω_β^{Y/D_1} , virtual count of representations of Q_β^{Y/D_1} of dimension vector $\mathbf{d}(\beta)$. A priori, Ω_β^{Y/D_1} is only defined if $\mathbf{d}(\beta) \in \mathbb{N}^{(Q_\beta^{Y/D_1})_0}$. If $\mathbf{d}(\beta) \notin \mathbb{N}^{(Q_\beta^{Y/D_1})_0}$, we set $\Omega_\beta^{Y/D_1} := 0$.

Our main result is the following theorem.

Theorem 1.1. *If the quiver Q_β^{Y/D_1} is acyclic, that is, does not contain any oriented cycle, then we have*

$$\Omega_\beta^{Y/D_1} = (-1)^{\beta \cdot D_1 + 1} N_{0,\beta}^{Y/D_1}.$$

In the previous result, D_1 and D_2 do not play symmetric roles. Indeed, we are counting curves with a tangency condition along D_1 whereas D_2 only appears in an indirect way in the number $\beta \cdot D_2$ of point conditions. If we exchange the roles of D_1 and D_2 , we get a new example.

Example: Let $Y = \mathbb{P}^2$, D_1 a line and D_2 a smooth conic not tangent to D_1 . Let d be a positive integer. Taking $\beta = d \in \mathbb{Z} = H_2(\mathbb{P}^2, \mathbb{Z})$, the quiver Q_β^{Y/D_1} consists of $2d + 1$ vertices, $i_1, \dots, i_{2d}$ and j , and $2d$ arrows $\alpha_k: i_k \rightarrow j$ for all $k = 1, \dots, 2d$. The dimension vector $\mathbf{d}(\beta)$ is 1 on the vertices $i_1, \dots, i_{2d}$, and d on the vertex j .

This quiver is acyclic and so Theorem 1.1 applies. In this case, Theorem 1.1 reduces to the main result obtained by [RW18]. More generally, Theorem 1.1 gives a way to produce a large number of new examples, see Section 6.

We will in fact prove a more general version of Theorem 1.1, involving higher genus Gromov-Witten invariants and refined Donaldson-Thomas invariants of the quiver Q_β^{Y/D_1} . The moduli space of genus g stable maps in Y of class β , intersecting D_1 in a unique point with maximal tangency and passing through $\beta \cdot D_2$ prescribed points, has virtual dimension g . We define Gromov-Witten invariants $N_{g,\beta}^{Y/D_1}$ by integrating $(-1)^g \lambda_g$ over the moduli space, where λ_g is the top Chern class of the Hodge bundle. On the other hand, refined Donaldson-Thomas theory for quivers defines a Laurent polynomial $\Omega_\beta^{Y/D_1}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm \frac{1}{2}}]$, refined virtual count of representations of Q_β^{Y/D_1} of dimension vector $\mathbf{d}(\beta)$.

Theorem 1.2. *If the quiver Q_β^{Y/D_1} is acyclic, then we have an equality of formal power series in $\hbar$:*

$$\Omega_\beta^{Y/D_1}(q^{\frac{1}{2}}) = (-1)^{\beta \cdot D_1 + 1} \left(2 \sin \left(\frac{\hbar}{2} \right) \right) \left(\frac{\hbar}{2 \sin \left(\frac{\hbar}{2} \right)} \right)^{\beta \cdot D_2} \left(\sum_{g \geq 0} N_{g,\beta}^{Y/D_1} \hbar^{2g-1} \right),$$

where $q = e^{i\hbar} = \sum_{n \geq 0} \frac{(i\hbar)^n}{n!}$.

We recover Theorem 1.1 from Theorem 1.2 in the limit $\hbar \rightarrow 0$, $q^{\frac{1}{2}} \rightarrow 1$. Theorem 1.2 is new even in the simplest case of $Y = \mathbb{P}^2$, D_1 a line and D_2 a smooth conic not tangent to D_1 .

1.2. Origin of the result. The present paper comes from a search for a “natural explanation” of the result of [RW18]. The source of our explanation is the correspondence existing between log Gromov-Witten invariants of log Calabi-Yau surfaces and quiver Donaldson-Thomas invariants. This known correspondence comes from the combination of the work of [Rei10] on the quiver side and of the work of [GPS10] on the Gromov-Witten side. Examples of this correspondence are discussed in [GS10] and in [RW17] under the name of the Gromov-Witten/Kronecker correspondence. The most general version of this correspondence, that we will use in this paper, can be found in Section 8.5 of [Bou20a]. In fact, in Section 8.5 of [Bou20a], a generalization of the original genus 0/DT correspondence is given, involving higher genus Gromov-Witten invariants with lambda class insertion and refined DT invariants.

Reineke and Weist observed the close similarity between their result and the Gromov-Witten/Kronecker correspondence but remarked that a direct connection does not seem obvious. The core of the present paper is to explain such connection.

The input of the general form of the Gromov-Witten/Kronecker correspondence described in Section 8.5 of [Bou20a] is a log Calabi-Yau surface, that is, the pair of a smooth projective surface Z and of an anticanonical divisor D in Z . We then consider Gromov-Witten counts of rational curves in Z in a fixed curve class β_Z , intersecting D in only one point with maximal tangency. This seems different from our $N_{0,\beta}^{Y/D_1}$, counting rational curves in Y of class β , intersecting D_1 in a unique point and passing through $\beta \cdot D_2$ points in general position in Y . The key geometric idea of the present paper is the following: in order to go from Y to some Z , we move the $\beta \cdot D_2$ points onto D_2 and we blow them up.

So, let Z be the surface obtained from Y by blowing-up $\beta \cdot D_2$ points on D_2 , distinct from each other and distinct from $D_1 \cap D_2$. Denote $\pi_Y: Z \rightarrow Y$ the blow-up morphism, $F_1, \dots, F_{\beta \cdot D_2}$ the exceptional divisors and $\beta_Z := \pi_Y^* \beta - \sum_{j=1}^{\beta \cdot D_2} [F_j] \in H_2(Z, \mathbb{Z})$. We still denote D_1 and D_2 the strict transforms of D_1 and D_2 in Z . Let $N_{0,\beta_Z}^{Z/D_1}$ be the log Gromov-Witten count of rational curves in Z of class β_Z , intersecting D_1 in a unique point with maximal tangency and not intersecting D_2 .

The intuitive picture of moving the $\beta \cdot D_2$ points from a general configuration in Y to a special configuration on D_2 suggests the following result.

Theorem 1.3. *We have $N_{0,\beta}^{Y/D_1} = N_{0,\beta_Z}^{Z/D_1}$.*

In absolute Gromov-Witten theory, trading exceptional divisors for point insertions is quite familiar, see [Gat01] for some early example. The main novelty in Theorem 1.3 is that we are dealing with relative Gromov-Witten theory.

Theorem 1.1 follows from Theorem 1.3 in a natural fashion. Indeed, the invariants $N_{0,\beta_Z}^{Z/D_1}$ are exactly those entering in the general form of the Gromov-Witten/Kronecker correspondence and a quiver can be obtained from a toric model of the log Calabi-Yau surface $(Z, D_1 \cup D_2)$. We review the construction of such a quiver in Section 2.3. In the

particular case of $Y = \mathbb{P}^2$, D_1 a line and D_2 a smooth conic not tangent to D_1 , we recover the quiver considered by Reineke and Weist. A feature of our approach is that we do not have to guess the quiver, which contrasts with the situation in [RW18]. In particular, we can produce many new examples, some of them being presented in Section 6.

1.3. Technical content. The technical content of this paper is the proof of Theorem 1.3 and consists in two degeneration arguments in Gromov-Witten theory. The 1st step involves degenerating the $\beta \cdot D_2$ points on D_2 . The 2nd step involves blowing these points up. Because we will be really considering curves with tangency conditions along $D_1 \cup D_2$, which is a singular normal crossing divisor, we will be working with log Gromov-Witten invariants. In this context, our degeneration arguments will use the decomposition formula of [ACGS17] in a way similar to how we used it in [Bou19] and [Bou20a], and then an explicit treatment of the gluing. In particular, our setting is explicit enough so that a general theory of gluing for log Gromov-Witten invariants, such as recently developed in [Ran19] or [ACGS19], is not necessary.

Let us stress the most technical point, which is not present in [Bou19] or [Bou20a]. In all cases, the general decomposition formula requires to identify rigid tropical curves, indexing the terms contributing in the formula. In [Bou19] and [Bou20a], the components of the special fibers of the degenerations are toric, or close enough of toric, so that the tropical balancing condition gives enough constraints to identify exactly the relevant contributing rigid tropical curves. In the present paper, the degenerations considered are far enough from being toric and purely combinatorial arguments are not enough to rule out rigid tropical curves which eventually will contribute zero in the degeneration formula. To eliminate those, we have to argue at the level of stable log maps. The issue is that rigid tropical curves are not necessarily realizable as tropicalization of stable log maps (see Example 6.4 of [ACGS17] for an example). The solution to this problem is to study the possible tropicalizations of all the relevant stable log maps, even the ones giving non-rigid tropical curves, and then to identify the contributing rigid tropical curves as possible rigid limits of them.

1.4. Relation with other works. We mention briefly two works that are logically independent of the present paper but which consider some of the same objects from a different point of view.

If D_1 is nef, it follows from the main result of [vGGR17] that the relative genus zero Gromov-Witten invariants $N_{0,\beta}^{Y/D_1}$ are related to the absolute genus zero Gromov-Witten invariants with $\beta \cdot D_2$ point insertions of the non-compact three-fold total space of the line bundle $\mathcal{O}_Y(-D_1)$. Thus, if D_1 is nef, we can view Theorem 1.1 as establishing a relation between absolute genus 0 Gromov-Witten theory of the three-fold $\mathcal{O}_Y(-D_1)$ and Donaldson-Thomas invariants of the quiver Q_β^{Y/D_1} .

For $Y = \mathbb{P}^2$, D_1 a line and D_2 a smooth conic not tangent to D_1 , [RW18] use the quiver interpretation of N_β^{Y/D_1} to derive a recursion relation satisfied by the invariants N_β^{Y/D_1} . In work [FW19], this recursion is derived as a very particular case of a general WDVV formalism for genus zero Gromov-Witten invariants relative to a smooth divisor.

It would be interesting to study, either from the quiver or Gromov-Witten sides, if similar recursion relations exist for the examples presented in Section 6.

1.5. Notations. In order to keep the amount of notations as low as possible, we systematically use the convention to denote in the same way a divisor and its strict transform by some blow-up morphism, hoping that the context will be clear enough to specify the relevant geometry. We denote A_* and A^* for respectively Chow groups and operatorial cohomology Chow groups, as in [Ful98].

1.6. Plan of the paper. In Section 2, we state Theorem 2.1, cited as Theorem 1.2 in the Introduction. The following two sections are dedicated to the proof of Theorem 2.1. In Section 3, we apply a degeneration argument to show that we can transform the $\beta \cdot D_2$ absolute point conditions entering the definition of N_β^{Y/D_1} into $\beta \cdot D_2$ relative point conditions along the divisor D_2 . In Section 4, another degeneration argument shows that these $\beta \cdot D_2$ relative point conditions along D_2 can be transformed into no condition on the surface Z obtained from Y by blowing-up $\beta \cdot D_2$ points on D_2 . In Section 5, we finish the proof of Theorem 2.1. In Section 6, we present various examples.

1.7. Note (v2). The first arxiv version und unfortunately the published version of this paper contain incorrect normalization factors in the statement of Theorems 1.2, 2.1 and of Proposition 3.1, the mistake in the proof being in the very last part of Section 3. The statements and the proof are corrected in the present version. I thank Longting Wu for asking me a question leading to the realization of the need of the correction.

1.8. Acknowledgements. I made the key observation allowing this paper to exist (the fact that for $Y = \mathbb{P}^2$, D_1 a line and D_2 a conic, the quiver associated to the surface Z by the recipe of [Bou20a] coincides with the quiver of [RW18]) in a TGV Lyria train between Paris and Zurich. I thank Honglu Fan, Rahul Pandharipande, Richard Thomas and Longting Wu for useful discussions.

I acknowledge the support of Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zürich Foundation.

2. MAIN RESULT

2.1. The Gromov-Witten side. Let Y be a smooth projective complex surface and let D_1 and D_2 be two smooth non-empty divisors on Y , intersecting transversally, and such that the union $D_1 \cup D_2$ is anticanonical. Elementary theory of surfaces implies that D_1 and D_2 are necessarily rational curves and that the intersection $D_1 \cap D_2$ consists of two points.

We fix $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta \cdot D_1 > 0$ and $\beta \cdot D_2 > 0$. Let $\overline{M}_{g, \beta \cdot D_2}(Y/D_1, \beta)$ be the moduli space of $(\beta \cdot D_2)$ -pointed genus g class β stable maps to Y relative to D_1 , with contact condition along D_1 at a single point and so with contact order $\beta \cdot D_1$. When we say $(\beta \cdot D_2)$ -pointed, we refer to marked points in addition to the marked point keeping track of the tangency condition along D_1 . The moduli space $\overline{M}_{g, \beta \cdot D_2}(Y/D_1, \beta)$ is a proper Deligne-Mumford stack of virtual dimension

$$(1 - g)(\dim Y - 3) + \beta \cdot c_1(Y) + \beta \cdot D_2 - (\beta \cdot D_1 - 1) = g + 2(\beta \cdot D_2),$$

where we used that $D_1 \cup D_2$ is anticanonical and so $\beta \cdot c_1(Y) = \beta \cdot D_1 + \beta \cdot D_2$. It admits a virtual fundamental class

$$[\overline{M}_{g,\beta \cdot D_2}(Y/D_1, \beta)]^{\text{virt}} \in A_{g+2(\beta \cdot D_2)}(\overline{M}_{g,\beta \cdot D_2}(Y/D_1, \beta), \mathbb{Q}).$$

Let $\text{ev}_k: \overline{M}_{g,\beta \cdot D_2}(Y/D_1, \beta) \rightarrow Y$, for $1 \leq k \leq \beta \cdot D_2$, be the evaluation maps at the $\beta \cdot D_2$ marked points. We define

$$N_{g,\beta}^{Y/D_1} := \int_{[\overline{M}_{g,\beta \cdot D_2}(Y/D_1, \beta)]^{\text{virt}}} (-1)^g \lambda_g \prod_{k=1}^{\beta \cdot D_2} \text{ev}_k^*(pt) \in \mathbb{Q},$$

where $pt \in A^1(Y)$ is the class of a point and λ_g is the top Chern class of the Hodge bundle over $\overline{M}_{g,\beta \cdot D_2}(Y/D_1, \beta)$. Recall that if

$$\nu: \mathcal{C} \rightarrow \overline{M}_{g,\beta \cdot D_2}(Y/D_1, \beta)$$

denotes the universal source curve, then the Hodge bundle is the rank g vector bundle $\nu_* \omega_\nu$, where ω_ν is the dualizing line bundle relative to ν .

