An asymptotic approximation of the traveling salesman problem with uniform non-overlapping time windows

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Abstract— **We develop a continuous asymptotic approximation of the traveling salesman problem with time windows in the Euclidean plane, constructing upon the well-known Beardwood-Halton-Hemmersley theorem. The time windows are taken to be a partition of a given time horizon. Computational experiments on random TSP with time windows instances show that the proposed asymptotic approximations of tour lengths and arrival times are close to the actual optimal values.**

I. INTRODUCTION

The computation of the shortest Hamiltonian cycle along a number of points is an NP-hard problem, known as the Traveling Salesman Problem (TSP). For a large number of points, this operation becomes cumbersome even if heuristic solving approaches are used. In a great number of situations, only an approximation of the optimal tour length is needed. Being able to quickly provide a reliable approximation with reduced efforts is actually critical to the design of logistic and distribution systems of several services. For instance, some postal systems, which usually tend to have a large number of deliveries, rely on continuous approximations of tour lengths to partition the service territory [1], [2]. This has even led the United States Postal Service to significant cost savings [3]. Also, continuous approximations do not require precise data about the points to visit. This could be shown to be useful in several contexts where the locations are not known in advance, such as in natural disasters or for deliveries in a dynamic environment.

The contribution of this paper is to extend the continuous asymptotic treatment to the TSP with time windows, specifically for a model of time windows partition. Time windows are critical components of the practical routing operations in logistic and freight distribution. These constraints arise in traffic restrictions to freight loading zones, to city centers, or are merely imposed by clients and patients in logistic operations and medical transport. We consider time windows for macro-level planning, which are used by several delivery services, wherein time slots are wide time intervals and represent a partition of the day. Several services including postal systems and courier and delivery companies could benefit from the use of asymptotic approximations of routing problems with time windows, when it is necessary to estimate the cost of the routes before computing the routes themselves. Such applications include districting, facility location problem, and fleet sizing. Approximations have an advantage over heuristics and algorithms for the TSP to be computed in constant time $\mathcal{O}(1)$.

The remaining of the article is organized as follows. Section II presents a brief literature review on the topic of continuous approximations in routing problems. Section IV introduces the proposed asymptotic approximations accounting for the time windows, while the preliminaries of the study are stated in Section III. The computational results are provided in Section V. The paper is concluded thereafter.

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II. BRIEF LITERATURE REVIEW

The approximation of the routing problem is grounded on the famous formula of Beardwood, Halton, and Hammersley (BHH), published in 1959 [4]. This result states that when the number of points to visit goes to infinity on a compact area, the optimal tour length approaches a constant value. The theorem is stated in the preliminaries. Noting that the BHH formula underestimates tour lengths in elongated areas, Daganzo [5] proposed a strip strategy method, which efficiently computes the optimal tour length in those types of areas. These two models [4], [5] gave rise to several extensions accounting for multiple vehicles, accommodating to the area's shape and size, and to the variants of the transportation problem. BHH formula has also led to the development of solving heuristics such as 'Partition' [6]. For a review of the overall extensions deriving from the continuous asymptotic approximation of the TSP, see [1] and [7].

Continuous approximations of the routing with time-windows concern mainly the vehicle routing problem (VRP). Daganzo [8], [9] has developed a model wherein the day is divided into time periods and customers into rectangles. Using a cluster-first route-second method, he obtained an approximation for the total distance traveled by all vehicles under these considerations. Figliozzi [10] tested several VRP approximations, and proposed a probabilistic modeling of the approximation such that the number of routes for a given number of time windows is derived probabilistically. Using similar assumptions to [8], [9], Carlson and Behzoodi [11] studied the worst-case time window distribution in terms of routing costs, and found that it corresponds to a concentrated demand on a single time period when the number of customers is low, or to a uniform distribution over the time for a large number of customers. Although VRP asymptotic approximations are intuitive and simple to use, they are mainly grounded on empirical evidence as opposed to the analytical derivation of the BHH theorem.

There has been several recent applications, whether for districting [12], location problems [13], fleet sizing [14] or for accounting for pickups and deliveries [15].

Our model is based for the time windows considerations on similar assumptions to [8], [9], [11] in the sense of taking non-overlapping intervals. However, we differ from all the previously mentioned approaches by approximating a problem, although basic, but which has not been considered yet, which is the TSP with time windows.

