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► To cite this version:

Philippe Balbiani. Emptiness relations in property systems. 6th International Conference on Relational Methods in Computer Science (RelMiCS 2001), Oct 2001, Oisterwijk, Netherlands. pp.15-34, 10.1007/3-540-36280-0_2 . hal-03252550

HAL Id: hal-03252550

<https://hal.science/hal-03252550>

Submitted on 7 Jun 2021

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Emptiness relations in property systems

Philippe Balbiani

Institut de recherche en informatique de Toulouse
118 route de Narbonne, 31062 Toulouse Cedex 4, France

Abstract. In this paper, we carry out the modal analysis of emptiness through a modal logic which modalities correspond to emptiness relations in property systems. We mainly address the questions of axiomatization/completeness and decidability/complexity of our modal logic.

1 Introduction

The concept of informational relation has been introduced by Pawlak [10] within the context of attribute systems — knowledge-based systems which describe objects in terms of attributes — and Vakarelov [16] within the context of property systems — knowledge-based systems which describe objects in terms of properties. It has been furthered by Demri [2], Demri, Orłowska and Vakarelov [3], Orłowska [6, 7], Orłowska and Pawlak [9] and Vakarelov [13–16] who have considered the questions of axiomatization/completeness and decidability/complexity of modal logics which modalities correspond to informational relations. The aim of these logics is to provide a formal method for reasoning about uncertain knowledge discovered from attribute systems or property systems.

All the informational relations in attribute systems and property systems are either indistinguishability relations or distinguishability relations. Indistinguishability relations indicate the way objects share attributes or properties whereas distinguishability relations indicate the way attributes and properties differentiate objects. This paper is devoted to the modal analysis of emptiness relations in property systems. We give in section 2 the formal definition of emptiness relations in property systems. We provide a mathematical definition of the syntax and semantics of our modal logic in section 3 where we mainly address the questions of completeness and decidability. In section 4 we show that the satisfiability problem for any given formula of our modal logic requires polynomial space whereas in section 5 we show that the satisfiability problem for any given formula of our modal logic is decidable in polynomial space.

2 Emptiness relations

Adapted from Vakarelov [16], a property system will be any algebraic structure (Obj, Pro, f) where:

- Obj is a nonempty set of objects;

- Pro is a nonempty set of properties;
- f is a function with domain Obj and range the power set of Pro .

The property system (Obj, Pro, f) will be defined to be non-trivial if for all $x \in Obj$, $f(x) \neq \emptyset$ and $f(x) \neq Pro$. We should consider, for example, the property system $S = (Obj, Pro, f)$ defined as follows. Define:

- Obj is $\{Ann, Bob, Cindy, Daniel, Emma\}$;
- Pro is $\{Arabic, Bulgarian, Castilian, Dutch, English\}$;
- f is the function defined by table 1.

Ann	Bob	Cindy	Daniel	Emma
$\{Arabic, Bulgarian, Castilian\}$	$\{Arabic, Bulgarian, Castilian, Dutch\}$	$\{Arabic, Bulgarian\}$	$\{English\}$	$\{Castilian, Dutch, English\}$

Table 1. Example of a property system.

Property systems constitute the starting point for the formal examination of sentences of the form “object x is indistinguishable from object y ” or sentences of the form “object x is distinguishable from object y ”. In this respect, the emptiness relations might play an important role. Given any property system $S = (Obj, Pro, f)$, the emptiness relations over S are defined by the following universal conditions:

$$\begin{aligned}
 x \preceq_S y &\text{ iff } f(x) \cap \overline{f(y)} = \emptyset; & x \succeq_S y &\text{ iff } \overline{f(x)} \cap f(y) = \emptyset; \\
 x \triangle_S y &\text{ iff } f(x) \cap f(y) = \emptyset; & x \nabla_S y &\text{ iff } \overline{f(x)} \cap \overline{f(y)} = \emptyset.
 \end{aligned}$$

Within the context of attribute systems, Orłowska [8] has named “forward inclusion”, “backward inclusion”, “right orthogonality”, “left orthogonality” our emptiness relations $\preceq_S$, $\succeq_S$, $\triangle_S$, ∇_S . The property system of table 1 is such that $Ann \preceq_S Bob$, $Ann \succeq_S Cindy$, $Ann \triangle_S Daniel$ and $Ann \nabla_S Emma$. We leave it to the reader to prove the following lemma.

Lemma 1. *Let $S = (Obj, Pro, f)$ be a non-trivial property system. The following universal conditions are satisfied.*

$$\begin{aligned}
 &x \preceq_S x; & &x \succeq_S x; \\
 &\text{If } x \preceq_S y \text{ then } y \succeq_S x; & &\text{If } x \succeq_S y \text{ then } y \preceq_S x; \\
 &\text{If } x \preceq_S y \text{ and } y \preceq_S z \text{ then } x \preceq_S z; & &\text{If } x \succeq_S y \text{ and } y \succeq_S z \text{ then } x \succeq_S z; \\
 &\text{If } x \preceq_S y \text{ and } y \triangle_S z \text{ then } x \triangle_S z; & &\text{If } x \succeq_S y \text{ and } y \nabla_S z \text{ then } x \nabla_S z; \\
 &x \triangle_S x; & &x \nabla_S x; \\
 &\text{If } x \triangle_S y \text{ then } y \triangle_S x; & &\text{If } x \nabla_S y \text{ then } y \nabla_S x; \\
 &\text{If } x \triangle_S y \text{ and } y \succeq_S z \text{ then } x \triangle_S z; & &\text{If } x \nabla_S y \text{ and } y \preceq_S z \text{ then } x \nabla_S z; \\
 &\text{If } x \triangle_S y \text{ and } y \nabla_S z \text{ then } x \preceq_S z; & &\text{If } x \nabla_S y \text{ and } y \triangle_S z \text{ then } x \succeq_S z.
 \end{aligned}$$

Lemma 1 motivates the following definition. An emptiness structure will be any relational structure $(W, \preceq, \succeq, \Delta, \nabla)$ where:

- W is a nonempty set of possible worlds;
- $\preceq, \succeq, \Delta, \nabla$ are binary relations on W subject to the universal conditions of lemma 1.

The following theorem explains the connection between emptiness structures and non-trivial property systems.

Theorem 1. (Characterization theorem for emptiness structures) *Let $(W, \preceq, \succeq, \Delta, \nabla)$ be an emptiness structure. There is a non-trivial property system $S = (Obj, Pro, f)$ such that $Obj = W$ and the following universal conditions are satisfied.*

$$\begin{array}{ll} x \preceq_S y \text{ iff } x \preceq y; & x \succeq_S y \text{ iff } x \succeq y; \\ x \Delta_S y \text{ iff } x \Delta y; & x \nabla_S y \text{ iff } x \nabla y. \end{array}$$

Proof. It follows immediately from the definition that the following universal conditions are satisfied:

$$\begin{array}{ll} x \preceq x; & x \succeq x; \\ \text{If } x \preceq y \text{ then } y \succeq x; & \text{If } x \succeq y \text{ then } y \preceq x; \\ \text{If } x \preceq y \text{ and } y \preceq z \text{ then } x \preceq z; & \text{If } x \succeq y \text{ and } y \succeq z \text{ then } x \succeq z; \\ \text{If } x \overline{\Delta} y \text{ then } x \overline{\Delta} x; & \text{If } x \overline{\nabla} y \text{ then } x \overline{\nabla} x; \\ x \overline{\Delta} x \text{ or } x \preceq y; & x \overline{\nabla} x \text{ or } x \succeq y; \\ \text{If } x \overline{\Delta} y \text{ then } y \overline{\Delta} x; & \text{If } x \overline{\nabla} y \text{ then } y \overline{\nabla} x; \\ \text{If } x \overline{\Delta} y \text{ and } y \preceq z \text{ then } x \overline{\Delta} z; & \text{If } x \overline{\nabla} y \text{ and } y \succeq z \text{ then } x \overline{\nabla} z; \\ x \overline{\Delta} y \text{ or } y \overline{\nabla} z \text{ or } x \preceq z; & x \overline{\nabla} y \text{ or } y \overline{\Delta} z \text{ or } x \succeq z. \end{array}$$

By Vakarelov [16], there is a property system $S = (Obj, Pro, f)$ such that $Obj = W$ and the following universal conditions are satisfied:

$$\begin{array}{ll} x \preceq_S y \text{ iff } x \preceq y; & x \succeq_S y \text{ iff } x \succeq y; \\ x \Delta_S y \text{ iff } x \Delta y; & x \nabla_S y \text{ iff } x \nabla y. \end{array}$$

In view of the fact that for all $x \in W$, $x \overline{\Delta} x$ and $x \overline{\nabla} x$, for all $x \in Obj$, $f(x) \neq \emptyset$ and $f(x) \neq Pro$. Consequently S is non-trivial.

