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On the Zeros of the Stockwell and Morlet Wavelet Transforms

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Abstract

Recently a link between Gaussian analytic functions (GAFs) and time-frequency transforms has been established by Bardenet et al. This work was motivated by the earlier work performed by Flandrin on the zeros of the Spectrogram (squared modulus of the short-time Fourier transform) and their regular distribution which form a point process in the time-frequency plane. The aim of this paper is to extend these earlier studies to the Stockwell transform (ST) which is an hybrid transform between the Short Time Fourier Transform (STFT) and the Continuous Wavelet Transform (CWT). First, the factorization of a generalized form of the ST in term of Bargmann transform is given. Then the ST of white Gaussian noise is developed formally in order to give the link between ST and planar GAFs. Because of the close relation between the ST and the Morlet Wavelet transform the zeros of this specific CWT are also discussed in this study. Finally, examples of zeros detected from the ST domain is illustrated on synthetic and real signals.

1. Introduction

Time-Frequency (TF) analysis provides powerful methods and transforms to analyse non-stationary signals when their statistical properties depend on

time. Indeed, the Fourier transform maps signals into the frequency domain but without any information on time which can be a limitation for non-stationary signals. Jointly representing the temporal and frequency content of a signal allows to better understand the physical phenomenon that generates it. There are different methods to generate this qualitative representation, which can be essentially summarized by two different approaches: linear and quadratic time-frequency methods. The first family tries to decompose the time-frequency plane into elementary parts called atoms describing locally the time and frequency content and assuming that the signal in question is stationary inside the applied window. The second family, called non-linear or more precisely quadratic transforms, tries to describe the energy distribution of the signal in the time-frequency plane. While the first family suffers from poor time-frequency concentration which will require optimization schemes related to the form and the length of the applied window [1], the second family suffers from cross-terms in multi-component signals and may also suffer from inner interference for mono-component signals. There is no time-frequency method which can be considered as optimal for all applications.

Detect relevant information (e.g signal components, special occurred events, ...) in signal usually was performed based on the regions in the time-frequency plane containing the most energy or the local maxima [2], which is intuitive since information will be mostly concentrated in the high energy regions in the time-frequency plane. However, the study of Flandrin [3] paved the way to consider the zeros distribution in the time-frequency plane instead to focus on the high energy regions. Flandrin focused on the Spectrogram and

its zeros which completely characterize the transform in question. The intuition behind this new observation is based on the fact that the zeros of the Spectrogram applied on white noise tend to have a uniform distribution and repelled each other. This naturally suggests the link with certain analytical functions that are completely characterized by their zeros as the Gaussian Analytic Functions [4]. Bardenet et al. [5, 6] extended this work on other time-frequency methods as the analytic wavelets and formalized the link between these methods and some GAFs. In parallel, Koliander et al. [7] found a link between hyperbolic wavelet transform and hyperbolic GAFs and authors give a filtering application based on the zeros of corresponding time-scale representation.

The aim of this paper is to extend the works cited above to study the zeros the Stockwell transform [8]. The Stockwell Transform (ST) can be considered as a hybrid between the Short Time Frequency Transform (STFT) and the Continuous Wavelet Transform (CWT) [8]. It can be viewed as a frequency dependent STFT or a phase corrected CWT. The ST has shown high performance for classification and feature extraction problems applied on non-stationary signals, such as heart sounds [9], power quality signals [10], EEG signals [11] etc. Classically the ST uses a Gaussian window, whose standard deviation varies over frequency. Whatever the analyzed signal, the width of the Gaussian window decreases as the frequency increases. This produces a higher frequency resolution at lower frequencies and a higher time resolution at higher frequencies. The main advantage of the ST compared to other linear time-frequency methods is that it generates multi-resolution analysis as the continuous wavelet transform while keeping the phase infor-

mation exploitable. The main contributions of this paper can be summarized as:

- Establishing the link between the Stockwell transform and Bargmann transform (Theorem 1)
- Characterizing formally the zeros distribution of the Stockwell transform of white Gaussian noise in order to link th ST to GAFs (Theorem 2)
- Extend the results provided by first two points to the zeros of the Morlet wavelet transform which has a direct relation with the ST.