III. PRELIMINARIES

A. Definition of the TSP with Time Windows (TSP-TW)

The set of points to visit is denoted $\mathcal{P}$, and is a finite set $|\mathcal{P}| = n < \infty$. For each point $i \in \mathcal{P}$, we denote b_i and e_i the begin and the end of the time window associated with i. For each couple of points $i, j \in \mathcal{P}$, we denote d_{ij} the travel duration between i and j. The specifications of the time windows, *i.e.* the values of b_i, e_i, and the durations d_{ij} are discussed in the following subsection.

The goal is to choose a starting point $s \in \mathcal{P}$ and an order of visit for the other points that minimise the total tour duration. For a couple of points $i, j \in \mathcal{P}$, let x_{ij} be a binary variable equal to one if and only if j is visited after i. For all $i \in \mathcal{P}$, let t_i be the arrival time at i. A formulation is as follows:

$$\min \sum_{i,j \in \mathcal{P}} d_{ij} x_{ij} \tag{1}$$

$$\sum_{j \in \mathcal{P} \setminus \{i\}} x_{ij} = 1 \qquad \forall i \in \mathcal{P} \tag{2}$$

$$\sum_{j \in \mathcal{P} \setminus \{i\}} x_{ji} = 1 \qquad \forall i \in \mathcal{P} \tag{3}$$

$$b_i \leq t_i \leq e_i \qquad \forall i \in \mathcal{P} \setminus \{s\} \tag{4}$$

$$t_j - t_i \geq d_{ij} + M(x_{ij} - 1) \qquad \forall i \in \mathcal{P} \setminus \{s\} \forall j \in \mathcal{P} \tag{5}$$

$$t_j - b_s \geq d_{sj} + M(x_{sj} - 1) \qquad \forall j \in \mathcal{P} \tag{6}$$

$$s \in \mathcal{P} \tag{7}$$

$$b_s \leq b_i \qquad \forall i \in \mathcal{P} \tag{8}$$

$$t_i \geq 0 \qquad \forall i \in \mathcal{P} \tag{9}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in \mathcal{P} \tag{10}$$

Constraints (2) and (3) are the flow conservation constraints. Constraint (4) ensures that the arrival times satisfy the time windows. Constraints (5) and (6) track the arrival times, with M is a very large number. Both of them generalize the subtour elimination constraints of Miller, Tucker and Zemlin for the TSP. Constraints (7) and (8) ensure that the starting point belongs to the set $\mathcal{P}$. Constraints (9) and (10) represent the binary and the bounding restrictions for the decision variables. Notice that the stop duration is null.

B. Random generation of TSP-TW instances

We consider instances of the TSP-TW that are randomly generated. Input parameters of the model used to randomly generate instances are: the size n of $\mathcal{P}$, the number of time-windows m, the time horizon h, and the side size a of the square area. Given these input parameters, TSP-TW instances are generated as follows:

- The n points in $\mathcal{P}$ are distributed uniformly on the *Euclidean* plane $\mathbb{R}^2$ within a square area $\mathcal{R}$, *i.e.* $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < a, \quad \text{and} \quad 0 \leq y < a\}$. The random variables associated with these n points are supposed to be independent and identically distributed (i.i.d) and are denoted $X_1, \ldots, X_n$.
- The duration d_{ij} is the *Euclidean* distance between i and j (*i.e.*, we assume that the vehicle has a constant speed of one unit of space per unit of time)[1].
- m equally sized and non-overlapping time windows are defined by partitioning the time horizon h. Hence, each time window $k \in \{1, \ldots m\}$ begins at time $b_k = (k-1) \times h/m$ and ends at time $e_k = k \times h/m$.
- Points in $\mathcal{P}$ are uniformly distributed over the m time windows in an i.i.d. fashion.

The reason for using the uniform distribution is its simplicity. Giving an equal spatial treatment to each point is realistic if the covered area is not large, and is uniform in terms of land use. Additionally, this distribution is a base case distribution for the worst case study. In fact, the uniform distribution corresponds for both space and time to the worst case distribution according to the principle of maximum entropy in case no prior statistical information about the demand of $\mathcal{P}$ is provided. This makes its study essential for a first approximation.