2.2. Basics on log Calabi-Yau surfaces. To explain the construction of the quiver Q_β^{Y/D_1} given in the next Section 2.3, it is useful to recall some terminology related to log Calabi-Yau surfaces. We refer to [Fri15], [GHK15c], [GHK15b] for more details. A log Calabi-Yau surface with maximal boundary is a pair (Y, D) with Y a smooth projective surface and D a singular reduced normal crossing anticanonical divisor of Y . There are two basic operations on log Calabi-Yau surfaces with maximal boundary:

- **Corner blow-up.** If (Y, D) is a log Calabi-Yau surface with maximal boundary, then $(\tilde{Y}, \tilde{D})$ is a log Calabi-Yau surface with maximal boundary, where $\tilde{Y}$ is a blow-up of Y at one singular point of D and $\tilde{D}$ is the preimage of D in $\tilde{Y}$.
- **Interior blow-up.** If (Y, D) is a log Calabi-Yau surface with maximal boundary, then $(\tilde{Y}, \tilde{D})$ is a log Calabi-Yau surface with maximal boundary, where $\tilde{Y}$ is a blow-up of Y at some smooth points of D (with infinitely near points allowed) and $\tilde{D}$ is the strict transform of D in $\tilde{Y}$.

The simplest example of log Calabi-Yau surface with maximal boundary is $(\bar{Y}, \bar{D})$, with $\bar{Y}$ a smooth projective toric surface and $\bar{D}$ the union of toric divisors of $\bar{Y}$. Such log Calabi-Yau surface with maximal boundary is called toric.

A toric model of a log Calabi-Yau surface with maximal boundary (Y, D) is a morphism $\pi: (Y, D) \rightarrow (\bar{Y}, \bar{D})$, where $(\bar{Y}, \bar{D})$ is a toric log Calabi-Yau surface with maximal boundary, and where π is obtained from $(\bar{Y}, \bar{D})$ by a sequence of interior blow-ups.

We recall Proposition 1.3 of [GHK15b]:

Proposition 2.1. *For every (Y, D) log Calabi-Yau surface with maximal boundary, there exists $(\tilde{Y}, \tilde{D})$, obtained from (Y, D) by a sequence of corner blow-ups, and admitting a toric model $(\tilde{Y}, \tilde{D}) \rightarrow (\bar{Y}, \bar{D})$*

Given (Y, D) , there are in general many (often infinitely many) $(\tilde{Y}, \tilde{D})$ and $(\bar{Y}, \bar{D})$ as in Proposition 2.1. The non-uniqueness of toric models is related to the theory of cluster mutations, see [GHK15a].

2.3. Quiver construction. Given Y , D_1 , D_2 and β as in Section 2.1, we explain how to construct a quiver Q_β^{Y/D_1} , that is, a finite oriented graph, and a dimension vector $\mathbf{d}(\beta) \in \mathbb{Z}^{(Q_\beta^{Y/D_1})_0}$, where $(Q_\beta^{Y/D_1})_0$ denote the set of vertices of Q_β^{Y/D_1} . In fact, our construction of Q_β^{Y/D_1} and $\mathbf{d}(\beta)$ will depend on an extra choice, that for simplicity we do not include in the notation. Different choices will define mutation equivalent quivers.

Let Z be the surface obtained from Y by blowing-up $\beta \cdot D_2$ points on D_2 , distinct from each other and distinct from $D_1 \cap D_2$. Denote

$$\pi_Y: Z \rightarrow Y$$

the blow-up morphism, $F_1, \dots, F_{\beta \cdot D_2}$ the exceptional divisors and

$$\beta_Z := \pi_Y^* \beta - \sum_{j=1}^{\beta \cdot D_2} [F_j] \in H_2(Z, \mathbb{Z}).$$

We still denote D_1 and D_2 the strict transforms of D_1 and D_2 in Z . We denote $D := D_1 \cup D_2$.

The pair (Z, D) is a log Calabi-Yau surface with maximal boundary in the sense of Section 2.2. Indeed, it is already true for (Y, D) by assumption and (Z, D) is obtained from (Y, D) by interior blow-ups.

According to Proposition 2.1, there exists a diagram of log Calabi-Yau surfaces with maximal boundary

$$\begin{array}{ccc} & (\tilde{Z}, \tilde{D}) & \\ \pi_Z \swarrow & & \searrow \pi_{\tilde{Z}} \\ (Z, D) & & (\bar{Z}, \bar{D}), \end{array}$$

where $(\bar{Z}, \bar{D})$ is a toric log Calabi-Yau surface with maximal boundary, π_Z is a sequence of corner blow-ups of (Z, D) , and $\pi_{\tilde{Z}}$ is a sequence of interior blow-ups of $(\tilde{Z}, \tilde{D})$. We still denote D_1 and D_2 for the strict transforms of D_1 and D_2 in $\tilde{Z}$ and for the images in $\bar{Z}$ of these strict transforms.

From now on, we fix a choice of such diagram. Our construction of Q_β^{Y/D_1} will depend on this choice. Different choices will define mutation equivalent quivers.

As $(\bar{Z}, \bar{D})$ is toric, it admits a fan in $\mathbb{R}^2$, whose rays are in bijection with the irreducible components of $\bar{D}$. The morphism $\pi_{\tilde{Z}}$ is a sequence of interior blow-ups. As we will be ultimately interested only in the Gromov-Witten theory of (Y, D) , which is deformation invariant, we can assume that $\pi_{\tilde{Z}}$ is a blow-up of $\tilde{Z}$ in distinct smooth points $x_1, \dots, x_n$ of $\tilde{D}$. Indeed, a log Calabi-Yau surface for which one has to blow-up infinitely near points are non-generic and can always be deformed into one for which one has to only blow-up distinct points. Let $E_{x_1}, \dots, E_{x_n}$ be the corresponding exceptional divisors in $\tilde{Z}$.

We now construct the quiver Q_β^{Y/D_1} . By definition, Q_β^{Y/D_1} has n vertices, in natural bijection with the points $x_1, \dots, x_n$. For every $1 \leq j \leq n$, let $m_j \in \mathbb{Z}^2$ be the primitive generator, pointing away from 0, of the ray in the fan of $\bar{Z}$ dual to the irreducible

component of $\bar{D}$ containing x_j . Let

$$\begin{aligned} \langle -, - \rangle &\rightarrow \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{Z} \\ ((a, b), (a', b')) &\mapsto \det((a, b), (a', b')) = ab' - a'b \end{aligned}$$

be the standard non-degenerate skew-symmetric bilinear form on $\mathbb{Z}^2$. By definition, Q_β^{Y/D_1} has $\max(\langle m_j, m_k \rangle, 0)$ arrows from the vertex j to the vertex k .

We now construct the dimension vector $\mathbf{d}(\beta)$. A dimension vector for Q_β^{Y/D_1} is an element in $\mathbb{Z}^{(Q_\beta^{Y/D_1})_0}$, where $(Q_\beta^{Y/D_1})_0$ is the set of vertices of Q_β^{Y/D_1} . So in order to define $\mathbf{d}(\beta)$, we have to define a non-negative integer $\mathbf{d}(\beta)_j$ for every $1 \leq j \leq n$. We define

$$\mathbf{d}(\beta)_j := ((\pi_Z)^* \beta_Z) \cdot E_{x_j}.$$

We have $\beta_Z \in H_2(Z, \mathbb{Z})$, $(\pi_Z)^* \beta_Z \in H_2(\tilde{Z}, \mathbb{Z})$, and so the intersection number with the divisor E_{x_j} of $\tilde{Z}$ makes sense. We have $\mathbf{d}(\beta) \in \mathbb{Z}^{(Q_\beta^{Y/D_1})_0}$ but a priori one could have $\mathbf{d}(\beta) \notin \mathbb{N}^{(Q_\beta^{Y/D_1})_0}$.

Remark that the strict transforms in $\tilde{Z}$ of the exceptional divisors $F_1, \dots, F_{\beta \cdot D_2}$ naturally form a subset of the set of exceptional divisors $E_1, \dots, E_n$. As $\beta_Z \cdot D_j = 1$, it follows that there is always a subset of cardinal $\beta \cdot D_2$ of the set of vertices of Q_β^{Y/D_1} , over which the dimension vector $\mathbf{d}(\beta)$ takes the value 1. In particular, as $\beta \cdot D_2 > 0$, $\mathbf{d}(\beta)$ is always a primitive element of $\mathbb{Z}^{(Q_\beta^{Y/D_1})_0}$.

2.4. The quiver Donaldson-Thomas side. We refer to [KS18], [JS12], [Rei10], [Rei11], [MR17] for Donaldson-Thomas theory of quivers.

Let Q be an acyclic quiver, that is, a quiver without oriented cycles. We denote Q_0 its set of vertices. Every $\theta = (\theta_j)_{j \in Q_0} \in \mathbb{Z}^{Q_0}$ defines a notion of stability for representations of Q . For every dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, we then have a projective variety $M_{\mathbf{d}}^{\theta-ss}$, moduli space of θ -semistable representations of Q of dimension $\mathbf{d}$, containing the open smooth locus $M_{\mathbf{d}}^{\theta-st}$ of θ -stable representations. Denote $\iota: M_{\mathbf{d}}^{\theta-st} \hookrightarrow M_{\mathbf{d}}^{\theta-ss}$ the natural inclusion. The main result of [MR17] is that the Laurent polynomials

$$\begin{aligned} \Omega_{\mathbf{d}}^{Q_m, \theta}(q^{\frac{1}{2}}) &:= (-1)^{\dim M_{\mathbf{d}}^{\theta-ss}} q^{-\frac{1}{2} \dim M_{\mathbf{d}}^{\theta-ss}} \sum_{j=0}^{\dim M_{\mathbf{d}}^{\theta-st}} (\dim H^{2j}(M_{\mathbf{d}}^{\theta-ss}, \iota_{!*} \mathbb{Q})) q^j \\ &\in (-1)^{\dim M_{\mathbf{d}}^{\theta-ss}} q^{-\frac{1}{2} \dim M_{\mathbf{d}}^{\theta-ss}} \mathbb{N}[q] \end{aligned}$$

are the refined Donaldson-Thomas invariants of Q for the stability θ . In the above formula, $\iota_{!*}$ is the intermediate extension functor defined by ι and so $\iota_{!*} \mathbb{Q}$ is a perverse sheaf on $M_{\mathbf{d}}^{\theta-ss}$.

As Q is acyclic, we can choose a numbering $v_1, \dots, v_{|Q_0|}$ of the vertices of Q such that the existence of a non-trivial arrow from v_j to v_k implies $j \leq k$. Let e_j be the dimension vector assigning 1 to v_j and 0 to v_k if $k \neq j$. If $\theta_1 < \theta_2 < \dots < \theta_{|Q_0|}$, then $\Omega_{e_j}^{Q_m, \theta}(q^{\frac{1}{2}}) = 1$, for all $1 \leq j \leq n$, and $\Omega_{\mathbf{d}}^{Q_m, \theta}(q^{\frac{1}{2}}) = 0$ for $\mathbf{d} \in \mathbb{Z}^{Q_0} - \{e_1, \dots, e_{|Q_0|}\}$. We call such θ a trivial stability condition.

If $\theta_1 > \theta_2 > \dots > \theta_{|Q_0|}$, we call θ a maximally non-trivial stability condition. We denote $\Omega_{\mathbf{d}}^{Q_m}(q^{\frac{1}{2}})$ for $\Omega_{\mathbf{d}}^{Q_m, \theta}(q^{\frac{1}{2}})$ and θ a maximally non-trivial stability condition.

We apply the previous definitions to the quiver Q_{β}^{Y/D_1} constructed in Section 2.3. If Q_{β}^{Y/D_1} is acyclic and $\mathbf{d}(\beta) \in \mathbb{N}^{(Q_{\beta}^{Y/D_1})_0}$, we define

$$\Omega_{\beta}^{Y/D_1}(q^{\frac{1}{2}}) := \Omega_{\mathbf{d}(\beta)}^{Q_{\beta}^{Y/D_1}}(q^{\frac{1}{2}}).$$

If $\mathbf{d}(\beta) \notin \mathbb{N}^{(Q_{\beta}^{Y/D_1})_0}$, we set

$$\Omega_{\beta}^{Y/D_1}(q^{\frac{1}{2}}) := 0.$$

2.5. Statement of the main result. Using the notations introduced in the previous Sections, we can state our main result, cited as Theorem 1.2 in the Introduction.

Theorem 2.1. *If the quiver Q_{β}^{Y/D_1} is acyclic, then we have an equality of formal power series in $\hbar$:*

$$\Omega_{\beta}^{Y/D_1}(q^{\frac{1}{2}}) = (-1)^{\beta \cdot D_1 + 1} \left(2 \sin \left(\frac{\hbar}{2} \right) \right) \left(\frac{\hbar}{2 \sin \left(\frac{\hbar}{2} \right)} \right)^{\beta \cdot D_2} \left(\sum_{g \geq 0} N_{g, \beta}^{Y/D_1} \hbar^{2g-1} \right),$$

where $q = e^{i\hbar} = \sum_{n \geq 0} \frac{(i\hbar)^n}{n!}$.

Taking the leading order term on both sides of Theorem 2.1 in the limit $\hbar \rightarrow 0$, $q^{\frac{1}{2}} \rightarrow 1$, we get the following Corollary, cited as Theorem 1.1 in the Introduction.

Corollary 2.1. *If the quiver Q_{β}^{Y/D_1} is acyclic, then we have*

$$\Omega_{\beta}^{Y/D_1} = (-1)^{\beta \cdot D_1 + 1} N_{0, \beta}^{Y/D_1}.$$

3. EXCHANGE OF ABSOLUTE WITH RELATIVE POINT CONDITIONS

We start the proof of Theorem 2.1. Our goal is to fit the invariants $N_{g, \beta}^{Y/D_1}$ into the framework of Section 8.5 of [Bou20a]. In the present Section, we realize the first step: we exchange the $\beta \cdot D_2$ absolute point conditions entering the definition of $N_{g, d}^{Y/D_1}$ with $\beta \cdot D_2$ point conditions relative to the divisor D_2 .