3 Modal analysis

Let us be clear that our emptiness relations are nothing but the relations of informational inclusion and the complementary relations of the relations of similarity introduced by Vakarelov [16]. In this section, their modal analysis is presented for the very first time. An abstract structure will be any structure $(W, \preceq, \succeq, \Delta, \nabla, \equiv, R)$ where:

- W is a nonempty set of possible worlds;
- $\preceq, \succeq, \Delta, \nabla$ are binary relations on W subject to the universal conditions of lemma 1;

- $\equiv, R$ are binary relations on W subject to the following universal conditions:
 - $x \equiv y$ iff $x \preceq y$ and $x \succeq y$;
 - xRy iff $x \triangle y$ and $x \nabla y$.

The proof of the following lemma is left as an exercise for the reader.

Lemma 2. *Let $(W, \preceq, \succeq, \triangle, \nabla, \equiv, R)$ be an abstract structure. The following universal conditions are satisfied.*

<i>If $x \equiv y$ then $x \preceq y$;</i>	<i>If xRy then $x \triangle y$;</i>
<i>If $x \equiv y$ then $x \succeq y$;</i>	<i>If xRy then $x \nabla y$;</i>
<i>$x \equiv x$;</i>	
<i>If $x \equiv y$ then $y \equiv x$;</i>	<i>If xRy then yRx;</i>
<i>If $x \equiv y$ and $y \equiv z$ then $x \equiv z$;</i>	<i>If xRy and $y \equiv z$ then xRz;</i>
<i>If $x \equiv y$ and yRz then xRz;</i>	<i>If xRy and yRz then $x \equiv z$.</i>

Lemma 2 motivates the following definition. A nonstandard abstract structure will be any structure $(W, \preceq, \succeq, \triangle, \nabla, \equiv, R)$ where:

- W is a nonempty set of possible worlds;
- $\preceq, \succeq, \triangle, \nabla$ are binary relations on W subject to the universal conditions of lemma 1 but the universal conditions “ $x \triangle x$ ”, “ $x \nabla x$ ”;
- $\equiv, R$ are binary relations on W subject to the universal conditions of lemma 2.

The linguistic basis of our modal logic is the propositional calculus enlarged with the modalities $[\preceq]$, $[\succeq]$, $[\triangle]$, $[\nabla]$, $[\equiv]$, $[R]$. We define the set of all formulas as follows:

- $A ::= p \mid \neg A \mid (A \vee B) \mid [\preceq]A \mid [\succeq]A \mid [\triangle]A \mid [\nabla]A \mid [\equiv]A \mid [R]A$;

where p ranges over a countably infinite set of propositional variables. The other standard connectives are defined by the usual abbreviations. In particular, $\langle \preceq \rangle A$ is $\neg[\preceq]\neg A$, $\langle \succeq \rangle A$ is $\neg[\succeq]\neg A$, $\langle \triangle \rangle A$ is $\neg[\triangle]\neg A$, $\langle \nabla \rangle A$ is $\neg[\nabla]\neg A$, $\langle \equiv \rangle A$ is $\neg[\equiv]\neg A$, $\langle R \rangle A$ is $\neg[R]\neg A$. We follow the standard rules for omission of the parentheses. The number of occurrences of symbols in formula A is denoted $length(A)$. A model (respectively: a nonstandard model) will be any structure $(W, \preceq, \succeq, \triangle, \nabla, \equiv, R, V)$ where:

- $(W, \preceq, \succeq, \triangle, \nabla, \equiv, R)$ is an abstract structure (respectively: a nonstandard abstract structure);
- V is a function with domain the set of all propositional variables and range the power set of W .

Let $M = (W, \preceq, \succeq, \triangle, \nabla, \equiv, R, V)$ be either a model or a nonstandard model. We define the relation “formula A is true at possible world x in M ”, denoted $M, x \models A$, as follows:

- $M, x \models p$ iff $x \in V(p)$;
- $M, x \models \neg A$ iff $M, x \not\models A$;

- $M, x \models A \vee B$ iff $M, x \models A$ or $M, x \models B$;
- $M, x \models [\preceq]A$ iff for all $y \in W$, if $x \preceq y$ then $M, y \models A$;
- $M, x \models [\succeq]A$ iff for all $y \in W$, if $x \succeq y$ then $M, y \models A$;
- $M, x \models [\triangle]A$ iff for all $y \in W$, if $x \triangle y$ then $M, y \models A$;
- $M, x \models [\nabla]A$ iff for all $y \in W$, if $x \nabla y$ then $M, y \models A$;
- $M, x \models [\equiv]A$ iff for all $y \in W$, if $x \equiv y$ then $M, y \models A$;
- $M, x \models [R]A$ iff for all $y \in W$, if xRy then $M, y \models A$.

Let *LER* — logic of emptiness relations — be the smallest normal modal logic that contains the following axioms:

$$\begin{array}{ll}
[\preceq]A \rightarrow A; & [\succeq]A \rightarrow A; \\
A \rightarrow [\preceq]\langle \succeq \rangle A; & A \rightarrow [\succeq]\langle \preceq \rangle A; \\
[\preceq]A \rightarrow [\preceq][\preceq]A; & [\succeq]A \rightarrow [\succeq][\succeq]A; \\
[\triangle]A \rightarrow [\preceq][\triangle]A; & [\nabla]A \rightarrow [\succeq][\nabla]A; \\
A \rightarrow [\triangle]\langle \triangle \rangle A; & A \rightarrow [\nabla]\langle \nabla \rangle A; \\
[\triangle]A \rightarrow [\triangle][\succeq]A; & [\nabla]A \rightarrow [\nabla][\preceq]A; \\
[\preceq]A \rightarrow [\triangle][\nabla]A; & [\succeq]A \rightarrow [\nabla][\triangle]A; \\
[\preceq]A \rightarrow [\equiv]A; & [\triangle]A \rightarrow [R]A; \\
[\succeq]A \rightarrow [\equiv]A; & [\nabla]A \rightarrow [R]A; \\
[\equiv]A \rightarrow A; & \\
A \rightarrow [\equiv]\langle \equiv \rangle A; & A \rightarrow [R]\langle R \rangle A; \\
[\equiv]A \rightarrow [\equiv][\equiv]A; & [R]A \rightarrow [R][\equiv]A; \\
[R]A \rightarrow [\equiv][R]A; & [\equiv]A \rightarrow [R][R]A.
\end{array}$$

A typical result is the following.

Theorem 2. (Completeness theorem for *LER*) *LER is complete with respect to the class of all models, the class of all finite nonstandard models and the class of all nonstandard models, i.e. the following conditions are equivalent.*

1. *A is true at all possible worlds in all models;*
2. *A is true at all possible worlds in all finite nonstandard models;*
3. *A is true at all possible worlds in all nonstandard models;*
4. *A is a theorem of LER.*

Proof. (1 implies 2): Let $M = (W, \preceq, \succeq, \triangle, \nabla, \equiv, R, V)$ be a finite nonstandard model and $M' = (W', \preceq', \succeq', \triangle', \nabla', \equiv', R', V')$ be the structure defined as follows. Define:

- W' is $W \times \mathbb{Z}^* \times \mathbb{Z}^*$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, $(x, i_1, i_2) \preceq'$ (y, j_1, j_2) iff $x \preceq y$, $i_1 = j_1$, $i_2 \preceq j_2$ and if $i_2 \succeq j_2$ then $x \equiv y$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, $(x, i_1, i_2) \succeq'$ (y, j_1, j_2) iff $x \succeq y$, $i_1 = j_1$, $i_2 \succeq j_2$ and if $i_2 \preceq j_2$ then $x \equiv y$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, $(x, i_1, i_2) \triangle'$ (y, j_1, j_2) iff $x \triangle y$, $i_1 = -j_1$, $i_2 \preceq -j_2$ and if $i_2 \succeq -j_2$ then xRy ;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, $(x, i_1, i_2) \nabla'$ (y, j_1, j_2) iff $x \nabla y$, $i_1 = -j_1$, $i_2 \succeq -j_2$ and if $i_2 \preceq -j_2$ then xRy ;

- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, $(x, i_1, i_2) \equiv' (y, j_1, j_2)$ iff $x \equiv y$, $i_1 = j_1$ and $i_2 = j_2$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, $(x, i_1, i_2) R' (y, j_1, j_2)$ iff $x R y$, $i_1 = -j_1$ and $i_2 = -j_2$;
- For all propositional variables p , $V'(p)$ is $V(p) \times \mathbb{Z}^* \times \mathbb{Z}^*$.