The paper is organized as follows: Section 2 gives a brief description of the Stockwell transform and present a generalized form of the ST before introducing the Stockwell-Bargmann factorization and the link between ST and planar GAFs. Then Section 3 presents the ralation between ST and Morlet wavelet transform and deduce the Morlet-Bargmann factorization and the link between the Morlet wavelet and planar GAFs. Section 4 gives numerical examples based on synthetic and real signals for different time-frequency methods and their zero's distributions. Finally, Section 5 gives the conclusion and the future work.

2. On the zeros of the Stockwell transform

In this section we will follow the template theorem proposed by Bardenet et al. [6] to search the orthonormal basis (h_k) and the analytic functions (Λ_k) allowing the continuous linear mapping: $\mathcal{L} : \mathcal{H} \to \mathcal{A}(\Lambda)$ with $\mathcal{L}$ is this paper

will play the role of the Stockwell transform operator. The idea is to find the adequate orthonormal basis which will allow us to link the ST to GAFs. Before we develop the details of this analogy, we will present a brief survey on the ST, then we will give the proof of the Stockwell-Bargmann proposition in order to proceed with the link between the ST of Hermite functions and the planar GAFs.

2.1. The Stockwell Transform

Let $f \in L^2(\mathbb{R})$ be a signal. Stockwell et al. initially introduced the ST in [8], using a Gaussian window $g(t) = e^{-t^2/2}, t \in \mathbb{R}$. The ST with respect to the window g, denoted $S_g f$ can be given as:

$$(S_g f)(\tau, \xi) = \frac{|\xi|}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-(t-\tau)^2 \xi^2/2} e^{-2\pi i t \xi} dt, \quad \tau, \xi \in \mathbb{R} \tag{1}$$

An alternative formulation with respect to Fourier transform of f deduced by rewriting (1) as a convolution product, can be given as:

$$(S_g f)(\tau, \xi) = \int_{\mathbb{R}} \widehat{f}(\upsilon + \xi) e^{-2\pi^2 \upsilon^2/\xi^2} e^{2\pi i \tau \upsilon} d\upsilon, \quad \tau \in \mathbb{R}, \xi \in \mathbb{R}^* \tag{2}$$

Where $e^{-2\pi^2 \upsilon^2/\xi^2}$ is the Fourier transform of the window $g(t)$ and $\widehat{f}$ is the Fourier transform of the signal f which is given as:

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} dt, \quad \xi \in \mathbb{R}$$

Note that the width of the window in Eq. 1 is inversely proportional to the frequency which will induce a better frequency resolution for low frequencies and a better temporal resolution for high frequencies. This is the main difference between the ST and the Short Time Fourier Transform (STFT) which uses a fixed window width throughout the analysis.

2.2. The generalized Stockwell transform

Several generalized Gaussian windows have been proposed in the literature for ST [1]. The main idea is to give more degrees of freedom for the variation of the window by introducing several parameters to control the width as a function of the frequency to make the analysis as adapted as possible to the signal.

Let $\sigma(\xi) \in L^1(\mathbb{R}^*) \cap L^2(\mathbb{R}^*)$ be an arbitrary function that aims to control the width of the Gaussian window. Then the generalized Stockwell transform of f, denoted in this paper $(S_g^\sigma f)$, with respect to the window $g(t)$ and the width function $\sigma(\xi)$ can be rewritten as:

$$(S_g^\sigma f)(\tau, \xi) = (\sqrt{2\pi}\sigma(\xi))^{-1} \int_{\mathbb{R}} f(t) \overline{g\left(\frac{t-\tau}{\sigma(\xi)}\right)} e^{-2\pi i t\xi} dt \tag{3}$$