3.1. Statement. We refer to [GS13], [Che14], [AC14], [ACGS17] for the theory of stable log maps. We denote $D := D_1 \cup D_2$. We view Y as a smooth log scheme, with the divisorial log structure defined by D . We fix $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta \cdot D_1 > 0$ and $\beta \cdot D_2 > 0$. Let $\overline{M}_g(Y/D, \beta)$ be the moduli space of genus g class β stable log maps to Y , with contact order $\beta \cdot D_1$ to D_1 in a single point, and with contact order 1 to D_2 in $\beta \cdot D_2$ points. It is a proper Deligne-Mumford stack, admitting a virtual fundamental class

$$[\overline{M}_g(Y/D, \beta)]^{\text{virt}} \in A_{g + \beta \cdot D_2}(\overline{M}_g(Y/D, \beta), \mathbb{Q}).$$

Let $\text{ev}_k: \overline{M}_g(Y/D, \beta) \rightarrow D_2$, for $k = 1, \dots, \beta \cdot D_2$, be the evaluation maps at the $\beta \cdot D_2$ contact points with D_2 . We define

$$N_{g,\beta}^{Y/D} := \int_{[\overline{M}_g(Y/D, \beta)]^{\text{virt}}} (-1)^g \lambda_g \prod_{k=1}^{\beta \cdot D_2} \text{ev}_k^*(\text{pt}) \in \mathbb{Q},$$

where $\text{pt} \in A^1(Y)$ is the class of a point and λ_g is the top Chern class of the Hodge bundle over $\overline{M}_g(Y/D, \beta)$.

Proposition 3.1. *We have*

$$\sum_{g \geq 0} N_{g,\beta}^{Y/D_1} \hbar^{2g-1} = \left(\sum_{g \geq 0} N_{g,\beta}^{Y/D} \hbar^{2g-1+\beta \cdot D_2} \right) \left(\frac{1}{\hbar} \right)^{\beta \cdot D_2}.$$

Exchanging absolute and relative Gromov-Witten invariants by some degeneration argument is standard in Gromov-Witten theory relative to a smooth divisor, see for example [MP06]. In particular, Proposition 3.1 has exactly the form given by the usual degeneration formula [Li02] applied for invariants defined relatively to D_2 . The issue is that our invariants have also tangency conditions along D_1 and so we are really dealing with log Gromov-Witten invariants relative to the normal crossing divisor $D = D_1 \cup D_2$. In order to prove Proposition 3.1, we have to show that the a priori possible corrections to the naive degeneration formula coming from the intersection points $D_1 \cap D_2$ actually are not there.

3.2. Summary of the proof of Proposition 3.1. The proof of Proposition 3.1 takes the rest of Section 3. In Section 3.3, we describe the relevant degeneration to the normal cone of D_2 . The special fiber of the degeneration has a natural log structure whose tropicalization is described in Section 3.4. This gives us a way to describe the relevant log Gromov-Witten invariants of the special fiber in Section 3.5. The core of the proof is Section 3.6, which is a study of the possible tropicalizations of stable log maps contributing to these invariants. In Section 3.7, we apply to our specific context the general decomposition formula of [ACGS17], and in Section 3.8 we identify using the results of Section 3.6 the relevant rigid tropical curves. We conclude the proof in Section 3.9, referring to the gluing techniques already used in [Bou19] and [Bou20a].

3.3. Degeneration. Let X be the degeneration of Y to the normal cone of D_2 , that is, the blow-up of $D_2 \times \{0\}$ in $Y \times \mathbb{A}^1$. Let $\pi: X \rightarrow \mathbb{A}^1$ be the natural projection. We have $\pi^{-1}(t) \simeq Y$ if $t \neq 0$. Let $N_{D_2|Y}$ be the normal line bundle of D_2 in Y , and let $\mathbb{P}$ be the projective bundle over D_2 obtained as the projectivization of $N_{D_2|Y} \oplus \mathcal{O}_{D_2}$. If $p \in D_2$, we denote $\mathbb{P}_p^1$ the fiber of the projection $\mathbb{P} \rightarrow D_2$ over p . The embeddings $\mathcal{O}_{D_2} \hookrightarrow N_{D_2|Y} \oplus \mathcal{O}_{D_2}$ and $N_{D_2|Y} \hookrightarrow N_{D_2|Y} \oplus \mathcal{O}_{D_2}$ define two sections of $\mathbb{P} \rightarrow D_2$, that we denote $D_{2,0}$ and $D_{2,\infty}$ respectively. The special fiber $\pi^{-1}(0)$ has two irreducible components, Y and $\mathbb{P}$, with D_2 in Y glued with $D_{2,0}$ in $\mathbb{P}$.

We also degenerate the $\beta \cdot D_2$ point conditions. For $j = 1, \dots, \beta \cdot D_2$, let σ_j be a section of $Y \times \mathbb{A}^1$ such that $\sigma_j(t) \notin D_1 \cup D_2$ for $t \neq 0$ and $\sigma_j(0) \in D_2 - (D_1 \cap D_2)$. We choose these sections such that $\sigma_j(0) \neq \sigma_k(0)$ if $j \neq k$, and such that σ_j is transverse to $D_2 \times \{0\}$. For $j = 1, \dots, \beta \cdot D_2$, we denote $\tilde{\sigma}_j$ the section of $\pi: X \rightarrow \mathbb{A}^1$ obtained as strict transform of

σ_j . By construction, the $\beta \cdot D_2$ points $\tilde{\sigma}_j(0)$ are in distinct fibers of $\mathbb{P} \rightarrow D_2$ and are away from $D_{2,0} \cup D_{2,\infty} \cup \mathbb{P}_{p_1}^1 \cup \mathbb{P}_{p_2}^1$, where p_1 and p_2 are the two intersection points of D_1 with D_2 . For $1 \leq j \leq \beta \cdot D_2$, we denote $\mathbb{P}_{\tilde{\sigma}_j(0)}^1$ the fiber of $\mathbb{P} \rightarrow D_2$ passing through $\tilde{\sigma}_j(0)$.

Let $\widetilde{D_1 \times \mathbb{A}^1}$ be the divisor on X obtained as strict transform of the divisor $D_1 \times \mathbb{A}^1$ in $Y \times \mathbb{A}^1$. We view X as a log scheme for the divisorial log structure defined by the normal crossing divisor $\widetilde{D_1 \times \mathbb{A}^1} \cup \pi^{-1}(0)$. We view $\mathbb{A}^1$ as a log scheme for the toric divisorial log structure. Then π naturally defines a log smooth morphism. Remark that for $t \neq 0$, $\pi^{-1}(t)$ with the log structure restricted from X is Y with the divisorial log structure defined by D_1 .

We denote $X_0 := \pi^{-1}(0)$ with the log structure restricted from X . Let $\text{pt}_{\mathbb{N}}$ be the standard log point, that is, the point $0 \in \mathbb{A}^1$ with the log structure restricted from $\mathbb{A}^1$. Then π induces by restriction a log smooth morphism $\pi_0: X_0 \rightarrow \text{pt}_{\mathbb{N}}$.

3.4. Tropicalization of the special fiber. We refer to Appendix B of [GS13] and Section 2 of [ACGS17] for the general notion of tropicalization of a log scheme.

Tropicalizing the log morphism $\pi_0: X_0 \rightarrow \text{pt}_{\mathbb{N}}$, we get a morphism of cone complexes $\Sigma(\pi_0): \Sigma(X_0) \rightarrow \Sigma(\text{pt}_{\mathbb{N}})$. We have $\Sigma(\text{pt}_{\mathbb{N}}) = \mathbb{R}_{\geq 0}$ and $\Sigma(X_0)$ is naturally identified with the cone over the fiber $\Sigma(\pi_0)^{-1}(1)$ at $1 \in \mathbb{R}_{\geq 0}$. It is thus enough to describe the cone complex $\Sigma(\pi_0)^{-1}(1)$. We denote

$$X_0^{\text{trop}} := \Sigma(\pi_0)^{-1}(1).$$

The cone complex X_0^{trop} consists of:

- Vertices v_Y and $v_{\mathbb{P}}$ respectively dual to Y and $\mathbb{P}$.
- Unbounded edges e_{D_1} , $e_{\mathbb{P}_{p_1}^1}$, $e_{\mathbb{P}_{p_2}^1}$ respectively dual to D_1 , $\mathbb{P}_{p_1}^1$, $\mathbb{P}_{p_2}^1$.
- One bounded edge e_{D_2} dual to D_2 .
- Faces f_{p_1} and f_{p_2} dual to p_1 and p_2 .

It is important to realize that X_0^{trop} is an abstract cone complex, with no natural embedding in the plane. In particular, in X_0^{trop} , the unbounded edge e_{D_1} is parallel to both $e_{\mathbb{P}_{p_1}^1}$ and $e_{\mathbb{P}_{p_2}^1}$. We draw below a picture of X_0^{trop} , where the two copies of e_{D_2} and the two copies of $v_{\mathbb{P}}$ have to be identified.

Remark that $\Sigma(\pi_0)^{-1}(0)$ is the “asymptotic version” of X_0^{trop} , and is a ray

which can be identified with the tropicalization of Y , viewed as a log scheme for the divisorial log structure defined by D_1 .

Most of the tropical arguments of the following Sections will take place in X_0^{trop} and will use the above picture of X_0^{trop} . In particular, the terms “vertical”, “horizontal”, “ordinates” will refer to the corresponding notions in this picture.

Contact orders for stable log maps to a log scheme are integral points in the tropicalization of this log scheme (see Definition 2.3.12 of [ACGS17]). We denote u_p the contact order for stable log maps to X_0 defined by $\beta \cdot D_1$ times the integral generator (pointing away from v_Y^0) of the edge $e_{D_1}^0$ in $\Sigma(\pi_0)^{-1}(0)$.

3.5. Invariants of the special fiber. Let $\overline{M}_g(X_0, \beta)$ be the moduli space of $\beta \cdot D_2$ -pointed genus g class β stable log maps to $\pi_0: X_0 \rightarrow \text{pt}_{\mathbb{N}}$, with a single point of contact order u_p . Let

$$[\overline{M}_g(X_0, \beta)]^{\text{virt}} \in A_{g+2(\beta \cdot D_2)}(\overline{M}_g(X_0, \beta), \mathbb{Q})$$

be its virtual fundamental class.

For simplicity, we work below in the category of stacks. In general, one should work in the category of fs log stacks to correctly define point constraints in log Gromov-Witten theory, see Section 6.3 of [ACGS17]. In our case, it does not matter because the log morphisms $\tilde{\sigma}_j(0): \text{pt}_{\mathbb{N}} \rightarrow X_0$ are strict, that is, the log structure on $\text{pt}_{\mathbb{N}}$ is the pullback by $\tilde{\sigma}_j(0)$ of the log structure on X_0 .

The $\beta \cdot D_2$ marked points define an evaluation morphism

$$\text{ev}: \overline{M}_g(X_0, \beta) \rightarrow (X_0)^{\beta \cdot D_2}.$$

Denote

$$\iota_{\tilde{\sigma}}: \tilde{\sigma}(0) = \{(\tilde{\sigma}_1(0), \dots, \tilde{\sigma}_{\beta \cdot D_2}(0))\} \hookrightarrow (X_0)^{\beta \cdot D_2},$$

$$\overline{M}_g(X_0, \beta, \tilde{\sigma}) := \overline{M}_g(X_0, \beta) \times_{(X_0)^{\beta \cdot D_2}} \tilde{\sigma}(0),$$

and

$$[\overline{M}_g(X_0, \beta, \tilde{\sigma})]^{\text{virt}} := \iota_{\tilde{\sigma}}^! [\overline{M}_g(X_0, \beta)]^{\text{virt}} \in A_g(\overline{M}_g(X_0, \beta, \tilde{\sigma}), \mathbb{Q}).$$

By deformation invariance of log Gromov-Witten invariants, we have

$$N_{g, \beta}^{Y/D_1} = \int_{[\overline{M}_g(X_0, \beta, \tilde{\sigma})]^{\text{virt}}} (-1)^g \lambda_g.$$

3.6. **Tropical curves.** Let

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow \nu & & \downarrow \pi_0 \\ W & \xrightarrow{g} & \text{pt}_{\mathbb{N}}, \end{array}$$

be an element of $\overline{M}_g(X_0, \beta, \tilde{\sigma})$. Let

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X_0) \\ \downarrow \Sigma(\nu) & & \downarrow \Sigma(\pi_0) \\ \Sigma(W) & \xrightarrow{\Sigma(g)} & \Sigma(\text{pt}_{\mathbb{N}}) \end{array}$$

be its tropicalization. For every $b \in \Sigma(g)^{-1}(1)$, let

$$\Sigma(f)_b: \Sigma(C)_b \rightarrow \Sigma(\pi_0)^{-1}(1) = X_0^{\text{trop}}$$

be the fiber of $\Sigma(f)$ over b .

The definition of $\overline{M}_g(X_0, \beta, \tilde{\sigma})$ fixes the set of unbounded edges of $\Sigma(C)_b$. The $\beta \cdot D_2$ marked points are dual to $\beta \cdot D_2$ unbounded edges $E_1, \dots, E_{\beta \cdot D_2}$ of $\Sigma(C)_b$. The contact order u_p at a single point is dual to an unbounded edge E_0 of $\Sigma(C)_b$. Furthermore, $\Sigma(C)_b$ has no other unbounded edge.

The fact that the marked points have to be mapped to $\tilde{\sigma}_1(0), \dots, \tilde{\sigma}_{\beta \cdot D_2}(0)$ implies that the unbounded edges E_j , $1 \leq j \leq \beta \cdot D_2$, are all contracted by $\Sigma(f)_b$ onto the vertex $v_{\mathbb{P}}$. The contact order u_p implies that E_0 is mapped by $\Sigma(C)_b$ onto an unbounded ray in X_0^{trop} , going down in a way parallel to $e_{\mathbb{P}_{p_1}^1}$, e_{D_1} and $e_{\mathbb{P}_{p_2}^1}$. The fact that the image of E_0 by $\Sigma(C)_b$ can a priori be any ray parallel to e_{D_1} reflects the fact that a stable log map to Y with a single point of maximal tangency along D_1 degenerates in a stable log map to X_0 with “maximal tangency” (in some logarithmic sense) along $D_1 \cup \mathbb{P}_{p_1}^1 \cup \mathbb{P}_{p_2}^1$, and a priori not only along D_1 .

The tropical balancing condition has to be satisfied everywhere except at the vertices v_Y and $v_{\mathbb{P}}$ (the corresponding components do not have toric divisorial log structures). At $v_{\mathbb{P}}$, the tropical balancing condition is only modified in the horizontal direction: it still holds in the vertical direction. This follows from the general form of the balancing conditions for stable log maps given in Proposition 1.15 of [GS13].

For all $1 \leq j \leq \beta \cdot D_2$, we denote V_j the vertex of $\Sigma(C)_b$ to which the unbounded edge E_j is attached. A priori, one could have $V_j = V_k$ for some $j \neq k$. We have $\Sigma(f)_b(V_j) = v_{\mathbb{P}}$ for all $1 \leq j \leq \beta \cdot D_2$. We denote V_0 the vertex of Γ to which the unbounded edge E_0 is attached.