C. BHH theorem

To approximate the tour durations of the routing problems, we make use of the BHH theorem, stated below for any probability spatial distribution in a planar region:

Theorem 1: For a set of n random variables $\{X_1, ..., X_n\}$ $(0 < n < \infty)$ independently and identically distributed and a compact support $\mathcal{R} \subset \mathbb{R}^2$, then the length L_n^{tsp} under the *Euclidean* metric of the shortest Hamiltonian path linking X_i satisfies

$$\frac{L_n^{tsp}}{\sqrt{n}} \xrightarrow[n \to \infty]{} \beta^{tsp} \int_{\mathcal{R}} \sqrt{\overline{f}(x, y)}\, dx\, dy,$$

with $\overline{f}(.)$ the absolutely continuous part of the probability distribution of the X_i, and β^{tsp} a constant.

Under a uniform probability distribution of the random variables $\{X_1, ..., X_n\}$, the BHH formula becomes,

$$\frac{L_n^{tsp}}{\sqrt{n}} \xrightarrow[n \to \infty]{} \beta^{tsp} \sqrt{|\mathcal{R}|},$$

where $|\mathcal{R}|$ is the size of the planar area $\mathcal{R}$. In theory, the constant β^{tsp} does not depend on the number of points n. Several estimates are given for it. According to Arlotto and Steele [16], β^{tsp} varies in the interval: $0.62499 \leq \beta^{tsp} \leq 0.91996$. Stein [17] uses the estimate $\beta^{tsp} = 0.765$, while Applegate *et al.* [18] and Lei *et al.* [12] give practical estimates depending on the number n.

For the VRP, a similar empirical formula was provided by Eilon *et al.*. [19] and subsequently considered by Daganzo

[1]The given approximations are valid for any cost function at the condition to be proportional to the *Euclidean* distance metric.

[20], which states that for $l = \lceil n/q \rceil$ vehicles with capacity q, the optimal length L_n^{vrp} is approximated by the formula,

$$L_n^{vrp} = 2\ \bar{d}\ l + 0.57\sqrt{n\ |\mathcal{R}|},$$

where $\bar{d}$ is the average distance of the n points from a unique depot.

IV. Asymptotic Approximations

In this section, we derive an asymptotic approximation of the TSP-TW tour length, waiting time and arrival time. Let us first note that it may be possible that time windows cannot be satisfied in some cases. A feasibility condition is first given in Proposition 1 in order to give the boundaries of the feasible solution set. Thereafter, the approximations are stated in Proposition 2.

Proposition 1: For the TSP-TW random generation model defined in Section III.B, the Pareto front (n, m) of feasible tours linking the realisations of X_i under the *Euclidean* metric satisfies on average,

$$n \times m = \frac{h^2}{|\mathcal{R}|\ (\beta^{tsp})^2} \tag{11}$$

Proof: If N_{tw} is the random variable of the number of customers by time window, its expected value under the uniform distribution is equal to $E(N_{tw}) = n/m$.
In order to have at least a feasible tour of the realizations of X_i, the duration of the shortest path linking the points within a time window and reaching the first point of the next time window, which is a path of $\lceil n/m \rceil$ arcs on average, must be at most equal to the size of the time window, thus,

$$L_{n/m}^{tsp} \leq \frac{h}{m} \Longrightarrow \beta^{tsp}\ \sqrt{n\ m\ |\mathcal{R}|} \leq h$$
$$\Longrightarrow n \times m \leq \frac{h^2}{|\mathcal{R}|\ (\beta^{tsp})^2}.$$

In the asymptotic domain, a path of n/m arcs can be approximated by a tour of n/m points, since BHH formula does not require specific positions of the points to visit.

If the number of time windows m is fixed, the number of customers that can be served asymptotically is limited by the area's size $|\mathcal{R}|$ and the time horizon h, *i.e.* $n \leq n_{max} = h^2/(m\ |\mathcal{R}|\ (\beta^{tsp})^2)$. Similarly, if the number of customers to be served n is fixed, the upper bound of the number of time windows in the asymptotic domain is given by $m \leq m_{max} = h^2/(n\ |\mathcal{R}|\ (\beta^{tsp})^2)$. Hence, the equation (11) defines a Pareto front curve of the feasible region of tours linking the realisations of X_i under the expected value operator of N_{tw}.