By setting $\sigma(\xi) = \frac{1}{|\xi|}$ we find the classical version of the ST defined in Eq. 1. By denoting $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{R})$, Eq. 3 can be rewritten as :

$$(S_g^\sigma f)(\tau, \xi) = < f, M_\xi T_\tau D_{\sigma(\xi)} g > \tag{4}$$

Where $(M_\xi f)(t) = e^{2i\pi\xi} f(t)$, $(T_\tau f)(t) = f(t-\tau)$, and $(D_{\sigma(\xi)} f)(t) = (\sqrt{2\pi}\sigma(\xi))^{-1} f(t/\sigma(\xi))$ are the modulation, translation and dilatation operators, respectively.

2.3. The Stockwell-Bargmann factorization

Let's now consider the generalized definition of the ST presented in Eq. 3.

Theorem 1. *Let $f \in L^2(\mathbb{R})$, τ, $\xi \in \mathbb{R}^2$ and $z \in \mathbb{C}$. Denoting $(S_g^\sigma f)(\tau, \xi)$ the generalized Stockwell transform of the function f respecting to the Gaussian*

window g and the width function $\sigma(\xi)$, can be factorized in term of Bargmann transform $(Bf)(z)$ as follows :

$$(S_g^\sigma f)(\tau,\xi) = 2^{-1/4} a e^{-i\pi\tau\xi} e^{-\frac{\pi}{2}|z|^2} (Bf(\cdot/a))(z) \tag{5}$$

Where $a = (\sqrt{2\pi}\sigma(\xi))^{-1}$ and $(S_g^\sigma f)$ is given in Eq. 3 The Bargmann transform of the function f can be given as [12]:

$$(Bf)(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi t z - \pi t^2 - \frac{\pi}{2} z^2} dt \tag{6}$$

Proof. Replacing explicitly $g(t)$ in the Eq. of $(S_g f)$ we obtain :

$$\begin{aligned}
(S_g^\sigma f)(\tau,\xi) &= a \int_{\mathbb{R}} f(t) e^{-\pi a^2 (t-\tau)^2} e^{-2i\pi} dt \\
&= a \int_{\mathbb{R}} f(t) e^{-\pi a^2 t^2} e^{-\pi a^2 \tau^2} e^{2a^2 \pi t \tau} e^{-2i\pi\xi t} dt \\
&= a e^{-i\pi\tau\xi} e^{-\frac{\pi}{2}(a^2\tau^2 + \frac{\xi^2}{a^2})} \\
&= \int_{\mathbb{R}} f(t) e^{-\pi a^2 t^2} e^{2a\pi t(a\tau - i\frac{\xi}{a})} e^{-\frac{\pi}{2}(a\tau - i\frac{\xi}{a})^2} dt
\end{aligned}$$

Making the change of variable $s = at$, and denoting $z = a\tau - i\frac{\xi}{a}$ we obtain :

$$(S_g^\sigma f)(\tau,\xi) = a e^{-i\pi\tau\xi} e^{-\frac{\pi}{2}|z|^2} \int_{\mathbb{R}} f\left(\frac{s}{a}\right) e^{-\pi s^2} e^{2\pi s z} e^{-\frac{\pi}{2} z^2} ds \tag{7}$$

Which is equivalent to :

$$(S_g^\sigma f)(\tau,\xi) = 2^{-1/4} a e^{-i\pi\tau\xi} e^{-\frac{\pi}{2}|z|^2} (Bf(\cdot/a))(z) \tag{8}$$

□

Theorem 1 means that the zeros of the Stockwell transform $\tau, \xi \mapsto (S_g^\sigma f)(\tau,\xi)$ of f are the zeros of the Bargmann transform of $s \mapsto f(s/a)$. It means that the Stockwell transform can be completely characterized by the distribution of its zeros as the Bargmann transform [3].

