Generalized Cox Model for Default Times

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June 18, 2021

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1 Introduction

In this paper, we revisit the standard model of default time based on Cox process, introduced by Lando in [19], in which the default time is the first time when an increasing process adapted to a given filtration $\mathbb{F}$, absolutely continuous with respect to Lebesgue's measure, hits a level which is a positive random variable independent of the given filtration. It follows that this default time

*This research benefited from the support of the 'Chaire Marchés en Mutation', French Banking Federation and ILB, Labex ANR 11-LABX-0019.

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avoids all $\mathbb{F}$-stopping times. We relax the assumption that the increasing process that hits the threshold level is absolutely continuous and only assume that this process is adapted, increasing and continuous on right with limits on left (or continuous on left with limits on right). This leads us to a random time which does not avoid $\mathbb{F}$-stopping times. We compute the characteristics of this random time, i.e., its compensator, the associated Azéma supermartingales and the conditional distribution. We also study existence of conditional densities, in the sense of Jacod [13], Jiao & Li [17] and the extended Jacod's hypothesis introduced in Li & Rutkowski [21]. One of the advantages of our construction is that one can fix in advance the sequence of $\mathbb{F}$-stopping times not avoided by the random time. A first attempt to such a generalisation can be found in Bélanger et al. [3] where the increasing process is predictable and right-continuous. Some related works are those of Jiao & Li [17, 18], Gehmlich & Schmidt [9] and Fontana & Schmidt [8]. We have chosen, for ease of the reader, to give elementary proofs, instead of making use of general results based on dual predictable (resp. optional) projections of the default process, as it is done in Jeanblanc & Li [14].

In the first section, we introduce notation and basic notions. In the second section, we present our model. In the third section, we give many examples of our construction and we pay a particular attention to shot noise modeling. In the fourth section, we give closed form expression for the price of some defaultable claims and their dynamics.

2 Generalities

In this section, we recall well known facts about stochastic calculus and models of default times.

2.1 Facts on stochastic processes

In this first subsection, we recall, for the ease of the reader, some classical results, notation and definitions which will be used in the paper. We refer to [1, Chapter 1] for related proofs (or references for them) and more information.

In this subsection, we work on a probability space $(\Omega, \mathcal{G}, \mathbb{K}, \mathbb{P})$ endowed with a filtration $\mathbb{K}$, complete and continuous on right. We denote by $\mathcal{O}(\mathbb{K})$ (resp. $\mathcal{P}(\mathbb{K})$) the $\mathbb{K}$-optional (resp. predictable) σ-algebra on $\Omega \times \mathbb{R}_+$ and by $\mathcal{B}(\mathbb{R}_+)$ the Borelian sets of $\mathbb{R}_+ := \{x : x \geq 0\}$.

For a càdlàg process $X = (X_t)_{t\geq 0}$, we denote by $X_- = (X_{t-})_{t\geq 0}$ its left limit process and $\Delta X_t = X_t - X_{t-}, \forall t \geq 0$ its jump at time t (with $X_{0-} = 0$).

A process $K = (K_t)_{t\geq 0}$ is increasing (resp. decreasing) if, for $0 \leq s < t$, one has $K_s \leq K_t, a.s.$ (resp. $K_t \leq K_s, a.s.$). We set $K_{0-} = 0$. For an increasing (or decreasing) càdlàg process K, we note $K^c_t = K_t - \sum_{s\leq t} \Delta K_s, \forall t \geq 0$ its continuous part.

The Stieljes integral of a bounded càdlàg process φ with respect to a càdlàg increasing process K is, for $0 \leq s < t$, denoted $\int_s^t \varphi_u dK_u := \int_{]s,t]} \varphi_u dK_u$. Note that $\int_{]0,t]} dK_u = K_t - K_0$, $\forall t > 0$ and that $\int_{[0,t]} dK_u = K_t - K_0 + \Delta K_0 = K_t$, $\forall t \geq 0$.

We recall that any càdlàg $\mathbb{K}$-supermartingale[1] (resp. submartingale) Y admits a unique Doob-Meyer decomposition, i.e., $Y = M^Y - A^Y$ (resp. $Y = M^Y + A^Y$) where M^Y is a $\mathbb{K}$-martingale and A^Y an increasing $\mathbb{K}$-predictable process with $A^Y_0 = 0$. Any strictly positive càdlàg $\mathbb{K}$-supermartingale

[1] If the supermartingale is not càdlàg, one has to use its Doob-Meyer-Mertens-Gal'čuk decomposition, see, e.g., Th. 1.2 in [14]. We shall not enter in this kind of computations here and we refer the reader to [14].

Y admits a unique multiplicative decomposition as

$$Y = NC \tag{2.1}$$

where N is a $\mathbb{K}$-local martingale satisfying $N_0 = 1$ and C a decreasing $\mathbb{K}$-predictable process (see, e.g., [1, Pro. 1.32, Page 15]).
The $\mathbb{K}$-compensator of a $\mathbb{K}$-adapted process X with bounded variation is the $\mathbb{K}$-predictable process with bounded variation $X^{com,\mathbb{K}}$ such that $X - X^{com,\mathbb{K}}$ is a $\mathbb{K}$-martingale and $X_0^{com,\mathbb{K}} = 0$. If X is increasing (resp. decreasing), so is $X^{com,\mathbb{K}}$.

If $\mathbb{H}$ is a filtration satisfying $\mathbb{H} \subset \mathbb{K}$, and Y is a $\mathbb{K}$-adapted process such that $Y_\vartheta 1\!\!1_{\{\vartheta<\infty\}}$ is integrable for any $\mathbb{H}$-stopping time ϑ, the $\mathbb{H}$-*optional projection* of Y is the $\mathbb{H}$-optional process ${}^{o,\mathbb{H}}Y$ such that $\mathbb{E}[Y_\vartheta 1\!\!1_{\{\vartheta<\infty\}}|\mathcal{H}_\vartheta] = {}^{o,\mathbb{H}}Y_\vartheta\, 1\!\!1_{\{\vartheta<\infty\}}$, for any $\mathbb{H}$-predictable stopping time ϑ. This optional projection satisfies $\mathbb{E}[Y_t|\mathcal{H}_t] = {}^{o,\mathbb{H}}Y_t$, for all $t \geq 0$. If Y is a càdlàg $\mathbb{K}$-martingale, then ${}^{o,\mathbb{H}}Y$ is an $\mathbb{H}$-martingale.
Likewise, the $\mathbb{H}$-*predictable projection* of the process Y such that $Y_\vartheta 1\!\!1_{\{\vartheta<\infty\}}$ is integrable for any predictable $\mathbb{H}$-stopping time ϑ, is the unique $\mathbb{H}$-predictable process ${}^{p,\mathbb{H}}Y$ such that

$$\mathbb{E}[Y_\vartheta 1\!\!1_{\{\vartheta<\infty\}}|\mathcal{H}_{\vartheta-}] = {}^{p,\mathbb{H}}Y_\vartheta\, 1\!\!1_{\{\vartheta<\infty\}}\,,$$

for any $\mathbb{H}$-predictable stopping time ϑ.

A $\mathbb{K}$-stopping time ϑ is said to be $\mathbb{K}$-*predictable* if there exists an increasing sequence $(\vartheta_n)_{n\geq 1}$ of $\mathbb{K}$-stopping times converging to ϑ, such that $\vartheta_n < \vartheta$ on the set $\{\vartheta_n > 0\}$, for all $n \geq 1$. A $\mathbb{K}$-stopping time ϑ is *totally inaccessible* if it avoids $\mathbb{K}$-predictable stopping times (i.e., $\mathbb{P}(\vartheta = S < \infty) = 0$ for any $\mathbb{K}$-predictable stopping time S). A *random time* τ is a non-negative random variable, its *graph* is the subset $[\![\tau]\!]$ of $\mathbb{R}_+ \times \Omega$ defined as $[\![\tau]\!] = \{(t,\omega) : \tau(\omega) = t\}$.

For two filtrations $\mathbb{F}$ and $\mathbb{K}$ satisfying $\mathbb{F} \subset \mathbb{K}$, one says that $\mathbb{F}$ is *immersed* in $\mathbb{K}$ if any $\mathbb{F}$-martingale is a $\mathbb{K}$-martingale (see, e.g., Chapter 3 in [1] and the references therein).

2.2 Default times and characteristics

We work now on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a complete and continuous on right filtration, where $\mathcal{F}_0$ is trivial, and $\mathcal{G}$ a σ-algebra satisfying $\mathcal{F}_\infty \subset \mathcal{G}$. In what follows, the filtration $\mathbb{F}$ is always taken to be the reference filtration and in order to reduce notation, whenever there is no confusion, we will not explicitly write the dependence on the filtration $\mathbb{F}$ when writing the projections on $\mathbb{F}$ and referring to a predictable, optional or adapted process (e.g., X is a predictable process means X is an $\mathbb{F}$-predictable process).

We are given a random time τ defined on $(\Omega, \mathcal{G})$. The law of the random time τ is denoted by η, i.e., $\mathbb{E}[h(\tau)] = \int_{\mathbb{R}_+} h(u)\eta(du)$ for any bounded Borel function h defined on $\mathbb{R}^+$. We introduce the indicator default process of τ, denoted by A, as the right-continuous increasing process defined by $A_t = 1\!\!1_{\{\tau\leq t\}}, \forall t \geq 0$.

We denote by Z the càdlàg *Azéma supermartingale* associated with τ, which is the optional projection of $1 - A$ and satisfies

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t), \forall t \geq 0, \tag{2.2}$$

and the optional Azéma supermartingale $\widetilde{Z}$ which is the optional projection of $1 - A_-$, and satisfies

$$\widetilde{Z}_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t), \forall t \geq 0\,. \tag{2.3}$$

Note that $Z = \widetilde{Z}_+$ (see [1, page 20]) and $Z_t > 0$ on the set $\{t < \tau\}$ and $Z_{t-} > 0$ on the set $\{t \leq \tau\}$ (see [1, Lemma 1.51]). The Doob-Meyer decomposition of Z is $Z = M - A^p$, where M is a martingale and A^p an increasing predictable process[2]. We recall that the random time τ is said to *avoid* all $\mathbb{F}$-stopping times (resp. all predictable $\mathbb{F}$-stopping times) if $\mathbb{P}(\tau = \vartheta < \infty) = 0$ for any $\mathbb{F}$-stopping time (resp. for any predictable $\mathbb{F}$-stopping time) ϑ .

The filtration $\mathbb{G}$ is the smallest complete and right-continuous filtration that contains $\mathbb{F}$ and turns τ into a $\mathbb{G}$-stopping time. The *compensator* of A (we shall also say compensator of τ) is the $\mathbb{G}$-predictable increasing process $\Lambda^{\mathbb{G}}$ such that $A - \Lambda^{\mathbb{G}}$ is a $\mathbb{G}$-martingale (in fact, $\Lambda^{\mathbb{G}} = A^{com,\mathbb{G}}$). It is well known that there exists an $\mathbb{F}$-predictable increasing process Λ such that $\Lambda^{\mathbb{G}}_t = \Lambda_{t\wedge\tau}, \forall t \geq 0$ (see e.g., Pro. 2.11 b), Page 36 in [1]) and that $\Lambda^{\mathbb{G}}_t 1\!\!1_{\{t\leq\tau\}} = 1\!\!1_{\{t\leq\tau\}} \int_0^t \frac{dA^p_s}{Z_{s-}}$ (see, e.g., Pro. 2.15, page 37 in [1]). The process Λ is not uniquely defined after τ (except if $Z_- > 0$) and, hereafter, we choose

$$d\Lambda_t = \frac{dA^p_t}{Z_{t-}} 1\!\!1_{\{Z_{t-}>0\}},\ \forall t \geq 0,\ \Lambda_0 = 0\,. \tag{2.4}$$

We shall call Λ the $\mathbb{F}$-*predictable reduction of the compensator* of τ.