Since the repartition of points to visit on time windows follows an i.i.d uniform distribution, then for a chosen time window j, the affectation of a point i can be seen as a Bernoulli trial with a probability $p_b = 1/m$. Therefore, the random variable N_{tw} follows a binomial distribution with the parameters n and p_b. Asymptotically and under the central limit theorem, N_{tw} converges to a normal distribution. Being symmetric for a large n, half of the realizations of N_{tw} are over $E(N_{tw})$, and half are below. Following this argument, the formula of the feasibility (11) of the TSP-TW tours is given on average. ■

Proposition 2: For the TSP-TW random generation model defined in Section III.B, the asymptotic length $L_{n,m}^{tsptw}$, total waiting time $W_{n,m}^{tsptw}$, and arrival time $Z_{n,m}^{tsptw}$ of the shortest Hamiltonian path tour linking X_i under the *Euclidean* metric and the feasibility condition (11) satisfy,

$$L_{n,m}^{tsptw} = \beta^{tsp}\ \sqrt{n\ m\ |\mathcal{R}|} + o\Big(\sqrt{n\ m}\Big).$$
$$W_{n,m}^{tsptw} = \frac{m-1}{m}\Big(h - \beta^{tsp}\ \sqrt{n\ m\ |\mathcal{R}|}\Big) + o\Big(\sqrt{\frac{n}{m}}\ (m-1)\Big).$$
$$Z_{n,m}^{tsptw} = \frac{m-1}{m}\ h + \beta^{tsp}\ \sqrt{\frac{n}{m}\ |\mathcal{R}|} + o\Big(\sqrt{\frac{n}{m}}\Big).$$

Proof: Since time windows are non-overlapping intervals, the Hamiltonian path linking the realizations of X_i can be seen as a summation of three terms. The first one is a sum over the open TSP tour lengths (without returning to the starting point) of the first $(m-1)$ time windows. It has a value of $(m-1)\ (1 - \frac{m}{n})L_{n/m}^{tsp}$, as each open tour has $E(N_{tw}) = n/m$ points, and subsequently $\frac{n}{m} - 1$ edges. Note that a closed tour of x points has x edges, thus the average length of one tour edge can be seen as L_x/x where L_x is the length of the tour, and the path length without returning to the depot can be considered to be equal to $(1 - \frac{1}{x})\ L_x$. The second term represents the connections between time windows. Similarly to Proposition 1, we consider that each connection is equal to one edge of a TSP tour. The term has then a value of $(m-1)\ \frac{m}{n}\ L_{n/m}^{tsp}$. The final term is a complete tour of n/m points for the last time window, which mimics returning to the starting point. Hence,

$$L_{n,m}^{tsptw} = (m-1)\ (1 - \frac{m}{n})L_{n/m}^{tsp} + (m-1)\ \frac{m}{n}\ L_{n/m}^{tsp} + L_{n/m}^{tsp}. \tag{12}$$

The freedom concerning the positions of the points to visit in the BHH theorem allows us to pin down this expression. An tour example is displayed in Figure 1. According to Theorem 1, the asymptotic approximation $L_{n/m}^{tsp}$ is equal to

$$L_{n/m}^{tsp} = \beta^{tsp}\ \sqrt{\frac{n}{m}\ |\mathcal{R}|} + o\Big(\sqrt{\frac{n}{m}}\Big), \tag{13}$$

which leads to the following expression equivalent to (12),

$$L_{n,m}^{tsptw} = \beta^{tsp}\ \sqrt{n\ m\ |\mathcal{R}|} + o\Big(\sqrt{n\ m}\Big).$$

The total waiting time $W_{n,m}^{tsptw}$ is the summation of the durations the transporter has to wait in the current time window before serving the customers of the following time window. It can pinned down as

$$\begin{aligned} W_{n,m}^{tsptw} &= (m-1)\ (\frac{h}{m} - (1 - \frac{m}{n})\ L_{n/m}^{tsp} + \frac{m}{n}\ L_{n/m}^{tsp}) \\ &= (m-1)\ (\frac{h}{m} - L_{n/m}^{tsp}). \end{aligned} \tag{14}$$