We refer to Definition 2.5.3 of [ACGS17] for details on edge marking. For V a vertex of $\Sigma(C)_b$ and E an edge adjacent to V , we denote $v_{V,E}$ the slope measured from V of the restriction of $\Sigma(f)_b$ to E . Given our picture of X_0^{trop} , we can identify $v_{V,E}$ with a well-defined element of $\mathbb{Z}^2$, except if $\Sigma(f)_b(E)$ is contained in e_{D_2} , in which case $v_{V,E}$ defines two elements of $\mathbb{Z}^2$ corresponding to the two copies of e_{D_2} and identified by the map gluing the two copies of e_{D_2} .

Using the same identification with $\mathbb{Z}^2$, we can write $u_p = (-\beta \cdot D_1, 0)$, and we will use the standard scalar product of vectors in $\mathbb{Z}^2$. In particular, $v_{V,E} \cdot u_p = 0$ is equivalent to $\Sigma(f)_b$ horizontal, $v_{V,E} \cdot u_p > 0$ is equivalent to $\Sigma(f)_b$ pointing downward from V , and $v_{V,E} \cdot u_p < 0$ is equivalent to $\Sigma(f)_b$ pointing upward from V .

Proposition 3.2. *Let*

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow \nu & & \downarrow \pi_0 \\ W & \xrightarrow{g} & \text{pt}_{\mathbb{N}}, \end{array}$$

be an element of $\overline{M}_g(X_0, \beta, \tilde{\sigma})$. Let

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X_0) \\ \downarrow \Sigma(\nu) & & \downarrow \Sigma(\pi_0) \\ \Sigma(W) & \xrightarrow{\Sigma(g)} & \Sigma(\text{pt}_{\mathbb{N}}) \end{array}$$

be its tropicalization. Let b be in the interior of $\Sigma(g)^{-1}(1)$. Assume that $\Sigma(C)_b$ is a graph of genus 0. Then:

- For every V vertex of $\Sigma(C)_b$, we have $\Sigma(f)_b(V) \in (e_{D_1} \cup e_{D_2})$.
- Let V and V' be vertices of $\Sigma(C)_b$ such that $\Sigma(f)_b(V)$ is contained in the interior of e_{D_1} and $\Sigma(f)_b(V')$ is contained in the interior of e_{D_2} . Then, there is no edge of $\Sigma(C)_b$ connecting V and V' .
- We have $E_0 \subset e_{D_1}$.

The proof of Proposition 3.2 is given by the following sequence of Lemmas and takes the rest of Section 3.6.

Lemma 3.1. *Let V be a vertex of $\Sigma(C)_b$ and let E be an edge of $\Sigma(C)_b$ adjacent to V such that $v_{V,E} \cdot u_p > 0$. Assume that $E \neq E_0$. Then there exists a path $\mathbf{c}_{V,E}$ in $\Sigma(C)_b$, starting at V , of first edge E , ending at V_0 , and whose vertices have images by $\Sigma(f)_b$ of strictly decreasing ordinates.*

Proof. The vertical projection of the balancing condition holds everywhere in X_0^{trop} except at v_Y . It follows that, starting with V and E such that $v_{V,E} \cdot u_p > 0$, there exists a path in $\Sigma(C)_b$, starting at V , of first edge E and whose vertices have images by $\Sigma(f)_b$ of strictly decreasing ordinates. Indeed, all the vertices of the path except maybe the first one have ordinates strictly less than the ordinate of v_Y and so in particular are distinct of v_Y . This path can only end at V_0 . $\square$

Lemma 3.2. *There is no vertex V of $\Sigma(C)_b$ such that $\Sigma(f)_b(V)$ has an ordinate strictly less than the ordinate of $\Sigma(f)_b(V_0)$.*

Proof. Let V be such vertex. We have necessarily $\Sigma(f)_b(V) \neq v_Y$ and so the vertical projection of the balancing condition applies to V . It follows from Lemma 3.1 that every edge E adjacent to V has $v_{V,E} = 0$. In particular, every vertex adjacent to V has the same ordinate than V and the previous argument applies to it. It follows that the vertices of

$\Sigma(C)_b$ whose images by $\Sigma(f)_b$ have ordinates equal to the ordinate of $\Sigma(f)_b(V)$ define a subgraph of $\Sigma(C)_b$, union of connected components. This contradicts the fact that $\Sigma(C)_b$ is connected. $\square$

Lemma 3.3. *Let V be a vertex of $\Sigma(C)_b$ such that $\Sigma(f)_b(V) \in e_{D_2}$ and $\Sigma(f)_b \neq v_Y$. Let E be an edge of $\Sigma(C)_b$ adjacent to V . Then E is horizontal, that is, $v_{V,E} \cdot u_p = 0$.*

Proof. If $v_{V,E} \cdot u_p > 0$ (resp. < 0), then, by the vertical balancing condition (applied through the gluing map of our two copies of e_{D_2}), there would exist an edge E' , adjacent to V such that $v_{V,E'} \cdot u_p < 0$ (resp. > 0). Applying Lemma 3.1 to both E and E' , one would get two distinct paths $\mathbf{c}_{V,E}$ and $\mathbf{c}_{V',E'}$ starting at V and ending at V_0 . The union $\mathbf{c}_{V,E} \cup \mathbf{c}_{V',E'}$ would be a non-trivial closed cycle in $\Sigma(C)_b$, in contradiction with our assumption that $\Sigma(C)_b$ has genus 0. $\square$

Recall that $\mathbb{P}$ has a natural fibration structure $\mathbb{P} \rightarrow D_2$.

Lemma 3.4. *Let C' be an irreducible component of C such that $f(C') \subset \mathbb{P}$. Then $f(C')$ is contained in a fiber of $\mathbb{P} \rightarrow D_2$.*

Proof. Let V be the vertex of $\Sigma(C)_b$ dual to C' . We have $\Sigma(f)_b(V) \in e_{D_2}$ and $\Sigma(f)_b(V) \neq v_Y$. Assume by contradiction that $f(C')$ is not contained in a fiber of $\mathbb{P} \rightarrow D_2$. Then $f(C')$ intersects $\mathbb{P}_{p_1}^1$ and $\mathbb{P}_{p_2}^1$. So there are edges E_{p_1} and E_{p_2} adjacent to V such that $\Sigma(f)_b(E_{p_1})$ goes inside f_{p_1} with $v_{V,E_{p_1}} \cdot u_p > 0$ and similarly $\Sigma(f)_b(E_{p_2})$ goes inside f_{p_2} with $v_{V,E_{p_2}} \cdot u_p > 0$. This contradicts Lemma 3.3. $\square$

Let $\Gamma_{\mathbb{P}}$ be the subgraph of $\Sigma(C)_b$ consisting of vertices mapped by $\Sigma(f)_b$ on $e_{\mathbb{P}_{p_1}^1} \cup e_{\mathbb{P}_{p_2}^1}$. Let τ be a connected component of $\Gamma_{\mathbb{P}}$ and let C_{τ} be the curve dual to Γ_{τ} . By Lemma 3.4, $f(C_{\tau})$ is contained in a fiber of $\mathbb{P} \rightarrow D_2$. We can write $[f(C_{\tau})] = d_{\tau}[\mathbb{P}^1] \in H_2(\mathbb{P}, \mathbb{Z})$, for some $d_{\tau} \in \mathbb{N}$, where $[\mathbb{P}^1]$ is the class of a fiber of $\mathbb{P} \rightarrow D_2$. As C is connected, C_{τ} cannot be entirely contracted by f , and so $d_{\tau} \geq 1$. In fact, d_{τ} is the sum of weights of edges of $\Sigma(C)_b$ adjacent to τ and mapped non-trivially in e_{D_2} .

For all $1 \leq j \leq \beta \cdot D_2$, let τ_j be the connected component of $\Gamma_{\mathbb{P}}$ containing V_j . As the points $\tilde{\sigma}_1(0), \dots, \tilde{\sigma}_{\beta \cdot D_2}(0)$ are in different fibers of $\mathbb{P} \rightarrow D_2$, we have $\tau_j \neq \tau_k$ if $j \neq k$. In particular, we have $V_j \neq V_k$ if $j \neq k$.

Let $\mathcal{V}_Y$ be the set of vertices V of $\Sigma(C)_b$ such that $\Sigma(f)_b(V) = v_Y$, and which are connected to some vertex in some connected component of $\Gamma_{\mathbb{P}}$ by some path mapped in e_{D_2} by $\Sigma(f)_b$. Let C_Y be the union of irreducible components of C dual to vertices in $\mathcal{V}_Y$. Denote $\beta_Y := [f(C_Y)] \in H_2(Y, \mathbb{Z})$.

Lemma 3.5. *We have:*

- $\beta_Y \cdot D_2 = \beta \cdot D_2$.
- *The connected components of $\Gamma_{\mathbb{P}}$ are exactly the τ_j , $1 \leq j \leq \beta \cdot D_2$. In particular, no vertex of $\Sigma(C)_b$ is mapped by $\Sigma(f)_b$ in the interior of $e_{\mathbb{P}_{p_1}^1}$ or the interior of $e_{\mathbb{P}_{p_2}^1}$.*
- $d_{\tau_j} = 1$ for every $1 \leq j \leq \beta \cdot D_2$.

Proof. It follows from Lemma 3.3 and from the horizontal balancing condition, satisfied on e_{D_2} away from v_Y and $v_{\mathbb{P}}$, that

$$\beta_Y \cdot D_2 \geq \sum_{\tau} d_{\tau},$$

where the sum is over the connected components of $\Gamma_{\mathbb{P}}$.

By Lemma 3.4, no irreducible component of C is mapped onto D_2 , so the image of every irreducible component of C intersects D_2 non-negatively and so $\beta \cdot D_2 \geq \beta_Y \cdot D_2$. On the other hand, we have $\sum_{\tau} d_{\tau} \geq \sum_{j=1}^{\beta \cdot D_2} d_{\tau_j} \geq \beta \cdot D_2$.

The combination of the previous inequalities gives

$$\beta \cdot D_2 \geq \beta_Y \cdot D_2 \geq \sum_{\tau} d_{\tau} \geq \sum_{j=1}^{\beta \cdot D_2} d_{\tau_j} \geq \beta \cdot D_2,$$

and so all these inequalities are in fact equalities. Lemma 3.5 follows. $\square$

Lemma 3.6. *Let C' be an irreducible component of C such that $f(C') \subset Y$ and C' is not contracted by f . Then*

$$(f(C') \cap D_2) \subset \{\mathbb{P}_{\tilde{\sigma}_j(0)}^1 \cap D_2, 1 \leq j \leq \beta \cdot D_2\}.$$

Proof. By Lemma 3.4, no irreducible component of C is mapped onto D_2 , so the image of every irreducible component of C intersects D_2 non-negatively. But according to Lemma 3.5, the total intersection number $\beta \cdot D_2$ of images by f components of C mapping to Y with D_2 is already accounted by the intersection points $\mathbb{P}_{\tilde{\sigma}_j(0)}^1 \cap D_2, 1 \leq j \leq \beta \cdot D_2$. $\square$

Lemma 3.7. *There is no vertex V of $\Sigma(C)_b$ such that $\Sigma(f)_b(V)$ is contained in the interior of f_{p_1} or the interior of f_{p_2} .*

Proof. It follows from Lemma 3.5 and Lemma 3.6 that no non-contracted component of C has an image by f intersecting p_1 or p_2 . As C is connected, this implies that no component of C can be contracted by f on p_1 or p_2 . $\square$

The first two points of Proposition 3.2 follow from Lemma 3.5, Lemma 3.6 and Lemma 3.7. In particular, we have $\Sigma(f)_b(V_0) \in e_{D_1} \cup e_{D_2}$. As $\beta \cdot D_1 > 0$, there is at least one non-contracted component of C mapping to Y and intersecting D_1 , and so there is at least one vertex V and an edge E such that $v_{V,E} \cdot u_p > 0$. Using Lemma 3.2, it follows that $\Sigma(f)_b(V_0) \in e_{D_2}$ is impossible, and so $\Sigma(f)_b(V_0) \in e_{D_1}$, proving the third point of Proposition 3.2.

3.7. Decomposition formula. We refer to Definitions 4.2.1 and 4.3.1 of [ACGS17] for the notions of decorated parametrized tropical curve and rigid decorated parametrized tropical curves. We say that a decorated parametrized tropical curve $h: \Gamma \rightarrow X_0^{\text{trop}}$ is of type u_p if

- The sum over vertices of curve classes decorations is equal to β .
- Γ contains unbounded edges $E_1, \dots, E_{\beta \cdot D_2}$ contracted by h onto $v_{\mathbb{P}}$.
- Γ contains an unbounded edge E_0 mapped by h with weight $\beta \cdot D_1$ onto an unbounded ray in X_0^{trop} , going down and parallel to e_{D_1} .

- E_0 and $E_1, \dots, E_{\beta \cdot D_2}$ are the only unbounded edges of Γ .

Let $h: \Gamma \rightarrow X_0^{\text{trop}}$ be a rigid genus g decorated parametrized tropical curve of type u_p . Let $\overline{M}_g^h(X_0, \beta)$ be the moduli space of genus g stable log maps of type u_p marked by h , that is, equipped of a retraction of their tropicalization onto h . According to Proposition 4.4.2 of [ACGS17], it is a proper Deligne-Mumford stack, equipped with a virtual fundamental class $[\overline{M}_g^h(X_0, \beta)]^{\text{virt}}$. The $\beta \cdot D_2$ marked points define an evaluation morphism

$$\text{ev}: \overline{M}_g^h(X_0, \beta) \rightarrow (X_0)^{\beta \cdot D_2}.$$

Denote

$$\begin{aligned} \iota_{\tilde{\sigma}}^h: \tilde{\sigma}(0) &= \{(\tilde{\sigma}_1(0), \dots, \tilde{\sigma}_{\beta \cdot D_2}(0))\} \hookrightarrow (X_0)^{\beta \cdot D_2}, \\ \overline{M}_g^h(X_0, \beta, \tilde{\sigma}) &:= \overline{M}_g^h(X_0, \beta) \times_{(X_0)^{\beta \cdot D_2}} \tilde{\sigma}(0), \end{aligned}$$

and

$$[\overline{M}_g^h(X_0, \beta, \tilde{\sigma})]^{\text{virt}} := \iota_{\tilde{\sigma}}^! [\overline{M}_g^h(X_0, \beta)]^{\text{virt}} \in A_g(\overline{M}_g^h(X_0, \beta, \tilde{\sigma}), \mathbb{Q}).$$

Forgetting the marking by h gives a morphism

$$j_h: \overline{M}_g^h(X_0, \beta, \tilde{\sigma}) \rightarrow \overline{M}_g(X_0, \beta, \tilde{\sigma}).$$

According to the decomposition formula, Theorem 4.8.1 of [ACGS17], we have

$$[\overline{M}_g(X_0, \beta, \tilde{\sigma})]^{\text{virt}} = \sum_h \frac{n_h}{|\text{Aut}(h)|} (j_h)_* [\overline{M}_g^h(X_0, \beta, \tilde{\sigma})]^{\text{virt}},$$

where the sum is over rigid genus g decorated parametrized tropical curves $h: \Gamma \rightarrow X_0^{\text{trop}}$ of type u_p , n_h is the smallest positive integer such that after scaling by n_h , h gets integral vertices and integral lengths, and $|\text{Aut}(h)|$ is the order of the automorphism group of h .