It is a simple matter to check that M' is a model. Moreover, M is a p-morphic image of M' , i.e., the following conditions are satisfied:

- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, if $(x, i_1, i_2) \preceq' (y, j_1, j_2)$ then $x \preceq y$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, if $(x, i_1, i_2) \succeq' (y, j_1, j_2)$ then $x \succeq y$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, if $(x, i_1, i_2) \triangle' (y, j_1, j_2)$ then $x \triangle y$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, if $(x, i_1, i_2) \nabla' (y, j_1, j_2)$ then $x \nabla y$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, if $(x, i_1, i_2) \equiv' (y, j_1, j_2)$ then $x \equiv y$;
- For all $x, y \in W$, for all $i_1, i_2 \in \mathbb{Z}^*$ and for all $j_1, j_2 \in \mathbb{Z}^*$, if $(x, i_1, i_2) R' (y, j_1, j_2)$ then $x R y$;

and the following conditions are satisfied:

- For all $x, y \in W$ and for all $i_1, i_2 \in \mathbb{Z}^*$, there is $j_1, j_2 \in \mathbb{Z}^*$ such that if $x \preceq y$ then $(x, i_1, i_2) \preceq' (y, j_1, j_2)$;
- For all $x, y \in W$ and for all $i_1, i_2 \in \mathbb{Z}^*$, there is $j_1, j_2 \in \mathbb{Z}^*$ such that if $x \succeq y$ then $(x, i_1, i_2) \succeq' (y, j_1, j_2)$;
- For all $x, y \in W$ and for all $i_1, i_2 \in \mathbb{Z}^*$, there is $j_1, j_2 \in \mathbb{Z}^*$ such that if $x \triangle y$ then $(x, i_1, i_2) \triangle' (y, j_1, j_2)$;
- For all $x, y \in W$ and for all $i_1, i_2 \in \mathbb{Z}^*$, there is $j_1, j_2 \in \mathbb{Z}^*$ such that if $x \nabla y$ then $(x, i_1, i_2) \nabla' (y, j_1, j_2)$;
- For all $x, y \in W$ and for all $i_1, i_2 \in \mathbb{Z}^*$, there is $j_1, j_2 \in \mathbb{Z}^*$ such that if $x \equiv y$ then $(x, i_1, i_2) \equiv' (y, j_1, j_2)$;
- For all $x, y \in W$ and for all $i_1, i_2 \in \mathbb{Z}^*$, there is $j_1, j_2 \in \mathbb{Z}^*$ such that if $x R y$ then $(x, i_1, i_2) R' (y, j_1, j_2)$.

As a consequence, see Hughes and Cresswell [4], if A is true at some possible world in M then A is true at some possible world in M' .

(2 implies 3): Let $M = (W, \preceq, \succeq, \triangle, \nabla, \equiv, R, V)$ be a nonstandard model and $M' = (W', \preceq', \succeq', \triangle', \nabla', \equiv', R', V')$ be the structure defined as follows. We define the smallest set Γ_A of formulas such that:

- $A \in \Gamma_A$;
- Γ_A is closed under subformulas.

It should be remarked that $\text{Card}(\Gamma_A) \leq \text{length}(A)$. Let $=_A$ be the equivalence relation on W defined as follows. For all $x, y \in W$, define:

- $x =_A y$ iff for all formulas B , if $B \in \Gamma_A$ then $M, x \models B$ iff $M, y \models B$.

For all $x \in W$, the equivalence class of x modulo $=_A$ is denoted $x|_{=_A}$. The quotient set of W modulo $=_A$ is denoted $W|_{=_A}$. For all propositional variables p , the quotient set of $V(p)$ modulo $=_A$ is denoted by $V(p)|_{=_A}$. Define:

- W' is $W|_{=_A}$;
- For all $x, y \in W$, $x|_{=_A} \preceq' y|_{=_A}$ iff for all formulas B , if $B \in \Gamma_A$ then:
 - If $M, x \models [\preceq]B$ then $M, y \models [\preceq]B$;
 - If $M, x \models [\triangle]B$ then $M, y \models [\triangle]B$;
 - If $M, y \models [\succeq]B$ then $M, x \models [\succeq]B$;
 - If $M, y \models [\nabla]B$ then $M, x \models [\nabla]B$;
- For all $x, y \in W$, $x|_{=_A} \succeq' y|_{=_A}$ iff for all formulas B , if $B \in \Gamma_A$ then:
 - If $M, x \models [\succeq]B$ then $M, y \models [\succeq]B$;
 - If $M, x \models [\nabla]B$ then $M, y \models [\nabla]B$;
 - If $M, y \models [\preceq]B$ then $M, x \models [\preceq]B$;
 - If $M, y \models [\triangle]B$ then $M, x \models [\triangle]B$;
- For all $x, y \in W$, $x|_{=_A} \triangle' y|_{=_A}$ iff for all formulas B , if $B \in \Gamma_A$ then:
 - If $M, x \models [\preceq]B$ then $M, y \models [\nabla]B$;
 - If $M, x \models [\triangle]B$ then $M, y \models [\succeq]B$;
 - If $M, y \models [\preceq]B$ then $M, x \models [\nabla]B$;
 - If $M, y \models [\triangle]B$ then $M, x \models [\succeq]B$;
- For all $x, y \in W$, $x|_{=_A} \nabla' y|_{=_A}$ iff for all formulas B , if $B \in \Gamma_A$ then:
 - If $M, x \models [\succeq]B$ then $M, y \models [\triangle]B$;
 - If $M, x \models [\nabla]B$ then $M, y \models [\preceq]B$;
 - If $M, y \models [\succeq]B$ then $M, x \models [\triangle]B$;
 - If $M, y \models [\nabla]B$ then $M, x \models [\preceq]B$;
- For all $x, y \in W$, $x|_{=_A} \equiv' y|_{=_A}$ iff for all formulas B , if $B \in \Gamma_A$ then:
 - If $M, x \models [\preceq]B$ then $M, y \models [\preceq]B$;
 - If $M, x \models [\succeq]B$ then $M, y \models [\succeq]B$;
 - If $M, x \models [\triangle]B$ then $M, y \models [\triangle]B$;
 - If $M, x \models [\nabla]B$ then $M, y \models [\nabla]B$;
 - If $M, x \models [\equiv]B$ then $M, y \models [\equiv]B$;
 - If $M, x \models [R]B$ then $M, y \models [R]B$;
 - If $M, y \models [\preceq]B$ then $M, x \models [\preceq]B$;
 - If $M, y \models [\succeq]B$ then $M, x \models [\succeq]B$;
 - If $M, y \models [\triangle]B$ then $M, x \models [\triangle]B$;
 - If $M, y \models [\nabla]B$ then $M, x \models [\nabla]B$;
 - If $M, y \models [\equiv]B$ then $M, x \models [\equiv]B$;
 - If $M, y \models [R]B$ then $M, x \models [R]B$;
- For all $x, y \in W$, $x|_{=_A} R' y|_{=_A}$ iff for all formulas B , if $B \in \Gamma_A$ then:
 - If $M, x \models [\preceq]B$ then $M, y \models [\nabla]B$;
 - If $M, x \models [\succeq]B$ then $M, y \models [\triangle]B$;
 - If $M, x \models [\triangle]B$ then $M, y \models [\succeq]B$;
 - If $M, x \models [\nabla]B$ then $M, y \models [\preceq]B$;
 - If $M, x \models [\equiv]B$ then $M, y \models [R]B$;
 - If $M, x \models [R]B$ then $M, y \models [\equiv]B$;

- If $M, y \models [\preceq]B$ then $M, x \models [\nabla]B$;
- If $M, y \models [\succeq]B$ then $M, x \models [\Delta]B$;
- If $M, y \models [\Delta]B$ then $M, x \models [\succeq]B$;
- If $M, y \models [\nabla]B$ then $M, x \models [\preceq]B$;
- If $M, y \models [\equiv]B$ then $M, x \models [R]B$;
- If $M, y \models [R]B$ then $M, x \models [\equiv]B$;
- For all propositional variables p , $V'(p)$ is $V(p)|_{=A}$.

It is a rather remarkable fact that M' is a finite nonstandard model. The interesting result is that M' is a filtration of M , i.e., the following conditions are satisfied:

- For all $x, y \in W$, if $x \preceq y$ then $x|_{=A} \preceq' y|_{=A}$;
- For all $x, y \in W$, if $x \succeq y$ then $x|_{=A} \succeq' y|_{=A}$;
- For all $x, y \in W$, if $x \Delta y$ then $x|_{=A} \Delta' y|_{=A}$;
- For all $x, y \in W$, if $x \nabla y$ then $x|_{=A} \nabla' y|_{=A}$;
- For all $x, y \in W$, if $x \equiv y$ then $x|_{=A} \equiv' y|_{=A}$;
- For all $x, y \in W$, if xRy then $x|_{=A} R'y|_{=A}$;

and the following conditions are satisfied:

- For all $x, y \in W$ and for all formulas B , if $x|_{=A} \preceq' y|_{=A}$, $B \in \Gamma_A$ and $M, x \models [\preceq]B$ then $M, y \models B$;
- For all $x, y \in W$ and for all formulas B , if $x|_{=A} \succeq' y|_{=A}$, $B \in \Gamma_A$ and $M, x \models [\succeq]B$ then $M, y \models B$;
- For all $x, y \in W$ and for all formulas B , if $x|_{=A} \Delta' y|_{=A}$, $B \in \Gamma_A$ and $M, x \models [\Delta]B$ then $M, y \models B$;
- For all $x, y \in W$ and for all formulas B , if $x|_{=A} \nabla' y|_{=A}$, $B \in \Gamma_A$ and $M, x \models [\nabla]B$ then $M, y \models B$;
- For all $x, y \in W$ and for all formulas B , if $x|_{=A} \equiv' y|_{=A}$, $B \in \Gamma_A$ and $M, x \models [\equiv]B$ then $M, y \models B$;
- For all $x, y \in W$ and for all formulas B , if $x|_{=A} R'y|_{=A}$, $B \in \Gamma_A$ and $M, x \models [R]B$ then $M, y \models B$.