The $\mathbb{F}$-*conditional cumulative function* of τ is defined, for any $(t, u) \in \mathbb{R}^2_+$ by

$$F_t(u) = \mathbb{P}(\tau \leq u|\mathcal{F}_t)\,. \tag{2.5}$$

The family $(F(u), u \in \mathbb{R}_+)$ is a family of $\mathbb{F}$-martingales, valued in $[0, 1]$, increasing w.r.t. the parameter u (i.e., $F_t(u) \leq F_t(v), a.s.$ for $u < v, \forall t \geq 0$). The family of processes $(G(u), u \in \mathbb{R}_+)$ defined as $G(u) = 1 - F(u)$ is the family of conditional survival processes.

2.3 Conditional densities

In this subsection, we recall some definitions on conditional densities. Later on, we shall examine if, in our model, these conditional densities exist.

Definition 2.1 *The random time τ admits*

- *a conditional density in the sense of Jacod [13, Condition (A)](we shall say a J-conditional density) if there exist a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $(\omega, t, u) \to p_t(\omega, u)$ càdlàg in t and a non-negative σ-finite measure ρ on $\mathbb{R}_+$ such that*
  (J1) for every u, the process $(p_t(u))_{t\geq 0}$ is a non-negative $\mathbb{F}$-martingale,
  (J2) for every $t \geq 0$, the measure $p_t(u)\rho(du)$ equals $\mathbb{P}(\tau \in du\,|\,\mathcal{F}_t)$, in other words, for any Borel bounded function h, for any $t \geq 0$

  $$\mathbb{E}[h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)p_t(u)\rho(du)\,.$$

  In that case $A^p_t = \int_{[0,t]} p_{u-}(u)\rho(du)$, for any $t \geq 0$ (see [1, Cor. 5.27]).
  Note that $\int_{\mathbb{R}_+} p_t(u)\rho(du) = 1$, $\forall t \geq 0$, a.s.

- *a generalized density in the sense of Jiao & Li [17] (we shall say a JL-conditional density) if there exist a family $(\tau_i, i = 1, \cdots, n)$ of $\mathbb{F}$-stopping times and a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $(\omega, t, u) \to \alpha_t(\omega, u)$ càdlàg in t such that*

[2] In the literature, the predictable part of the Doob-Meyer decomposition of Z is shown to be the $\mathbb{F}$-dual predictable projection of A, this is why we keep the notation A^p. The process $\widetilde{Z}$ is not càdlàg, hence one can not define as usual its Doob-Meyer decomposition.

(JL1) for every u, the process $(\alpha_t(u))_{t\geq 0}$ is a non-negative $\mathbb{F}$-martingale,
(JL2) for every $t \geq 0$, for any bounded Borel function h

$$\mathbb{E}[h(\tau)\prod_{i=1}^{n}\mathbb{1}_{\{\tau\neq\tau_i\}}|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\pi(du)$$

where π is a non-negative, non atomic measure on $\mathbb{R}_+$.

Definition 2.2 *The random time τ satisfies the extended Jacod's hypothesis (introduced by Li & Rutkowski in [21, Pro. 2.5], we shall say LR-condition) if there exist a non-negative $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable function $(\omega,t,u)\to m_t(\omega,u)$ càdlàg in t and an $\mathbb{F}$-adapted increasing process D such that*
(LR1) for every u, the process $(m_t(u))_{t\geq u}$ is a non-negative $\mathbb{F}$-martingale,
(LR2) for $u \leq t$

$$F_t(u) = \int_{[0,u]} m_t(s)dD_s\,.$$

Comments 2.3 a) If the J-conditional density exists, one can always choose $\rho = \eta$ where η is the law of τ (see Jacod [13, Pro. 1.5]). Moreover, if η is non atomic, τ avoids all $\mathbb{F}$-stopping times [7, Cor. 2.2]. If η has an atom at t^*, then $\mathbb{P}(\tau = t^*) > 0$ and the constant stopping time t^* is not avoided by τ.
b) If τ avoids all $\mathbb{F}$-stopping times, the existence of a JL-conditional density is equivalent to the existence of a J-conditional density, and one can chose $\pi = \eta$.
c) If the J-conditional density exists, then LR-condition holds. We shall see that the LR-condition may hold when J and JL-conditional densities do not exist.
d) Under LR-condition, if D is predictable, $A^p = \int_0^{\cdot} {}^p m_u(u)dD_u$, where ${}^p m$ is the $\mathbb{F}$-predictable projection of m (see [21, Pro. 5.5.1]).
e) Note that, under LR-condition, the conditional cumulative distribution $F_t(u)$ given for $u \leq t$ in (LR2) can be obtained for any pair (t,u). To do that, we use the fact that for any u, the process $F(u)$ is a martingale. Hence, we set $F_t(u) = \mathbb{E}[F_u(u)|\mathcal{F}_t]$ for $t < u$. To check that indeed, $F_t(u)$ is increasing w.r.t. u for any t, we note that the martingale property of $F(u)$ implies $F_t(u) = \mathbb{E}[F_s(u)|\mathcal{F}_t]$ for any $s > t$ so that $F_s(u)$ being increasing w.r.t. u for $u < s$ leads to $F_t(u) = \mathbb{E}[F_s(u)|\mathcal{F}_t] \leq \mathbb{E}[F_s(v)|\mathcal{F}_t] = F_t(v)$ for $u < v$.

Proposition 2.4 *If $\mathbb{F}$ is immersed in $\mathbb{G}$, then LR-condition holds.*

PROOF: We recall that, if $\mathbb{F}$ is immersed in $\mathbb{G}$, then Z is a decreasing process. Under immersion, for $u \leq t$, one has $F_t(u) = 1 - Z_u$ and Z is decreasing. Hence, LR condition holds, with $D = 1 - Z, D_{0-} = 0$ and $m(s) \equiv 1$, since $\int_{[0,u]} dD_s = D_u = 1 - Z_u, \forall u \in \mathbb{R}_+$. □

3 Generalized Cox model

We now assume that the σ-algebra $\mathcal{G}$ is large enough to support a random variable Θ with unit exponential law, independent from $\mathcal{F}_\infty$. We study the generalized Cox model where K is an increasing $\mathbb{F}$-adapted process such that $K_0 = 0$. More precisely, if K is càdlàg, one defines a random time τ (called a generalized Cox time hereafter) as

$$\tau = \inf\{t \geq 0\,:\, K_t \geq \Theta\} \tag{3.1}$$

and, if K is càglàd, we modify the definition (we shall see why later), setting

$$\tau = \inf\{t \geq 0\,:\, K_t > \Theta\}\,.$$

Note that τ is finite if and only if $K_{\infty-} = \infty$.

In both cases, immersion holds between the reference filtration $\mathbb{F}$ and $\mathbb{G}$, its progressive enlargement with τ, since, obviously $\mathbb{F}$ is immersed in $\mathbb{F} \vee \sigma(\Theta)$ and $\mathbb{F} \subset \mathbb{G} \subset \mathbb{F} \vee \sigma(\Theta)$. The conditional survival process is

$$\begin{aligned} \mathbb{P}(\tau > u|\mathcal{F}_t) &= Z_u, \quad \text{for}\, u < t \\ &= \mathbb{E}[Z_u|\mathcal{F}_t], \quad \text{for}\, t \le u\,. \end{aligned}$$

In a generalized Cox model, we shall call the quadruplet $Z, \widetilde{Z}$ (defined in (2.2), (2.3)), A^p and $F(u)$ (defined in (2.5)) the *characteristics* of τ (see [14] for the general definition of the characteristics of a default time).

We recall that, for any t and any $\mathcal{F}_t$-measurable r.v. ζ_t, one has

$$\mathbb{P}(\zeta_t > \Theta|\mathcal{F}_t) = \mathbb{P}(\zeta_t \ge \Theta|\mathcal{F}_t) = e^{-\zeta_t}\,.$$

3.1 Case where K is continuous

Let K be an increasing $\mathbb{F}$-adapted continuous process, with $K_0 = 0$. We define $\tau := \inf\{t\,:\, K_t \ge \Theta\}$.

We obtain immediately that $Z = e^{-K}$. Furthermore, as a particular case of the following Lemma 3.1, τ avoids all $\mathbb{F}$-stopping times, and, in particular $\widetilde{Z} = Z$ (for the $\mathbb{F}$-stopping time $\vartheta = t$, one has $\mathbb{P}(\tau = t|\mathcal{F}_t) = 0$). The Doob-Meyer decomposition of Z is $Z = 1 - (1 - e^{-K})$, hence $A^p = 1 - Z$. From (2.4), the $\mathbb{F}$-predictable reduction of the compensator of τ is $\Lambda = K$, and the multiplicative decomposition (see (2.1)) of Z is $Z = e^{-K} = e^{\Lambda}$ (with a local martingale part equal to 1). The law of τ is given by $\mathbb{P}(\tau > u) = \mathbb{E}[e^{-K_u}]$, so that $\mathbb{E}[h(\tau)] = \mathbb{E}\big[\int_{\mathbb{R}_+} h(u) e^{-K_u} dK_u\big]$.

Moreover, if K is absolutely continuous w.r.t. Lebesgue's measure, i.e., $K_t = \int_0^t k_u du$, $\forall t \ge 0$, with k a non-negative process, then, τ admits a density f w.r.t. Lebesgue's measure satisfying $f(u) = \mathbb{E}[k_u e^{-K_u}]$, $\forall u \ge 0$ and a J-conditional density given by (choosing $\rho(du) = du$):

$$\begin{aligned} p_t(u) &= k_u e^{-K_u}, \quad \text{for}\, u < t, \\ &= \mathbb{E}[k_u e^{-K_u}|\mathcal{F}_t], \quad \text{for}\, t \le u\,. \end{aligned}$$

If K is continuous but not absolutely continuous, the J-density may fail to exist, as we show now. Let K be the continuous increasing process defined by $K_t = -\ln(1 - L_{t\wedge 1}) + 1\!\!1_{\{t>1\}}(t-1)$, $t \ge 0$, where L is the local time at level 0 of a standard Brownian motion. Then $\mathbb{P}(\tau > t) = \mathbb{E}[1 - L_{t\wedge 1}]e^{(t-1)^+}$ and, from $\mathbb{E}[L_t] = \mathbb{E}[|W_t|] = \frac{\sqrt{2t}}{\sqrt{\pi}}$, we deduce that τ has a density f w.r.t. Lebesgue's measure. Therefore, if the J-conditional hypothesis is satisfied then, one would have, for $u < t$

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = Z_u = \int_u^\infty p_t(s) f(s) ds$$

and Z would be absolutely continuous w.r.t. Lebesgue's measure, which is not the case.