3.8. Contributing rigid tropical curves. We first explain how to construct a particular class of decorated parametrized tropical curves. Let $\vec{g} = (g_0, g_1, \dots, g_{\beta \cdot D_2})$ be a $(\beta \cdot D_2 + 1)$ -tuple of non-negative integers such that $|\vec{g}| := g_0 + \sum_{j=1}^{\beta \cdot D_2} g_j = g$. Let $\Gamma_{\vec{g}}$ be the genus 0 graph consisting of vertices V_0 and V_j , $1 \leq j \leq \beta \cdot D_2$, bounded edges $E_j^{D_2}$, $1 \leq j \leq \beta \cdot D_2$, connecting V_0 and V_j , unbounded edges E_j , $1 \leq j \leq \beta \cdot D_2$, attached to V_j and one unbounded edge E_0 attached to V_0 . We define a structure of tropical curve on $\Gamma_{\vec{g}}$ by assigning:

- Genera to the vertices. We assign g_0 to V_0 and g_j to V_j .
- The length $\ell(E_j^{D_2}) = 1$ to the bounded edge $E_j^{D_2}$, for all $1 \leq j \leq \beta \cdot D_2$.

Finally, we define a decorated parametrized tropical curve

$$h_{\vec{g}}: \Gamma_{\vec{g}} \rightarrow X_0^{\text{trop}}$$

by the following data:

- We define $h_{\vec{g}}(V_0) := v_Y$ and $h_{\vec{g}}(V_j) = v_{\mathbb{P}}$ for all $1 \leq j \leq \beta \cdot D_2$.
- For all $1 \leq j \leq \beta \cdot D_2$, the bounded edge $E_j^{D_2}$ is mapped by $h_{\vec{g}}$ to the bounded edge e_{D_2} with weight 1.
- For all $1 \leq j \leq \beta \cdot D_2$, the unbounded edge E_j is contracted by $h_{\vec{g}}$ to the vertex $v_{\mathbb{P}}$.

- The unbounded edge E_0 is mapped by h to the unbounded edge e_{D_1} with weight $\beta \cdot D_1$.
- We decorate V_0 with the curve class $\beta \in H_2(Y, \mathbb{Z})$.
- For all $1 \leq j \leq \beta \cdot D_2$, we decorate the vertex V_j with the curve class $[\mathbb{P}^1]$, fiber class of $\mathbb{P} \rightarrow D_2$.

Figure: the tropical curve $\Gamma_{\tilde{g}}$.

Lemma 3.8. *For every $\tilde{g}$, the genus g decorated parametrized tropical curve $h_{\tilde{g}}: \Gamma_{\tilde{g}} \rightarrow X_0^{\text{trop}}$ is rigid and of type u_p .*

Proof. The fact that $h_{\tilde{g}}$ is of type u_p is immediate from its definition. The rigidity follows from the facts that $h_{\tilde{g}}$ has no contracted bounded edge and all vertices of $\Gamma_{\tilde{g}}$ are mapped to vertices of X_0^{trop} : it is not possible to deform h without changing its combinatorial type. $\square$

Proposition 3.3. *Let $h: \Gamma \rightarrow X_0^{\text{trop}}$ be a genus g rigid decorated parametrized tropical curve of type u_p . Assume that there exists*

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow \nu & & \downarrow \pi_0 \\ W & \xrightarrow{g} & \text{pt}_{\mathbb{N}}, \end{array}$$

element of $\overline{M}_g^h(X_0, \beta, \tilde{\sigma})$, such that the dual graph of C has genus 0. Then there exists $\tilde{g}$ with $|\tilde{g}| = g$ such that $h = h_{\tilde{g}}$.

Proof. Consider the tropicalization

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X_0) \\ \downarrow \Sigma(\nu) & & \downarrow \Sigma(\pi_0) \\ \Sigma(W) & \xrightarrow{\Sigma(g)} & \Sigma(\text{pt}_{\mathbb{N}}). \end{array}$$

For every $b \in \Sigma(g)^{-1}(1)$, let

$$\Sigma(f)_b: \Sigma(C)_b \rightarrow \Sigma(\pi_0)^{-1}(1) = X_0^{\text{trop}}$$

be the fiber of $\Sigma(f)$ over b . It follows from the definition of marking by h (Definition 4.4.1 of [ACGS17]) that h is a retraction of $\Sigma(f)_b: \Sigma(C)_b \rightarrow X_0^{\text{trop}}$. By assumption $\Sigma(C)_b$

has genus 0 and so we can apply Proposition 3.2 to get a relatively explicit description of $\Sigma(f)_b$. It follows from this description that any rigid retraction of $\Sigma(f)_b$ is of the form $h_{\vec{g}}$ for some $\vec{g}$. $\square$

Lemma 3.9. *Let $h: \Gamma \rightarrow X_0^{\text{trop}}$ be a rigid genus g decorated parametrized tropical curve of type u_p . Assume that for every*

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow \nu & & \downarrow \pi_0 \\ W & \xrightarrow{g} & \text{pt}_{\mathbb{N}}, \end{array}$$

element of $\overline{M}_g^h(X_0, \beta, \tilde{\sigma})$, the dual graph of C has positive genus. Then we have

$$\int_{[\overline{M}_g^h(X_0, \beta, \tilde{\sigma})]^{\text{virt}}} (-1)^g \lambda_g = 0.$$

Proof. It is a general property of λ_g that it vanishes on families of curves containing cycles of irreducible components (see for example Lemma 8 of [Bou19]). $\square$

3.9. End of the proof of Proposition 3.1. Combining Proposition 3.3 and Lemma 3.9, the decomposition formula of Section 3.7 implies that

$$N_{g, \beta}^{Y/D_1} = \sum_{\vec{g}, |\vec{g}|=g} \int_{[\overline{M}_g^h(X_0, \beta, \tilde{\sigma})]^{\text{virt}}} (-1)^g \lambda_g.$$

The last step of the proof of Proposition 3.1 is the expression of

$$\int_{[\overline{M}_g^h(X_0, \beta, \tilde{\sigma})]^{\text{virt}}} (-1)^g \lambda_g$$

as product of factors indexed by the vertices of V , and the evaluation of each factor. This is obtained by the same gluing argument used in Section 7 of [Bou19] and Section 5.5 of [Bou20a]. The factors $\frac{1}{h}$ are the contributions of the vertices V_j of $h_{\vec{g}}$, $1 \leq j \leq \beta \cdot D_2$, and come from the relative Gromov-Witten theory of $\mathbb{P}_{\tilde{\sigma}_j(0)}^1 \simeq \mathbb{P}^1$. Indeed, let M be the moduli space of degree 1 stable maps to $\mathbb{P}^1$ relative to a point $\infty \in \mathbb{P}^1$, $\nu: C \rightarrow M$ the universal curve, $f: C \rightarrow \mathbb{P}^1$ the universal stable map, and $[M]^{\text{virt}}$ the natural virtual fundamental class. As the normal bundle of $\mathbb{P}_{\tilde{\sigma}_j(0)}^1$ in $\mathbb{P}$ is trivial, the virtual fundamental class on M coming from the surface $\mathbb{P}$ is obtained by intersecting $[M]^{\text{virt}}$ with $e(R^1\nu_* f^* \mathcal{O}) = (-1)^g \lambda_g$. Therefore, the contribution of V_j is $\sum_{g \geq 1} \left(\int_{[M]^{\text{virt}}} \lambda_g^2 \right) h^{2g-1}$. By Mumford's formula [M83], we have $\lambda_g^2 = 0$ for $g > 0$ and $\lambda_0^2 = 1$.

4. EXCHANGE OF RELATIVE POINT CONDITIONS WITH BLOW-UPS

4.1. Statement. We continue the proof of Theorem 2.1. Given Proposition 3.1, it is enough to fit the invariants $N_{g, \beta}^{Y/D}$ into the framework of Section 5.8 of [Bou20a]. This will be done by exchanging the $\beta \cdot D_2$ relative points conditions along D_2 with no condition on a surface obtained by blowing-up $\beta \cdot D_2$ points on D_2 .

Let Z be the surface obtained from Y by blowing-up $\beta \cdot D_2$ points on D_2 , which are distinct from each other and distinct from $D_1 \cap D_2$. Denote $\pi_Y: Z \rightarrow Y$ the blow-up morphism, $F_1, \dots, F_{\beta \cdot D_2}$ the exceptional divisors and

$$\beta_Z := \pi_Y^* \beta - \sum_{j=1}^{\beta \cdot D_2} [F_j] \in H_2(Z, \mathbb{Z}).$$

We still denote D_1 , D_2 and D the strict transforms of D_1 , D_2 and D in Z . We view Z as a smooth log scheme for the divisorial log structure defined by $D = D_1 \cup D_2$.

Let $\overline{M}_g(Z/D, \beta_Z)$ be the moduli space of genus g class β_Z stable log maps to Z , with contact order $\beta \cdot D_1$ to D_1 in a single point. It is a proper Deligne-Mumford stack, admitting a virtual fundamental class

$$[\overline{M}_g(Z/D, \beta_Z)]^{\text{virt}} \in A_g(\overline{M}_g(Z/D, \beta_Z), \mathbb{Q}).$$

We define

$$N_{g, \beta}^{Z/D} := \int_{[\overline{M}_g(Z/D, \beta_Z)]^{\text{virt}}} (-1)^g \lambda_g \in \mathbb{Q}.$$

Proposition 4.1. *We have*

$$\sum_{g \geq 0} N_{g, \beta}^{Z/D} h^{2g-1} = \left(\sum_{g \geq 0} N_{g, \beta}^{Y/D} h^{2g-1+\beta \cdot D_2} \right) \left(\frac{1}{2 \sin(\frac{h}{2})} \right)^{\beta \cdot D_2}.$$

4.2. Summary of the proof of Proposition 4.1. The proof of Proposition 4.1 takes the rest of Section 4 and is parallel to the proof of Proposition 3.1 given in Section 3. For this reason, we will often refer to the proof of Proposition 3.1 and only explain in detail the points specific to Proposition 4.1.

In Section 4.3, we describe the relevant degeneration. The special fiber of the degeneration has a natural log structure whose tropicalization is described in Section 4.4. This gives us a way to describe the relevant log Gromov-Witten invariants of the special fiber in Section 4.5. The core of the proof is Section 4.6, which is a study of the possible tropicalizations of stable log maps contributing to these invariants. We conclude the proof in Section 4.7.

4.3. Degeneration. As in Section 3.3, let X be the degeneration of Y to the normal cone of D_2 . For every $j = 1, \dots, \beta \cdot D_2$, we choose a point x_j on $D_2 - (D_2 \cap D_1)$ such that $x_j \neq x_k$ if $j \neq k$. Let s_j be the section of $\pi: X \rightarrow \mathbb{A}^1$ obtained as strict transform of the section $\{x_j\} \times \mathbb{A}^1$ of $Y \times \mathbb{A}^1$.

We blow-up the sections s_j of π to obtain a new family $\tilde{\pi}: \tilde{X} \rightarrow \mathbb{A}^1$. We have $\tilde{\pi}^{-1}(t) = Z$ if $t \neq 0$. Let $\tilde{\mathbb{P}}$ be the surface obtained from $\mathbb{P} = \mathbb{P}(N_{D_2|Y} \oplus \mathcal{O}_{D_2})$ by blowing-up the points $s_j(0)$, $1 \leq j \leq \beta \cdot D_2$. We still denote $D_{2,0}$, $D_{2,\infty}$, $\mathbb{P}_{p_1}^1$ and $\mathbb{P}_{p_2}^2$ the strict transforms in $\tilde{\mathbb{P}}$ of the divisors $D_{2,0}$, $D_{2,\infty}$, $\mathbb{P}_{p_1}^1$ and $\mathbb{P}_{p_2}^2$ of $\mathbb{P}$. The special fiber $\tilde{\pi}^{-1}(0)$ has two irreducible components, Y and $\tilde{\mathbb{P}}$, with D_2 in Y glued along $D_{2,0}$ in $\tilde{\mathbb{P}}$.

Let $\widetilde{D_1 \times \mathbb{A}^1}$ be the divisor on $\tilde{X}$ obtained as strict transform of the divisor $D_1 \times \mathbb{A}^1$ in $Y \times \mathbb{A}^1$. Similarly, let $\widetilde{D_2 \times \mathbb{A}^1}$ be the divisor on $\tilde{X}$ obtained as strict transform of the divisor $D_2 \times \mathbb{A}^1$ in $Y \times \mathbb{A}^1$. We view $\tilde{X}$ as a log scheme for the divisorial log structure

defined by the normal crossing divisor $\widetilde{D_1 \times \mathbb{A}^1} \cup \widetilde{D_2 \times \mathbb{A}^1} \cup \tilde{\pi}^{-1}(0)$. We view $\mathbb{A}^1$ as a log scheme for the toric divisorial log structure. Then $\tilde{\pi}$ naturally define a log smooth morphism. Remark that for $t \neq 0$, $\tilde{\pi}^{-1}(t)$ with the log structure restricted from $\tilde{X}$ is Z with the divisorial log structure defined by $D = D_1 \cup D_2$.

We denote $\tilde{X}_0 := \tilde{\pi}^{-1}(0)$ with the log structure restricted from $\tilde{X}$, and $\tilde{\pi}_0: \tilde{X}_0 \rightarrow \text{pt}_{\mathbb{N}}$ the log smooth morphism to the standard log point induced by $\tilde{\pi}$.

4.4. Tropicalization of the special fiber. Tropicalizing the log morphism $\tilde{\pi}_0: \tilde{X}_0 \rightarrow \text{pt}_{\mathbb{N}}$, we get a morphism of cone complexes $\Sigma(\tilde{\pi}_0): \Sigma(\tilde{X}_0) \rightarrow \Sigma(\text{pt}_{\mathbb{N}})$. We have $\Sigma(\text{pt}_{\mathbb{N}}) = \mathbb{R}_{\geq 0}$ and $\Sigma(\tilde{X}_0)$ is naturally identified with the cone over the fiber $\Sigma(\pi_0)^{-1}(1)$ at $1 \in \mathbb{R}_{\geq 0}$. It is thus enough to describe the cone complex $\Sigma(\tilde{\pi}_0)^{-1}(1)$. We denote

$$\tilde{X}_0^{\text{trop}} := \Sigma(\tilde{\pi}_0)^{-1}(1).$$

The cone complex $\tilde{X}_0^{\text{trop}}$ consists of:

- Vertices v_Y and $v_{\tilde{\mathbb{P}}}$ respectively dual to Y and $\tilde{\mathbb{P}}$.
- unbounded edges $e_{D_1}, e_{\mathbb{P}_{p_1}^1}, e_{\mathbb{P}_{p_2}^1}, e_{D_{2,\infty}}$ respectively dual to $D_1, \mathbb{P}_{p_1}^1, \mathbb{P}_{p_2}^1, D_{2,\infty}$.
- One bounded edge $e_{D_{2,0}}$ dual to $D_{2,0}$.
- Faces $f_{p_{1,0}}, f_{p_{2,0}}, f_{p_{1,\infty}}, f_{p_{2,\infty}}$ respectively dual to $p_{1,0}, p_{2,0}, p_{1,\infty}, p_{2,\infty}$.