The arrival time Z_n^{tsptw} with a start time equals to zero is computed as

$$Z_{n,m}^{tsptw} = (m-1)\ \frac{h}{m} + L_{n/m}^{tsp}. \tag{15}$$

By substituting $L^{tsp}_{n/m}$ of (13) into (14) and (15) we obtain the expressions of $W^{tsptw}_{n,m}$ and $Z^{tsptw}_{n,m}$. ∎

The approximations of Proposition 2 for one time window $m = 1$ are equal to,

$$W^{tsptw}_{n,1} = 0,$$

$$L^{tsptw}_{n,1} = Z^{tsptw}_{n,1} = \beta^{tsp}\,\sqrt{n\,|\mathcal{R}|} + o\Big(\sqrt{n}\Big) = L^{tsp}_{n},$$

which corresponds to the result of the BHH Theorem. Under the feasibility condition (11), we obtain,

$$L^{tsptw}_{n,m} = Z^{tsptw}_{n,m} = h,\; W^{tsptw}_{n,m} = 0,$$

which is consistent with the limiting behavior of a feasible TSP-TW tour for a maximum number of customers and time windows. Considering distributions other than the uniform one is naturally possible. The spatial distribution only intervenes in the computation of the integral $\int_{\mathcal{R}} \sqrt{f(x,y)}\,dx\,dy$, and the temporal distribution in the expected value $E(N_{tw})$.

V. COMPUTATIONAL RESULTS

In this section, we provide results of a computational experiment comparing the asymptotic approximations to actual realized tours of the random TSP-TW problem.

A. Label Setting Algorithm

We use dynamic programming (DP) for the solving approach [21], which can be easily extended to handle constraints such as time windows [22], [23]. In the presence of time windows, *i.e.*, constraint (iii), the states of DP are not all explored. Our objective function is the shortest travel duration linking the set of points P. We consider a square $\mathcal{R}$ of a diameter equal to one hour duration, *i.e.* $a = 3600/\sqrt{2}$, and a time horizon set to ten hours, $h = 10 \times 3600$. The constant β^{tsp} is fixed to $\beta^{tsp} = 0.765$, according to Stein [17]. One hundred TSP-TW instances are randomly generated for each (n, m) value, where $n \in \{30, 32, ..., 58, 60\}$ and $m \in \{2, 3, 4, ..9, 10\}$. The uniform distribution is used for the random generation of both the temporal and the spatial demand distributions. Some combinations (n, m) are very difficult to solve to optimality, thus the corresponding results are not reported. They consist of instances where $m = 2$ and $n \geq 38$, and instances where $m = 3$ and $n \geq 52$. Figure 2 shows the average running time of a DP resolution for the (n, m) combinations. A low value of n or/and a high value of m induces a reduction of the number of states explored by DP. This is reflected in the average running times. The experiments are performed on an *Intel(R) Core(TM) i7-8750H CPU @ 2.20GHz* processor with 32 GB RAM memory machine.

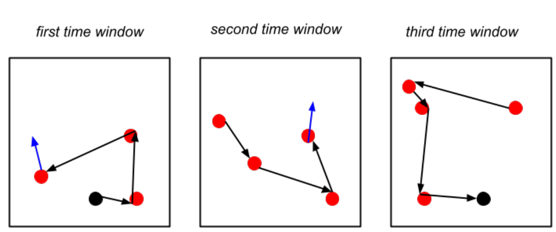

Fig. 1. A TSP-TW toy example with $n = 12$ and $m = 3$. The starting point s is designated by the black dot and the remaining points by red dots. The blue edges represent connections between the TSP open tours for the first $m - 1 = 2$ time windows.

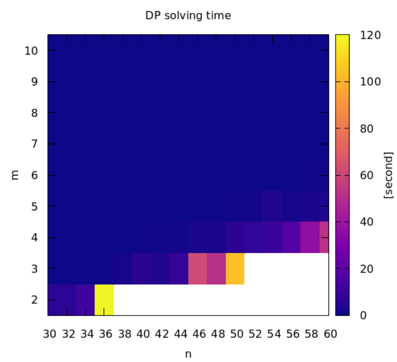

Fig. 2. The average running time to solve each combination of (n, m) instances, for both feasible and unfeasible cases. The white area corresponds to instances that cannot be solved within our CPU time limit.