3.2 Case where K is càdlàg

Let K be an increasing $\mathbb{F}$-adapted càdlàg process, with $K_0 = 0$, $K_t < \infty$ for all $t \ge 0$ and[3] $K_\infty = \infty$. We do not assume that K is predictable (a basic example is a Poisson process).

We define

$$\tau = \inf\{t\,:\, K_t \ge \Theta\}\,. \tag{3.2}$$

[3]If $\mathbb{P}(K_\infty < \infty) > 0$, the r.v. τ takes the value $+\infty$ with strictly positive probability. If there exists ϑ such that $\mathbb{P}(K_\vartheta = \infty) > 0$, then $\tau \le \vartheta$ on $\{K_\vartheta = \infty\}$.

Lemma 3.1 *For ω fixed, the set $\{t : K_t \geq \Theta\}$ is of the form $[t_0, \infty[$ with $K_{t_0} \geq \Theta$ and $\tau = t_0$, hence*

$$\{K_t < \Theta\} = \{\tau > t\}.$$

For any finite $\mathbb{F}$-stopping time ϑ, one has $\mathbb{P}(\tau = \vartheta | \mathcal{F}_t) = \mathbb{E}[e^{-K_{\vartheta -}} - e^{-K_\vartheta} | \mathcal{F}_t]$. In particular, $\mathbb{P}(\tau = \vartheta) = \mathbb{E}[e^{-K_{\vartheta -}} - e^{-K_\vartheta}]$, hence, τ avoids all $\mathbb{F}$-stopping times if and only if K is continuous. In particular, if K has jumps and η is non-atomic, the J-conditional density does not exist.

Proof: Due to the increasing property of K, for any ω fixed, the set $\{t : K_t \geq \Theta\}$ is of the form $[t_0, \infty[$ or $]t_0, \infty[$ (where t_0 depends on ω). In the case $\{t : K_t \geq \Theta\} =]t_0, \infty[$, one has $K_{t_0} < \Theta$ and $K_{t_0 + \epsilon} \geq \Theta$ for any $\epsilon > 0$. The right-continuity of K yields to $K_{t_0} \geq \Theta$, which is a contradiction. Hence $\{t : K_t \geq \Theta\} = [t_0, \infty[$ and $K_{t_0} \geq \Theta$ and $\tau = t_0$.
It follows that $\{s < \tau \leq t\} = \{K_s < \Theta \leq K_t\}$, and, for a finite $\mathbb{F}$-stopping time ϑ, one obtains, denoting $\vartheta(\epsilon) = (\vartheta - \epsilon) \vee 0)$, that $\{\vartheta(\epsilon) < \tau \leq \vartheta\} = \{K_{\vartheta(\epsilon)} < \Theta \leq K_\vartheta\}$ and

$$\mathbb{P}(\vartheta(\epsilon) < \tau \leq \vartheta | \mathcal{F}_t) = \mathbb{E}[e^{-K_{\vartheta(\epsilon)}} - e^{-K_\vartheta} | \mathcal{F}_t] \, .$$

Passing to the limit when ϵ goes to 0 leads to the result. It K is continuous, then, τ avoids any $\mathbb{F}$-stopping time. If τ avoids $\mathbb{F}$-stopping times, then $\mathbb{E}[e^{-K_{\vartheta -}} - e^{-K_\vartheta}] = 0$ for any ϑ which, due to the increasing property of K implies that K is continuous. If η has no atoms, then the J-conditional density does not exist if K has jumps see Comments 2.3 a). □

3.2.1 Doob-Meyer decomposition of Z

From Lemma 3.1

$$Z_t := \mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(K_t < \Theta | \mathcal{F}_t) = e^{-K_t} \, ,$$

so that, in particular, under our assumption $K_t < \infty, \forall t \geq 0$, one has $Z_t > 0, \forall t \geq 0$.

Lemma 3.2 *Let I be the càdlàg $\mathbb{F}$-submartingale $I_t = \sum_{s \leq t} (1 - e^{-\Delta K_s})$ which admits a Doob-Meyer decomposition $I = M^I + A^I$. Then, Z admits a DM decomposition $Z = M - A^p$ where $dM_t = e^{-K_{t-}} dM^I_t$ and*

$$dA^p_t = e^{-K_{t-}} (dK^c_t + dA^I_t), \; A^p_0 = 0 \, ,$$

where K^c is the continuous part of the increasing process K. Furthermore, the $\mathbb{F}$-predictable reduction of the compensator of τ is $\Lambda = K^c + A^I$.

Proof: From $Z = e^{-K}$, one has, using Stieljes' integration

$$dZ_t \;\; = \;\; -e^{-K_{t-}} dK^c_t + e^{-K_{t-}} (e^{-\Delta K_t} - 1), \forall t \geq 0 \, ,$$

where K^c is the continuous part of K, i.e., $K^c_t = K_t - \sum_{s \leq t} \Delta K_s, \forall t \geq 0$.
The process K^c being increasing and continuous is a submartingale with Doob-Meyer decomposition with no martingale part, i.e., $K^c = 0 + K^c$. The process I being increasing (indeed $1 - e^{-\Delta K} \geq 0$) is a submartingale and admits a Doob-Meyer decomposition $I = M^I + A^I$. Finally

$$\begin{aligned} dZ_t &= -e^{-K_{t-}} dK^c_t - e^{-K_{t-}} dI_t && (3.3) \\ &= -e^{-K_{t-}} (dK^c_t + dM^I_t + dA^I_t) \\ &= -e^{-K_{t-}} dM^I_t - e^{-K_{t-}} (dK^c_t + dA^I_t) \, . && (3.4) \end{aligned}$$

Therefore

$$dA^p_t = e^{-K_{t-}} (dK^c_t + dA^I_t) \qquad (3.5)$$

and $(A_t - \Lambda_{t\wedge\tau})_{t\geq 0}$ is a $\mathbb{G}$-martingale where, from (2.4) and the fact that $Z_- > 0$,

$$d\Lambda_t = \frac{1}{Z_{t-}} dA^p_t = dK^c_t + dA^I_t\,.$$

The martingale part in the Doob-Meyer decomposition of Z is the process M given by $dM_t = -Z_{t-}dM^I_t$, $M_0 = Z_0$ (which is a true martingale due to the fact that Z is bounded). $\square$

Remark 3.3 Let us go back to a general setting where τ is such that immersion holds between $\mathbb{F}$ and $\mathbb{G}$ where $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ with τ (here, τ is not necessarily obtained by a generalized Cox model). Then, the càdlàg Azéma supermartingale Z is decreasing, and there exists K, an $\mathbb{F}$-adapted càdlàg increasing process such that $Z = e^{-K}$. Defining ϑ from K as in (3.2) leads to a random time with Azéma supermartingale equal to Z. Furthermore

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{E}[Z_u|\mathcal{F}_t] = \mathbb{P}(\vartheta > u|\mathcal{F}_t)$$

for all pairs (t, u) (due to immersion). Of course, the fact that τ and ϑ have the same characteristics does not imply that $\mathbb{P}(\tau = \vartheta) = 1$, as explained in the trivial following example. Let $\tau = \inf\{t, K_t \geq \Sigma\}$ where K is an $\mathbb{F}$-adapted increasing continuous process and Σ a unit exponential random variable independent from $\mathbb{F}$ and $\vartheta = \inf\{t, K_t \geq \Theta\}$ defined as in (3.2). If $\tau = \vartheta$, one should have $K_\tau = K_\vartheta$, hence $\Sigma = \Theta$ which is not requested.

Comment 3.4 If K^c and $A^{I,c}$ are absolutely continuous w.r.t. Lebesgue's measure, and if the sequence $(\tau_i, i \geq 1)$ of jumps times of K is increasing, writing $\sum_{s\leq t} \Delta A^I_s = \int_0^t \int_{\mathbb{R}_+} x\delta_{(\tau_i,\theta_i)}(ds, dx)$ where δ is the Dirac measure and $\theta_i = \Delta A^I_{\tau_i}$, we recover the form of the $\mathbb{F}$-predictable reduction of the compensator of τ presented in [9, eq. (7)].

Proposition 3.5 *The optional Azéma supermartingale is* $\widetilde{Z} = e^{-K_-}$.

PROOF: As before, we note $t(\epsilon) = t \vee \epsilon$. From $\{t(\epsilon) < \tau \leq t\} = \{K_{t(\epsilon)} < \Theta \leq K_t\}$, we obtain, letting ϵ go to 0 $\mathbb{P}(\tau = t|\mathcal{F}_t) = e^{-K_{t(\epsilon)}} - e^{-K_t}$ and $\widetilde{Z}_t = Z_t + \mathbb{P}(\tau = t|\mathcal{F}_t) = e^{-K_{t-}}$. The result is also a consequence of [1, Pro.3.9a and Pro.1.46c]. The fact that $\widetilde{Z}$ is predictable can also be obtained by immersion (see [1, Th. 5.35 f]). $\square$

3.2.2 Multiplicative decomposition

Lemma 3.6 *The multiplicative decomposition of Z is*[4]

$$\mathcal{E}(Y)_t\mathcal{E}(-\Lambda)_t = \mathcal{E}(Y)_t e^{-\Lambda_t} \prod_{s\leq t}(1 - \Delta A^I_s)e^{\Delta A^I_s}, \forall t \geq 0\,,$$

where $Y_t = \displaystyle\int_0^t \frac{1}{Z_{s-}(1 - \Delta A^I_s)} dM_s, \forall t \geq 0$, *with M being the martingale part of Z in its Doob-Meyer decomposition and A^I, Λ being given in Lemma 3.2.*

[4]We recall that the Doléans-Dade exponential of a càdlàg semimartingale X is

$$\mathcal{E}(X)_t = e^{X_t - \frac{1}{2}\langle X^{(c)}, X^{(c)}\rangle_t} \prod_{0<s\leq t}(1 + \Delta X_s)e^{-\Delta X_s}\,,$$

where $X^{(c)}$ is the continuous martingale part of X (see, e.g., [1, Page 8 and 14].