We draw below a picture of $\tilde{X}_0^{\text{trop}}$, where the two copies of $e_{D_{2,0}}$, the two copies of $v_{\tilde{\mathbb{P}}}$ and the two copies of $e_{D_{2,\infty}}$ have to be identified.

Remark that $\Sigma(\tilde{\pi}_0)^{-1}(0)$ is the “asymptotic version” of $\tilde{X}_0^{\text{trop}}$, given by

which can be identified with the tropicalization of Z , viewed as a log scheme for the divisorial log structure defined by $D = D_1 \cup D_2$.

Contact orders for stable log maps to a log scheme are integral points in the tropicalization of this log scheme (see Definition 2.3.12 of [ACGS17]). We denote $\tilde{u}_p$ the contact order for stable log maps to $\tilde{X}_0$ defined by $\beta \cdot D_1$ times the integral generator (pointing away from v_Z^0) of the edge $e_{D_1}^0$ in $\Sigma(\tilde{\pi}_0)^{-1}(0)$.

4.5. Invariants of the special fiber. Let $\overline{M}_g(\tilde{X}_0, \beta_Z)$ be the moduli space of genus g class β_Z stable log maps to $\tilde{\pi}_0: \tilde{X}_0 \rightarrow \text{pt}_{\mathbb{N}}$, with a single point of contact order $\tilde{u}_p$. Let

$$[\overline{M}_g(\tilde{X}_0, \beta_Z)]^{\text{virt}} \in A_g(\overline{M}_g(\tilde{X}_0, \beta_Z), \mathbb{Q})$$

be its virtual fundamental class. By deformation invariance of log Gromov-Witten invariants, we have

$$N_{g, \beta}^{Z/D} = \int_{[\overline{M}_g(\tilde{X}_0, \beta_Z)]^{\text{virt}}} (-1)^g \lambda_g.$$

4.6. Tropical curves. Let

$$\begin{array}{ccc} C & \xrightarrow{f} & \tilde{X}_0 \\ \downarrow \nu & & \downarrow \tilde{\pi}_0 \\ W & \xrightarrow{g} & \text{pt}_{\mathbb{N}}, \end{array}$$

be an element of $\overline{M}_g(\tilde{X}_0, \beta_Z)$. Let

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(\tilde{X}_0) \\ \downarrow \Sigma(\nu) & & \downarrow \Sigma(\tilde{\pi}_0) \\ \Sigma(W) & \xrightarrow{\Sigma(g)} & \Sigma(\text{pt}_{\mathbb{N}}) \end{array}$$

be its tropicalization. For every $b \in \Sigma(g)^{-1}(1)$, let

$$\Sigma(f)_b: \Sigma(C)_b \rightarrow \Sigma(\tilde{\pi}_0)^{-1}(1) = \tilde{X}_0^{\text{trop}}$$

be the fiber of $\Sigma(f)$ over b .

The definition of $\overline{M}_g(\tilde{X}_0, \beta_Z)$ fixes the set of unbounded edges of $\Sigma(C)_b$. The contact order $\tilde{u}_p$ at a single point is dual to an unbounded edge E_0 of $\Sigma(C)_b$. Furthermore, $\Sigma(C)_b$ has no other unbounded edge. The contact order $\tilde{u}_p$ implies that E_0 is mapped by $\Sigma(C)_b$ onto an unbounded ray in $\tilde{X}_0^{\text{trop}}$, going down in a way parallel to $e_{\mathbb{P}_1^1}$, e_{D_1} and $e_{\mathbb{P}_2^1}$. The fact that the image of E_0 by $\Sigma(C)_b$ can a priori be any ray parallel to e_{D_1} reflects the fact that a stable log map to Y with a single point of maximal tangency along D_1 degenerates in a stable log map to $\tilde{X}_0$ with maximal tangency along $D_1 \cup \mathbb{P}_{p_1}^1 \cup \mathbb{P}_{p_2}^1$, and a priori not only along D_1 . We denote V_0 the vertex of Γ to which the unbounded edge E_0 is attached.

The tropical balancing condition has to be satisfied everywhere except at the vertices v_Y and $v_{\mathbb{P}}$ (the corresponding components do not have toric divisorial log structures). At $v_{\mathbb{P}}$, the tropical balancing condition is only modified in the horizontal direction: it still

holds in the vertical direction. We refer to Proposition 1.15 of [GS13] for the general form of the balancing conditions for stable log maps.

For V a vertex of $\Sigma(C)_b$ and E an edge adjacent to V , we denote $v_{V,E}$ the slope measured from V of the restriction of $\Sigma(f)_b$ to E . Given our picture of $\tilde{X}_0^{\text{trop}}$, we can identify $v_{V,E}$ with a well-defined element of $\mathbb{Z}^2$, except if $\Sigma(f)_b(E)$ is contained in $e_{D_{2,0}} \cup e_{D_{2,\infty}}$, in which case $v_{V,E}$ defines two elements of $\mathbb{Z}^2$ corresponding to the two copies of $e_{D_{2,0}} \cup e_{D_{2,\infty}}$ and identified by the map gluing the two copies of $e_{D_{2,0}} \cup e_{D_{2,\infty}}$.

Using the same identification with $\mathbb{Z}^2$, we can write $\tilde{u}_p = (-\beta \cdot D_1, 0)$, and we will use the standard scalar product of vectors in $\mathbb{Z}^2$. In particular, $v_{V,E} \cdot \tilde{u}_p = 0$ is equivalent to $\Sigma(f)_b$ horizontal, $v_{V,E} \cdot \tilde{u}_p > 0$ is equivalent to $\Sigma(f)_b$ pointing downward from V , and $v_{V,E} \cdot \tilde{u}_p < 0$ is equivalent to $\Sigma(f)_b$ pointing upward from V .

The following Proposition is the analogue for the present degeneration of the Proposition 3.2 for the degeneration considered in Section 3.

Proposition 4.2. *Let*

$$\begin{array}{ccc} C & \xrightarrow{f} & \tilde{X}_0 \\ \downarrow \nu & & \downarrow \tilde{\pi}_0 \\ W & \xrightarrow{g} & \text{pt}_{\mathbb{N}}, \end{array}$$

be an element of $\overline{M}_g(\tilde{X}_0, \beta_Z)$. Let

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(\tilde{X}_0) \\ \downarrow \Sigma(\nu) & & \downarrow \Sigma(\tilde{\pi}_0) \\ \Sigma(W) & \xrightarrow{\Sigma(g)} & \Sigma(\text{pt}_{\mathbb{N}}) \end{array}$$

be its tropicalization. Let b be in the interior of $\Sigma(g)^{-1}(1)$. Assume that $\Sigma(C)_b$ is a graph of genus 0. Then:

- For every V vertex of $\Sigma(C)_b$, we have $\Sigma(f)_b(V) \in (e_{D_1} \cup e_{D_{2,0}})$.
- Let V and V' be vertices of $\Sigma(C)_b$ such that $\Sigma(f)_b(V)$ is contained in the interior of e_{D_1} and $\Sigma(f)_b(V')$ is contained in the interior of $e_{D_{2,0}}$. Then, there is no edge of $\Sigma(C)_b$ connecting V and V' .
- We have $E_0 \subset e_{D_1}$.

The proof of Proposition 4.2 is parallel to the proof of Proposition 3.2. We first remark that the natural analogues of Lemmas 3.1, 3.2, 3.3 still hold, with the same proofs.

Lemma 4.1. *There is no vertex V of $\Sigma(C)_b$ such that $\Sigma(f)_b(V) \in e_{D_{2,\infty}}$ and $\Sigma(f)_b(V) \neq v_{\tilde{\mathbb{P}}}$.*

Proof. Let V be a vertex of $\Sigma(C)_b$ such that $\Sigma(f)_b(V) \in e_{D_{2,\infty}}$ and $\Sigma(f)_b(V) \neq v_{\tilde{\mathbb{P}}}$. Arguing as in the proof of Lemma 3.3, we get that any edge E adjacent to V should be horizontal. Look at V the leftmost (or rightmost, in our picture of $\tilde{X}_0^{\text{trop}}$) of such vertices. Then, using the balancing condition, there should be an unbounded edge adjacent to V and whose image by $\Sigma(f)_b$ is contained in $e_{D_{2,\infty}}$. This contradicts the fact that E_0 is the only unbounded edge of $\Sigma(C)_b$. $\square$

Recall that $\tilde{\mathbb{P}}$ is obtained from the projective bundle $\mathbb{P} \rightarrow D_2$ by blowing-up the $\beta \cdot D_2$ points $s_j(0) \in D_{2,\infty}$, $1 \leq j \leq \beta \cdot D_2$. Let F_j , $1 \leq j \leq \beta \cdot D_2$, be the exceptional divisor and let $\mathbb{P}_j^1$, $1 \leq j \leq \beta \cdot D_2$, be the strict transforms of the $\mathbb{P}^1$ fibers of $\mathbb{P} \rightarrow D_2$ passing through $s_j(0)$. In particular, for every $1 \leq j \leq \beta \cdot D_2$, $\mathbb{P}_j^1$ and F_j intersect in one point. We denote $\tilde{\mathbb{P}} \rightarrow D_2$ the composition of the blow-up morphism $\tilde{\mathbb{P}} \rightarrow \mathbb{P}$ with the $\mathbb{P}^1$ -fibration $\mathbb{P} \rightarrow D_2$.

Lemma 4.2. *Let C' be an irreducible component of C such that $f(C') \subset \tilde{\mathbb{P}}$. Then $f(C')$ is contained in a fiber of $\tilde{\mathbb{P}} \rightarrow D_2$.*

Proof. Given Lemma 4.1, the analogue of the proof of Lemma 3.4 applies. $\square$

Lemma 4.3. *Let C' be an irreducible component of C such that $f(C') \subset \tilde{\mathbb{P}}$. Then, either there exists j , $1 \leq j \leq \beta \cdot D_2$, such that $f(C') \subset \mathbb{P}_j^1$, or $f(C') \subset \mathbb{P}_{p_1}^1$ or $f(C') \subset \mathbb{P}_{p_2}^1$.*

Proof. By Lemma 4.2, we already know that $f(C')$ is contained in a fiber of $\tilde{\mathbb{P}} \rightarrow D_2$. To conclude, it is enough to show that $f(C') \cap D_{2,\infty}$ is included in $(D_{2,\infty} \cap \mathbb{P}_{p_1}^1) \cup (D_{2,\infty} \cap \mathbb{P}_{p_2}^1)$. If it were not the case, then there would exist an edge of $\Sigma(C)_b$ which is mapped non-trivially in $e_{D_{2,\infty}}$, in contradiction with the proof of Lemma 4.1. $\square$

Let $\Gamma_{\tilde{\mathbb{P}}}$ be the subgraph of $\Sigma(C)_b$ consisting of vertices mapped by $\Sigma(f)_b$ on $e_{\mathbb{P}_{p_1}^1} \cup e_{\mathbb{P}_{p_2}^1}$. Let τ be a connected component of $\Gamma_{\tilde{\mathbb{P}}}$ and let C_τ be the curve dual to Γ_τ . By Lemma 4.2, $f(C_\tau)$ is contained in a fiber of $\tilde{\mathbb{P}} \rightarrow D_{2,0}$. In particular, as C is connected, we have $d_\tau := [f(C_\tau)] \cdot D_{2,0} \geq 1$. In fact, d_τ is the sum of weights of edges of $\Sigma(C)_b$ adjacent to τ and mapped non-trivially in $e_{D_{2,0}}$.

It follows from Lemma 4.3 and from $[f(C)] = \beta_Z$ (in particular $[f(C)] \cdot [F_j] = 1$), that, for every $1 \leq j \leq \beta \cdot D_2$, the vertices of $\Sigma(C)_b$ dual to components C' such that $f(C') \subset \mathbb{P}_j^1$ define a connected component τ_j of $\Gamma_{\tilde{\mathbb{P}}}$ with $d_{\tau_j} = 1$.

Let $\mathcal{V}_Y$ be the set of vertices V of $\Sigma(C)_b$ such that $\Sigma(f)_b(V) = v_Y$, and which are connected to some vertex in some connected component of $\Gamma_{\tilde{\mathbb{P}}}$ by some path mapped in $e_{D_{2,0}}$ by $\Sigma(f)_b$. Let C_Y be the union of irreducible components of C dual to vertices in $\mathcal{V}_Y$. Denote $\beta_Y := [f(C_Y)] \in H_2(Y, \mathbb{Z})$.

Lemma 4.4. *We have $\beta_Y \cdot D_{2,0} = \beta \cdot D_{2,0}$, and every connected component of $\Gamma_{\tilde{\mathbb{P}}}$ is of the form τ_j for some $1 \leq j \leq \beta \cdot D_{2,0}$. In particular, no vertex of $\Sigma(C)_b$ is mapped by $\Sigma(f)_b$ in the interior of $e_{\mathbb{P}_{p_1}^1}$ or the interior of $e_{\mathbb{P}_{p_2}^1}$.*

Proof. This is parallel to the proof of Lemma 3.5. It follows from the analogue of Lemma 3.3 and from the horizontal balancing condition, satisfied on $e_{D_{2,0}}$ away from v_Y and $v_{\tilde{\mathbb{P}}}$, that

$$\beta_Y \cdot D_2 \geq \sum_{\tau} d_{\tau},$$

where the sum is over the connected components of $\Gamma_{\tilde{\mathbb{P}}}$.

By Lemma 4.2, no irreducible component of C is mapped onto D_2 , so the image of every irreducible component of C intersects D_2 non-negatively and so $\beta \cdot D_2 \geq \beta_Y \cdot D_2$. On the other hand, we have $\sum_{\tau} d_{\tau} \geq \sum_{j=1}^{\beta \cdot D_2} d_{\tau_j} \geq \beta \cdot D_2$.