2.4. The ST of Gaussian white noise

The Hermite functions have the advantage of a simple closed-form Bargmann transform [13]. Thus, if we decompose the white noise onto the basis formed the Hermite functions, this allow us to compute easily the ST of the closed-form Bargmann transforms, thanks to theorem 1.

2.4.1. Definitions

Let $(H_k)_{k\in\mathbb{N}}$ be the Hermite orthonormal polynomials with respect to the Gaussian window $g(t) = e^{-t^2/2}$, $t \in \mathbb{R}$ and $(h_k)_{k\in\mathbb{N}}$ the associated Hermite functions [14]. Let X a random variable with white Gaussian distribution on Hilbert spaces $\mathcal{H}$ associated with orthonormal basis (h_k) as follows:

$$X := \sum_{k\in\mathbb{N}} X_k h_k \tag{9}$$

2.4.2. The ST of X

We can define the ST of X as a random function:

$$\tau, \xi \to \left\langle X, M_\xi T_\tau D_{\sigma(\xi)} g \right\rangle$$

Theorem 2. *Let $\tau, \xi \in \mathbb{R}^2$, and $z = a\tau - i\frac{\xi}{a} \in \mathbb{C}$. Then the ST of Gaussian noise X can be given as :*

$$\left\langle X, M_\xi T_\tau D_{\sigma(\xi)} g \right\rangle = 2^{-1/4} a e^{i\pi\tau\xi} e^{-\frac{\pi}{2}|z|^2} \sum_{k=0}^{\infty} \langle X, h_k \rangle \frac{\pi^{k/2} z^k}{\sqrt{k!}} \tag{10}$$

where (h_k) denote the orthonormal Hermite functions.

Proof. Let us decompose $M_\xi T_\tau D_{\sigma(\xi)} g$ in the Hermite basis (h_k) of $L^2(\mathbb{R})$:

$$\left\langle X, M_\xi T_\tau D_{\sigma(\xi)} g \right\rangle = \sum_{k=0}^{\infty} \langle X, h_k \rangle \left\langle M_\xi T_\tau D_{\sigma(\xi)} g, h_k \right\rangle \tag{11}$$

$$= \sum_{k=0}^{\infty} \langle X, h_k \rangle \, (S_g^\sigma h_k)(\tau, \xi) \tag{12}$$

Let us now calculate $(S_g^\sigma h_k)(\tau,\xi)$, by using the Eq. 5, we can find directly:

$$(S_g^\sigma h_k)(\tau,\xi) = 2^{-1/4} a e^{-i\pi\tau\xi} e^{-\frac{\pi}{2}|z|^2} B(h_k(./a))(z) \tag{13}$$

where the Bargmann transform of an Hermite function is the following monomial related to the planar GAFs [4, 15, 3] :

$$B(h_k(./a))(z) = \frac{\pi^{k/2} z^k}{\sqrt{k!}} \tag{14}$$

□

3. On the zeros of the Morlet wavelet Transform

3.1. The Morlet wavelet transform

The continuous wavelet transform of a signal $f \in L^2(\mathbb{R})$ is defined at each time τ and scale s for an admissible mother wavelet ψ as follows:

$$(W_\Psi f)(\tau,s) = \frac{1}{\sqrt{|s|}} \int_{\mathbb{R}} f(t) \Psi \left(\frac{(t-\tau)}{s} \right)^* dt \tag{15}$$

where Ψ and its Fourier transform $\hat{\Psi}$ are satisfying the admissibility condition of mother wavelet :

$$C_\Psi = \int_{\mathbb{R}} \frac{|\hat{\Psi}|(\omega)}{\omega} d\omega < \infty \tag{16}$$

By defining the scale $s = \frac{\sqrt{2\pi}}{\xi}$, Eq. 15 can be expressed as time-frequency transform as follows :

$$(W_\Psi f)(\tau,\xi) = \frac{\sqrt{|\xi|}}{\xi_0} \int_{\mathbb{R}} f(t) \Psi \left(\frac{\xi}{\xi_0}(t-\tau) \right)^* dt \tag{17}$$

The Morlet wavelet is defined as [16] :

$$\Psi(t) = \frac{\pi^{-1/4}}{\sqrt{T}} e^{\frac{-t^2}{2T^2}} e^{i\sqrt{2\pi}t} \tag{18}$$

where T plays the role of time-spread parameter [17]. By introducing Eq. 18 in Eq. 15 we obtain the definition of the Morlet wavelet transform of signal f denoted MW_Ψ.