PROOF: This result is established in full generality in [14], where the authors "guess" this form and check it. However, it seems interesting to recover that result without guessing this decomposition, and we present a proof. One knows that there exists a predictable increasing process Γ, so that, from the last equality above, $Z = Ne^{-\Gamma}$ where N is a local martingale. The integration by parts formula using Yoeurp's lemma [1, Pro. 1.16] leads to

$$dZ_t \quad = \quad e^{-\Gamma_t} dN_t + N_{t-} de^{-\Gamma_t} \,.$$

By identifying this decomposition of Z and the DM decomposition of Z given in (3.4), one gets

$$e^{-\Gamma} dN = dM, \quad N_- de^{-\Gamma} = -Z_-(dK^c + dA^I)\,,$$

hence $d\Gamma^c + 1 - e^{-\Delta\Gamma} = dK^c + dA^I$ which leads to

$$d\Gamma^c = dK^c + dA^{I,c} \text{ and } 1 - e^{-\Delta\Gamma} = \Delta A^I \,.$$

Finally, using the fact that $\Lambda = K^c + A^I = K^c + A^{I,c} + \sum_{s\le \cdot} \Delta A^I_s$,

$$e^{-\Gamma_t} = e^{-\Gamma^c_t} e^{-\sum_{s\le t}\Delta\Gamma_s} = e^{-\Lambda_t} \prod_{s\le t} (1 - \Delta A^I_s) e^{\Delta A^I_s}, \forall t \ge 0\,. \tag{3.6}$$

The equality $dN = e^{\Gamma} dM$ can be written as $dN = N_- Y dM$ with $Y = \frac{e^{\Gamma}}{N_-} = \frac{e^{\Delta\Gamma}}{Z_-} = \frac{1}{Z_-(1-\Delta A^I)}$, hence, using the fact that $\Lambda = K^c + A^I = K^c + A^{I,c} + \sum_{s\le\cdot} \Delta A^I_s = \Gamma^c + \sum_{s\le\cdot}\Delta A^I_s$, the proof is done. $\square$

Comment 3.7 Note that, from (3.6) and the fact that Z is non-negative, one has $1 - \Delta A^I \ge 1$. Another proof can be done noting that $\Delta A^I = {}^p(\Delta I) = {}^p(1 - e^{-\Delta K}) = 1 - {}^p(e^{-\Delta K}) \le 1$, where the first equality comes from [1, Pro. 1.36 b]. Note that this implies that $\Delta\Lambda \le 1$.
One can also check that $1 + \Delta N > 0$. Indeed,

$$\Delta N = -e^{-(K_- + \Gamma)} \Delta M^I = -e^{-(K_- + \Gamma)} (1 - e^{-\Delta K} - \Delta A^I) > -1\,.$$

3.2.3 Conditional densities

Proposition 3.8 *If the continuous part of K is absolutely continuous w.r.t. Lebesgue's measure, the JL-conditional density exists with $\alpha_t(u) = \mathbb{E}[k_u e^{-K_u} | \mathcal{F}_t]$ and π is the Lebesgue measure.*

PROOF: Let $(\tau_i)_{i\ge 1}$ be the sequence of jump times of K. For any bounded Borel function h, the process

$$X_t = h(t) \prod_{i\ge 1} 1\!\!1_{\{t \ne \tau_i\}} = h(t) \prod_{i\ge 1} (1\!\!1_{\{t<\tau_i\}} + 1\!\!1_{\{t>\tau_i\}}) \tag{3.7}$$

is $\mathbb{F}$-optional (indeed, $1\!\!1_{\{t<\tau_i\}}$ is càd and $1\!\!1_{\{t>\tau_i\}}$ càg hence predictable therefore optional). Hence

$$\begin{aligned}
\mathbb{E}[h(\tau) \prod_{i\ge 1} 1\!\!1_{\{\tau\ne\tau_i\}} | \mathcal{F}_t] &= \mathbb{E}[X_\tau | \mathcal{F}_t] = \mathbb{E}[\int_0^\infty X_s d\widetilde{Z}_s | \mathcal{F}_t] = -\mathbb{E}[\int_0^\infty X_s dZ^c_s | \mathcal{F}_t] \\
&= \mathbb{E}[\int_0^\infty h(s) k_s e^{-K_s} ds | \mathcal{F}_t] = \int_0^\infty h(s) \mathbb{E}[k_s e^{-K_s} | \mathcal{F}_t] ds\,.
\end{aligned}$$

The second equality is due to (4.1) (We shall present this well known result later in section 4), the third equality is due to the fact that X vanishes on the discontinuities of $\widetilde{Z}$ (which are the discontinuities of Z, i.e., of K, since $\widetilde{Z}_t = e^{-K_{t-}}$), so that $\int_0^\infty X_s d\widetilde{Z}_s = \int_0^\infty X_s dZ^c_s$, then we use the fact that, from (3.3), $Z^c_t = 1 - \int_0^t e^{-K_{s-}} dK^c_s = 1 - \int_0^t e^{-K_s} dK^c_s = 1 - \int_0^t e^{-K_s} k_s ds$. $\square$

Remark 3.9 In the case where $K^c = 0$, one has $[\![\tau]\!] \subset \cup_i [\![\tau_i]\!]$, and τ is a thin time (see, e.g., definition 1.40, Page 18 in [1]). In that case, the JL density is null. This can be viewed as a consequence of the previous proposition, but this follows directly from the definition. Indeed, since $\sum_{i\geq 1} \mathbb{P}(\tau = \tau_i) = 1$, one has

$$\mathbb{E}[h(\tau)\prod_{i\geq 1} 1\!\!1_{\{\tau\neq\tau_i\}}|\mathcal{F}_t] = 0$$

hence $\alpha(u) \equiv 0$.

3.2.4 Other computations

From Lemma 3.1, if ϑ is an $\mathbb{F}$-stopping time not avoided by τ, then $\mathbb{P}(\Delta K_\vartheta > 0) > 0$ and K has a jump at time ϑ. Let $(\tau_i)_{i\geq 1}$ be the sequence of jump times of K. The conditional probability that the default occurs at time τ_i is

$$p_t^i := \mathbb{P}(\tau = \tau_i|\mathcal{F}_t) \quad = \quad \mathbb{E}\left[e^{-K_{\tau_i -}}(1 - e^{-\Delta K_{\tau_i}})|\mathcal{F}_t\right], \forall t \geq 0\,.$$

Note that $p_t^i = p_{t\wedge\tau_i}^i$, a result due to immersion pointed out in [17, Pro. 5.1]. This implies that $p_{t\vee\tau_i}^i = p_{\tau_i}^i$. The following result is an immediate consequence of Proposition 2.7 in [17].

Proposition 3.10 *For a bounded Borel function h, if the JL conditional density exists,*

$$\mathbb{E}[h(\tau)|\mathcal{F}_t] = \int_0^\infty h(u)\alpha_t(u)\pi(du) + \sum_{i\geq 1}\mathbb{E}[h(\tau_i)p_{\tau_i}^i|\mathcal{F}_t], \forall t \geq 0\,.$$

The conditional survival probability is given by

$$\begin{aligned}\mathbb{P}(\tau > u|\mathcal{F}_t) &= e^{-K_u}, \quad \text{for } t \geq u \\ &= \int_u^\infty \alpha_t(y)\pi(dy) + \sum_{i\geq 1}\mathbb{E}[1\!\!1_{\{\tau_i > u\}}p_{\tau_i}^i|\mathcal{F}_t], \quad \text{for } t < u\end{aligned}$$

3.2.5 Particular case: K predictable

In the case where K is càdlàg and predictable, the set of jump times of K, which are all predictable, exhaust the sequence of $\mathbb{F}$-stopping times not avoided by τ. The same model was presented in [3], in a slightly more general setting where Θ is not a unit exponential r.v. (but is still independent of $\mathbb{F}$). Let

$$K_t = \int_0^t k_s ds + \sum_{i\geq 1} 1\!\!1_{\{\tau_i\leq t\}}\theta_i \tag{3.8}$$

where $(\tau_i)_{i\geq 1}$ is an increasing sequence of predictable stopping times, k is an $\mathbb{F}$-adapted non-negative process and $(\theta_i)_{i\geq 1}$ a sequence of non-negative random variables with $\theta_i \in \mathcal{F}_{\tau_i -}$. Note that I (defined in Lemma 3.2) is predictable, hence $A^I = I = \sum_{i\geq 1} 1\!\!1_{\{\tau_i\leq \cdot\}}(1 - e^{-\theta_i})$ and $M^I = 0$. As a check, one can see, from the previous computations (3.5) with $dA_t^I = dI_t$, that $dA_t^p = -dZ_t$.
Then, $\Lambda_t = \int_0^t k_s ds + \sum_{i\geq 1} 1\!\!1_{\{\tau_i\leq t\}}(1 - e^{-\theta_i})$. The multiplicative decomposition obtained in Lemma 3.6 is the one obtained in [3, equation 2.4], i.e., $Z_t = \exp(-\Lambda_t^c)\prod_{s\leq t}(1 - \Delta\Lambda_s)$.

3.3 Examples of càdlàg processes K

3.3.1 Brownian filtration

We construct a simple example where K is càdlàg (but not continuous) and predictable. Let $\mathbb{F}$ be a Brownian filtration, $(\tau_i)_{i\geq 1}$ an increasing sequence of finite stopping times (e.g., $\tau_i = \inf\{t \,:\, S_t = b_i\}$

where S is an $\mathbb{F}$-diffusion going to $+\infty$ when t goes to $+\infty$ and $(b_i)_{i\geq 1}$ an increasing sequence of real numbers with $b_1 > S_0$) and define $K_t = \int_0^t k_s ds + \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}}\theta_i$ where $(\theta_i)_{i\geq 1}$ is a sequence of non-negative random variables, with $\theta_i \in \mathcal{F}_{\tau_i}$, i.e., $\theta_i = c + \int_0^{\tau_i} \psi_s^{(i)} dW_s$ for an $\mathbb{F}$-optional process $\psi^{(i)}$ (such that $\theta_i \geq 0$). In that case, results of Subsection 3.2.5 can be applied, and, under the condition $K_\infty = \infty$, the random time τ is finite.
Furthermore, predictable representation property holds in $\mathbb{G}$ with respect to $(W, M^{\mathbb{G}})$, where $M^{\mathbb{G}} = A - \Lambda_{\cdot\wedge\tau}$ (See [16, Rem. 6.1]).

3.3.2 Jiao & Li Model

We can recover the model of Jiao & Li [18].

Proposition 3.11 *We consider $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ a filtered probability space, Γ a continuous increasing $\mathbb{F}$-adapted process and $(\tau_i)_{i\geq 1}$ a strictly increasing sequence of $\mathbb{F}$-stopping times and we set $\tau_0 = 0$. We introduce the process A^i setting $A^i_t = 1\!\!1_{\{\tau_i \leq t\}}, \forall t \geq 0$ and its $\mathbb{F}$-compensator J^i. One denotes by Ψ an increasing deterministic function such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. We set*

$$K_t = \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}}[\Psi(\tau_i) - \Psi(\tau_{i-1})] + \Gamma_t, \forall t \geq 0\,.$$

Then,

$$A^p_t = \Gamma_t + \sum_{i\geq 1} \int_0^t \left(e^{-\Psi(\tau_{i-1})} - e^{-\Psi(s)}\right) dJ^i_s,\ \forall t \geq 0\,.$$