As a consequence, see Hughes and Cresswell [4], if A is true at some possible world in M then A is true at some possible world in M' .

(3 implies 4): The proof can be obtained by the canonical model construction.

(4 implies 1): The proof is trivial because models satisfy the conditions which are needed to verify the axioms of LER .

We now turn our attention to the decidability of the satisfiability problem for any given formula of LER .

Theorem 3. (Decidability theorem for LER) *The satisfiability problem for any given formula of LER is decidable.*

Proof. By theorem 2, LER is a finitely axiomatizable normal modal logic which has the finite model property. As a consequence, the satisfiability problem for any given formula of LER is decidable.

A less obvious result concerns the inherent difficulty of the satisfiability problem for any given formula of LER . What we have in mind is to prove that the satisfiability problem for any given $\{[\equiv], [R]\}$ -free formula A of LER is $PSPACE$ -complete in $length(A)$. We outline how this result can be proved, but leave the details to the rest of the paper. In section 4, we show how the satisfiability problem for any given formula of $S4$ can be linearly reduced to the satisfiability problem for any given $\{[\equiv], [R]\}$ -free formula of LER . In section 5, we show how the satisfiability problem for any given $\{[\equiv], [R]\}$ -free formula of LER can be solved by means of a polynomial-space bounded nondeterministic algorithm.

4 $PSPACE$ -hardness

We define the set of all formulas of $S4$ as follows:

- $A ::= p \mid \neg A \mid (A \vee B) \mid \Box A$;

where p ranges over a countably infinite set of propositional variables. We follow the standard rules for omission of the parentheses. The number of occurrences of symbols in formula A of $S4$ is denoted $length(A)$. A model of $S4$ is a structure $(W, |, V)$ where:

- W is a nonempty set of possible worlds;
- $|$ is a binary relation on W such that for all $x, y \in W$:
 - $x | x$;
 - If $x | y$ and $y | z$ then $x | z$;
- V is a function with domain the set of all propositional variables and range the power set of W .

Let $M = (W, |, V)$ be a model of $S4$. We define the relation “formula A of $S4$ is true at possible world x in M ”, denoted $M, x \models A$, as follows:

- $M, x \models p$ iff $x \in V(p)$;
- $M, x \models \neg A$ iff $M, x \not\models A$;
- $M, x \models A \vee B$ iff $M, x \models A$ or $M, x \models B$;
- $M, x \models \Box A$ iff for all $y \in W$, if $x | y$ then $M, y \models A$.

Let t be the function that assigns to each formula A of $S4$ the $\{[\equiv], [R]\}$ -free formula $t(A)$ of LER as follows:

- $t(p) = p$;
- $t(\neg A) = \neg t(A)$;
- $t(A \vee B) = t(A) \vee t(B)$;
- $t(\Box A) = [\preceq]t(A)$.

The interesting result is the following.

Lemma 3. *For all formulas A of $S4$, the following conditions are equivalent.*

1. A is true at some possible world in some model of $S4$;

2. $t(A)$ is true at some possible world in some model of LER .

Proof. (1 implies 2): Let $M = (W, |, V)$ be a model of $S4$ and $M' = (W', \preceq', \succeq', \Delta', \nabla', V')$ be the structure defined as follows. Define:

- W' is W ;
- For all $x, y \in W$, $x \preceq' y$ iff $x | y$;
- For all $x, y \in W$, $x \succeq' y$ iff $y | x$;
- For all $x, y \in W$, $x \Delta' y$;
- For all $x, y \in W$, $x \nabla' y$;
- For all propositional variables p , $V'(p)$ is $V(p)$.

It is a simple matter to check that M' is a model of LER . The reader may easily verify by induction on the *length* of formula A of $S4$ that if A is true at some possible world in M then $t(A)$ is true at some possible world in M' .

(2 implies 1): Let $M = (W, \preceq, \succeq, \Delta, \nabla, V)$ be a model of LER and $M' = (W', |, V')$ be the structure defined as follows. Define:

- W' is W ;
- For all $x, y \in W$, $x | y$ iff $x \preceq y$;
- For all propositional variables p , $V'(p)$ is $V(p)$.

It is a rather remarkable fact that M' is a model of $S4$. The reader may easily verify by induction on the *length* of formula A of $S4$ that if $t(A)$ is true at some possible world in M then A is true at some possible world in M' .

Seeing that the satisfiability problem for any given formula A of $S4$ is $PSPACE$ -hard in $length(A)$, see Ladner [5], we infer immediately the following.

Theorem 4. ($PSPACE$ -hardness theorem for LER) *The satisfiability problem for any given $\{[\equiv], [R]\}$ -free formula A of LER is $PSPACE$ -hard in $length(A)$.*

5 $PSPACE$ -completeness

Following the line of reasoning suggested by Spaan [12], let Cl be the function that assigns to each string $\omega \in \{\preceq, \succeq, \Delta, \nabla\}^*$ and to each $\{[\equiv], [R]\}$ -free formula A of LER the smallest set $Cl(\omega, A)$ of $\{[\equiv], [R]\}$ -free formulas of LER as follows:

- $A \in Cl(\epsilon, A)$;
- $Cl(\omega, A)$ is closed under subformulas;
- For all $\{[\equiv], [R]\}$ -free formulas B of LER :
 - If $[\preceq]B \in Cl(\omega, A)$ then $[\preceq]B \in Cl(\omega \preceq, A)$ and $[\nabla]B \in Cl(\omega \Delta, A)$;
 - If $[\succeq]B \in Cl(\omega, A)$ then $[\succeq]B \in Cl(\omega \succeq, A)$ and $[\Delta]B \in Cl(\omega \nabla, A)$;
 - If $[\Delta]B \in Cl(\omega, A)$ then $[\Delta]B \in Cl(\omega \preceq, A)$ and $[\succeq]B \in Cl(\omega \Delta, A)$;
 - If $[\nabla]B \in Cl(\omega, A)$ then $[\nabla]B \in Cl(\omega \succeq, A)$ and $[\preceq]B \in Cl(\omega \nabla, A)$;

where ϵ denotes the empty string. The following lemma is basic.

Lemma 4. *Let A be a $\{\equiv, [R]\}$ -free formula of LER.*

- $Card(Cl(\epsilon, A)) \leq length(A)$;
- $Card(Cl(\preceq, A)) \leq length(A)$;
- $Card(Cl(\succeq, A)) \leq length(A)$;
- $Card(Cl(\Delta, A)) \leq length(A)$;
- $Card(Cl(\nabla, A)) \leq length(A)$;
- For all strings $\omega \in \{\preceq, \succeq, \Delta, \nabla\}^*$:
 - $Card(Cl(\omega \preceq \preceq, A)) = Card(Cl(\omega \preceq, A))$;
 - $Card(Cl(\omega \preceq \succeq, A)) < Card(Cl(\omega \preceq, A))$;
 - $Card(Cl(\omega \preceq \Delta, A)) = Card(Cl(\omega \preceq, A))$;
 - $Card(Cl(\omega \preceq \nabla, A)) < Card(Cl(\omega \preceq, A))$;
 - $Card(Cl(\omega \succeq \preceq, A)) < Card(Cl(\omega \succeq, A))$;
 - $Card(Cl(\omega \succeq \succeq, A)) = Card(Cl(\omega \succeq, A))$;
 - $Card(Cl(\omega \succeq \Delta, A)) < Card(Cl(\omega \succeq, A))$;
 - $Card(Cl(\omega \succeq \nabla, A)) = Card(Cl(\omega \succeq, A))$;
 - $Card(Cl(\omega \Delta \preceq, A)) < Card(Cl(\omega \Delta, A))$;
 - $Card(Cl(\omega \Delta \succeq, A)) = Card(Cl(\omega \Delta, A))$;
 - $Card(Cl(\omega \Delta \Delta, A)) < Card(Cl(\omega \Delta, A))$;
 - $Card(Cl(\omega \Delta \nabla, A)) = Card(Cl(\omega \Delta, A))$;
 - $Card(Cl(\omega \nabla \preceq, A)) = Card(Cl(\omega \nabla, A))$;
 - $Card(Cl(\omega \nabla \succeq, A)) < Card(Cl(\omega \nabla, A))$;
 - $Card(Cl(\omega \nabla \Delta, A)) = Card(Cl(\omega \nabla, A))$;
 - $Card(Cl(\omega \nabla \nabla, A)) < Card(Cl(\omega \nabla, A))$.