The combination of the previous inequalities gives

$$\beta \cdot D_2 \geq \beta_Y \cdot D_2 \geq \sum_{\tau} d_{\tau} \geq \sum_{j=1}^{\beta \cdot D_2} d_{\tau_j} \geq \beta \cdot D_2,$$

and so all these inequalities are in fact equalities. Lemma 4.4 follows. $\square$

Lemma 4.5. *Let C' be an irreducible component of C such that $f(C') \subset Y$ and C' is not contracted by f . Then*

$$(f(C') \cap D_2) \subset \{\mathbb{P}_j^1 \cap D_{2,0}, 1 \leq j \leq \beta \cdot D_2\}.$$

Proof. By Lemma 4.2, no irreducible component of C is mapped onto D_2 , so the image of every irreducible component of C intersects D_2 non-negatively. But according to Lemma 4.4, the total intersection number $\beta \cdot D_2$ of images by f components of C mapping to Y with D_2 is already accounted by the intersection points $\mathbb{P}_j^1 \cap D_2, 1 \leq j \leq \beta \cdot D_2$. $\square$

Lemma 4.6. *There is no vertex V of $\Sigma(C)_b$ such that $\Sigma(f)_b(V)$ is contained in the interior of either $f_{p_1,0}$, $f_{p_2,0}$, $f_{p_1,\infty}$ or $f_{p_2,\infty}$.*

Proof. It follows from Lemma 4.4 and Lemma 4.5 that no non-contracted component of C has an image by f intersecting p_1 , p_2 , $\mathbb{P}_{p_1}^1 \cap D_{2,\infty}$ or $\mathbb{P}_{p_2}^1 \cap D_{2,\infty}$. As C is connected, this implies that no component of C can be contracted by f on p_1 , p_2 , $\mathbb{P}_{p_1}^1 \cap D_{2,\infty}$ or $\mathbb{P}_{p_2}^1 \cap D_{2,\infty}$. $\square$

The first two points of Proposition 4.2 follow from Lemma 4.4, Lemma 4.5 and Lemma 4.6. In particular, we have $\Sigma(f)_b(V_0) \in e_{D_1} \cup e_{D_{2,0}}$. As $\beta \cdot D_1 > 0$, there is at least one non-contracted component of C mapping to Y and intersecting D_1 , and so there is at least one vertex V and an edge E such that $v_{V,E} \cdot u_p > 0$. Using the analogue of Lemma 3.2, it follows that $\Sigma(f)_b(V_0) \in e_{D_{2,0}}$ is impossible, and so $\Sigma(f)_b(V_0) \in e_{D_1}$, proving the third point of Proposition 4.2.

4.7. End of the proof of Proposition 4.1. Given Proposition 4.2, the rest of the proof of Proposition 4.1 is parallel to the proof of Proposition 3.1. Proposition 4.2 gives a way to identify the rigid decorated parametrized tropical curves contributing to the degeneration formula after integration of λ_g : they are the “obvious ones” from the point of view of the usual degeneration formula in relative Gromov-Witten theory, with V_0 a vertex mapping to v_Y and V_j vertices mapping to $v_{\tilde{\mathbb{P}}}$, for $1 \leq j \leq \beta \cdot D_2$. The factors $\frac{1}{2 \sin(\frac{\hbar}{2})}$ are the contributions of the vertices V_j , $1 \leq j \leq \beta \cdot D_2$, and comes from relative Gromov-Witten theory of $\mathbb{P}_j^1 \simeq \mathbb{P}^1$. Indeed, let M be the moduli space of degree 1 stable maps to $\mathbb{P}^1$ relative to a point $\infty \in \mathbb{P}^1$, $\nu: C \rightarrow M$ the universal curve, $f: C \rightarrow \mathbb{P}^1$ the universal stable map, and $[M]^{\text{virt}}$ the natural virtual fundamental class. As the normal bundle of $\mathbb{P}_j^1$ in $\tilde{\mathbb{P}}$ is $\mathcal{O}(-1)$, the virtual fundamental class on M coming from the surface $\tilde{\mathbb{P}}$ is obtained by intersecting $[M]^{\text{virt}}$ with $e(R^1 \nu_* f^* \mathcal{O}(-1))$. Therefore, the contribution of V_j is $\sum_{g \geq 1} \left(\int_{[M]^{\text{virt}}} (-1)^g \lambda_g e(R^1 \nu_* f^* \mathcal{O}(-1)) \right) \hbar^{2g-1}$, which is equal to $\frac{1}{2 \sin(\frac{\hbar}{2})}$ by [BP05] (see the proof of Theorem 5.1).

5. END OF THE PROOF OF THEOREM 2.1

By combination of Proposition 3.1 and Proposition 4.1, we have

$$N_{g,\beta}^{Z/D} = N_{g,\beta}^{Y/D_1},$$

result stated as Theorem 1.3 in the Introduction.

If $\mathbf{d}(\beta) \in \mathbb{N}^{(Q_\beta^{Y/D_1})_0}$, then, according to Theorem 8.13 of [Bou20a], we have

$$\Omega_\beta^{Y/D_1}(q^{\frac{1}{2}}) = (-1)^{\beta \cdot D_1 + 1} \left(2 \sin \left(\frac{\hbar}{2} \right) \right) \left(\sum_{g \geq 0} N_{g,\beta}^{Z/D} \hbar^{2g-1} \right),$$

where $q = e^{i\hbar}$. Indeed, as $\beta_Z \cdot E_j = 1$, for all $1 \leq j \leq \beta \cdot D_2$, the class β_Z is primitive in $H_2(Z, \mathbb{Z})$ and so the integral log BPS invariant appearing in Theorem 8.13 of [Bou20a] coincides with the rational log BPS invariant defined in Conjecture 8.1 of [Bou20a]. Replacing $N_{g,\beta}^{Z/D}$ by $N_{g,\beta}^{Y/D_1}$ in the above formula for $\Omega_\beta^{Y/D_1}(q^{\frac{1}{2}})$, we get exactly the statement of Theorem 2.1.

If $\mathbf{d}(\beta) \notin \mathbb{N}^{(Q_\beta^{Y/D_1})_0}$, it follows from the proof of Proposition 33 of [Bou20b] that $N_{g,\beta}^{Z/D} = 0$, for every $g \geq 0$. By definition, we also have $\Omega_\beta^{Y/D}(q^{\frac{1}{2}}) = 0$ and so Theorem 2.1 is still valid in this case.

6. EXAMPLES

In this Section, we consider various examples of Y , D_1 , D_2 as in Section 2.1, and we construct a corresponding quiver following the recipe of Section 2.3. Each time this quiver is acyclic, Theorem 2.1 applies. To index our examples, we use the notation $Y(D_1^2, D_2^2)$. In total, we give 7 examples with an acyclic quiver, and 3 examples with a non-acyclic quiver. The reader is invited to try other examples.

According to Section 2.3, the main step in the quiver construction is the identification of a diagram

$$\begin{array}{ccc} & (\tilde{Z}, \tilde{D}) & \\ \swarrow \pi_Z & & \searrow \pi_{\tilde{Z}} \\ (Z, D) & & (\bar{Z}, \bar{D}), \end{array}$$

with $(\bar{Z}, \bar{D})$ toric, π_Z is a sequence of corner blow-ups of (Z, D) , and $\pi_{\tilde{Z}}$ is a sequence of interior blow-ups of $(\tilde{Z}, \tilde{D})$.

For practical computations, it is useful to know that, given a log Calabi-Yau surface with maximal boundary $(\bar{Z}, \bar{D})$, if the sequence of self intersection numbers of irreducible components of D can be realized as the sequence of self intersection numbers of toric divisors of a toric surface, then in fact $(\bar{Z}, \bar{D})$ is toric. See Lemma 2.10 of [Fri15] for a proof of this result. Thus, the identification of a diagram as above mostly requires only numerical work.

6.1. $\mathbb{P}^2(1, 4)$. We consider $Y = \mathbb{P}^2$, D_1 a line and D_2 a smooth conic not tangent to D_1 , so that $D_1^2 = 1$ and $D_2^2 = 4$. We first explain how to construct a toric model of (Y, D) , where $D = D_1 \cup D_2$. Denote p_1 and p_2 the two intersection points of D_1 and D_2 . Denote L the line tangent to D_2 at the point p_1 .

We first blow-up p_1 . Let F_1 be the exceptional divisor and let p'_1 be the intersection point of F_1 and D_2 . The line L passes through p'_1 . We then blow-up p'_1 . Let F_2 be the exceptional divisor. The line L is now a (-1) -curve, which can be contracted. We get a log Calabi-Yau surface, with boundary $D_1 \cup F_1 \cup F_2 \cup D_2$, where $D_1^2 = 0$, $F_1^2 = -2$, $F_2^2 = 0$, $D_2^2 = 2$, that is, a Hirzebruch surface $\mathbb{F}_2$. The fan of $\mathbb{F}_2$ is given by four rays generated by $(-1, 0)$, $(0, -1)$, $(0, 1)$ and $(1, 2)$:

We wrote near each ray the corresponding divisor and in parenthesis its self intersection number.

Let $\beta = d \in \mathbb{Z} = H_2(\mathbb{P}^2, \mathbb{Z})$. Following the class β through the previous blow-ups and blow-down, we get the class $\beta_{\mathbb{F}_2} \in H_2(\mathbb{F}_2, \mathbb{Z})$ such that $\beta_{\mathbb{F}_2} \cdot F_2 = d$, $\beta_{\mathbb{F}_2} \cdot F_1 = 0$, $\beta_{\mathbb{F}_2} \cdot D_1 = d$ and $\beta_{\mathbb{F}_2} \cdot D_2 = 2d$. Remark that the balancing collection

$$d(-1, 0) + 2d(0, -1) + d(1, 2) = 0$$

is indeed satisfied.

The surface Z is obtained by blowing-up $\beta \cdot D_2 = 2d$ points on D_2 . It follows that we get $\tilde{Z}$ starting from $\mathbb{F}_2$ and blowing-up $2d$ points on the divisor dual to the ray of direction $(0, -1)$ and one point on the divisor dual to the ray of direction $(-1, 0)$. As

$$\langle (-1, 0), (0, -1) \rangle = 1,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with $2d$ vertices of dimension one on the left. This quiver is acyclic and so Theorem 2.1 applies. This quiver coincides with the quiver considered in [RW18].

6.2. $\mathbb{P}^2(4, 1)$. We consider $Y = \mathbb{P}^2$, D_1 a smooth conic and D_2 a line not tangent to D_1 . Let $\beta = d \in \mathbb{Z} = H_2(\mathbb{P}^2, \mathbb{Z})$. The surface Z is obtained by blowing-up $\beta \cdot D_2 = d$ points on D_2 . It follows from the above discussion of $\mathbb{P}^2(1, 4)$ that we get $\tilde{Z}$ starting from $\mathbb{F}_2$ and blowing-up d points on the divisor dual to the ray of direction $(1, 2)$ and one point of the ray of direction $(-1, 0)$. As

$$\langle (1, 2), (-1, 0) \rangle = 2,$$

it follows that the corresponding quiver and dimension vectors are given by

with d vertices of dimension one on the left. This quiver is acyclic and so Theorem 2.1 applies.

6.3. $\mathbb{F}_0(2, 2)$. We consider $Y = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and D_1, D_2 smooth divisors of degree $(1, 1)$. We have $D_1^2 = D_2^2 = 2$.

We first explain how to construct a toric model for (Y, D) , where $D = D_1 \cup D_2$. We blow-up one of the two intersection points of L_1 and L_2 . Let F be the exceptional divisor. The strict transforms of the horizontal (degree $(1, 0)$) and vertical (degree $(0, 1)$) $\mathbb{P}^1$ s passing through this point are disjoint (-1) -curves, which can be contracted. The resulting log Calabi-Yau surface is $\mathbb{P}^2$ with its toric boundary. The intersection numbers of β with these two interior (-1) -curves are d_1 and d_2 . The fan of $\mathbb{P}^2$ is given by three rays generated by $(-1, 0)$, $(0, -1)$, $(1, 1)$.

We wrote near each ray the corresponding divisor and in parenthesis its self intersection number.

Let $\beta = (d_1, d_2) \in \mathbb{Z}^2 = H_2(\mathbb{F}_0, \mathbb{Z})$. We have $\beta \cdot D = 2(d_1 + d_2)$ and $\beta \cdot D_1 = \beta \cdot D_2 = d_1 + d_2$. Following the class β through the previous blow-ups and blow-down, we get the class $\beta_{\mathbb{P}^2} \in H_2(\mathbb{P}^2, \mathbb{Z})$ such that $\beta_{\mathbb{P}^2} \cdot D_1 = \beta_{\mathbb{P}^2} \cdot D_2 = d_1 + d_2$ and $\beta_{\mathbb{P}^2} \cdot F = d_1 + d_2$. Remark that the balancing condition

$$(d_1 + d_2)(-1, 0) + (d_1 + d_2)(0, -1) + (d_1 + d_2)(1, 1) = 0$$

is indeed satisfied.

The surface $\tilde{Z}$ is obtained by blowing-up $d_1 + d_2$ points on the divisor dual to the ray of direction $(0, -1)$ and two points on the divisor dual to the ray of direction $(1, 1)$. As

$$\langle (0, -1), (1, 1) \rangle = 1,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with $d_1 + d_2$ vertices of dimension 1 on the left. This quiver is acyclic and so Theorem 2.1 applies.

6.4. $\mathbb{F}_1(0, 4)$. We consider $Y = \mathbb{F}_1$. We denote $C_{-1}, C_1, f_0, f_\infty$ the toric divisors of $\mathbb{F}_1$ so that $C_{-1}^2 = -1, C_1^2 = 1, f_0^2 = f_\infty^2 = 0$. We have the linear equivalence relations $f_0 \sim f_\infty$ and $C_1 \sim C_{-1} + f_0$.

We take $D_1 = f_0$ and D_2 a smooth rational curve linearly equivalent to $C_{-1} + f_\infty + C_1 \sim C_{-1} + 2f_0$, and intersecting D_1 transversally. We have $D_1^2 = 0$ and $D_2^2 = 4$.

We first explain how to construct a toric model for (Y, D) , where $D = D_1 \cup D_2$. We remark that C_{-1} is a (-1) -curve disjoint from D_2 and so can be contracted. The resulting log Calabi-Yau surface is $\mathbb{P}^2$, with D_1 being a line and D_2 being a conic. So it is enough

to take the known toric model for $(\mathbb{P}^2, D_1 \cup D_2)$, which is given by $\mathbb{F}_2$ as described in Section 6.1.

The fan of $\mathbb{F}_2$ is given by four rays generated by $(-1, 0)$, $(0, -1)$, $(0, 1)$ and $(1, 2)$:

We wrote near each ray the corresponding divisor and in parenthesis its self intersection number.