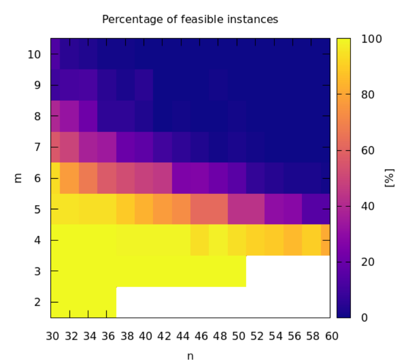

Fig. 3. The percentage of feasible instances for each combination (n, m). The white area corresponds to instances that are difficult to solve.

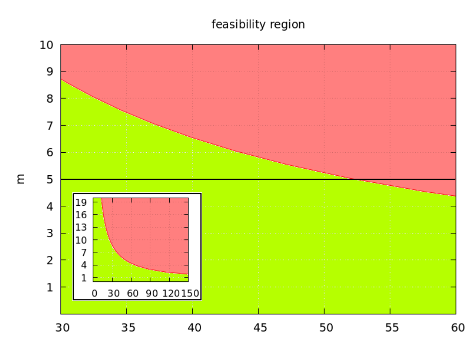

Fig. 4. The feasibility region in green delimited by the feasibility condition (11) in the plane (n, m) for square value and the horizon set in the experiment. The red area corresponds to the region of unfeasible instances.

B. Feasibility

Figure 3 shows the percentage of feasible instances. For example, for $m = 8$ the percentage is equal to 39%, 21% 6%, 0% when $n = 30, 34, 38, 42$, respectively. This suggests that instances generated for $m = 8$ and $n \geq 40$ lie outside the feasible region. By drawing the feasibility condition established in Equation (11) in Figure 4, we could indeed confirm this observation. Most instances with $6 \leq m \leq 10$ lie outside the feasibility region, which explains the results obtained in Figure 3. In the subsequent, we only consider feasible instances and, therefore, we restrict the number of time windows to $m = 2, 3, 4$ and 5.

C. Quality gap of the approximations

An important aspect of the proposed approximation is to assess the error term to the realized tours, which is a quality indicator of how close the approximations can be to the actual values under the assumption of uniform demand and *Euclidean* distances. More precisely, the quality gap for the tour duration of a given instance is equal to $|L^{tsptw}_{n,m} - L_{n,m}|/L_{n,m}$ where $L_{n,m}$ is the actual optimal tour duration for this instance. Similarly, we define the quality gaps for the total waiting time $W^{tsptw}_{n,m}$ and the arrival time $Z^{tsptw}_{n,m}$. Figure 5 shows the average quality gaps across instances of each (n, m), for the three approximated values. For $L^{tsptw}_{n,m}$, the gap varies from a maximum of 8.15% to a minimum of 2.95%, with a standard deviation of 1.04. Most average absolute gaps lie between the percentages 4% and 6%. For a fixed number of time windows, the average gaps tend to decrease for an increase of the number of points to visit n. For instance, the average gap for $m = 4$ is equal to $6.30, 4.11, 4.70,$ and 3.39% when $n = 30, 40, 50,$ and 60, respectively. Concerning waiting times $W^{tsptw}_{n,m}$, the average gaps have overall high percentages. The approximation $W^{tsptw}_{n,m}$ provides one value for each (n, m) couple, while we observe much variations in the real waiting times across instances, especially for a large values of n and m. For instance, the average gap for $m = 4$ is equal to $14.04, 17.20, 25.54,$ and 33.0% when $n = 30, 40, 50,$ and 60, respectively. This variation is due to the fact that the optimization criterion is solely based on tour duration and does not account for waiting times. The gaps of the arrival times $Z^{tsptw}_{n,m}$ are quite small. This is expected since variations are solely due to the tour length of last time windows, *i.e.* Equation (15). Absolute averages alone can be misleading. To understand the variations for a fixed value m, we draw the distribution of the gaps without the absolute value for $n = 30, 50$ and 60, and $m = 2, 3, 4$ and 5, in Figure 6. These boxplots show that the distribution of the gaps tends to narrow with an increase of the points to visit n centering around a value close to the zero gap. For $n = 60$, all the gaps are comprised in the interval $[-13, 10.7\%]$. We draw also Figure 7 to have a sense of the regions where the approximation over- and underestimate the actual tour lengths. The figure shows that the approximation tends to overestimate (resp. underestimate) the tour length for a large (small) number of time windows.