Proof. We first observe that $Cl(\epsilon, A)$ is the set of all subformulas of A . As a consequence $Card(Cl(\epsilon, A)) \leq length(A)$. It is a simple matter to check that $Cl(\preceq, A)$ does not contain more formulas than $Cl(\epsilon, A)$. Hence we have $Card(Cl(\preceq, A)) \leq length(A)$. Similarly $Card(Cl(\succeq, A)) \leq length(A)$, $Card(Cl(\Delta, A)) \leq length(A)$ and $Card(Cl(\nabla, A)) \leq length(A)$. It is a rather remarkable fact that $Cl(\omega \preceq \preceq, A)$ does not contain less formulas than $Cl(\omega \preceq, A)$. Hence we have $Card(Cl(\omega \preceq \preceq, A)) = Card(Cl(\omega \preceq, A))$. Similarly $Card(Cl(\omega \preceq \Delta, A)) = Card(Cl(\omega \preceq, A))$, $Card(Cl(\omega \succeq \succeq, A)) = Card(Cl(\omega \succeq, A))$, $Card(Cl(\omega \succeq \nabla, A)) = Card(Cl(\omega \succeq, A))$, $Card(Cl(\omega \Delta \succeq, A)) = Card(Cl(\omega \Delta, A))$, $Card(Cl(\omega \Delta \nabla, A)) = Card(Cl(\omega \Delta, A))$, $Card(Cl(\omega \nabla \preceq, A)) = Card(Cl(\omega \nabla, A))$ and $Card(Cl(\omega \nabla \Delta, A)) = Card(Cl(\omega \nabla, A))$. The reader can readily check that $Cl(\omega \preceq \succeq, A)$ contains less formulas than $Cl(\omega \preceq, A)$. Hence we have $Card(Cl(\omega \preceq \succeq, A)) < Card(Cl(\omega \preceq, A))$. Similarly $Card(Cl(\omega \preceq \nabla, A)) < Card(Cl(\omega \preceq, A))$, $Card(Cl(\omega \succeq \preceq, A)) < Card(Cl(\omega \succeq, A))$, $Card(Cl(\omega \succeq \Delta, A)) < Card(Cl(\omega \succeq, A))$, $Card(Cl(\omega \Delta \preceq, A)) < Card(Cl(\omega \Delta, A))$, $Card(Cl(\omega \Delta \Delta, A)) < Card(Cl(\omega \Delta, A))$, $Card(Cl(\omega \nabla \succeq, A)) < Card(Cl(\omega \nabla, A))$ and $Card(Cl(\omega \nabla \nabla, A)) < Card(Cl(\omega \nabla, A))$.

To continue we introduce the function *depth* that assigns to each string $\omega \in \{\preceq, \succeq, \Delta, \nabla\}^*$ the positive integer *depth*(ω) as follows:

- $depth(\epsilon) = 0$;
- $depth(\preceq) = 0$;
- $depth(\succeq) = 0$;
- $depth(\triangle) = 0$;
- $depth(\nabla) = 0$;
- For all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$:
 - $depth(\omega \preceq \preceq) = depth(\omega \preceq)$;
 - $depth(\omega \preceq \succeq) = depth(\omega \preceq) + 1$;
 - $depth(\omega \preceq \triangle) = depth(\omega \preceq)$;
 - $depth(\omega \preceq \nabla) = depth(\omega \preceq) + 1$;
 - $depth(\omega \succeq \preceq) = depth(\omega \succeq) + 1$;
 - $depth(\omega \succeq \succeq) = depth(\omega \succeq)$;
 - $depth(\omega \succeq \triangle) = depth(\omega \succeq) + 1$;
 - $depth(\omega \succeq \nabla) = depth(\omega \succeq)$;
 - $depth(\omega \triangle \preceq) = depth(\omega \triangle) + 1$;
 - $depth(\omega \triangle \succeq) = depth(\omega \triangle)$;
 - $depth(\omega \triangle \triangle) = depth(\omega \triangle) + 1$;
 - $depth(\omega \triangle \nabla) = depth(\omega \triangle)$;
 - $depth(\omega \nabla \preceq) = depth(\omega \nabla)$;
 - $depth(\omega \nabla \succeq) = depth(\omega \nabla) + 1$;
 - $depth(\omega \nabla \triangle) = depth(\omega \nabla)$;
 - $depth(\omega \nabla \nabla) = depth(\omega \nabla) + 1$.

For our purpose, the crucial property of the function $depth$ is the following.

Lemma 5. *For all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$ and for all $\{[\equiv], [R]\}$ -free formulas A of LER , if $depth(\omega) \geq length(A)$ then $Cl(\omega, A) = \emptyset$.*

Proof. The proof is conducted by induction on the *length* of $\{[\equiv], [R]\}$ -free formula A of LER .

Basis: Let p be a propositional variable. For all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$, if $depth(\omega) \geq length(p)$ then $depth(\omega) \geq 1$. Hence $Cl(\omega, p) = \emptyset$.

Hypothesis: Let B, C be $\{[\equiv], [R]\}$ -free formulas of LER such that for all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$, if $depth(\omega) \geq length(B)$ then $Cl(\omega, B) = \emptyset$ and if $depth(\omega) \geq length(C)$ then $Cl(\omega, C) = \emptyset$.

Step: For all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$, if $depth(\omega) \geq length(\neg B)$ then $depth(\omega) > length(B)$. It follows that $Cl(\omega, B) = \emptyset$ and $Cl(\omega, \neg B) \subseteq Cl(\omega, B)$. Therefore $Cl(\omega, \neg B) = \emptyset$.

For all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$, if $depth(\omega) \geq length(B \vee C)$ then $depth(\omega) > length(B)$ and $depth(\omega) > length(C)$. It follows that $Cl(\omega, B) = \emptyset$, $Cl(\omega, C) = \emptyset$ and $Cl(\omega, B \vee C) \subseteq Cl(\omega, B) \cup Cl(\omega, C)$. Therefore $Cl(\omega, B \vee C) = \emptyset$.

For all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$, if $depth(\omega) \geq length([\preceq]B)$ then $depth(\omega) > length(B)$. It follows that there is a string $\omega' \in \{\preceq, \succeq, \triangle, \nabla\}^*$ such that $\omega = \omega' \preceq \preceq$ or $\omega = \omega' \preceq \nabla$ or $\omega = \omega' \succeq \preceq$ or $\omega = \omega' \succeq \triangle$ or $\omega = \omega' \triangle \preceq$ or $\omega = \omega' \triangle \triangle$ or $\omega = \omega' \nabla \preceq$ or $\omega = \omega' \nabla \nabla$. Suppose that $\omega = \omega' \preceq \preceq$. Hence $depth(\omega' \preceq) \geq length(B)$. It follows that $Cl(\omega' \preceq, B) = \emptyset$ and $Cl(\omega, [\preceq]B) \subseteq Cl(\omega' \preceq, B)$. Therefore $Cl(\omega, [\preceq]B) = \emptyset$. The same line of reasoning applies if $\omega = \omega' \preceq \nabla$ or $\omega = \omega' \succeq \preceq$ or $\omega = \omega' \succeq \triangle$ or $\omega = \omega' \triangle \preceq$ or $\omega = \omega' \triangle \triangle$ or $\omega = \omega' \nabla \preceq$ or $\omega = \omega' \nabla \nabla$.

For all strings $\omega \in \{\preceq, \succeq, \Delta, \nabla\}^*$, the same line of reasoning applies if $\text{depth}(\omega) \geq \text{length}([\succeq]B)$ or $\text{depth}(\omega) \geq \text{length}([\Delta]B)$ or $\text{depth}(\omega) \geq \text{length}([\nabla]B)$.

Let S be a set of $\{[\equiv], [R]\}$ -free formulas of LER and A be a $\{[\equiv], [R]\}$ -free formula of LER . A $(\preceq, A)$ -witness of S will be any set T of $\{[\equiv], [R]\}$ -free formulas of LER such that $A \notin T$ and for all $\{[\equiv], [R]\}$ -free formulas B of LER :

- If $[\preceq]B \in S$ then $[\preceq]B \in T$;
- If $[\Delta]B \in S$ then $[\Delta]B \in T$;
- If $[\succeq]B \in T$ then $[\succeq]B \in S$;
- If $[\nabla]B \in T$ then $[\nabla]B \in S$.

A $(\succeq, A)$ -witness of S will be any set T of $\{[\equiv], [R]\}$ -free formulas of LER such that $A \notin T$ and for all $\{[\equiv], [R]\}$ -free formulas B of LER :

- If $[\succeq]B \in S$ then $[\succeq]B \in T$;
- If $[\nabla]B \in S$ then $[\nabla]B \in T$;
- If $[\preceq]B \in T$ then $[\preceq]B \in S$;
- If $[\Delta]B \in T$ then $[\Delta]B \in S$.

A (Δ, A) -witness of S will be any set T of $\{[\equiv], [R]\}$ -free formulas of LER such that $A \notin T$ and for all $\{[\equiv], [R]\}$ -free formulas B of LER :

- If $[\Delta]B \in S$ then $[\succeq]B \in T$;
- If $[\preceq]B \in S$ then $[\nabla]B \in T$;
- If $[\Delta]B \in T$ then $[\succeq]B \in S$;
- If $[\preceq]B \in T$ then $[\nabla]B \in S$.