Let $\beta = d_1[C_{-1}] + d_2[f_0] \in H_2(\mathbb{F}_1, \mathbb{Z})$. We have $\beta \cdot D = d_1 + 2d_2$, $\beta \cdot D_1 = d_1$, $\beta \cdot D_2 = 2d_2$. Following the class β through the previous blow-ups and blow-down, we get the class $\beta_{\mathbb{F}_2} \in H_2(\mathbb{F}_2, \mathbb{Z})$ such that $\beta_{\mathbb{F}_2} \cdot D_1 = d_1 + \beta \cdot C_{-1} = d_1 + (d_2 - d_1) = d_2$, $\beta_{\mathbb{F}_2} \cdot D_2 = 2d_2$, $\beta_{\mathbb{F}_2} \cdot F_2 = d_2$, $\beta_{\mathbb{F}_2} \cdot F_1 = 0$. Remark that the balancing condition

$$d_2(-1, 0) + 2d_2(0, -1) + d_2(1, 2) = 0$$

is indeed satisfied.

The surface $\tilde{Z}$ is obtained from $\mathbb{F}_2$ by blowing-up $2d_2$ points on the divisor dual to the ray of direction $(0, -1)$, one point on the divisor dual to the ray of direction $(-1, 0)$, and one point on the divisor dual to the ray of direction $(1, 2)$. As

$$\langle (-1, 0), (0, -1) \rangle = 1, \langle (0, -1), (1, 2) \rangle = 1, \langle (1, 2), (-1, 0) \rangle = 2,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with $2d_2$ vertices of dimension 1 on the left. This quiver contains oriented cycles and so Theorem 2.1 does not apply.

6.5. $\mathbb{F}_1(4, 0)$. We consider $Y = \mathbb{F}_1$. We denote $C_{-1}, C_1, f_0, f_\infty$ the toric divisors of $\mathbb{F}_1$ so that $C_{-1}^2 = -1, C_1^2 = 1, f_0^2 = f_\infty^2 = 0$. We have the linear equivalence relations $f_0 \sim f_\infty$ and $C_1 \sim C_{-1} + f_0$.

We take $D_2 = f_0$ and D_1 a smooth rational curve linearly equivalent to $C_{-1} + f_\infty + C_1 \sim C_{-1} + 2f_0$, and intersecting D_2 transversally. We have $D_2^2 = 0$ and $D_1^2 = 4$.

It follows from the above discussion of $\mathbb{F}_1(0, 4)$ that $\tilde{Z}$ is obtained from $\mathbb{F}_2$ by blowing-up $d_1 + 1$ points on the divisor dual to the ray of direction $(1, 2)$, one point on the divisor dual to the ray of direction $(-1, 0)$.

As

$$\langle (1, 2), (-1, 0) \rangle = 2,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with d_1 vertices of dimension 1 on the left. This quiver is acyclic and so Theorem 2.1 applies. Remark that for $d_1 = d_2$, this quiver reduces to the quiver describing the $\mathbb{P}^2(4, 1)$ example, that is, $\mathbb{P}^2$ relatively to a conic. Indeed, in this case, $\beta \cdot C_{-1} = 0$, and the geometry can be directly reduced from $\mathbb{F}_1$ to $\mathbb{P}^2$.

6.6. $\mathbb{F}_1(1, 3)$. We consider $Y = \mathbb{F}_1$. We denote $C_{-1}, C_1, f_0, f_\infty$ the toric divisors of $\mathbb{F}_1$ so that $C_{-1}^2 = -1, C_1^2 = 1, f_0^2 = f_\infty^2 = 0$. We have the linear equivalence relations $f_0 \sim f_\infty$ and $C_1 \sim C_{-1} + f_0$.

We take $D_1 = C_1$ and D_2 a smooth rational curve linearly equivalent to $C_{-1} + f_0 + f_\infty \sim C_{-1} + 2f_0$. We have $D_1^2 = 1$ and $D_2^2 = 3$.

We first explain how to construct a toric model for (Y, D) , where $D = D_1 \cup D_2$. We blow-up the two intersection points of D_1 and D_2 . Let F_1 and F_2 be the two exceptional divisors. The strict transforms of the $\mathbb{P}^1$ fibers (of degree $[f_0] = [f_\infty]$) passing through these points are disjoint (-1) -curves, which can be contracted. The resulting log Calabi-Yau surface is $\mathbb{F}_1$ with its toric boundary.

The fan of $\mathbb{F}_1$ is given by the four rays generated by $(-1, 0), (0, -1), (0, 1)$ and $(1, 1)$:

We wrote near each ray the corresponding divisor and in parenthesis its self intersection number.

Let $\beta = d_1[C_{-1}] + d_2[f_0] \in H_2(\mathbb{F}_1, \mathbb{Z})$. We have $\beta \cdot D = d_1 + 2d_2$, $\beta \cdot D_1 = d_2$, $\beta \cdot D_2 = d_1 + d_2$. Following the class β through the previous blow-ups and blow-downs, we get the class $\beta_{\mathbb{F}_1} \in H_2(\mathbb{F}_1, \mathbb{Z})$ such that $\beta_{\mathbb{F}_1} \cdot D_1 = d_2$, $\beta_{\mathbb{F}_1} \cdot D_2 = d_1 + d_2$, $\beta_{\mathbb{F}_1} \cdot F_1 = \beta_{\mathbb{F}_1} \cdot F_2 = \beta \cdot f_0 = d_1$. Remark that the balancing condition

$$d_2(0, 1) + (d_1 + d_2)(0, -1) + d_1(-1, 0) + d_1(1, 1) = 0$$

is indeed satisfied.

The surface $\tilde{Z}$ is obtained from $\mathbb{F}_1$ by blowing-up $d_1 + d_2$ points on the divisor dual to the ray of direction $(0, -1)$, one point on the divisor dual to the ray of direction $(-1, 0)$, and one point on the divisor dual to the ray of direction $(1, 1)$. As

$$\langle (-1, 0), (0, -1) \rangle = 1, \langle (1, 1), (-1, 0) \rangle = 1, \langle (0, -1), (1, 1) \rangle = 1,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with $d_1 + d_2$ vertices of dimension 1 on the left. This quiver contains oriented cycles and so Theorem 2.1 does not apply.

6.7. $\mathbb{F}_1(3, 1)$. We consider $Y = \mathbb{F}_1$. We denote C_{-1} , C_1 , f_0 , f_∞ the toric divisors of $\mathbb{F}_1$ so that $C_{-1}^2 = -1$, $C_1^2 = 1$, $f_0^2 = f_\infty^2 = 0$. We have the linear equivalence relations $f_0 \sim f_\infty$ and $C_1 \sim C_{-1} + f_0$.

We take $D_2 = C_1$ and D_1 a smooth rational curve linearly equivalent to $C_{-1} + f_0 + f_\infty \sim C_{-1} + 2f_0$. We have $D_2^2 = 1$ and $D_1^2 = 3$.

It follows from the above discussion of $\mathbb{F}_1(1, 3)$ that $\tilde{Z}$ is obtained from $\mathbb{F}_2$ by blowing-up d_2 points on the divisor dual to the ray of direction $(0, 1)$, one point on the divisor dual

to the ray of direction $(-1, 0)$ and one point on the divisor dual to the ray of direction $(1, 1)$. As

$$\langle (0, 1), (-1, 0) \rangle = 1, \langle (1, 1), (-1, 0) \rangle = 1, \langle (1, 1), (0, 1) \rangle = 1,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with d_2 vertices of dimension 1 on the left. This quiver is acyclic and so Theorem 2.1 applies.

6.8. $\mathbb{F}_2(2, 2)$. We consider $Y = \mathbb{F}_2$. We denote $C_{-2}, C_2, f_0, f_\infty$ the toric divisors of $\mathbb{F}_2$ so that $C_{-2}^2 = -2, C_2^2 = 2, f_0^2 = f_\infty^2 = 0$. We have the linear equivalence relations $f_0 \sim f_\infty$ and $C_2 \sim C_{-2} + 2f_0$.

We take $D_1 = C_2$ and D_2 a smooth rational curve linearly equivalent to $C_{-2} + f_0 + f_\infty \sim C_2$, intersecting C_2 transversally. We have $D_1^2 = D_2^2 = 2$.

We first explain how to construct a toric model for (Y, D) , where $D = D_1 \cup D_2$. We blow-up the two intersection points of D_1 and D_2 . Let F_1 and F_2 be the two exceptional divisors. The strict transforms of the $\mathbb{P}^1$ fibers (of degree $[f_0] = [f_\infty]$) passing through these points are disjoint (-1) -curves, which can be contracted. The resulting log Calabi-Yau surface is $\mathbb{F}_0 = \mathbb{P}_1 \times \mathbb{P}^1$ with its toric boundary.

The fan of $\mathbb{F}_0$ is given by the four rays generated by $(-1, 0), (0, -1), (0, 1)$ and $(1, 0)$:

We wrote near each ray the corresponding divisor and in parenthesis its self intersection number.

Let $\beta = d_1[C_{-2}] + d_2[f_0] \in H_2(\mathbb{F}_2, \mathbb{Z})$. We have $\beta \cdot D = 2d_2$ and $\beta \cdot D_1 = \beta \cdot D_2 = d_2$. Following the class β through the previous blow-ups and blow-downs, we get the class $\beta_{\mathbb{F}_0} \in H_2(\mathbb{F}_0, \mathbb{Z})$ such that $\beta_{\mathbb{F}_0} \cdot D_1 = \beta_{\mathbb{F}_0} \cdot D_2 = d_2$ and $\beta_{\mathbb{F}_0} \cdot F_1 = \beta_{\mathbb{F}_0} \cdot F_2 = \beta \cdot f_0 = d_1$. Remark that the balancing condition

$$d_2(0, 1) + d_2(0, -1) + d_1(1, 0) + d_1(-1, 0) = 0$$

is indeed satisfied.

The surface $\tilde{Z}$ is obtained from $\mathbb{F}_0$ by blowing-up d_2 points on the divisor dual to the ray of direction $(0, -1)$, one point on the divisor dual to the ray of direction $(-1, 0)$ and one point on the divisor dual to the ray of direction $(1, 0)$. As

$$\langle (-1, 0), (0, -1) \rangle = 1, \langle (0, -1), (1, 0) \rangle = 1,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with d_2 vertices of dimension 1 on the left. This quiver is acyclic and so Theorem 2.1 applies.

6.9. $\mathbb{F}_N(-N, N+4)$. We consider $Y = \mathbb{F}_N$. We denote C_{-N} , C_N , f_0 , f_∞ the toric divisors of $\mathbb{F}_N$ so that $C_{-N}^2 = -N$, $C_N^2 = N$, $f_0^2 = f_\infty^2 = 0$. We have the linear equivalence relations $f_0 \sim f_\infty$ and $C_N \sim C_{-N} + Nf_0$.

We take $D_1 = C_{-N}$ and D_2 a smooth rational curve linearly equivalent to $C_N + f_0 + f_\infty \sim C_N + 2f_0$, intersecting C_{-N} transversally. We have $D_1^2 = -N$ and $D_2^2 = N+4$.

We first explain how to construct a toric model for (Y, D) , where $D = D_1 \cup D_2$. We blow-up the two intersection points of D_1 and D_2 . Let F_1 and F_2 be the two exceptional divisors. The strict transforms of the $\mathbb{P}^1$ fibers (of degree $[f_0] = [f_\infty]$) passing through these points are disjoint interior (-1) -curves, which can be contracted. The resulting log Calabi-Yau surface is $\mathbb{F}_{N+2}$ with its toric boundary.

The fan of $\mathbb{F}_{N+2}$ is given by the four rays generated by $(-1, 0)$, $(0, -1)$, $(0, 1)$ and $(1, N+2)$.

Let $\beta = d_1[C_{-N}] + d_2[f_0] \in H_2(\mathbb{F}_N, \mathbb{Z})$. We have $\beta \cdot D = -Nd_1 + 2(d_1 + d_2)$, $\beta \cdot D_1 = -Nd_1 + d_2$ and $\beta \cdot D_2 = 2d_1 + d_2$. Following the class β through the previous blow-ups and blow-downs, we get the class $\beta_{\mathbb{F}_{N+2}}$ such that $\beta_{\mathbb{F}_{N+2}} \cdot D_1 = -Nd_1 + d_2$, $\beta_{\mathbb{F}_{N+2}} \cdot D_2 = 2d_1 + d_2$, $\beta_{\mathbb{F}_{N+2}} \cdot F_1 = \beta_{\mathbb{F}_{N+2}} \cdot F_2 = \beta \cdot f_0 = d_1$. Remark that the balancing condition

$$(-Nd_1 + d_2)(0, 1) + (2d_1 + d_2)(0, -1) + d_1(-1, 0) + d_1(1, N+2) = 0$$

is indeed satisfied.

The surface $\tilde{Z}$ is obtained from $\mathbb{F}_{N+2}$ by blowing-up $2d_1 + d_2$ points on the divisor dual to the ray of direction $(0, -1)$, one point on the divisor dual to the ray of direction $(-1, 0)$

and one point on the divisor dual to the ray of direction $(1, N+2)$. As

$$\langle (1, N+2), (-1, 0) \rangle = N+2, \langle (-1, 0), (0, -1) \rangle = 1, \langle (0, -1), (1, N+2) \rangle = 1,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with $2d_1 + d_2$ vertices of dimension 1 on the left and $N+2$ vertical arrows between the two vertices on the right. Remark that this quiver contains oriented cycles and so Theorem 2.1 does not apply.

6.10. $\mathbb{F}_N(N+4, -N)$. We consider $Y = \mathbb{F}_N$. We denote $C_{-N}, C_N, f_0, f_\infty$ the toric divisors of $\mathbb{F}_N$ so that $C_{-N}^2 = -N, C_N^2 = N, f_0^2 = f_\infty^2 = 0$. We have the linear equivalence relations $f_0 \sim f_\infty$ and $C_N \sim C_{-N} + Nf_0$.

We take $D_2 = C_{-N}$ and D_1 a smooth rational curve linearly equivalent to $C_N + f_0 + f_\infty \sim C_N + 2f_0$, intersecting C_{-N} transversally. We have $D_2^2 = -N$ and $D_1^2 = N+4$.

It follows from the above discussion of $\mathbb{F}_N(-N, N+4)$ that the surface $\tilde{Z}$ is obtained from $\mathbb{F}_{N+2}$ by blowing-up $-Nd_1 + d_2$ points on the divisor dual to the ray of direction $(0, 1)$, one point on the divisor dual to the ray of direction $(-1, 0)$ and one point on the divisor dual to the ray of direction $(1, N+2)$. As

$$\langle (1, N+2), (-1, 0) \rangle = N+2, \langle (0, 1), (-1, 0) \rangle = 1, \langle (1, N+2), (0, 1) \rangle = 1,$$

it follows from Section 2.3 that the corresponding quiver and dimension vectors are given by

with $-Nd_1 + d_2$ vertices of dimension 1 on the left and $N + 2$ vertical arrows between the two vertices on the right. This quiver is acyclic and so Theorem 2.1 applies.

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