Proof: Note that K is càdlàg, with continuous part $K^c = \Gamma$ and has jumps at time τ_i with size $\Psi(\tau_i) - \Psi(\tau_{i-1})$.
Moreover, the support of J^i is $[\tau_{i-1}, \tau_i]$. Indeed, the process $M^i := A^i - J^i$ is a martingale with null initial value and, due to the fact that $A^i_{\tau_{i-1}} = 0$, one obtains

$$0 = \mathbb{E}[M^i_{\tau_{i-1}}] = \mathbb{E}[A^i_{\tau_{i-1}} - J^i_{\tau_{i-1}}] = -\mathbb{E}[J^i_{\tau_{i-1}}]$$

hence $J^i_{\tau_{i-1}} = 0$ which implies that $J^i_t = 0$ on $\{t \leq \tau_{i-1}\}$. We recall that, for any bounded $\mathbb{F}$-predictable process H, the compensator of the increasing process $\int_0^\cdot H_s dA^i_s$ is $\int_0^\cdot H_s dJ^i_s$. As an immediate application, the $\mathbb{F}$-compensator of the increasing process

$$Y^{1,i}_t := 1\!\!1_{\{\tau_i \leq t\}} e^{-\Psi(\tau_i)} = \int_0^t e^{-\Psi(s)} dA^i_s, \forall t \geq 0\,,$$

is $\int_0^\cdot e^{-\Psi(s)} dJ^i_s$, i.e., $Y^{1,i}_t - \int_0^t e^{-\Psi(s)} dJ^i_s, \forall t \geq 0$ is an $\mathbb{F}$-martingale and the compensator of the increasing process

$$Y^{2,i}_t := 1\!\!1_{\{\tau_{i+1} \leq t\}} e^{-\Psi(\tau_i)} = \int_0^t e^{-\Psi(\tau_i)} dA^{i+1}_s = \int_0^t e^{-\Psi(\tau_i)} 1\!\!1_{\{\tau_i < s\}} dA^{i+1}_s, \forall t \geq 0$$

is $\int_0^\cdot 1\!\!1_{\{\tau_i < s\}} e^{-\Psi(\tau_i)} dJ^{i+1}_s = \int_0^\cdot e^{-\Psi(\tau_i)} dJ^{i+1}_s$, where we have used that the condition on the support of J^{i+1} and the fact that, being adapted and càg, the process $(1\!\!1_{\{\tau_i < t\}} e^{-\Psi(\tau_i)})_{t\geq 0}$ is predictable. Let us denote by U the increasing process $U_t = \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}}(\Psi(\tau_i) - \Psi(\tau_{i-1})), \forall t \geq 0$. Then

$$e^{-U_t} = \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t < \tau_{i+1}\}} e^{-\Psi(\tau_i)} = \sum_{i\geq 1} (Y^{1,i}_t - Y^{2,i}_t)\,.$$

The supermartingale $e^{-U} = \sum_{i\geq 1}(Y^{1,i} - Y^{2,i})$ admits the DM decomposition

$$e^{-U_t} = \Upsilon_t + \sum_{i\geq 1}\int_0^t (e^{-\Psi(s)} - e^{-\Psi(\tau_i)})dJ_s^{i+1} = \Upsilon_t - \zeta_t, \forall t \geq 0\,,$$

where Υ is a martingale and ζ is the predictable increasing process $\sum_{i\geq 0}\int_0^\cdot (e^{-\Psi(\tau_i)} - e^{-\Psi(s)})dJ_s^{i+1}$. Applying integration by parts formula to $Z = e^{-U}e^{-\Gamma}$, one obtains

$$dZ_t = e^{-\Gamma_t}d\Upsilon_t - (Z_t d\Gamma_t + e^{-\Gamma_t}d\zeta_t), \forall t \geq 0\,,$$

hence $dA_t^p = Z_t d\Gamma_t + e^{-\Gamma_t}d\zeta_t, \forall t \geq 0$. If the stopping times $(\tau_i)_{i\geq 1}$ are predictable, Z is predictable, and $A^p = 1 - Z$.
□

This result extends to the case where Ψ is an increasing $\mathbb{F}$-adapted process and $K_t = \sum_{i=1}^\infty \mathbb{1}_{\{\tau_i \leq t\}}[\Psi_{\tau_i} - \Psi_{\tau_{i-1}}] + \Gamma_t$ where Γ is $\mathbb{F}$-adapted increasing and continuous to give that

$$dA_t^p = Z_t d\Gamma_t + e^{-\Gamma_t}d\zeta_t, \forall t \geq 0\,,$$

with $\zeta = \sum_{i\geq 1}\int_0^\cdot (e^{-\Psi_{\tau_{i-1}}} - e^{-\Psi_s})dJ_s^i$.

3.3.3 Subordinator

Let K be a subordinator with null drift (i.e. an increasing Lévy process with null continuous part (see, e.g., Prop 3.10 and Section 4.2.2 in [6] and [15, Section 11.6]) with Lévy's measure ν, and let $\mathbb{F}$ be its filtration. Then, setting $\Psi(u) = \int_{\mathbb{R}_+}(1 - e^{-ux})\nu(dx)$ and $\kappa = \Psi(1)$, the process $n_t = e^{-K_t + t\kappa}, \forall t \geq 0$ is an $\mathbb{F}$-martingale and the multiplicative decomposition of Z is $Z_t = n_t e^{-t\kappa}, \forall t \geq 0$, leading to $dZ_t = -\kappa Z_t dt + e^{-t\kappa}dn_t$, hence $dA_t^p = Z_t\kappa dt$ and $\Lambda_t = t\kappa, \forall t \geq 0$.
This can be also obtained from the previous results noting that the process I, defined in Lemma 3.2 is also a subordinator, and from the compensation formula (see, e.g., [15, Proposition 11.2.2.3]) the process $(I_t - t\kappa, t \geq 0)$ is a martingale. Therefore $A_t^I = t\kappa$, $\forall t \geq 0$ and, from Lemma 3.2, we recover the form of the process Λ.
Furthermore,using the independence and stationarity of increments of the Lévy process K, one gets

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{E}[Z_u|\mathcal{F}_t] = e^{-K_t}\mathbb{E}[e^{-K_{u-t}}] = e^{-K_t}e^{-\kappa(u-t)}, \text{ for all } u \geq t \geq 0.$$

The law of τ is exponential with parameter κ.
Let us consider the particular case where X is a Compound Poisson Process (CCP) with non-negative jumps, i.e., $X_t = \sum_{n=1}^{N_t} Y_n, \forall t \geq 0$ where N is a Poisson process with intensity λ with jump times $(T_n, n \geq 1)$ and $(Y_n, n \geq 1)$ are non-negative random variables, i.i.d. with cumulative distribution function F, and independent from N. The process X being a subordinator with Lévy's measure $\nu(dx) = \lambda F(dx)$, we obtain that $\kappa = \Psi(1) = \lambda(1 - \mathbb{E}[e^{-Y_1}])$.

We consider now the case where $\mathbb{F}$ is the filtration generated by a subordinator X and a Brownian motion W independent from X, and set $K_t = \int_0^t k_s ds + X_t, \forall t \geq 0$,where k is $\mathbb{F}^W$ adapted and $e^{-X_t} = n_t e^{-\kappa t}$. Then,

$$dZ_t = -k_t Z_t dt + e^{-\int_0^t k_s ds}(-\kappa e^{-X_t}dt + e^{-t\kappa}dn_t) = M_t - Z_t(k_t + \kappa)dt$$

and $\Lambda_t = \int_0^t k_s ds + \kappa t, \forall t \geq 0$.

3.3.4 Marked point process

If an increasing sequence $(\tau_i)_{i\geq 1}$ of $\mathbb{F}$-stopping times and a sequence of non-negative random variables variables $(\theta_i \in \mathcal{F}_{\tau_i}, i \geq 1)$ are given, as well as a non-negative $\mathbb{F}$-adapted process k, one can construct an increasing càdlàg process K as $K_t = \int_0^t k_s ds + \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}}\theta_i$. This framework covers many cases of generalized Cox model.

The associated random time τ does not avoid the stopping random times $(\tau_i)_{i\geq 1}$ and we set $\tau_0 = 0$. We have, using the notation of section 3.2.1 where $I_t = \sum_{i\geq 1}(1-e^{-\theta_i})1\!\!1_{\{\tau_i \leq t\}} = \sum_{i\geq 1}\gamma_i 1\!\!1_{\{\tau_i \leq t\}} = M_t^I + A_t^I$. We consider the marked point process $(\gamma_i, \tau_i)_{i\geq 1}$ with jump measure μ defined as

$$\mu(\omega, [0,t], A) = \sum_{i\geq 1} 1\!\!1_{\{\tau_i(\omega)\leq t\}} 1\!\!1_{\{\gamma_i(\omega)\in A\}}, \forall t \geq 0$$

and its compensator ν given by

$$\nu(dt, dx) = \sum_{i\geq 0} 1\!\!1_{\{\tau_i < t \leq \tau_{i+1}\}} \frac{\mathbb{P}(\tau_{i+1} \in dt,\ \gamma_{i+1} \in dx | \mathcal{F}_{\tau_i})}{\mathbb{P}(\tau_{i+1} > t | \mathcal{F}_{\tau_i})}$$

(see Last and Brandt [20, Section 1.10]). Then

$$I_t = \int_0^t \int_{\mathbb{R}_+} x\mu(ds, dx) = M_t^I + A_t^I,$$

where M^I is the martingale $\int_0^\cdot \int_{\mathbb{R}_+} x(\mu(ds,dx) - \nu(ds,dx))$ and $A^I = \int_0^\cdot \int_{\mathbb{R}_+} x\nu(ds,dx)$. Furthermore, $dA_t^p = e^{-K_{t-}}(dK_t^c + dA_t^I)$.

As a particular case of marked point process, we consider the case where K is a point process, i.e., $K_t = \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}}$ for an increasing sequence of $(\tau_i)_{i\geq 1}$ and $\mathbb{F}$ is the natural filtration of K. One obtains $dA_t^p = e^{K_{t-}} dA_t^I$ where $I = (1-e^{-1})K$ so that $A^I = (1-e^{-1})\sum_{i\geq 1} J^i$, where, for any i, the process J^i is the $\mathbb{F}$-compensator of $A^i = 1\!\!1_{\{\tau_i \leq \cdot\}}$.
In the case where all the J^i are continuous, the process Ze^Λ is a martingale, where $\Lambda_t = \int_0^t \frac{dA_s^p}{Z_{s-}} = A_t^I$. Hence, $e^{-K + (1-e^{-1})\sum_{i\geq 1} J^i}$ is a martingale.
One can recover the result of Giesecke [10, Pro. 3.1] that, setting $\psi(u) = 1 - e^{-u}$, the process $e^{-uK + \psi(u)\sum_{i\geq 1} J^i}$ is a martingale by considering the increasing process $\widehat{K} = uK$.

Note that if all the $(\tau_i)_{i\geq 1}$ are predictable, K is predictable, hence $\sum_{i\geq 1} J^i = K$. Furthermore, as noticed in Subsection 3.2.5, $dA^p = dI = -dZ$. One has also $A^I = I$, so that $\Lambda = (1-e^{-1})K$.

