Microlocal partition of energy for linear wave or Schrödinger equations

Jean-Marc Delort

Microlocal partition of energy for linear wave or Schrödinger equations

Jean-Marc Delort*
Université Paris XIII (Sorbonne Paris-Nord),
LAGA, CNRS (UMR 7539),
99, Avenue J.-B. Clément,
F-93430 Villetaneuse

Abstract

We prove a microlocal partition of energy for solutions to linear half-wave or Schrödinger equations in any space dimension. This extends well-known (local) results valid for the wave equation outside the wave cone, and allows us in particular, in the case of even dimension, to generalize the radial estimates due to Côte, Kenig and Schlag to non radial initial data.

0 Introduction

The goal of this paper is to revisit the property of space partition of energy when time goes to infinity for solutions of linear wave equations that has been uncovered by Duyckaerts, Kenig and Merle [7, 8] in odd dimensions and by Côte, Kenig and Schlag [4] in even dimensions, and to extend it to other dispersive equations.
Recall that if w solves the linear wave equation on $\mathbb{R} \times \mathbb{R}^d$

$$\begin{aligned}(\partial_t^2 - \Delta_x)w &= 0\\ w|_{t=0} &= w_0\\ \partial_t w|_{t=0} &= w_1\end{aligned}$$

and if one defines the energy at time t outside the wave cone by

$$E^{\mathrm{W}}(w_0, w_1, t) = \int_{|x|>|t|} \big[|\partial_t w(t,x)|^2 + |\nabla_x w(t,x)|^2\big]\,dx, \tag{1}$$

then it has been proved in [7, 8] that, if d *is odd*, either

$$\begin{aligned}&\forall t \geq 0,\ E^{\mathrm{W}}(w_0, w_1, t) \geq \frac{1}{2}\big[\|w_1\|_{L^2}^2 + \|\nabla_x w_0\|_{L^2}^2\big]\\ &\text{or}\\ &\forall t \leq 0,\ E^{\mathrm{W}}(w_0, w_1, t) \geq \frac{1}{2}\big[\|w_1\|_{L^2}^2 + \|\nabla_x w_0\|_{L^2}^2\big].\end{aligned} \tag{2}$$

2020 *Mathematics Subject Classification*: 35L05, 35Q41. Keywords: Wave equation, Schrödinger equation, microlocal analysis

*Part of this material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while the author participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2021 semester.

Since $t \to E^{\mathrm{W}}(w_0, w_1, \pm t)$ is non increasing on $[0, +\infty[$, the above property is actually a consequence of

$$\lim_{t \to +\infty} \left[E^{\mathrm{W}}(w_0, w_1, t) + E^{\mathrm{W}}(w_0, w_1, -t)\right] = \|w_1\|_{L^2}^2 + \|\nabla_x w_0\|_{L^2}^2. \tag{3}$$

In even dimension, it has been shown by Côte, Kenig and Schlag that (2) does *not* hold in general (see nevertheless Proposition 1 in [9] for a qualitative version of the result in even dimension). Though, for *radial* data, Côte *et al.* have been able to compute the limit in (3) by an explicit formula that, for *special* classes of radial initial data, provides estimates of the form (2).
Inequalities of the form (2) are important as a tool used to study solutions of energy critical wave equations. They have been initially introduced for such a purpose in [7, 8] in odd dimensions. Their version for even dimension have been applied to equivariant wave maps by Côte, Kenig, Lawrie and Schlag in [1, 2] and to the focusing energy critical wave equation in four dimension by the same authors in [3].
Our goal in this paper is to address the following questions:
(i) May one refine (3) replacing the energy truncated in the complement of the wave cone by a similar microlocal expression, cut-off in a smaller domain, so that an equality of the form (3) still holds?
(ii) Is it possible to prove a property of the form (3) for *other* dispersive equations? In particular, this raises the question of determining what would be the natural set that should replace the wave cone in the definition of the sharp cut-off in (1).
(iii) Is it possible to extend the result of equipartition in even dimension of Côte, Kenig and Schlag [4] to non radial data?
We shall study these three questions in a relatively unified framework. Actually, we shall deal simultaneously with the half-wave equation and the Schrödinger one. Following some heuristics explained in subsection 1.2 below, we define a microlocalized cut-off energy for a solution v of the half-wave equation $(D_t - |D_x|)v = 0$ (where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D_x = \frac{1}{i}\frac{\partial}{\partial x}$) in the following way: for any frequency direction $\omega = \frac{\xi}{|\xi|}$, we cut-off microlocally the solution intersecting the outside of the wave cone $|x| > |t|$ with an angular sector around $-\omega$, of angle $t^{-\frac{1}{2}+0}$ (see the statement of Theorem 1.1.1 in subsection 1.1 and Figure 1). We show then that, in *any* space dimension, a property of the form (3) holds, so that a significant part of the energy stays in the microlocal domain in which our energy has been cut-off. We obtain a similar result for the Schrödinger equation $\left(D_t - \frac{|D_x|^2}{2}\right)v = 0$, with two differences: first, the wave cone $|x| > |t|$ has to be replaced by the "Schrödinger cone" $|x| > |t||\xi|$; second, the limit of the analogous of (3) is no longer the whole initial energy $\|v(0, \cdot)\|_{L^2}^2$ but only half of it. Let us mention that though we limit ourselves here to the discussion of half-wave or Schrödinger equations, our method could be used for more general equations, of the form $(D_t - p(D_x))v = 0$, with some radial function $p(\xi)$, the sharp cut-off involved in the definition of the microlocalized energy being then replaced by $|x| > |t||p'(\xi)|$.
As a byproduct of our method, we extend the results of Côte, Kenig and Schlag [4] to non radial data. Actually, one may express the left hand side of (3) from the quantities that appear in the study of the half-wave equation, and from other terms, that are the ones responsible for the gap between odd and even space dimension.
The main theorems of the paper are stated in subsection 1.1 and some heuristics explaining the choice of our microlocal cut-offs are given in subsection 1.2. We devote section 2 to the study of the L^2 boundedness of the sharp cut-off operators that define our microlocal energies. Such boundedness properties are trivial for the (half)-wave equation but not for the Schrödinger one, as in this case the sharp cut-off depends both on space and frequency. In section 3 we give the proof of our main theorems and gather in an appendix some more technical stationary phase results that are used in sections 2 and 3, as well as some other technical points.

We thank Thomas Duyckaerts for providing us some bibliographical references.