VI. CONCLUSIONS

We propose a continuous asymptotic approximation for the TSP with time windows, which up to our best knowledge has not been considered yet. The approximation concerns the value of the tour length, the total waiting time and the arrival time. Computational results on random TSP-TW instances on a square area show a low range of the gap of the approximation of travel times to the actual values, with only one digit for most cases. For a larger number of points the distribution of these gaps tend to even narrow. The proposed continuous approximation seems to be accurate under the assumption of non-overlapping time windows, *Euclidean* distances and a uniform distribution for the demand, in the same fashion of the Beardwood-Halton-Hemmersley formula

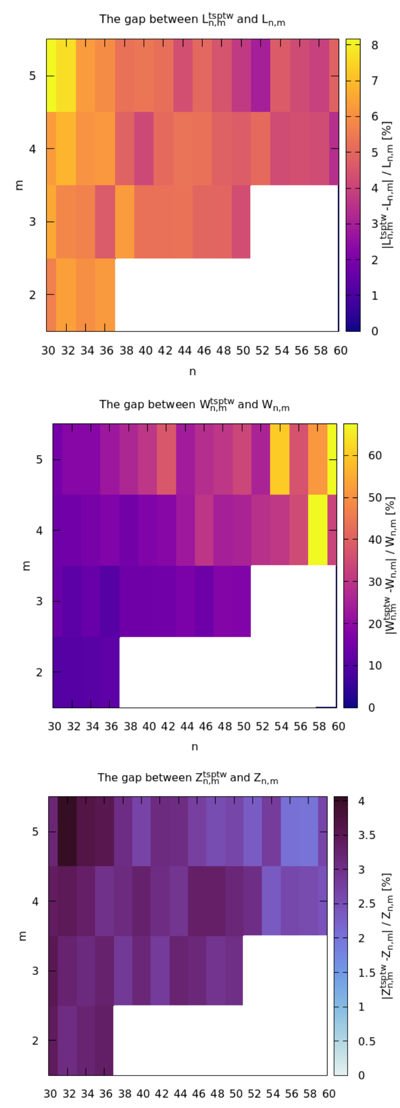

Fig. 5. The average quality gap between the approximations $L^{tsptw}_{n,m}$, $W^{tsptw}_{n,m}$ $Z^{tsptw}_{n,m}$ of Proposition 2 and the realized travel duration $L_{n,m}$, waiting time $W_{n,m}$, and arrival time $Z_{n,m}$ for feasible instances of (n, m) combinations. The white area corresponds to instances that are not solved within our CPU time limit.

for the TSP [18]. For future work, we intend to study in more details the theoretical and the empirical implications of relaxing the assumption of the uniform distribution of the customers' demand distributions, in terms of both space and time, which is the main limitation of our study.

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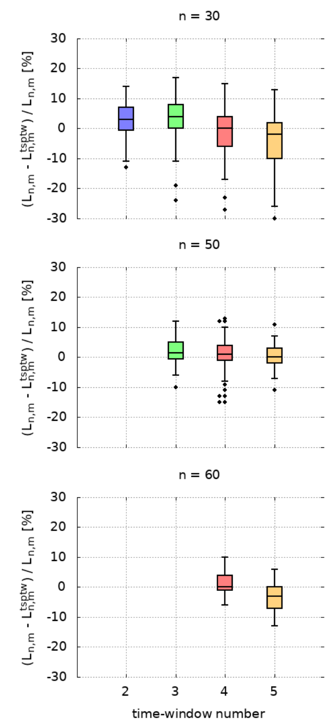

Fig. 6. The distribution of the gaps of $L_{n,m}^{tsptw}$ to the actual travel duration $L_{n,m}$, for $n = 30, 50$ and 60, and $m = 2, 3, 4$ and 5.

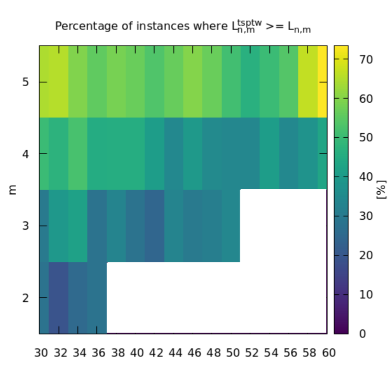

Fig. 7. The percentage of instances for which the approximation overestimates the actual tour length.

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