3.3.5 Shot noise

Let $\mathbb{F}$ be a given filtration, $(\tau_i)_{i\geq 1}$ be a strictly increasing sequence of $\mathbb{F}$-stopping times with $\tau_1 > 0$, and $(\gamma_i)_{i\geq 1}$ a sequence of random variables with $\gamma_i \in \mathcal{F}_{\tau_i}$. We consider the random jump measure μ of the marked point process $(\tau_i, \gamma_i)_{i\geq 1}$, defined as $\mu(\omega, [0,t], A) = \sum_{i\geq 1} 1\!\!1_{\{\tau_i(\omega)\leq t\}} 1\!\!1_{\{\gamma_i(\omega)\in A\}}$ for $A \in \mathcal{B}(\mathbb{R})$ and ν its compensator. We denote by $\widetilde{\mu}$ the compensated random measure $\widetilde{\mu} = \mu - \nu$. We define an increasing continuous on right process $K = (K_t)_{t\geq 0}$ (called a shot-noise process, see [26]) by

$$K_t = \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}} G(t - \tau_i, \gamma_i) = \int_0^t \int_{\mathbb{R}} G(t-s, x)\mu(ds, dx), \forall t \geq 0\,, \tag{3.9}$$

where G is a function $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$. Note that $\Delta K_{\tau_i} = G(0, \gamma_i)$ and $K_t^c = \sum_{i\geq 1} 1\!\!1_{\{\tau_i \leq t\}}[G(t - \tau_i, \gamma_i) - G(0, \gamma_i)], \forall t \geq 0$. We assume that

$$G(t, x) = G(0, x) + \int_0^t g(s, x) ds, \quad \forall t \geq 0,\ x \in \mathbb{R}, \tag{3.10}$$

where g is a non-negative Borel function on $\mathbb{R}_+ \times \mathbb{R}$, so that G is increasing with respect to its first variable. We also assume that

$$\int_0^T \int_{\mathbb{R}} g^2(s,x)\nu(ds,dx) < \infty, \forall T\infty \tag{3.11}$$

and that there exists a non negative function φ on $\mathbb{R}_+ \times \mathbb{R}$ such that

$$|g(s,x)| \leq \varphi(x), \forall (s,x), \quad \text{with } \int_0^T \int_{\mathbb{R}} \varphi(x)\nu(ds,dx) < \infty, \, \forall T < \infty. \tag{3.12}$$

Comment 3.12 In the case where G does not depend of t, the shot-noise process is a marked point process.

In the next lemma, we give a decomposition of the submartingale K that will be used later. Our result is similar to the one of the proof of Lemma 2 in [26][5].

Lemma 3.13 *The shot-noise process K admits the Doob-Meyer decomposition $K = M^K + A^K$ where the $\mathbb{F}$-predictable increasing part A^K is*

$$A_t^K = \int_{u=0}^t \Big(\int_{s=0}^u \int_{\mathbb{R}} g(u-s,x)\mu(ds,dx) \Big) du + \int_{s=0}^t \int_{\mathbb{R}} G(0,x)\nu(ds,dx), \forall t \geq 0$$

and the $\mathbb{F}$-martingale part M^K has the following form :

$$M_t^K = \int_0^t \int_{\mathbb{R}} G(0,x)\tilde{\mu}(ds,dx), \forall t \geq 0\,.$$

PROOF: We extend G to $\mathbb{R} \times \mathbb{R}$ setting $G(u,x) = G(0,x)$ for $u < 0$. For any $a \in \mathbb{R}^+$, we define

$$Y_t(a) = \int_0^t \int_{\mathbb{R}} G(a-s,x)\tilde{\mu}(ds,dx), \forall t \geq 0,$$

one can write

$$K_t = Y_t(t) + \int_0^t \int_{\mathbb{R}} G(t-s,x)\nu(ds,dx)\,.$$

We apply the Itô-Ventzell formula as developed in [23, Theorem 3.1] to the process $Y_t(a)$ with parameter a, where a will be replaced by t (note that in our setting, since we integrate only deterministic functions w.r.t. the compensated random measure, forward integrals in [23] are usual stochastic integrals). The first derivative of $Y_t(a)$ with respect to the parameter a is (see [22]) , due to condition (3.12),

$$Y_t'(a) = \int_0^t \int_{\mathbb{R}} g(a-s,x)\tilde{\mu}(ds,dx).$$

Using Theorem 3.1 of [23], it follows that,

$$Y_t(t) = \int_{u=0}^t Y_u'(u)du + \int_{s=0}^t \int_{\mathbb{R}} G(0,x)\tilde{\mu}(ds,dx), \forall t \geq 0.$$

Note that, being continuous, the process $\int_{u=0}^{\cdot} Y_u'(u)du$ is predictable. Moreover, it is with bounded variation. Hence

$$\begin{aligned} K_t &= \int_{u=0}^t \Big(\int_{s=0}^u \int_{\mathbb{R}} g(u-s,x)\tilde{\mu}(ds,dx) \Big) du + \int_0^t \int_{\mathbb{R}} G(0,x)\tilde{\mu}(ds,dx) + \int_0^t \int_{\mathbb{R}} G(t-s,x)\nu(ds,dx) \\ &= M_t^K + \int_0^t \int_{\mathbb{R}} G(t-s,x)\nu(ds,dx) + \int_{u=0}^t \Big(\int_{s=0}^u \int_{\mathbb{R}} g(u-s,x)\tilde{\mu}(ds,dx) \Big) du, \forall t \geq 0\,. \end{aligned}$$

[5]There is a missprint in the formula given in the proof of this lemma.

where $M_t^K = \int_0^t \int_{\mathbb{R}} G(0,x)\tilde{\mu}(ds,dx)$. Using (3.10), one has

$$\int_0^t \int_{\mathbb{R}} G(t-s,x)\nu(ds,dx) = \int_{s=0}^t \int_{\mathbb{R}} \int_{u=0}^{t-s} g(u,x)du\, \nu(ds,dx) + \int_0^t \int_{\mathbb{R}} G(0,x)\nu(ds,dx),$$

hence, applying a change of variable and stochastic Fubini's theorem (see [24, Th. 65]) valid under condition (3.11), we obtain,

$$\begin{aligned} \int_{s=0}^t \int_{\mathbb{R}} \int_{u=0}^{t-s} g(u,x)du\, \nu(ds,dx) &= \int_{s=0}^t \int_{\mathbb{R}} \int_{u=s}^{t} g(u-s,x)du\, \nu(ds,dx) \\ &= \int_{u=0}^t \Big(\int_{\mathbb{R}} \int_{s=0}^{u} g(u-s,x)\, \nu(ds,dx)\Big)du \end{aligned} \tag{3.13}$$

for any $t \geq 0$, hence

$$\int_0^t \int_{\mathbb{R}} G(t-s,x)\nu(ds,dx) = \int_{u=0}^t \Big(\int_{\mathbb{R}} \int_{s=0}^{u} g(u-s,x)\, \nu(ds,dx)\Big)du + \int_0^t \int_{\mathbb{R}} G(0,x)\nu(ds,dx)\,.$$

Setting $k_u = \int_{\mathbb{R}} \int_{s=0}^u g(u-s,x)\ \mu(ds,dx)$, we see that the process $\int_0^\cdot k_u du$ is continuous, hence predictable. Therefore

$$A^K = \int_0^\cdot k_u du + \int_0^\cdot \int_{\mathbb{R}} G(0,x)\nu(ds,dx)\,.$$

□

Proposition 3.14 *The Doob-Meyer decomposition of Z is $Z = M - A^p$ with*

$$\begin{aligned} M_t &= \int_0^t Z_{s-}\Big(\int_{\mathbb{R}} (e^{-G(0,x)} - 1)\widetilde{\mu}(ds,dx) \\ A_t^p &= \int_{u=0}^t Z_{u-} k_u du + \int_0^t \int_{\mathbb{R}} (e^{-G(0,x)} - 1)\nu(ds,dx)\,, \end{aligned}$$

where $k_u = \int_{\mathbb{R}} \int_{s=0}^u g(u-s,x)\ \mu(ds,dx)$. The random time τ admits a JL-conditional density. The $\mathbb{F}$-predictable reduction of the compensator of τ is Λ where

$$d\Lambda_t = k_t\, dt + \int_{\mathbb{R}} (1-e^{-G(0,x)})\nu(dt,dx),\ \Lambda_0 = 0.$$

Proof: In the one hand, the process I defined in (3.2) is $I_t = \int_0^t \int_{\mathbb{R}} (1-e^{-G(0,x)})\mu(ds,dx)$ and it follows that

$$A_t^I = \int_0^t \int_{\mathbb{R}} (1-e^{-G(0,x)})\nu(ds,dx)\,.$$

In the other hand

$$\begin{aligned} K_t^c &= \sum_{i\geq 1} \mathbb{1}_{\{\tau_i \leq t\}}[G(t-\tau_i,\gamma_i) - G(0,\gamma_i)] = \sum_{i\geq 1} \mathbb{1}_{\{\tau_i \leq t\}} \int_0^{t-\tau_i} g(u,\gamma_i)du \\ &= \int_{s=0}^t \Big(\int_{\mathbb{R}} \int_{u=0}^{t-s} g(u,x)du\Big)\mu(ds,dx) \\ &= \int_{u=0}^t \Big(\int_{\mathbb{R}} \int_{s=0}^{u} g(u-s,x)\mu(ds,dx)\Big)du = \int_0^t k_u du\,, \end{aligned}$$

where we have used (3.13). This proves that K^c admits a density w.r.t. Lebesgue's measure, hence the existence of a JL-conditional density. The $\mathbb{F}$-predictable reduction of the compensator of τ is $\Lambda = K^c + A^I$.

An easy computation leads to

$$\begin{aligned} dZ_t &= e^{-K_{t-}} dK_t + \Delta e^{-K_t} + e^{K_{t-}} \Delta K_t \\ &= Z_{t-}\Big(- dM_t^K - dA_t^K + \int_{\mathbb{R}} (e^{-G(0,x)} - 1 + G(0,x)\mu(ds,dx)\Big) \\ &= Z_{t-}\Big(\int_{\mathbb{R}} (e^{-G(0,x)} - 1)\widetilde{\mu}(ds,dx) - k_t dt + \int_{\mathbb{R}} (e^{-G(0,x)} - 1)\nu(dt,dx)\Big) \end{aligned}$$

and the predictable part in the Doob-Meyer decomposition is

$$dA_t^p = Z_{t-}\Big(k_t dt + \int_{\mathbb{R}} (1 - e^{-G(0,x)})\nu(dt,dx)\Big)\,.$$

We recover the form of the $\mathbb{F}$-predictable reduction of the compensator. □

The increasing process I given by $I_t = \sum_{i\geq 1}(1 - e^{-G(t-\tau_i,\gamma_i)})1\!\!1_{\{\tau_i\leq t\}}$ is also a shot-noise process. The function $H(t,x) = 1 - e^{-G(t,x)}$ verifies for any $0 \leq t \leq T$, with T fixed,

$$H(t,x) = (1 - e^{-G(0,x)}) + \int_0^t g(s,x)e^{-G(s,x)} ds = H(0,x) + \int_0^t h(s,x) ds$$

where $h(t,x) = g(t,x)e^{-G(t,x)}$ satisfies (3.11) and (3.12). From the above, I admits a decomposition $I = M^I + A^I$ where $M_t^I = \int_0^t \int_{\mathbb{R}} H(0,x)\tilde{\mu}(ds,dx)$ and A^I is the increasing predictable process defined as

$$A_t^I \;=\; \int_{u=0}^t \Big(\int_{\mathbb{R}} \int_{s=0}^u h(u-s,x)\;\mu(ds,dx)\Big) du + \int_0^t \int_{\mathbb{R}} H(0,x)\nu(ds,dx)\,.$$

Proposition 3.15 *If ν is deterministic, the survival conditional law of τ is*

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = c(u)L_t(u), \quad \text{for } u \geq t \tag{3.14}$$

with, $c(u) = \exp\big(\int_0^u \int_{\mathbb{R}} (e^{-G(u-s,x)} - 1)\nu(ds,dx)\big)$ and $L(u)$ is the exponential $\mathbb{F}$-martingale $L(u) = \mathcal{E}(\int_0^\cdot \int_{\mathbb{R}} \big(e^{-G(u-s,x)} - 1\big)\widetilde{\mu}(ds,dx))$, for any $u \in \mathbb{R}_+$. In particular, the survival function of τ is $\mathbb{P}(\tau > u) = c(u)$.
For $t > u$, immersion property leads to $\mathbb{P}(\tau > u|\mathcal{F}_t) = c(u)L_u(u)$.