1 Statement of the main results

1.1 Lower bounds for energy channels

For $(x,\xi) \to a(x,\xi)$ a bounded (or locally integrable in ξ for any fixed x) function on $\mathbb{R}^d \times \mathbb{R}^d$, with values in $\mathbb{C}$, and u in $\mathcal{S}(\mathbb{R}^d)$, we set

$$a(x,D_x)u = \mathrm{Op}(a)u = \frac{1}{(2\pi)^d}\int e^{ix\xi} a(x,\xi)\hat{u}(\xi)\,d\xi. \tag{1.1.1}$$

Let χ be in $C_0^\infty(\mathbb{R}^d)$ radial, with small enough support, equal to one close to zero and let $\tilde{\chi}$ be a similar function on $\mathbb{R}$. For t in $\mathbb{R}$, $\delta \in [0,\frac{1}{2}]$, we introduce the following cut-offs:

$$a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,x,\xi) = \chi\Big(\Big(x + t\frac{\xi}{|\xi|}\Big)|t|^{-\delta-\frac{1}{2}}\Big)\tilde{\chi}\big(|t|^{-\delta}(|t|-|x|)\big)\mathbb{1}_{|x|>|t|} \tag{1.1.2}$$

and

$$a^{\mathrm{Schr}}_{\chi,\delta}(t,x,\xi) = \chi\left(\frac{x+t\xi}{|t||\xi|\langle\sqrt{|t|}|\xi|\rangle^{-\frac{1}{2}+\delta}}\right)\mathbb{1}_{|x|>|t||\xi|}. \tag{1.1.3}$$

For u_0 in $L^2(\mathbb{R}^d)$, we define the following micro-localized energies of the solution of the half-wave equation and of the Schrödinger equation respectively:

$$\begin{aligned} E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) &= \big\|\mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}\big)\big[e^{it|D_x|}u_0\big]\big\|^2_{L^2} \\ E^{\mathrm{Schr}}_{\chi,\delta}(u_0,t) &= \big\|\mathrm{Op}\big(a^{\mathrm{Schr}}_{\chi,\delta}\big)\big[e^{it\frac{|D_x|^2}{2}}u_0\big]\big\|^2_{L^2}, \end{aligned} \tag{1.1.4}$$

where $D_x = \frac{1}{i}\frac{\partial}{\partial x}$. Our main theorem asserts that the above quantities are finite and that, asymptotically when t goes to $+\infty$, $E_*(u_0,t) + E_*(u_0,-t)$ is larger than a significant fraction of the total energy of u_0, where E_* stands for $E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}$ or $E^{\mathrm{Schr}}_{\chi,\delta}$.

Theorem 1.1.1 *Let $d \geq 2$ for the half-wave equation and $d \geq 1$ for the Schrödinger equation. Then if the supports of χ, $\tilde{\chi}$ have been taken small enough, the operators $\mathrm{Op}(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta})$ and $\mathrm{Op}(a^{\mathrm{Schr}}_{\chi,\delta})$ are bounded on $L^2(\mathbb{R}^d)$ uniformly for $|t| \geq 1$. Moreover, for any u_0 in $L^2(\mathbb{R}^d)$,*
(i) For the half wave equation, if $\delta \in]0,\frac{1}{2}]$, one gets

$$\lim_{t\to+\infty}\big[E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) + E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,-t)\big] = \|u_0\|^2_{L^2}. \tag{1.1.5}$$

(ii) For the Schrödinger equation, if $\delta \in [0,\frac{1}{2}[$, one gets

$$\lim_{t\to+\infty}\big[E^{\mathrm{Schr}}_{\chi,\delta}(u_0,t) + E^{\mathrm{Schr}}_{\chi,\delta}(u_0,-t)\big] = \frac{1}{2}\|u_0\|^2_{L^2}. \tag{1.1.6}$$

Remarks: • The cut-off in (1.1.2) means that, if one takes a direction ω in $\mathbb{S}^{d-1}$ and the point of the wave cone $(t,-t\omega)$, then we compute in (1.1.4) the microlocal energy of $e^{it|D_x|}u_0$ at time $t > 0$ in a domain given by the intersection of the complement of the wave cone $\{|x| > t\}$, of a neighborhood of the slice of the wave cone at time t of the form $\{x \in \mathbb{R}^d; |t-|x|| \ll t^0\}$ and of an angular neighborhood of the direction $-\omega$ of aperture $o(t^{-\frac{1}{2}+0})$.

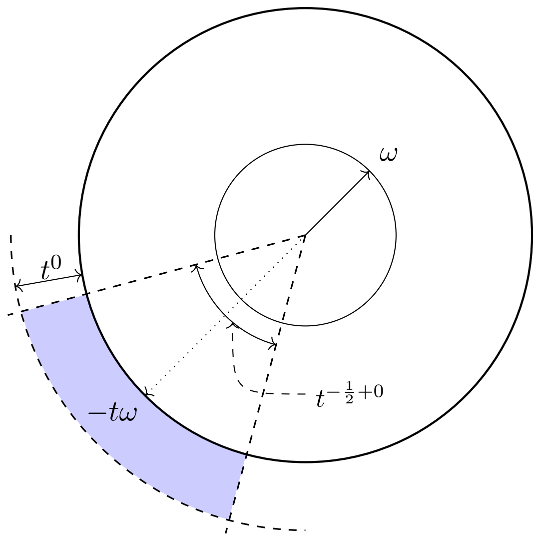

Figure 1

The shaded area above is essentially the smallest set, outside the wave cone, in which we may expect to find a significant fraction of $\|u_0\|_{L^2}^2$ when we compute the limit for t going to $+\infty$ of the sum of the truncated energies at t and $-t$. Actually, if one takes a smooth cut-off outside the ball of center $-t\omega$, of radius $|t|^{\frac{1}{2}+0}$, or outside the neighborhood of the wave cone $||x| - |t|| \leq c|t|^0$, the stationary phase formula shows that the corresponding localized energy goes to zero when t goes to infinity.

• In the case (1.1.6) of the Schrödinger equation, it would be natural to expect that the result holds true if the cut-off in (1.1.3) were localizing for x in a neighborhood of $-t\xi$ of radius $\sim t^{\frac{1}{2}+0}$ when t goes to $+\infty$ (that would correspond to $\delta > -\frac{1}{2}$ in (1.1.3)), while in our statement we have only a larger radius $t^{\frac{3}{4}}$ (corresponding to $\delta \geq 0$). This comes from the fact that we are unable prove L^2 boundedness of operator $\mathrm{Op}(a^{\mathrm{Schr}}_{\chi,\delta})$ for smaller radii. This might be actually just a technical problem.

• On the other hand, the restriction $\delta < \frac{1}{2}$ might be unavoidable. For $\delta = \frac{1}{2}$, there would be points (x, ξ) with x orthogonal to ξ in the support of the cut-off. The estimates we use in our proof of L^2 boundedness break at such points. One may not exclude that, without such cut-offs, the L^2 boundedness result be false. Actually, if one replaces the usual quantization of symbols we use here by the Weyl one (which, for non smooth symbols, might not be innocent) it has been proved recently by Lerner [11] that the sharp symbol $\mathbb{1}_E(x, \xi)$ gives rise to an operator that is *unbounded* on L^2 for "almost all" subsets E of $\mathbb{R}^d \times \mathbb{R}^d$. See Theorem 6.21 of [11]. Of course, our goal in this paper is to obtain the smallest possible subset that contains a positive amount of energy, so that the smaller δ in (1.1.3), the better.

• We have indicated in the introduction that our method could be used for more general equations that the Schrödinger one, namely for $(D_t - p(D_x))u = 0$, where $p(\xi)$ is a radial strictly convex function. Actually, in subsection 3.1, when we study several phase integrals that imply (1.1.6), we consider a general symbol $p(\xi)$ instead of just $\frac{|\xi|^2}{2}$, as the analysis is not more difficult in that case. What would have to be done in order to get conclusion (1.1.6) also in this more general case would be to prove the boundedness of a cut-off operator of the form (1.1.3) adapted to such a p. This is certainly possible, but would make the paper longer and more technical. This is why we limit ourselves to the Schrödinger framework.