3.2. Relation with the Stockwell transform

In this paper, in the same fashion of Eq. 3, T can be replaced in Eq. 18 by the arbitrary function $\sigma(\xi)$ which control the width of Gaussian window in function of the frequency ξ as follows:

$$\Psi(t) = \frac{\pi^{-1/4}}{\sqrt{\sigma(\xi)}} e^{\frac{-t^2}{2\sigma(\xi)^2}} e^{i\sqrt{2\pi}t} \tag{19}$$

This leads us to write the Morlet wavelet transform ($MW_\Psi f$) as a generalized ST, ($S_g^\sigma f$), formulated in this paper using the proof given in [18] as follows:

$$(MW_\Psi f)(\tau, \xi) = \sqrt{\frac{2^{3/2}\pi\sigma(\xi)}{|\xi|}} e^{i\xi t} (S_g^\sigma f)(\tau, \xi) \tag{20}$$

Eq. 20 will give us the two follows corollaries corresponding to Theorem 1 and Theorem 2, respectively:

Corollary 1. *Let $f \in L^2(\mathbb{R})$, τ, $\xi \in \mathbb{R}^2$ and $z \in \mathbb{C}$. Denoting $(MW_\Psi f)(\tau, \xi)$ the Morlet wavelet transform of the function f, it can be factorized in term of Bargmann transform $Bf(z)$ as follows :*

$$(MW_\Psi f)(\tau, \xi) = \frac{e^{-i\xi(\pi\tau - t)}}{\sqrt{\xi\sigma(\xi)}} e^{-\frac{\pi}{2}|z|^2} (Bf(\cdot/a))(z) \tag{21}$$

Corollary 2. *Let $\tau, \xi \in \mathbb{R}^2$, and $z = a\tau - i\frac{\xi}{a} \in \mathbb{C}$. Let X a random variable as defined in section D.1. Then the Morlet wavelet transform of X, $(MW_\Psi X)$, can be given as :*

$$(MW_\Psi X)(\tau, \xi) = \frac{e^{-i\xi(\pi\tau - t)}}{\sqrt{\xi\sigma(\xi)}} e^{-\frac{\pi}{2}|z|^2} \sum_{k=0}^{\infty} \langle X, h_k \rangle \frac{\pi^{k/2} z^k}{\sqrt{k!}} \tag{22}$$

4. Numerical Experiments

The proofs provided in this paper have been based on the continuous domain. We are aware that the transition to the discrete domain must be interpreted carefully. However, we will show an example of a synthetic signal to observe the distribution of zeros in the time-frequency plane for the Short Time Fourier Transform and an optimized Stockwell transform [1]. We generate a multi-component synthetic signal with sampling frequency 2 KHz, composed of a linear chirp and a sinusoidally modulated component in the time-frequency plane. A white Gaussian noise is added to the generated signal with Signal-to-Noise Ratio (SNR) equal to 5 dB. We tested also a real cardiac signal acquired from accelerometer (BallistoCardioGram - BCG) with 1 KHz sampling frequency [19]. The detection of zeros is done by applying a sliding 2D window on the time-frequency matrix and by calculating the local minimum.