PROOF: Let $X_t(u) = -\int_0^t \int_{\mathbb{R}} G(u-s,x)\mu(ds,dx) - \int_0^t \int_{\mathbb{R}} (e^{-G(u-s,x)} - 1)\nu(ds,dx)$. From Itô's formula for semimartingales, setting $L_t(u) = e^{X_t(u)}$, one obtains, denoting for u fixed, L_t and X_t instead of $L_t(u)$ and $X_t(u)$

$$dL_t = L_{t-} dX_t + e^{X_{t-}}(e^{\Delta X_t} - 1 - \Delta(X_t))\,,$$

i.e., writing $e^{\Delta X_t} - 1 - \Delta(X_t) = \int_{\mathbb{R}} (e^{-G(u-t,x)} - 1 + G(u-t,x))\mu(dt,dx)$, and simplifying

$$dL_t = L_{t-} \int_{\mathbb{R}} \big(e^{-G(u-t,x)} - 1\big)\widetilde{\mu}(dt,dx) \tag{3.15}$$

which proves that L is a local martingale. From (3.14), it is a true martingale. If ν is deterministic, using the fact that $-K_u = X_u + \int_0^u \int_{\mathbb{R}} (e^{-G(u-s,x)} - 1)\nu(ds,dx)$, we obtain, for $u > t$

$$\mathbb{E}[e^{-K_u}|\mathcal{F}_t] = cL_t$$

with $c = \mathbb{E}[e^{-K_u}] = \exp\big(\int_0^u \int_{\mathbb{R}} (e^{-G(u-s,x)} - 1\big)\nu(ds,dx))$. We recover the result given in the proof of Proposition 2.1 in [25]. □

Remark 3.16 One can observe that $\Delta L_t(u) = e^{X_{t-}(u)}(e^{\Delta X_t(u)} - 1), \forall t \geq 0$.

Comment 3.17 It is proved (Th. 6.2.1 and Definition 6.2.1) in [12] that if ν is deterministic, then μ has independent increments.

Example 3.18 If $(\gamma_i)_{i\geq 1}$ are i.i.d. and $(\tau_i)_{i\geq 1}$ are the jump times of a Poisson process, independent of $(\gamma_i)_{i\geq 1}$, the random measure ν is deterministic.

3.4 Case where K is càglàd

In that case, defining τ as in (3.2), the equality $\{K_t \geq \Theta\} = \{\tau \leq t\}$ does not hold. In order to define a random time with nice properties, we set

$$\tau = \inf\{t \,:\, K_t > \Theta\}\,.$$

We define K_{t+} the right-limit of K at time t and $\Delta^+ K_t = K_{t_+} - K_t$.

Lemma 3.19 *Assume that K is càglàd. The set $\{t \,:\, K_t > \Theta\}$ is of the form $]t_0, \infty[$ with $K_{t_0} \leq \Theta$ and $\tau = t_0$. In particular*

$$\{K_t > \Theta\} = \{\tau < t\}.$$

For any $\mathbb{F}$-stopping time ϑ, one has $\mathbb{P}(\tau = \vartheta) = \mathbb{E}[e^{-K_\vartheta} - e^{-K_{\vartheta+}}]$.

PROOF: Due to the increasing property of K, the set $\{t \,:\, K_t > \Theta\}$ is of the form $[t_0, \infty[$ or $]t_0, \infty[$. If $\{t \,:\, K_t > \Theta\} = [t_0, \infty[$, one should have $K_{t_0} > \Theta$ and $K_{t_0 - \epsilon} \leq \Theta$ for any $\epsilon > 0$. The left-continuity of K yields to $K_{t_0} \leq \Theta$, which is a contradiction. Hence $\{t \,:\, K_t > \Theta\} =]t_0, \infty[$ and $K_{t_0} \leq \Theta$ and $\tau = t_0$. □

We deduce that $\{K_t \leq \Theta\} = \{\tau \geq t\}$, hence $\widetilde{Z}_t = e^{-K_t}$.

Proposition 3.20 *If K is càglàd, and $\tau = \inf\{t \,:\, K_t > \Theta\}$, one has $Z = e^{-K_+}$, $d\Lambda = e^{-\Delta^+ K}(dK^c + dA^C)$ where A^C is the predictable part of the supermartingale $C_t = \sum_{s\leq t} e^{-\Delta^+ K_s} - 1$.*

PROOF:

Let $\kappa_t := K_{t+} = K_t + \Delta^+ K_t$. Note that κ is continuous on right and that $\Delta \kappa_t = \Delta^+ K_t$. Let K^c_t be the continuous part of K, defined as $K^c_t = K_t - \sum_{s<t} \Delta^+ K_s, \forall t \geq 0$. Note that $K^c = \kappa^c$.

Then, from $\widetilde{Z}_t = e^{-K_t}$ and $Z_t = \widetilde{Z}_{t+} = e^{-\kappa_t}$, one has

$$\begin{aligned} dZ_t &= -e^{-\kappa_{t-}} d\kappa_t + (e^{-\kappa_t} - e^{-\kappa_{t-}} + e^{-\kappa_{t-}} \Delta\kappa_t) \\ &= -e^{-\kappa_{t-}} dK^c_t + e^{-\kappa_{t-}}(e^{-\Delta\kappa_t} - 1) \end{aligned}$$

The decreasing càdlàg process C defined as $C_t := \sum_{s\leq t} e^{-\Delta\kappa_s} - 1 = \sum_{s\leq t} e^{-\Delta^+ K_s} - 1$ admits a Doob-Meyer decomposition $C = M^C - A^C$ and

$$\begin{aligned} dZ_t &= e^{-\kappa_{t-}} dM^C_t - e^{-\kappa_{t-}}(dK^c_t + dA^C_t) \\ &= e^{-K_t} dM^C_t - e^{-K_t}(dK^c_t + dA^C_t) \end{aligned}$$

and $dA^p_t = e^{-K_t}(dK^c_t + dA^C_t)$, $d\Lambda_t = e^{-\Delta^+ K_t}(dK^c_t + dA^C_t)$. □

The multiplicative decomposition of Z can be deduced from Lemma 3.6, using that $Z = e^{-\kappa}$ with a right-continuous process κ.

The JL-conditional density exists if K^c is absolutely continuous

$$\mathbb{E}[h(\tau) \prod_i 1\!\!1_{\{\tau \neq \tau_i\}} | \mathcal{F}_t] = \int_0^\infty h(s) \mathbb{E}[k_s e^{-K_s} | \mathcal{F}_t] ds\,.$$

In conclusion, the case where K is càglàd reduces to the right-continuous case, making use of κ.

4 Pricing of Defaultable Bonds

A *defaultable claim* of maturity T is a pair (X,R) where $X \in \mathcal{F}_T$ is an integrable random variable and R is an $\mathbb{F}$-optional bounded process. The payoff of this claim is made of two parts: X paid at time T if $T < \tau$ and R_τ paid at time τ if $\tau \le T$. In the case of constant interest rate r, $\mathbb{P}$ being the pricing measure, the price of this claim at time t is

$$P_t(T) = e^{rt}\mathbb{E}[Xe^{-rT}\mathbb{1}_{\{T<\tau\}} + e^{-r\tau}R_\tau \mathbb{1}_{\{t<\tau\le T\}}|\mathcal{G}_t]\,.$$

A *zero coupon defaultable bond* is a defaultable claim with $R = 0$ and $X = 1$.
According to [4, Proposition 5.1.1], for R predictable such that R_τ is integrable, one has

$$\begin{aligned} P_t(T)e^{-rt} &= \mathbb{1}_{\{t<\tau\}}\frac{1}{Z_t}\mathbb{E}[Xe^{-rT}Z_T - \int_t^T e^{-ru}R_u dZ_u|\mathcal{F}_t] \\ &= \mathbb{1}_{\{t<\tau\}}\frac{1}{Z_t}\mathbb{E}[Xe^{-rT}Z_T + \int_t^T e^{-ru}R_u dA^p_u|\mathcal{F}_t],\ \forall t\ge 0\,. \end{aligned}$$

There exists an $\mathbb{F}$-adapted process $\widetilde{P}(T)$ such that $P_t(T)\mathbb{1}_{\{t<\tau\}} = \widetilde{P}_t(T)\mathbb{1}_{\{t<\tau\}}$, and if $\mathbb{F}$ is a continuous filtration (i.e., all $\mathbb{F}$-martingales are continuous) and A^p continuous, then $\widetilde{P}(T)$ is continuous. If $\mathbb{F}$ is continuous and A^p has jumps at predictable times $\tau_i, i\ge 1$, $\widetilde{P}(T)$ may have jumps at times τ_i. As an example of such a case, let $\mathbb{F}$ be the trivial filtration, $K_t = t\mathbb{1}_{\{t<1\}} + 2t\mathbb{1}_{\{1\le t\}}$, $r = 0, R = 1, X = 0$ and $T > 1$. Then $Z_t = e^{-K_t}$, $A^p_t = 1 - e^{-K_t}$ and $\tilde{P}_t(T) = \frac{e^{-K_t} - e^{-K_T}}{1-e^{-K_t}}$ has a jump at $t = 1$.

The *forward intensity rate*, is, when it exists, the family of $\mathbb{F}$-adapted process $\lambda(u) = (\lambda_t(u))_{t\ge 0}$ such that for any pair (t,T) with $t \le T$, one has $B_t(T) := \mathbb{P}(\tau > T|\mathcal{F}_t) = \exp\left(-\int_0^T \lambda_t(u)du\right)$ [7, Remark 2.3]. It is given by $\lambda_t(u) = \partial_u \ln B_t(u)$.

It is not difficult to show that, mimicking the proof of [4, Proposition 5.1.1] and taking into account that optional processes are generated by right-continuous processes of the form $\mathbb{1}_{\{u\le\tau<v\}}C_u$ with $C_u \in \mathcal{F}_u$, one obtains, for R optional, that

$$\mathbb{E}[X\mathbb{1}_{\{T<\tau\}} + R_\tau \mathbb{1}_{\{t\le\tau<T\}}|\mathcal{G}_t] = \mathbb{1}_{\{t<\tau\}}\frac{1}{Z_t}\mathbb{E}[XZ_T - \int_t^T R_u d\widetilde{Z}_u|\mathcal{F}_t],\ \forall t\ge 0\,. \tag{4.1}$$

We now restrict our attention to the case where K is càd. Our goal is to give an explicit form for the price of some defaultable assets, and to exhibit their dynamics.