Our method applies also to the usual wave equation and allows us to recover the result of Duyckaerts, Kenig and Merle [7, 8]:

Theorem 1.1.2 *Let w be a solution to the linear wave equation*

$$
\begin{aligned}
(\partial_t^2 - \Delta_x)w &= 0\\
w|_{t=0} &= w_0\\
\partial_t w|_{t=0} &= w_1
\end{aligned}
\tag{1.1.7}
$$

with data $(w_0, w_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Define the truncated energy at time t by

$$
E^{\mathrm{W}}_{c,\delta}(w_0, w_1, t) = \int_{|t|<|x|<|t|+c|t|^\delta} \big(|\partial_t w(t,x)|^2 + |\nabla_x w(t,x)|^2\big)\,dx \tag{1.1.8}
$$

where $c > 0, \delta > 0$. Then, if d is odd, *one has*

$$
\lim_{t\to+\infty} \big[E^{\mathrm{W}}_{c,\delta}(w_0, w_1, t) + E^{\mathrm{W}}_{c,\delta}(w_0, w_1, -t)\big] = \|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}. \tag{1.1.9}
$$

Remarks: • The above theorem will be a corollary of the case of the half-wave equation in Theorem 1.1.1. Actually, we shall express the left hand side of (1.1.9) from energies of the form $E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}$ as in (1.1.4) and from other terms that, *in odd dimension*, converge to zero when t goes to infinity.

• As already mentioned in the introduction, in [8] the authors state an (apparently) stronger property than (1.1.9). Actually, the quantity $t \to \int_{|x|>t}\big(|\partial_t w(t,x)|^2 + |\nabla_x w(t,x)|^2\big)\,dx$ is non increasing on $[0,+\infty[$, so that (1.1.9) implies that either for any $t \geq 0$

$$
\int_{|x|>t} \big(|\partial_t w(t,x)|^2 + |\nabla_x w(t,x)|^2\big)\,dx \geq \frac{1}{2}\big(\|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}\big),
$$

or for any $t \leq 0$

$$
\int_{|x|>|t|} \big(|\partial_t w(t,x)|^2 + |\nabla_x w(t,x)|^2\big)\,dx \geq \frac{1}{2}\big(\|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}\big),
$$

which is the result of [8].

In the case of even dimensions, one may get instead of (1.1.9) a lower bound under extra assumptions on the initial data. We shall actually prove the following result, that extends to the non radial case Corollary 2 of the paper [4] of Côte, Kenig and Schlag.

Theorem 1.1.3 *Assume that d is even. Then*

$$
\begin{aligned}
\lim_{t\to+\infty} \big[E^{\mathrm{W}}_{c,\delta}(w_0, w_1, t) + E^{\mathrm{W}}_{c,\delta}(w_0, w_1, -t)\big] &= \|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}\\
&+ \frac{2(-1)^{\frac{d}{2}}}{(2\pi)^{d+1}} \mathrm{Re} \int_{\mathbb{S}^{d-1}} \Big[\int_0^{+\infty} \big[H(\rho^{\frac{d-1}{2}}\hat{u}_0(\rho\omega))\big](\rho')\rho'^{\frac{d-1}{2}}\hat{u}_0(\rho'\omega)\,d\rho'\Big]\,d\omega
\end{aligned}
\tag{1.1.10}
$$

where $u_0 = -iw_1 + |D_x|w_0$ and where for $f \in L^2([0,+\infty[, d\rho)$ the Hankel transformation H is defined by

$$
Hf(\rho') = \int_0^{+\infty} \frac{f(\rho)}{\rho + \rho'}\,d\rho \tag{1.1.11}
$$

(which is a bounded operator from $L^2([0,+\infty[, d\rho)$ to itself).
Moreover, if one assumes either

$$
(d \equiv 0 \mod 4, w_0 \text{ even}, w_1 \text{ odd}) \tag{1.1.12}
$$

or

$$(d \equiv 2 \mod 4, w_0 \text{ odd}, w_1 \text{ even}), \tag{1.1.13}$$

then the left hand side of (1.1.10) is bounded from below by $\|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}$.
Actually, under (1.1.12) or (1.1.13), one has even the stronger statement

$$\lim_{t\to+\infty} E^{\mathrm{W}}_{c,\delta}(w_0, w_1, \pm t) \geq \frac{1}{2}\big[\|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}\big] \tag{1.1.14}$$

for both signs.

Remarks: • By [4], one knows that when (1.1.12), (1.1.13) do not hold, one may construct sequences of initial data such that $\|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}$ is equal to one but for which the left hand side of (1.1.10) goes to zero.
• After completion of this work, formula (1.1.10) has been also obtained by Côte and Laurent in [5].

1.2 Comments on the preceding results

In this subsection, we want to explain heuristically why one may expect the results of Theorem 1.1.1 to hold true.
We consider first the half-wave flow $e^{it|D_x|}$. Let $a_\pm\big(\frac{x}{t}, \xi\big)$ be some symbol to be defined and using notation (1.1.1) consider for $t > 0$ the quantity

$$\Big\|a_+\Big(\frac{x}{t}, D_x\Big)\big[e^{it|D_x|}u_0\big]\Big\|^2_{L^2} + \Big\|a_-\Big(\frac{x}{t}, D_x\Big)\big[e^{-it|D_x|}u_0\big]\Big\|^2_{L^2} \tag{1.2.1}$$

for u_0 in $L^2(\mathbb{R}^d)$. Let us rewrite this expression under a "semi-classical form" i.e. set $h = t^{-1}$ (for $t \geq 1$) and define $v_{0,h}$ by $u_0(t,x) = t^{-\frac{d}{2}}v_{0,h}\big(\frac{x}{t}\big)$ i.e. $v_{0,h}(x) = h^{-\frac{d}{2}}u_0\big(\frac{x}{h}\big)$, so that $\|v_{0,h}\|_{L^2} = \|u_0\|_{L^2}$. Then

$$\Big\|a_\pm\Big(\frac{x}{t}, D_x\Big)\big[e^{\pm it|D_x|}u_0\big]\Big\|_{L^2} = \big\|a_\pm(x, hD_x)\big[e^{\pm\frac{i}{h}|hD_x|}v_{0,h}\big]\big\|_{L^2} \tag{1.2.2}$$

where $a_\pm(x, hD_x)$ is defined as in (1.1.1) but with $a(x,\xi)$ replaced by $a_\pm(x, h\xi)$ under the integral.
One may rewrite (1.2.1) under the form

$$\begin{aligned}\Big\langle e^{-\frac{i}{h}|hD_x|}a_+(x,hD_x)^*a_+(x,hD_x)e^{\frac{i}{h}|hD_x|}v_{0,h}, v_{0,h}\Big\rangle&\\ + \Big\langle e^{\frac{i}{h}|hD_x|}a_-(x,hD_x)^*a_-(x,hD_x)e^{-\frac{i}{h}|hD_x|}v_{0,h}, v_{0,h}\Big\rangle&.\end{aligned} \tag{1.2.3}$$