Fig. 1 shows the zeros detected on the representation of synthetic signal computed by the STFT with Gaussian window. For the optimized Stockwell transform (see Fig. 2), the energy concentration in the time-frequency plane is improved compared to Fig. 1. Theoretically the distribution properties of zeros should be independent from the quality of the time-frequency transform in terms of energy concentration. But, intuitively speaking, the better the energy concentration, the more efficient will be the filtering based on the zeros distribution, since the signal components should be already well separated in time-frequency domain. Fig. 3 shows the classical ST representation with the detected zeros for the real tested signal.

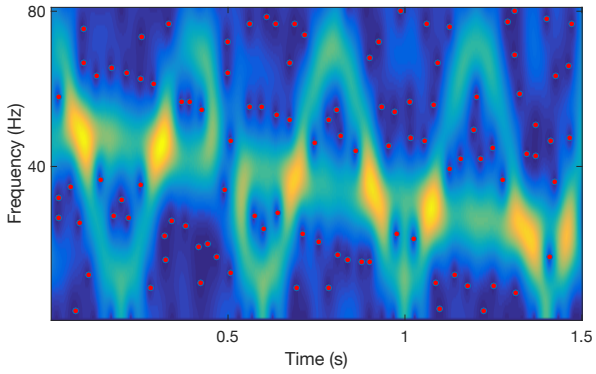


Figure 1: Detected zeros (red dots) based on the STFT.

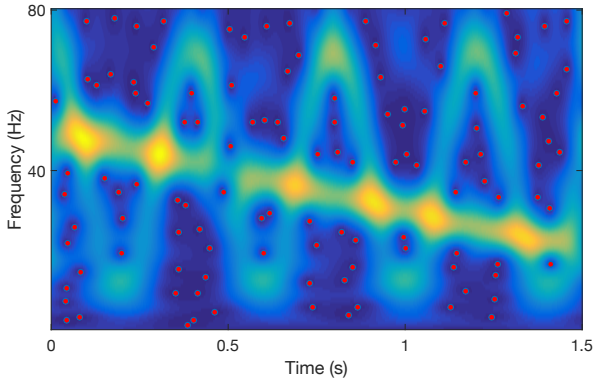


Figure 2: Detected zeros (red dots) based on the ST.

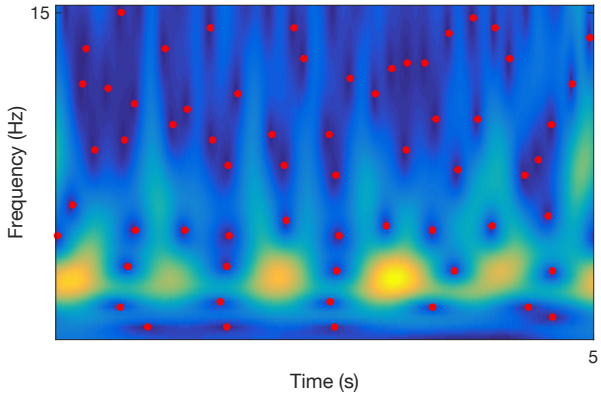


Figure 3: Detected zeros (red dots) based on the ST for real signal (Ballistocardiogram).

5. Conclusion

We characterized in this paper the zeros of two time-frequency methods : the Stockwell transform and the Morlet wavelet transform. First, we defined the generalized ST with Gaussian window $g(t)$ with an arbitrary function to control the window's width in function of frequency denoted $\sigma(\xi) \in L^1(\mathbb{R}^*) \cap L^2(\mathbb{R}^*)$. Then, we presented the Stockwell-Bargmann factorization which allowed us to establish the relation between the ST and the Bargmann transform (theorem 1). This allowed us to formalize the link between the generalized ST and the planar GAFs (theorem 2) by calculating the ST of white Gaussian noise associated with orthonormal Hermite functions basis. The known direct link between the ST and the Morlet wavelet transform allowed us to extend these results to the zeros of the MW. Future work, will focus on filtering methods based on zeros distribution of different time-frequency methods and the possibility of extension of the zeros characterization on the Discrete Orthonomal Stockwell transform (DOST) [20, 21].

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