A (∇, A) -witness of S will be any set T of $\{[\equiv], [R]\}$ -free formulas of LER such that $A \notin T$ and for all $\{[\equiv], [R]\}$ -free formulas B of LER :

- If $[\nabla]B \in S$ then $[\preceq]B \in T$;
- If $[\succeq]B \in S$ then $[\Delta]B \in T$;
- If $[\nabla]B \in T$ then $[\preceq]B \in S$;
- If $[\succeq]B \in T$ then $[\Delta]B \in S$.

To test the satisfiability problem for any given $\{[\equiv], [R]\}$ -free formula A of LER , let $SAT(A)$ be the nondeterministic algorithm defined as follows. Define:

```
function  $SAT(A)$  returns Boolean
begin
for all subsets  $S$  of  $Cl(\epsilon, A)$  such that  $A \in S$  do
     $WORLD(S, \epsilon, A)$ 
if all these calls return false then return false
return true
end
```

where $WORLD(\mathcal{L}, \omega, A)$ is the nondeterministic algorithm defined as follows. Our nondeterministic algorithm $WORLD$ is similar to Spaan's nondeterministic algorithm $S4_t - WORLD$ — a procedure developed by Spaan [12] to investigate

the inherent difficulty of the satisfiability problem for any given formula of $S4_t$ — in the sense that we keep track of a list $\mathcal{L}$ of sets of $\{\equiv, [R]\}$ -free formulas and a string $\omega \in \{\preceq, \succeq, \Delta, \nabla\}^*$. Define:

```

function WORLD( $\mathcal{L}, \omega, A$ ) returns Boolean
begin
  let  $S$  be last( $\mathcal{L}$ )
  for all  $\{\equiv, [R]\}$ -free formulas  $B$  of LER such that  $\neg B \in Cl(\omega, A)$  do
    if  $(\neg B \in S \text{ and } B \in S) \text{ or } (\neg B \notin S \text{ and } B \notin S)$  then return false
  for all  $\{\equiv, [R]\}$ -free formulas  $B, C$  of LER such that  $B \vee C \in Cl(\omega, A)$  do
    if  $(B \vee C \in S \text{ and } B \notin S \text{ and } C \notin S) \text{ or } (B \vee C \notin S \text{ and } B \in S) \text{ or } (B \vee C \notin S \text{ and } C \in S)$  then return false
  for all  $\{\equiv, [R]\}$ -free formulas  $B$  of LER such that  $[\preceq]B \in Cl(\omega, A)$  do
    begin
      if  $[\preceq]B \in S \text{ and } B \notin S$  then return false
      if  $[\preceq]B \notin S$  and there is no  $T$  in  $\mathcal{L}$  such that  $T$  is a  $(\preceq, B)$ -witness of  $S$  then
        begin
          for all subsets  $T$  of  $Cl(\omega \preceq, A)$  such that  $T$  is a  $(\preceq, B)$ -witness of  $S$  do
            if  $\omega = \epsilon$  or  $\omega$  is ended by  $\succeq$  or  $\Delta$  then WORLD( $T, \omega \preceq, A$ ) else
              WORLD( $\mathcal{L} \circ T, \omega \preceq, A$ )
          if all these calls return false then return false
        end
      end
    end
  for all  $\{\equiv, [R]\}$ -free formulas  $B$  of LER such that  $[\succeq]B \in Cl(\omega, A)$  do
    begin
      if  $[\succeq]B \in S \text{ and } B \notin S$  then return false
      if  $[\succeq]B \notin S$  and there is no  $T$  in  $\mathcal{L}$  such that  $T$  is a  $(\succeq, B)$ -witness of  $S$  then
        begin
          for all subsets  $T$  of  $Cl(\omega \succeq, A)$  such that  $T$  is a  $(\succeq, B)$ -witness of  $S$  do
            if  $\omega = \epsilon$  or  $\omega$  is ended by  $\preceq$  or  $\nabla$  then WORLD( $T, \omega \succeq, A$ ) else
              WORLD( $\mathcal{L} \circ T, \omega \succeq, A$ )
          if all these calls return false then return false
        end
      end
    end
  for all  $\{\equiv, [R]\}$ -free formulas  $B$  of LER such that  $[\Delta]B \in Cl(\omega, A)$  do
    if  $[\Delta]B \notin S$  and there is no  $T$  in  $\mathcal{L}$  such that  $T$  is a  $(\Delta, B)$ -witness of  $S$  then
      begin
        for all subsets  $T$  of  $Cl(\omega \Delta, A)$  such that  $T$  is a  $(\Delta, B)$ -witness of  $S$  do
          if  $\omega = \epsilon$  or  $\omega$  is ended by  $\succeq$  or  $\Delta$  then WORLD( $T, \omega \Delta, A$ ) else
            WORLD( $\mathcal{L} \circ T, \omega \Delta, A$ )
          if all these calls return false then return false
        end
      end
    end
  for all  $\{\equiv, [R]\}$ -free formulas  $B$  of LER such that  $[\nabla]B \in Cl(\omega, A)$  do
    if  $[\nabla]B \notin S$  and there is no  $T$  in  $\mathcal{L}$  such that  $T$  is a  $(\nabla, B)$ -witness of  $S$  then
      begin
        for all subsets  $T$  of  $Cl(\omega \nabla, A)$  such that  $T$  is a  $(\nabla, B)$ -witness of  $S$  do

```

```

    if  $\omega = \epsilon$  or  $\omega$  is ended by  $\preceq$  or  $\nabla$  then  $WORLD(T, \omega \nabla, A)$  else
       $WORLD(\mathcal{L} \circ T, \omega \nabla, A)$ 
    if all these calls return false then return false
  end
return true
end

```

where $last(\mathcal{L})$ denotes the last element of $\mathcal{L}$ and $\mathcal{L} \circ T$ denotes the list obtained by extending $\mathcal{L}$ with T . The following lemma is basic.

Lemma 6. *For all $\{[\equiv], [R]\}$ -free formulas A of LER and for all subsets S of $Cl(\epsilon, A)$, the following conditions are equivalent.*

1. $WORLD(S, \epsilon, A)$ succeeds;
2. There is a model $M = (W, \preceq, \succeq, \Delta, \nabla, V)$ of LER and there is a possible world $x \in W$ such that for all $\{[\equiv], [R]\}$ -free formulas B of LER, if $B \in Cl(\epsilon, A)$ then $M, x \models B$ iff $B \in S$.

Proof. (1 implies 2): Suppose $WORLD(S, \epsilon, A)$ succeeds and let $M = (W, \preceq, \succeq, \Delta, \nabla, V)$ be the structure defined as follows. Define:

- W is the set of all pairs (T, ω) for which there is a finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$ such that:
 - $\mathcal{L}_0 = S$ and $\omega_0 = \epsilon$;
 - $last(\mathcal{L}_K) = T$ and $\omega_K = \omega$;
 - For all positive integers k , if $k \leq K$ then $WORLD(\mathcal{L}_k, \omega_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
 - For all positive integers k , if $k < K$ then $WORLD(\mathcal{L}_{k+1}, \omega_{k+1}, A)$ is called by $WORLD(\mathcal{L}_k, \omega_k, A)$;
- For all $(T, \omega), (T', \omega') \in W$, $(T, \omega) \ll (T', \omega')$ iff there is a finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$ such that:
 - $\mathcal{L}_0 = S$ and $\omega_0 = \epsilon$;
 - $last(\mathcal{L}_K) = T$ and $\omega_K = \omega$;
 - For all positive integers k , if $k \leq K$ then $WORLD(\mathcal{L}_k, \omega_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
 - For all positive integers k , if $k < K$ then $WORLD(\mathcal{L}_{k+1}, \omega_{k+1}, A)$ is called by $WORLD(\mathcal{L}_k, \omega_k, A)$;

and there is a finite sequence $(\mathcal{L}'_0, \omega'_0), \dots, (\mathcal{L}'_{K'}, \omega'_{K'})$ such that:

- $\mathcal{L}'_0 = S$ and $\omega'_0 = \epsilon$;
- $last(\mathcal{L}'_{K'}) = T'$ and $\omega'_{K'} = \omega'$;
- For all positive integers k , if $k \leq K'$ then $WORLD(\mathcal{L}'_k, \omega'_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
- For all positive integers k , if $k < K'$ then $WORLD(\mathcal{L}'_{k+1}, \omega'_{k+1}, A)$ is called by $WORLD(\mathcal{L}'_k, \omega'_k, A)$;

and either $(\mathcal{L}_K, \omega_K) = (\mathcal{L}'_{K'}, \omega'_{K'})$ or $WORLD(\mathcal{L}'_{K'}, \omega'_{K'}, A)$ is called by $WORLD(\mathcal{L}_K, \omega_K, A)$ in the $[\preceq]$ -segment or the finite sequence $(\mathcal{L}'_0, \omega'_0), \dots, (\mathcal{L}'_{K'}, \omega'_{K'})$ is a prefix of the finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$, the call $WORLD(\mathcal{L}_K, \omega_K, A)$ enters in the $[\preceq]$ -segment and there is a $\{[\equiv], [R]\}$ -free formula B of LER such that $[\preceq]B \in Cl(\omega_K, A)$, $[\preceq]B \notin T$ and T' is a $(\preceq, B)$ -witness of T ;