To proceed, let us recall some ideas of symbolic calculus for semi-classical pseudo-differential operators. We shall not give rigorous statements as our symbols will *not* satisfy the assumptions that are necessary in order to apply classical theorems of symbolic calculus. The reader may refer to the books of Dimassi-Sjöstrand [6] or of Zworski [13] for such topics. Let us just say that, under convenient assumptions on the symbols $a_\pm(x,\xi)$, one may prove that, if one sets

$$b_\pm(x,\xi) = |a_\pm(x,\xi)|^2 \tag{1.2.4}$$

then

$$a_\pm(x,hD_x)^* \circ a_\pm(x,hD_x) = b_\pm(x,hD_x) + R \tag{1.2.5}$$

where $\|R\|_{\mathcal{L}(L^2)} = o(1)$ when h goes to zero (i.e. t goes to infinity). Modulo a remainder going to zero, we may thus rewrite (1.2.3) as

$$\Big\langle e^{-\frac{i}{h}|hD_x|} b_+(x, hD_x) e^{\frac{i}{h}|hD_x|} v_{0,h}, v_{0,h} \Big\rangle + \Big\langle e^{\frac{i}{h}|hD_x|} b_-(x, hD_x) e^{-\frac{i}{h}|hD_x|} v_{0,h}, v_{0,h} \Big\rangle. \tag{1.2.6}$$

Moreover, we have formally

$$e^{\mp\frac{i}{h}|hD_x|} b_\pm(x, hD_x) e^{\pm\frac{i}{h}|hD_x|} \sim c_\pm(x, hD_x) \tag{1.2.7}$$

modulo negligible remainders, where

$$c_\pm(x, \xi) = b_\pm\Big(x \mp \frac{\xi}{|\xi|}, \xi\Big) \tag{1.2.8}$$

so that, up to a $o(1)$ contribution when h goes to zero, one may rewrite (1.2.6) as

$$\langle \big(c_+(x, hD_x) + c_-(x, hD_x)\big) v_{0,h}, v_{0,h} \rangle. \tag{1.2.9}$$

If $b_\pm$ are chosen so that $c_+(x,\xi) + c_-(x,\xi) \equiv 1$, one would get that, modulo a remainder going to zero when t goes to infinity, (1.2.1) would be converging to $\|v_{0,h}\|^2_{L^2} = \|u_0\|^2_{L^2}$ when t goes to $+\infty$. In other words, defining a localized energy of the form (1.1.4) by

$$E^{\mathrm{HW}}(u_0, t) = \Big\| a_+\Big(\frac{x}{t}, D_x\Big) e^{it|D_x|} u_0 \Big\|^2_{L^2} \tag{1.2.10}$$

and taking $a_-(x,\xi) = a_+(-x,\xi)$, so that (1.2.1) would be $E^{\mathrm{HW}}(u_0, t) + E^{\mathrm{HW}}(u_0, -t)$, we would obtain

$$\lim_{t\to+\infty} \big[E^{\mathrm{HW}}(u_0, t) + E^{\mathrm{HW}}(u_0, -t)\big] = \|u_0\|^2_{L^2}. \tag{1.2.11}$$

The question now is to choose $a_\pm$ such that the symbols $c_\pm$ defined from $a_\pm$ by (1.2.4), (1.2.8) satisfy $c_+(x,\xi) + c_-(x,\xi) \equiv 1$, and to try to take these $a_\pm$ with the smallest possible support, this support being located outside the wave cone, in order to get a channel of energy estimate. As a first try, set

$$a_+(x,\xi) = \mathbb{1}_{x\cdot\frac{\xi}{|\xi|}<-1}, \ a_-(x,\xi) = a_+(-x,\xi) = \mathbb{1}_{x\cdot\frac{\xi}{|\xi|}>1}. \tag{1.2.12}$$

Then $x \to a_+(x,\xi)$ (resp. $x \to a_-(x,\xi)$) is the characteristic function of the shaded half-plane in the left (resp. right) picture below:

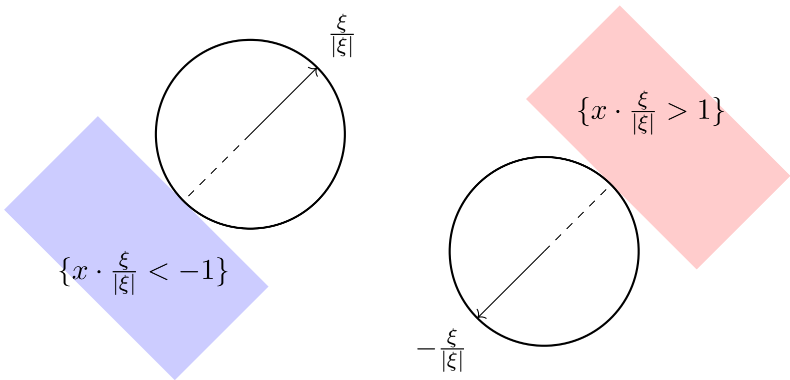

Figure 2

By (1.2.4), $b_\pm = a_\pm$ and by (1.2.8)

$$c_+(x,\xi) = \mathbb{1}_{x\cdot\frac{\xi}{|\xi|}<0}, \; c_-(x,\xi) = \mathbb{1}_{x\cdot\frac{\xi}{|\xi|}>0}$$

so that their respective supports are obtained pulling the support of a_+ (resp. a_-) of one unit in the direction of $\frac{\xi}{|\xi|}$ (resp. $-\frac{\xi}{|\xi|}$). In that way, the union of those two supports exactly covers the whole plane, so that $c_+ + c_- \equiv 1$:

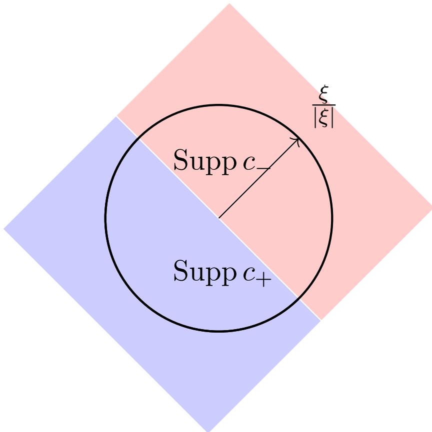

Figure 3

Consequently, we would have obtained a channels of energy estimate (1.2.11) where, in the left hand side, unlike in [8, 4], the cut-off energies are taken not on the whole complement of the wave cone, but only on the subset of it given at each fixed frequency by the shaded area on Figure 2. Our result for the half-wave equation in Theorem 1.1.1 is of that type, except that we cannot make rigorous the above heuristics if we define $a_\pm$ as in (1.2.12). Actually, in order to just start a rigorous proof, we would have to know that operators with symbols $a_\pm$ are bounded on L^2, so that quantities (1.2.10) would be finite. We show in Appendix A.2 below that this is *not* the case. We have thus to modify the definition of $a_\pm$ in order that the associated operators be bounded on L^2, while retaining as much as possible of the support properties indicated in figures 2 and 3. In the statement of Theorem 1.1.1, we replace a_+ by (1.1.2), which cut-offs along the shaded area in Figure 1. If one looks at it at time $t = 1$, in order to compare it with the left picture in Figure 2, one should concentrate on a neighborhood of point $-\omega$. We see that instead of taking as support of our symbol an half-space whose boundary is tangent to the unit circle at $-\omega$, we consider the complement of the unit disc, intersected with a convenient neighborhood of $-\omega$. If one translates the shaded domain of Figure 1 by $t\omega$, and the symmetric one relatively to the origin by $-t\omega$, one obtains instead of Figure 3 the following picture:

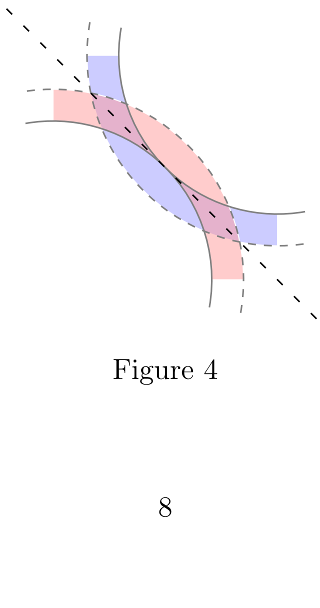

Figure 4

In this case, the union of the two shaded areas does not cover the whole plane as in Figure 3, but it contains a ball of center 0, radius t^δ for some $\delta > 0$, so that when $t \to +\infty$, this domain will nevertheless allow one to recover the whole L^2 norm of the initial condition.
As indicated earlier, the above argument does not provide a rigorous proof, because our symbols (1.2.12) or (1.1.2) are not smooth, and do not obey symbolic calculus. We shall have thus to give direct proofs, using systematically stationary phase formula. The fact that we may further localize through the cut-offs $\chi, \tilde{\chi}$ in (1.1.2), instead of taking into account the complement of the whole wave cone as in [8, 4], is not a surprise: actually, forgetting the sharp cut-off in (1.1.2), it is easy to see that the energy outside the neighborhood given by the intersection of the supports of these two functions goes to zero when time goes to infinity.
The interest of the above heuristics is that they are not limited to the half-wave equation and work as well for other dispersive models. For instance, if we replace in (1.2.2) $e^{it|D_x|}$ by $e^{it\frac{|D_x|^2}{2}}$, the same reasoning as above would lead us to define $a_\pm$ replacing (1.2.12) by

$$a_+(x,\xi) = \mathbb{1}_{x\cdot\xi<-1},\ a_-(x,\xi) = \mathbb{1}_{x\cdot\xi>1} \tag{1.2.13}$$

for which we have the analogous of Figure 2, where $\frac{\xi}{|\xi|}$ has been replaced by ξ. Again, in order to avoid issues with L^2 boundedness of the associated operators, we replace $a_+\left(\frac{x}{t},\xi\right)$ by the symbol in (1.1.3), that cut-offs sharply for $|x| > |t||\xi|$ and smoothly around a ball centered at $x = -t\xi$, with radius $|t||\xi|\langle\sqrt{|t|}|\xi|\rangle^{-\frac{1}{2}+0}$. Unlike in the case (1.1.2), the L^2 boundedness of the operator with symbol (1.1.3) uniformly for $|t| \geq 1$, is not trivial, and will occupy most of section 2 of that paper. Moreover, we are able to prove this boundedness only when, at a fixed frequency, the radius of the ball on which we truncate is in $t^{\frac{3}{4}+0}$, while in the case of the half-wave equation (1.1.2), we could take instead a ball of radius $t^{\frac{1}{2}+0}$, which is essentially optimal in order to get (1.1.5). Another difference with the wave equation is that the part of the energy that remains outside the "Schrödinger cone" $|x| > t|\xi|$ is *half* of the initial energy according to (1.1.6), instead of the whole of it in (1.1.5). This shows the limitation of our heuristics illustrated by Figures 2 and 3 and the need for a precise proof.
To finish this subsection, let us comment on the case of the "real wave equation" of Theorem 1.1.2 and Theorem 1.1.3. For this last result, one has to discuss relatively to the residue of the dimension modulo 4, while for the half-wave equation conclusion (1.1.5) holds in any dimension. This is due to the fact that the truncated energy defined in (1.1.8) may be expressed on the one hand from quantities of the form (1.2.6), for which the above analysis applies, and on the other hand from other expressions like

$$\left\langle e^{\epsilon\frac{i}{h}|hD_x|}b(x,hD_x)e^{\epsilon\frac{i}{h}|hD_x|}v'_{0,h}, v''_{0,h}\right\rangle \tag{1.2.14}$$

where the sign $\epsilon \in \{-,+\}$ is the *same* on each exponential factor. The functions $v'_{0,h}, v''_{0,h}$ are given from $w, D_t w$. Expression (1.2.14) is not the intertwining of $b(x,hD_x)$ by $e^{\pm\frac{i}{h}|hD_x|}$ unlike in (1.2.6), so that the preceding heuristics do not apply. Nevertheless, our computations in the proof of the results for the half-wave equation allow us to treat expressions of the form (1.2.14) and show that these terms give a zero contribution to (1.1.9) in odd dimension (as already known) and provide the extra term involving the Hankel transformation in (1.1.10) (as was also known in the radial case).

2 L^2 boundedness of sharp microlocal cut-offs

In Theorem 1.1.1, we stated that the operators with symbols (1.1.2), (1.1.3) are bounded on L^2, uniformly in time. We shall prove this statement first for the half wave equation, for which this is

almost immediate, and then for the Schrödinger case, that is more involved. Let us describe our strategy in this last case. We shall first use a space Littlewood-Paley decomposition in order to reduce ourselves to operators $a(x, D_x)$ with symbols $a(x, \xi)$ supported for x in some ring. These symbols involve a sharp cut-off $\mathbb{1}_{|\xi|<|x|}$, so that they do not fall inside the usual framework of pseudo-differential calculus. We decompose them as sums $a = a' + a''$ where a' is the part of the symbol that corresponds to the convolution kernel of $a(x, D_x)$ cut-off close to the diagonal. We show that symbol a' satisfies the assumptions of the Calderón-Vaillancourt theorem. We are thus reduced to the L^2 boundedness of $a''(x, D_x)$. The symbol $a''(x, \xi)$ is not smooth, but if we use polar coordinates both in x and ξ, it turns out that a'' is smooth relatively to the angular variables. Using the classical Hörmander generalization of the Hausdorff-Young inequality, we are able to obtain an $\mathcal{L}(L^2)$ bound relatively to these angular variables. The contribution of radial variables is then treated by Schur's lemma.
In the rest of the paper, if χ is some radial function on $\mathbb{R}^d$, we shall denote by abuse of notation $\chi(r)$ for $\chi(x)$ if $r = |x|$.