We consider a zero coupon defaultable bond (DB) with maturity T, which delivers 1 monetary unit at maturity if and only if the default did not occur before T. We assume that the riskless interest rate is null. We also assume that K is càdlàg and $\mathbb{F}$-adapted, hence optional and that e^{K_t} is integrable, for any $t \ge 0$. The price of the DB, if $\mathbb{P}$ is the pricing measure, is

$$D_t(T) = \mathbb{P}(\tau > T|\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\frac{\mathbb{E}[Z_T|\mathcal{F}_t]}{Z_t} =: \mathbb{1}_{\{t<\tau\}}\frac{Y_t}{Z_t} = \mathbb{1}_{\{t<\tau\}}Y_t(T)e^{K_t} = \mathbb{1}_{\{t<\tau\}}\widetilde{D}_t(T)$$

where $Y(t)$ is the $\mathbb{F}$-martingale, valued in $[0,1]$, defined by $Y_t(T) = \mathbb{E}[Z_T|\mathcal{F}_t]$ and $\widetilde{D}(T)$ is the predefault price $\widetilde{D}(T) = Ye^K$.
When there are no ambiguity, we delete T in the notation.
The jump's sizes of $\widetilde{D}$ are $\Delta\widetilde{D} = \Delta(Ye^K) = \widetilde{D}_-(e^{\Delta K}\frac{Y}{Y_-} - 1)$.
The dynamics of $\widetilde{D}$ is

$$\begin{aligned} d\widetilde{D}_t &= Y_{t-}de^{K_t} + e^{K_{t-}}dY_t + d[Y,e^K]_t \\ &= Y_{t-}e^{K_{t-}}\left(dK^c_t + dJ_t\right) + e^{K_{t-}}dY_t + d[Y,e^K]_t,\ \forall t\ge 0 \end{aligned}$$

where J is the submartingale $J_t = \sum_{s\leq t}(e^{\Delta K_s}-1), \forall t \geq 0$ with Doob-Meyer decomposition M^J+A^J. The covariation process $[Y, e^K] = \int_0^\cdot \Delta Y_s de^{K_s}$ is a special semimartingale [6], with decomposition $[Y, e^K] = M^* + A^*$ where M^* is a local martingale and A^* a predictable process.

Therefore, the local martingale part of $\widetilde{D}$ is $\int_0^\cdot Y_{s-}e^{K_{s-}}dM^J_s + \int_0^\cdot e^{K_{s-}}dY_s + M^*$ and the predictable bounded variation part is $\int_0^\cdot Y_{s-}e^{K_{s-}}\big(dK^c_s + dA^J_s\big) + A^*$.
Note that $e^{K_{s-}}(dK^c_s + dA^J_s) = e^{K_s}(dK^c_s + dA^I_s) = e^{K_s}d\Lambda_s$.

Due to the fact that Y is orthogonal to $(1-A)e^K$ (see, e.g., prop 2.18 b, page38 in [1]), setting $U = (1-A)e^K$ (which is a $\mathbb{G}$-martingale)

$$dD = d(UY) = U_- dY + Y_- dU + d[Y, U]$$

where the three terms on the right hand side are $\mathbb{G}$-local martingales (due to immersion property and the orthogonality between U and Y).
The dynamics of Y is not explicit. In the case where $\mathbb{F}$ is a Brownian filtration, no closed form for its diffusion part is known, and one can note that the pair (Y, Θ) is solution of an elementary BSDE

$$dY_t = \Theta_t dW_t,\, Y_T = Z_T = e^{-K_T}\,.$$

Example 4.1 Two particular cases lead to more explicit formulae
1) If $\mathbb{F}$ is a continuous filtration, the $\mathbb{F}$-martingale Y is continuous, hence $[Y, e^K] = 0$ and $A^* = 0$. Furthermore, since optional processes are predictable[7], J is predictable and $M^J = 0, A^J = J$. Hence,

$$d\widetilde{D}_t = Y_t de^{K_t} + e^{K_{t-}}dY_t = Y_t e^{K_{t-}}\big(dK^c_t + dJ_t\big) + e^{K_{t-}}dY_t = Y_t e^{K_t}d\Lambda_t + e^{K_{t-}}dY_t$$

has a (continuous) local martingale part $\int_0^\cdot e^{K_{s-}}dY_s$, and a predictable part

$$\int_0^\cdot Y_s e^{K_s}d\Lambda_s\,.$$

The jump times of $\widetilde{D}$ are the predictable jump times of K, i.e., the jump times of K, and $\Delta D_{\tau_i} = D_{\tau_i -}(e^{\Delta K_{\tau_i}} - 1)$, so that the jumps are non negative. The impact of the jumps of K is then visible on the predefault price of defaultable bonds, and the size of the jump of K can be deduced from the jump's size of $\widetilde{D}$.

2) If K is predictable, $M^J \equiv 0$ and $A^J = J$, and by Yoeurp's lemma $[Y, e^K]_t = \int_0^t \Delta e^{K_s}dY_s$ which is the local martingale that we have denoted M^*, and $A^* = 0$. Then $e^{K_{s-}}dY_s + dM^*_s = (e^{K_{s-}} + \Delta e^{K_s})dY_s = e^{K_s}dY_s$ and the local martingale part of the semimartingale $\widetilde{D}_t$ is $\int_0^\cdot e^{K_s}dY_s$. The predictable bounded variation part is $\int_0^\cdot Y_{s-}e^{K_{s-}}\big(dK^c_s + dJ_s\big) = \int_0^\cdot Y_{s-}e^{K_s}d\Lambda_s$.
Here, the jump's sizes are $\widetilde{D}_-(e^{\Delta K}\frac{Y}{Y_-} - 1)$ and can be negative or positive.

Example 4.2 We assume to be in the case of shot-noise model presented in subsection 3.3.5, and ν is deterministic. From the results of subsection 3.3.5, it follows that $Y_t = c(T)L_t(T)$, hence, with simple computations that

$$\widetilde{D}_t = \exp\Big(\int_t^T\int_{\mathbb{R}}(e^{-G(T-s,x)} - 1)\nu(ds, dx) - \int_0^t\int_{\mathbb{R}}[G(T-s,x) - G(t-s,x)]\mu(ds,dx)\Big), \forall t \geq 0\,.$$

In particular, $\widetilde{D}$ has jumps at times $(\tau_i)_{i\geq 1}$ with negative jump sizes :

$$\Delta\widetilde{D}_{\tau_i} = \widetilde{D}_{\tau_i -}(e^{-(G(T-\tau_i,\gamma_i) - G(0,\gamma_i))} - 1)\,.$$

[6] The special feature comes from the fact that, for any t, the random variable $\sup_{s\leq t}|[Y, e^K]_s| \leq 2e^{K_t}$ is integrable
[7] A filtration $\mathbb{H}$ is continuous if all martingales are continuous. This implies that the two σ-algebra $\mathcal{O}(\mathbb{H})$ and $\mathcal{P}(\mathbb{H})$ are equal (see page 512 in [15])

Note also that $\tilde{D}(T)$ is increasing between two jumps.

The dynamics of $\widetilde{D}$ can be deduced by Itô-Ventcell's formula. We prefer to make use of the general results presented in the first part of this section. The jumps of Y and e^K occur at times τ_i. One has, from (3.15)

$$\begin{aligned} \Delta Y_{\tau_i} &= Y_{\tau_i -}(e^{-G(T-\tau_i,\gamma_i)}-1) \\ \Delta(e^K)_{\tau_i} &= e^{K_{\tau_i -}}(e^{G(\tau_i-\tau_i,\gamma_i)}-1) = e^{K_{\tau_i -}}(e^{G(0,\gamma_i)}-1)\,. \end{aligned}$$

Hence

$$\begin{aligned} [Y,e^K]_t &= \sum_{s\le t}\Delta Y_s \Delta(e^K)_s = \int_0^t Y_{s-}e^{K_{s-}}\int_{\mathbb{R}}(e^{-G(T-s,x)}-1)(e^{G(0,x)}-1)\mu(ds,dx) \\ &= M_t^* + \int_0^t Y_{s-}e^{K_{s-}}\int_{\mathbb{R}}(e^{-G(T-s,x)}-1)(e^{G(0,x)}-1)\nu(ds,dx), \forall t\ge 0\,. \end{aligned} \tag{4.2}$$

As in section 3.3.5 for the computation of A^I, and noting that J is a shot noise process as $J_t = \sum \mathbb{1}_{\{\tau_i\le t\}}H^J(t-\tau_i,\gamma_i)$ with $H^J(t,x)=e^{G(t,x)}-1$, we obtain

$$A_t^J = \int_{u=0}^t \Big(\int_{\mathbb{R}}\int_{s=0}^u h^J(u-s,x)\,\mu(ds,dx)\Big)du + \int_0^t\int_{\mathbb{R}} H^J(0,x)\nu(ds,dx)\,,$$

where $h^J(t,x)=g(t,x)e^{G(t,x)}$. Therefore, the local martingale part of $\widetilde{D}_t(T)$ is $\int_0^t Y_{s-}e^{K_{s-}}dM_s^J + \int_0^t e^{K_{s-}}dY_s + M_t^*$ where

$$\begin{aligned} M_t^* &= \int_0^t Y_{s-}e^{K_{s-}}\int_{\mathbb{R}}(e^{-G(T-s,x)}-1)(e^{G(0,x)}-1)\widetilde{\mu}(ds,dx) \\ M_t^J &= \int_0^t\int_{\mathbb{R}} H^J(0,x)\widetilde{\mu}(ds,dx) \end{aligned}$$

so that the local martingale part reduces, after simple computation, to

$$\int_0^t Y_{s-}e^{K_{s-}}\int_{\mathbb{R}}(e^{G(0,x)-G(T-s,x)}-1)\widetilde{\mu}(ds,dx)$$

and the predictable bounded variation part is $\int_0^t Y_{s-}e^{K_{s-}}\big(dA_s^J + \int_{\mathbb{R}}(e^{-G(T-s,x)}-1)(e^{G(0,x)}-1)\nu(ds,dx)\big)$.

Example 4.3 In the case presented in Subsection 3.3.3, we find that

$$\begin{aligned} \widetilde{D}_t(T) &= \mathbb{E}[\exp\big(-\int_t^T k_s ds - (X_T - X_t)\big)|\mathcal{F}_t] \\ &= \mathbb{E}[\exp\big(-\int_t^T k_s ds\big)\mathbb{E}[\exp\big(-(X_T-X_t)\big)|\mathcal{F}_T^W\vee\mathcal{F}_t^X]|\mathcal{F}_t] \\ &= \mathbb{E}[\exp -X_{T-t}]\mathbb{E}[\exp\big(-\int_t^T k_s ds\big)|\mathcal{F}_t] = e^{-(T-t)\lambda\psi(-1)}\mathbb{E}[\exp\big(-\int_t^T k_s ds\big)|\mathcal{F}_t^W] \\ &= e^{-(T-t)\lambda\psi(-1)}\exp\big(\int_0^t k_s ds\big)\mathbb{E}[\exp\big(-\int_0^T k_s ds\big)|\mathcal{F}_t^W], \forall t\ge 0 \end{aligned}$$

and $\widetilde{D}(T)$ is a continuous process. Setting $\eta_t = \mathbb{E}[\exp\big(-\int_0^T k_s ds\big)|\mathcal{F}_t^W]$ and $\kappa_t = e^{-(T-t)\lambda\psi(-1)}\exp\big(\int_0^t k_s ds\big)$, the dynamics of $\widetilde{D}_t(T)$ is $d\widetilde{D}_t(T) = \kappa_t d\eta_t + \eta_t\kappa_t(\lambda\psi(-1)+k_t)dt$.

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