- For all $(T, \omega), (T', \omega') \in W$, $(T, \omega) \gg (T', \omega')$ iff there is a finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$ such that:
 - $\mathcal{L}_0 = S$ and $\omega_0 = \epsilon$;
 - $last(\mathcal{L}_K) = T$ and $\omega_K = \omega$;
 - For all positive integers k , if $k \leq K$ then $WORLD(\mathcal{L}_k, \omega_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
 - For all positive integers k , if $k < K$ then $WORLD(\mathcal{L}_{k+1}, \omega_{k+1}, A)$ is called by $WORLD(\mathcal{L}_k, \omega_k, A)$;
 and there is a finite sequence $(\mathcal{L}'_0, \omega'_0), \dots, (\mathcal{L}'_{K'}, \omega'_{K'})$ such that:
 - $\mathcal{L}'_0 = S$ and $\omega'_0 = \epsilon$;
 - $last(\mathcal{L}'_{K'}) = T'$ and $\omega'_{K'} = \omega'$;
 - For all positive integers k , if $k \leq K'$ then $WORLD(\mathcal{L}'_k, \omega'_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
 - For all positive integers k , if $k < K'$ then $WORLD(\mathcal{L}'_{k+1}, \omega'_{k+1}, A)$ is called by $WORLD(\mathcal{L}'_k, \omega'_k, A)$;
 and either $(\mathcal{L}_K, \omega_K) = (\mathcal{L}'_{K'}, \omega'_{K'})$ or $WORLD(\mathcal{L}'_{K'}, \omega'_{K'}, A)$ is called by $WORLD(\mathcal{L}_K, \omega_K, A)$ in the $[\succeq]$ -segment or the finite sequence $(\mathcal{L}'_0, \omega'_0), \dots, (\mathcal{L}'_{K'}, \omega'_{K'})$ is a prefix of the finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$, the call $WORLD(\mathcal{L}_K, \omega_K, A)$ enters in the $[\succeq]$ -segment and there is a $\{[\equiv], [R]\}$ -free formula B of LER such that $[\succeq]B \in Cl(\omega_K, A)$, $[\succeq]B \notin T$ and T' is a $(\succeq, B)$ -witness of T ;
- For all $(T, \omega), (T', \omega') \in W$, $(T, \omega) \smile (T', \omega')$ iff there is a finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$ such that:
 - $\mathcal{L}_0 = S$ and $\omega_0 = \epsilon$;
 - $last(\mathcal{L}_K) = T$ and $\omega_K = \omega$;
 - For all positive integers k , if $k \leq K$ then $WORLD(\mathcal{L}_k, \omega_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
 - For all positive integers k , if $k < K$ then $WORLD(\mathcal{L}_{k+1}, \omega_{k+1}, A)$ is called by $WORLD(\mathcal{L}_k, \omega_k, A)$;
 and there is a finite sequence $(\mathcal{L}'_0, \omega'_0), \dots, (\mathcal{L}'_{K'}, \omega'_{K'})$ such that:
 - $\mathcal{L}'_0 = S$ and $\omega'_0 = \epsilon$;
 - $last(\mathcal{L}'_{K'}) = T'$ and $\omega'_{K'} = \omega'$;
 - For all positive integers k , if $k \leq K'$ then $WORLD(\mathcal{L}'_k, \omega'_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
 - For all positive integers k , if $k < K'$ then $WORLD(\mathcal{L}'_{k+1}, \omega'_{k+1}, A)$ is called by $WORLD(\mathcal{L}'_k, \omega'_k, A)$;
 and either $WORLD(\mathcal{L}'_{K'}, \omega'_{K'}, A)$ is called by $WORLD(\mathcal{L}_K, \omega_K, A)$ in the $[\triangle]$ -segment or the finite sequence $(\mathcal{L}'_0, \omega'_0), \dots, (\mathcal{L}'_{K'}, \omega'_{K'})$ is a prefix of the finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$, the call $WORLD(\mathcal{L}_K, \omega_K, A)$ enters in the $[\triangle]$ -segment and there is a $\{[\equiv], [R]\}$ -free formula B of LER such that $[\triangle]B \in Cl(\omega_K, A)$, $[\triangle]B \notin T$ and T' is a $(\triangle, B)$ -witness of T ;
- For all $(T, \omega), (T', \omega') \in W$, $(T, \omega) \frown (T', \omega')$ iff there is a finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$ such that:
 - $\mathcal{L}_0 = S$ and $\omega_0 = \epsilon$;
 - $last(\mathcal{L}_K) = T$ and $\omega_K = \omega$;

- For all positive integers k , if $k \leq K$ then $WORLD(\mathcal{L}_k, \omega_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
 - For all positive integers k , if $k < K$ then $WORLD(\mathcal{L}_{k+1}, \omega_{k+1}, A)$ is called by $WORLD(\mathcal{L}_k, \omega_k, A)$;
- and there is a finite sequence $(\mathcal{L}'_0, \omega'_0), \dots, (\mathcal{L}'_{K'}, \omega'_{K'})$ such that:
- $\mathcal{L}'_0 = S$ and $\omega'_0 = \epsilon$;
 - $last(\mathcal{L}'_{K'}) = T'$ and $\omega'_{K'} = \omega'$;
 - For all positive integers k , if $k \leq K'$ then $WORLD(\mathcal{L}'_k, \omega'_k, A)$ is successfully called in $WORLD(S, \epsilon, A)$ at some depth of the recursion;
 - For all positive integers k , if $k < K'$ then $WORLD(\mathcal{L}'_{k+1}, \omega'_{k+1}, A)$ is called by $WORLD(\mathcal{L}'_k, \omega'_k, A)$;
- and either $WORLD(\mathcal{L}'_{K'}, \omega'_{K'}, A)$ is called in $WORLD(\mathcal{L}_K, \omega_K, A)$ in the $[\nabla]$ -segment or the finite sequence $(\mathcal{L}'_0, \omega'_0), \dots, (\mathcal{L}'_{K'}, \omega'_{K'})$ is a prefix of the finite sequence $(\mathcal{L}_0, \omega_0), \dots, (\mathcal{L}_K, \omega_K)$, the call $WORLD(\mathcal{L}_K, \omega_K, A)$ enters in the $[\nabla]$ -segment and there is a $\{\equiv, [R]\}$ -free formula B of LER such that $[\nabla]B \in Cl(\omega_K, A)$, $[\nabla]B \notin T$ and T' is a (∇, B) -witness of T ;
- For all propositional variables p , $V(p)$ is the set of all $(T, \omega) \in W$ such that $p \in T$.

Now the definition of M can be completed. Firstly define:

- $\preceq_0$ is $\ll \cup \gg^{-1}$;
- $\succeq_0$ is $\gg \cup \ll^{-1}$;
- $\triangle_0$ is $\smile \cup \frown^{-1}$;
- ∇_0 is $\frown \cup \smile^{-1}$.

Secondly define for all positive integers k :

- $\preceq_{k+1}$ is $(\preceq_k \circ \preceq_k) \cup (\triangle_k \circ \nabla_k)$;
- $\succeq_{k+1}$ is $(\succeq_k \circ \succeq_k) \cup (\nabla_k \circ \triangle_k)$;
- $\triangle_{k+1}$ is $(\preceq_k \circ \triangle_k) \cup (\triangle_k \circ \succeq_k)$;
- ∇_{k+1} is $(\succeq_k \circ \nabla_k) \cup (\nabla_k \circ \preceq_k)$.

Thirdly define:

- $\preceq$ is $\bigcup \{\preceq_k : k \text{ is a positive integer}\}$;
- $\succeq$ is $\bigcup \{\succeq_k : k \text{ is a positive integer}\}$;
- $\triangle$ is $\bigcup \{\triangle_k : k \text{ is a positive integer}\}$;
- ∇ is $\bigcup \{\nabla_k : k \text{ is a positive integer}\}$.

It is a simple matter to check that M is a model of LER . What is more one can establish by induction on the positive integer k the remarkable result that for all $\{\equiv, [R]\}$ -free formulas B of LER and for all $(T, \omega), (T', \omega') \in W$:

- If $[\preceq]B \in T$ and $(T, \omega) \preceq_k (T', \omega')$ then $[\preceq]B \in T'$;
- If $[\succeq]B \in T$ and $(T, \omega) \succeq_k (T', \omega')$ then $[\succeq]B \in T'$;
- If $[\triangle]B \in T$ and $(T, \omega) \preceq_k (T', \omega')$ then $[\triangle]B \in T'$;
- If $[\nabla]B \in T$ and $(T, \omega) \succeq_k (T', \omega')$ then $[\nabla]B \in T'$;
- If $[\triangle]B \in T$ and $(T, \omega) \triangle_k (T', \omega')$ then $[\succeq]B \in T'$;

- If $[\nabla]B \in T$ and $(T, \omega) \nabla_k (T', \omega')$ then $[\preceq]B \in T'$;
- If $[\preceq]B \in T$ and $(T, \omega) \triangle_k (T', \omega')$ then $[\nabla]B \in T'$;
- If $[\succeq]B \in T$ and $(T, \omega) \nabla_k (T', \omega')$ then $[\triangle]B \in T'$.