2.1 The half-wave equation

We want to prove:

Proposition 2.1.1 *Let $a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}$ be defined in (1.1.2). Then operator $\mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}\big)$ is bounded on $L^2(\mathbb{R}^d)$, uniformly in $t \in \mathbb{R}^*$.*

Proof: It suffices to show that the operator

$$u \to \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} \chi\Big(|t|^{-\delta'}\Big(x + t\frac{\xi}{|\xi|}\Big)\Big) \hat{u}(\xi)\, d\xi \tag{2.1.1}$$

is bounded on $L^2(\mathbb{R}^d)$ uniformly in time, for any exponent $\delta' \geq 0$ and χ in $C_0^\infty(\mathbb{R}^d)$. Writing $\chi = \mathcal{F}^{-1}\hat{\chi}$, we see that (2.1.1) is the operator

$$u \to \frac{1}{(2\pi)^d} \int \hat{\chi}(\zeta)(A_\zeta u)(x)\, d\zeta$$

where for any fixed ζ

$$\begin{aligned} A_\zeta u(x) &= \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi + i|t|^{-\delta'} x\cdot\zeta + it|t|^{-\delta'}\zeta\cdot\frac{\xi}{|\xi|}} \hat{u}(\xi)\, d\xi \\ &= e^{i|t|^{-\delta'} x\cdot\zeta} e^{it|t|^{-\delta'}\zeta\cdot\frac{D_x}{|D_x|}} u. \end{aligned}$$

Since A_ζ is bounded on $L^2(\mathbb{R}^d)$ uniformly in ζ, the conclusion follows. □

2.2 The Schrödinger equation

We want to prove:

Proposition 2.2.1 *Let $a^{\mathrm{Schr}}_{\chi,\delta}$ be defined by (1.1.3) with $\delta \in [0, \frac{1}{2}[$ and $\mathrm{Supp}\,\chi$ small enough. Then the operator $\mathrm{Op}\big(a^{\mathrm{Schr}}_{\chi,\delta}(t, \cdot)\big)$ is bounded on $L^2(\mathbb{R}^d)$ uniformly in $t \in \mathbb{R}^*$.*

Before starting the proof, let us make some reductions. It is enough to prove the result for $t > 0$, and since, for any symbol $b(x, \xi)$, any $\lambda > 0$, the symbol $b_\lambda(x, \xi) = b(\lambda x, \xi/\lambda)$ is such that

$\|\mathrm{Op}(b_\lambda)\|_{\mathcal{L}(L^2)} = \|\mathrm{Op}(b)\|_{\mathcal{L}(L^2)}$, one may reduce to the case $t = 1$. To simplify notation, we shall just set

$$a(x,\xi) = \chi\Big(\frac{x+\xi}{|\xi|\langle\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|x|>|\xi|} \tag{2.2.1}$$

and shall show that for some $C > 0$, any u in $L^2(\mathbb{R}^d)$ (or in $\mathcal{S}(\mathbb{R}^d)$),

$$\|\mathrm{Op}(a)u\|_{L^2} \leq C\|u\|_{L^2}. \tag{2.2.2}$$

If ψ is in $C_0^\infty(\mathbb{R}^d)$, it follows immediately from (2.2.1) that

$$|\psi(x)\mathrm{Op}(a)u(x)| \leq C|\psi(x)| \int_{|\xi|\leq M} |\hat{u}(\xi)|\,d\xi$$

for some M, so that (2.2.2) holds for the symbol $\psi(x)a(x,\xi)$. Take ψ equal to one close to zero, let κ be in $C_0^\infty(\mathbb{R})$, supported close to 1 and equal to one close to that point. Decompose

$$(1-\psi)(x)a(x,\xi) = \chi\Big(\frac{x+\xi}{|\xi|\langle\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|x|>|\xi|}\kappa\Big(\frac{|x|}{|\xi|}\Big)(1-\psi)(x) + e(x,\xi) \tag{2.2.3}$$

with

$$e(x,\xi) = (1-\psi)(x)\chi\Big(\frac{x+\xi}{|\xi|\langle\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|x|>|\xi|}(1-\kappa)\Big(\frac{|x|}{|\xi|}\Big). \tag{2.2.4}$$

Since χ is compactly supported, since κ is supported close to 1, equal to one on a neighborhood of 1, and since on the support of e, ξ cannot vanish, one sees that e is a smooth function satisfying

$$|\partial_x^\alpha\partial_\xi^\beta e(x,\xi)| \leq C_{\alpha,\beta}$$

for any α,β in $\mathbb{N}^d$. By the Calderón-Vaillancourt theorem, $\mathrm{Op}(e)$ is bounded on L^2, so that we just have to study the first term in the right hand side of (2.2.3)

Let φ be in $C_0^\infty(\mathbb{R}^d)$ and ψ be chosen so that we have a Littlewood-Paley decomposition $\psi(x) + \sum_{k=1}^{+\infty}\varphi(2^{-k}x) \equiv 1$. Assume moreover that φ,ψ are radial. For any k in $\mathbb{N}^*$, define if $f \in L^2(\mathbb{R}^d)$

$$A_k f(x) = \frac{2^{2kd}}{(2\pi)^d}\varphi(x)\int e^{i2^{2k}x\cdot\xi}\chi\Big(\frac{x+\xi}{|\xi|\langle 2^k\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|x|>|\xi|}\kappa\Big(\frac{|x|}{|\xi|}\Big)\hat{f}(2^{2k}\xi)\,d\xi. \tag{2.2.5}$$

Lemma 2.2.2 *Assume that we have proved that* $\sup_{k\geq 1}\|A_k\|_{\mathcal{L}(L^2)}$ *is finite. Then conclusion of Proposition 2.2.1 holds.*

Proof: We have seen that we need to prove boundedness of $\mathrm{Op}(a)$, with a given by (2.2.1), and that $\psi(x)\mathrm{Op}(a)$ and $\mathrm{Op}(e)$, with e given by (2.2.4), are bounded on L^2. By (2.2.3), we just have to obtain the boundedness of

$$\sum_{k=1}^{+\infty}\varphi(2^{-k}x)\mathrm{Op}\Big(a(x,\xi)\kappa\Big(\frac{|x|}{|\xi|}\Big)\Big)$$

and by almost orthogonality, to get a bound for

$$\sum_{k=1}^{+\infty}\Big\|\varphi(2^{-k}x)\mathrm{Op}\Big(a(x,\xi)\kappa\Big(\frac{|x|}{|\xi|}\Big)\Big)u\Big\|_{L^2}^2 \tag{2.2.6}$$

for any u in $L^2(\mathbb{R}^d)$. Let $\tilde{\varphi}$ be in $C_0^\infty(\mathbb{R}^d)$ be such that $x \in \operatorname{Supp}\varphi$ and $\frac{|x|}{|\xi|} \in \operatorname{Supp}\kappa$ implies $\tilde{\varphi}(\xi) \equiv 1$. We write

$$\begin{aligned}\varphi(2^{-k}x)\mathrm{Op}\Big(a(x,\xi)\kappa\Big(\frac{|x|}{|\xi|}\Big)\Big)u &= \varphi(2^{-k}x)\int e^{ix\cdot\xi}\kappa\Big(\frac{|x|}{|\xi|}\Big)a(x,\xi)\widehat{\tilde{\Delta}_k u}(\xi)\,d\xi \\ &= (A_k f_k)(2^{-k}x)\end{aligned}$$

where $\tilde{\Delta}_k = \tilde{\varphi}(2^{-k}D_x)$, A_k is given by (2.2.5) and f_k is defined by $\hat{f}_k(\xi) = 2^{-kd}\widehat{\tilde{\Delta}_k u}(2^{-k}\xi)$. According to the assumption, we may bound (2.2.6) by

$$C\sum_{k=1}^{+\infty} 2^{kd}\|f_k\|_{L^2}^2 \le C'\|u\|_{L^2}^2$$

using the almost orthogonality of the $\tilde{\Delta}_k$'s. This concludes the proof. $\square$

Our next task is to show that the assumption of Lemma 2.2.2 holds. We introduce the kernel

$$K_k(x,z) = \frac{2^{2kd}}{(2\pi)^d}\varphi(x)\int e^{i2^{2k}z\cdot\xi}\chi\Big(\frac{x+\xi}{|\xi|\langle 2^k\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|\xi|<|x|}\kappa\Big(\frac{|x|}{|\xi|}\Big)\,d\xi \tag{2.2.7}$$

so that

$$A_k f(x) = \int K_k(x, x-y)f(y)\,dy. \tag{2.2.8}$$

Let m be in $C_0^\infty(\mathbb{R}^d)$ radial, equal to one close to zero and decompose

$$K_k = K_k' + K_k'' \text{ with } K_k'(x,z) = m(2^k z)K_k(x,z). \tag{2.2.9}$$

We shall study successively the operators associated to the kernels K_k' and K_k''.

• **Study of operator with kernel K_k'**

Define the symbol

$$b_k'(x,\eta) = \frac{\varphi(2^{-k}x)}{(2\pi)^d}\int \hat{m}(\eta-\xi)\chi\Big(\frac{x+\xi}{|\xi|\langle\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|\xi|<|x|}\kappa\Big(\frac{|x|}{|\xi|}\Big)\,d\xi \tag{2.2.10}$$

and set $a_k'(x,\eta) = b_k'(2^k x, 2^{-k}\eta)$. Then the operator with convolution kernel K_k' is just $\mathrm{Op}(a_k')$ and moreover $\|\mathrm{Op}(a_k')\|_{\mathcal{L}(L^2)} = \|\mathrm{Op}(b_k')\|_{\mathcal{L}(L^2)}$. We shall prove:

Proposition 2.2.3 *For any α, β in $\mathbb{N}^d$, there is a constant $C > 0$ such that for any k*

$$|\partial_x^\alpha\partial_\eta^\beta b_k'(x,\eta)| \le C. \tag{2.2.11}$$

Moreover the operator $\mathrm{Op}(a_k')$ with kernel K_k' is bounded on L^2 uniformly in k.

Proof: Notice that the last statement follows from (2.2.11) and the Calderón-Vaillancourt theorem.
When $\alpha = 0$ in (2.2.11), the bound follows from (2.2.10). Since $b_k'(x,\eta)$ is supported for $|x| \ge c > 0$, setting $x = r\omega$ with $r > c$, $|\omega| = 1$, it is sufficient to get uniform estimates for ∂_r and $r^{-1}\partial_\omega$ derivatives of

$$\varphi(2^{-k}r)\int_0^{+\infty}\int_{\mathbb{S}^{d-1}}\hat{m}(\eta-\rho\theta)\chi\Big(\frac{r\omega+\rho\theta}{\rho\langle\rho\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{\rho<r}\kappa\Big(\frac{r}{\rho}\Big)\rho^{d-1}\,d\theta d\rho. \tag{2.2.12}$$

As $\delta \geq 0$, the case of $r^{-1}\partial_\omega$-derivatives is clear. If we take one ∂_r-derivative of (2.2.12), we get on the one hand an expression that has essentially the same form as (2.2.12) and on the other hand a contribution

$$\varphi(2^{-k}r)\int_{\mathbb{S}^{d-1}} \hat{m}(\eta - r\theta)\chi\big(\langle r\rangle^{\frac{1}{2}-\delta}(\omega+\theta)\big)r^{d-1}\,d\theta. \tag{2.2.13}$$

As $\sup_\eta \int_{\mathbb{S}^{d-1}} |\hat{m}(\eta - r\theta)|\,d\theta = O(r^{-(d-1)})$, one gets a $O(1)$ bound for (2.2.13). If one takes further ∂_r-derivatives of (2.2.12), (2.2.13), the same uniform bound holds. This concludes the proof of the proposition. □

• **Study of operator with kernel K_k''**

As we did in (2.2.10), we define

$$b_k''(x,\eta) = \frac{\varphi(2^{-k}x)}{(2\pi)^d}\int e^{iz\cdot(\xi-\eta)}\chi\Big(\frac{x+\xi}{|\xi|\langle\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|\xi|<|x|}\kappa\Big(\frac{|x|}{|\xi|}\Big)(1-m)(z)\,dzd\xi. \tag{2.2.14}$$

Of course, the above integral in dz should be interpreted as an oscillatory integral, i.e. as the limit when $\epsilon \to 0+$ of the absolutely convergent integral obtained inserting in the integrand a factor like $e^{-\epsilon|z|^2}$. Since this factor disappears at the limit after performing the integrations by parts that will be made below, bringing to absolutely convergent expressions, we shall not write it explicitly. Notice nevertheless that the symbol $b_k''(x,\eta)$ that we shall estimate in Proposition 2.2.6 below will not be bounded but will have a logarithmic singularity along $|x| = |\eta|$. If we define $a_k''(x,\eta) = b_k''(2^kx, 2^{-k}\eta)$, then the operator with kernel K_k'' is nothing but $\mathrm{Op}(a_k'')$. We shall prove:

Proposition 2.2.4 *The operator* $\mathrm{Op}(b_k'')$ *is bounded on* $L^2(\mathbb{R}^d)$ *uniformly in* k *in* $\mathbb{N}^*$.

Since $\|\mathrm{Op}(b_k'')\|_{\mathcal{L}(L^2)} = \|\mathrm{Op}(a_k'')\|_{\mathcal{L}(L^2)}$ the proposition implies that the operator with kernel K_k'' is bounded on L^2 uniformly in k. By Proposition 2.2.3 and (2.2.9), (2.2.8), this shows that the assumption of Lemma 2.2.2 holds, so that this will conclude the proof of Proposition 2.2.1.

The proof of Proposition 2.2.4 will be made splitting b_k'' into several pieces, each of them providing a bounded operator on L^2. These pieces will be studied in the following lemmas.

Lemma 2.2.5 *Let* χ_0 *be in* $C_0^\infty(\mathbb{R}^d)$ *radial, equal to one close to zero, with small enough support. Define*

$$\begin{aligned} b_{k,1}''(x,\eta) = \frac{\varphi(2^{-k}x)}{(2\pi)^d}\int e^{iz\cdot(\xi-\eta)}&\chi\Big(\frac{x+\xi}{|\xi|\langle\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\\ &\times(1-\chi_0)\Big(\frac{\xi-\eta}{(1+\xi^2+\eta^2)^{\frac{1}{2}}}\Big)\mathbb{1}_{|\xi|<|x|}\kappa\Big(\frac{|x|}{|\xi|}\Big)(1-m)(z)\,dzd\xi. \end{aligned} \tag{2.2.15}$$

Then $\mathrm{Op}(b_{k,1}'')$ *is bounded on* L^2, *uniformly in* $k \in \mathbb{N}^*$.