It follows immediately by induction on the *length* of $\{[\equiv], [R]\}$ -free formula B of LER that for all $(T, \omega) \in W$, if $B \in Cl(\omega, A)$ then:

- $M, (T, \omega) \models B$ iff $B \in T$.

Hence we have that for all $\{[\equiv], [R]\}$ -free formulas B of LER , if $B \in Cl(\epsilon, A)$ then:

- $M, (S, \epsilon) \models B$ iff $B \in S$.

(2 implies 1): The reader may easily prove by induction on the *depth* of string $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$ that for all sets T of $\{[\equiv], [R]\}$ -free formulas of LER , for all models $M = (W, \preceq, \succeq, \triangle, \nabla, V)$ of LER and for all possible worlds $x \in W$, if $T \subseteq Cl(\omega, A)$ and:

- For all $\{[\equiv], [R]\}$ -free formulas B of LER , if $B \in Cl(\omega, A)$ then $M, x \models B$ iff $B \in T$;

then $WORLD(T, \omega, A)$ succeeds. These considerations prove that if there is a model $M = (W, \preceq, \succeq, \triangle, \nabla, V)$ of LER and there is a possible world $x \in W$ such that:

- For all $\{[\equiv], [R]\}$ -free formulas B of LER , if $B \in Cl(\epsilon, A)$ then $M, x \models B$ iff $B \in S$;

then $WORLD(S, \epsilon, A)$ succeeds.

It follows immediately from lemma 6 that the nondeterministic algorithm $SAT(A)$ works correctly, i.e., $SAT(A)$ succeeds iff A is true at some possible world in some model of LER . Much more difficult than lemma 6 are the following important results.

Lemma 7. *For all $\{[\equiv], [R]\}$ -free formulas A of LER , for all lists $\mathcal{L}$ of sets of $\{[\equiv], [R]\}$ -free formulas and for all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$, if $WORLD(\mathcal{L}, \omega, A)$ is called in $WORLD(S, \epsilon, A)$ at some depth of the recursion then $length(\mathcal{L}) = \mathcal{O}(length(A)^2)$.*

Proof. Similar to the line of reasoning suggested by Demri [2] at pages 225–226.

Lemma 8. *For all $\{[\equiv], [R]\}$ -free formulas A of LER , for all lists $\mathcal{L}$ of sets of $\{[\equiv], [R]\}$ -free formulas and for all strings $\omega \in \{\preceq, \succeq, \triangle, \nabla\}^*$, if $WORLD(\mathcal{L}, \omega, A)$ is called in $WORLD(S, \epsilon, A)$ at some depth of the recursion then $depth(\omega) \leq length(A)$.*

Proof. Suppose that $depth(\omega) > length(A)$. It follows that there is a string $\omega' \in \{\preceq, \succeq, \triangle, \nabla\}^*$ such that $\omega = \omega' \preceq \succeq$ or $\omega = \omega' \preceq \nabla$ or $\omega = \omega' \succeq \preceq$ or $\omega = \omega' \succeq \triangle$ or $\omega = \omega' \triangle \preceq$ or $\omega = \omega' \triangle \triangle$ or $\omega = \omega' \nabla \succeq$ or $\omega = \omega' \nabla \nabla$. Suppose that $\omega = \omega' \preceq \succeq$. Hence $depth(\omega' \preceq) \geq length(A)$. It follows that $Cl(\omega' \preceq, A) = \emptyset$: a contradiction. Therefore $depth(\omega) \leq length(A)$. The same line of reasoning applies if $\omega = \omega' \preceq \nabla$ or $\omega = \omega' \succeq \preceq$ or $\omega = \omega' \succeq \triangle$ or $\omega = \omega' \triangle \preceq$ or $\omega = \omega' \triangle \triangle$ or $\omega = \omega' \nabla \succeq$ or $\omega = \omega' \nabla \nabla$.

By applying lemma 7 and lemma 8, we infer immediately that the nondeterministic algorithm $SAT(A)$ is polynomial-space bounded in $length(A)$. We can summarize the results proved above as follows for all $\{[\equiv], [R]\}$ -free formulas A of LER :

- $SAT(A)$ succeeds iff A is true at some possible world in some model of LER ;
- The total space to run $SAT(A)$ is polynomial in $length(A)$.

We get that the satisfiability problem for any given $\{[\equiv], [R]\}$ -free formula A of LER is in $NPSPACE$ of $length(A)$. Seeing that $NPSPACE = PSPACE$, see Savitch [11], we therefore conclude the following.

Theorem 5. (*PSPACE-completeness theorem for LER*) *The satisfiability problem for any given $\{[\equiv], [R]\}$ -free formula A of LER is PSPACE-complete in $length(A)$.*

6 Conclusion

The key feature of this paper is the modal analysis of emptiness relations in property systems. The proof that the satisfiability problem for any given $\{[\equiv], [R]\}$ -free formula A of LER is $PSPACE$ -complete in $length(A)$ goes through a line of reasoning similar to that applied by Spaan [12] to investigate the inherent difficulty of the satisfiability problem for any given formula of $S4_t$. Let us note that our proof does not consider the full language of our modal logic, so it might not be suitable for studying the inherent difficulty of the satisfiability problem for any given formula of LER , a question that remains unsolved.

Previous modal analyses of informational relations in property systems have been given by Vakarelov [16] who has mainly considered the relations of informational inclusion — our emptiness relations $\preceq$ and $\succeq$ — and the relations of similarity — the complementary relations of our emptiness relations $\triangle$ and ∇ . On that occasion, Vakarelov [16] gave a completeness theorem and a decidability theorem for a modal logic which modalities correspond to the relations of informational inclusion and the relations of similarity in property systems. Modal analyses of our emptiness relations in property systems together with other informational relations like similarity relations are not known.

References

1. Balbiani, P., Vakarelov, D.: First-order characterization and modal analysis of indiscernibility and complementarity in information systems. In Benferhat, S., Besnard, P. (Editors): *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*. Springer-Verlag, Lecture Notes in Artificial Intelligence **2143** (2001) 772–781.
2. Demri, S.: The nondeterministic information logic NIL is PSPACE-complete. *Fundamenta Informaticæ* **42** (2000) 211–234.
3. Demri, S., Orłowska, E., Vakarelov, D.: Indiscernibility and complementarity relations in information systems. In Gerbrandy, J., Marx, M., de Rijke, M., Venema, Y. (Editors): *JFAK: Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday*. Amsterdam University Press (1999) <http://turing.wins.uva.nl/~j50/cdrom/contribs/demri/index.html>.
4. Hughes, G., Cresswell, M.: *A Companion to Modal Logic*. Methuen (1984).
5. Ladner, R.: The computational complexity of provability in systems of modal propositional logic. *SIAM Journal on Computing* **6** (1977) 467–480.
6. Orłowska, E.: Logic of nondeterministic information. *Studia Logica* **44** (1985) 91–100.
7. Orłowska, E.: Kripke semantics for knowledge representation logics. *Studia Logica* **49** (1990) 255–272.
8. Orłowska, E. (Editor): *Incomplete Information: Rough Set Analysis*. Physica-Verlag, Studies in Fuzziness and Soft Computing **13** (1998).
9. Orłowska, E., Pawlak, Z.: Representation of nondeterministic information. *Theoretical Computer Science* **29** (1984) 27–39.
10. Pawlak, Z.: Information systems — theoretical foundations. *Information Systems* **6** (1981) 205–218.
11. Savitch, W.: Relationships between nondeterministic and deterministic tape complexities. *Journal of Computer and System Sciences* **4** (1970) 177–192.
12. Spaan, E.: The complexity of propositional tense logics. In de Rijke, M. (Editor): *Diamonds and Defaults: Studies in Pure and Applied Intensional Logic*. Kluwer, Synthese Library **229** (1993) 287–307.
13. Vakarelov, D.: A modal logic for similarity relations in Pawlak knowledge representation systems. *Fundamenta Informaticæ* **15** (1991) 61–79.
14. Vakarelov, D.: Modal logics for knowledge representation systems. *Theoretical Computer Science* **90** (1991) 433–456.
15. Vakarelov, D.: A duality between Pawlak’s knowledge representation systems and BI-consequence systems. *Studia Logica* **55** (1995) 205–228.
16. Vakarelov, D.: Information systems, similarity relations and modal logics. In Orłowska, E. (Editor): *Incomplete Information: Rough Set Analysis*. Physica-Verlag, Studies in Fuzziness and Soft Computing **13** (1998) 492–550.