Proof: Since on the support of the integrand $|\xi-\eta| \geq c(1+\xi^2+\eta^2)^{\frac{1}{2}}$, making ∂_z-integrations by parts in the oscillatory integral, we gain rapid decay in z and an arbitrary negative power of $(1+|\xi|+|\eta|)$. We may thus write

$$b_{k,1}''(x,\eta) = \frac{\varphi(2^{-k}x)}{(2\pi)^d}\int \chi\Big(\frac{x+\xi}{|\xi|\langle\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|\xi|<|x|}\kappa\Big(\frac{|x|}{|\xi|}\Big)U(\xi,\eta)\,d\xi$$

with $|U(\xi,\eta)| \leq C(1+|\xi|+|\eta|)^{-d-1}$. We write then

$$\mathrm{Op}(b''_{k,1})u = \varphi(2^{-k}x)\int \chi\Big(\frac{x+\xi}{|\xi|\langle\xi\rangle^{-\frac{1}{2}+\delta}}\Big)\mathbb{1}_{|\xi|<|x|}\kappa\Big(\frac{|x|}{|\xi|}\Big)U(\xi,D_x)u\,d\xi.$$

Since $\|U(\xi,D_x)u\|_{L^2(dx)} \leq C(1+|\xi|)^{-d-1}\|u\|_{L^2}$, the conclusion follows. $\square$

We shall have next to study the symbol $b''_{k,2}$ obtained replacing $(1-\chi_0)$ by χ_0 in (2.2.15). Setting $x = r\omega$, $\eta = r'\omega'$, $z = \rho\theta$, $\xi = \rho'\theta'$ with $\omega, \omega', \theta, \theta'$ in $\mathbb{S}^{d-1}$, we may write

$$\begin{aligned}(2.2.16)\quad b''_{k,2}(x,\eta) = \frac{\varphi(2^{-k}r)}{(2\pi)^d}&\int_{(\mathbb{R}^*_+)^2\times(\mathbb{S}^{d-1})^2} e^{i\rho\theta\cdot(\rho'\theta'-r'\omega')}\chi\Big(\frac{r\omega+\rho'\theta'}{\rho'\langle\rho'\rangle^{-\frac{1}{2}+\delta}}\Big)\\ &\times \chi_0\Big(\frac{\rho'\theta'-r'\omega'}{(1+\rho'^2+r'^2)^{\frac{1}{2}}}\Big)\mathbb{1}_{\rho'<r}\kappa\Big(\frac{r}{\rho'}\Big)(1-m)(\rho)(\rho\rho')^{d-1}\,d\rho d\rho' d\theta d\theta'.\end{aligned}$$

Notice that inside the integral, we have $r \sim \rho' \sim r'$ because of the cut-offs. Our main result in the rest of this section is:

Proposition 2.2.6 *Set*

$$(2.2.17)\qquad \mu(r') = \langle r'\rangle^{-\frac{1}{2}+\delta} = \langle\eta\rangle^{-\frac{1}{2}+\delta} \sim \langle\rho'\rangle^{-\frac{1}{2}+\delta}.$$

Then (2.2.16) may be written as a sum of functions of the following two forms:
- *On the one hand, functions*

$$(2.2.18)\qquad \varphi(2^{-k}r)\Omega(\omega,\omega',r,r')$$

smooth in ω, ω' and satisfying for any β, β' in $\mathbb{N}^{d-1}$, any N in $\mathbb{N}$ estimates

$$(2.2.19)\qquad |\partial_\omega^\beta\partial_{\omega'}^{\beta'}\Omega(\omega,\omega',r,r')| \leq C_{\beta,\beta'}\mu(r')^{-|\beta|-|\beta'|}\Big\langle\frac{d(-\omega,\omega')}{\mu(r')}\Big\rangle^{-N}|\log|r-r'||\langle r-r'\rangle^{-2}$$

(where d is the distance on $\mathbb{S}^{d-1}$) and supported for $d(-\omega,\omega') \ll 1$ and $C^{-1} \leq \frac{r}{r'} \leq C$.
- *On the other hand, functions*

$$(2.2.20)\qquad \varphi(2^{-k}r)\Omega^R(\omega,\omega',r,r')$$

that satisfy bounds of the form

$$(2.2.21)\qquad |\Omega^R(\omega,\omega',r,r')| \leq C\langle r\omega+r'\omega'\rangle^{-d-1}.$$

We shall prove the above proposition in several steps. We first study the $d\theta'$ integral in (2.2.16), namely

$$(2.2.22)\qquad \int_{\mathbb{S}^{d-1}} e^{i\rho\rho'\theta\cdot\theta'}\chi_0\Big(\frac{\rho'\theta'-r'\omega'}{(1+\rho'^2+r'^2)^{\frac{1}{2}}}\Big)\chi\Big(\Big(\frac{r}{\rho'}\omega+\theta'\Big)\langle\rho'\rangle^{\frac{1}{2}-\delta}\Big)\,d\theta'.$$

As χ has small support, we have $\big|\frac{r}{\rho'}-1\big| \ll \mu(r') \leq 1$, so that, since in (2.2.16) $r \geq c2^k \geq 1$, we may assume that $\rho' \geq c > 0$. Moreover, because of the cut-off $1-m$ in (2.2.16), we may also assume that $\rho \geq c > 0$. In addition to notation (2.2.17), we define

$$(2.2.23)\qquad \nu = \nu(r,\rho') = \frac{r}{\rho'}$$

which will be close to 1. Notice also that since $\delta \geq 0$ and $\rho' \sim r'$, we shall have $\rho'\mu(r') \geq c > 0$. To simplify notation, we shall replace in the argument of χ in (2.2.22) $\langle\rho'\rangle^{-\frac{1}{2}+\delta}$ by the independent variable $\mu = \mu(\rho')$, so that we shall consider

$$\int_{\mathbb{S}^{d-1}} e^{i\rho\rho'\theta\cdot\theta'}\chi_0\Big(\frac{\rho'\theta' - r'\omega'}{(1+\rho'^2+r'^2)^{\frac{1}{2}}}\Big)\chi\Big(\Big(\frac{r}{\rho'}\omega+\theta'\Big)\mu^{-1}\Big)\,d\theta'. \tag{2.2.24}$$

Since we have seen that $\rho' \sim r'$, this new parameter μ will be equivalent to $\mu(r')$ defined in (2.2.17) when we replace μ by its value $\langle\rho'\rangle^{-\frac{1}{2}+\delta}$.

Lemma 2.2.7 *The integral (2.2.24) may be written as the sum of*
- *The following two expressions, for the two possible signs*

$$e^{\pm i\rho\rho'}\mu^{d-1}S^{\pm}_{-\frac{d-1}{2}}(\theta,\omega,\omega',\rho',r',\nu,\mu;\rho\rho'\mu^2) \tag{2.2.25}$$

where $S^{\pm}_{-\frac{d-1}{2}}(\theta,\omega,\omega',\rho',r',\nu,\mu;\zeta)$ is a smooth function of $(\theta,\omega,\omega',\rho',r',\nu,\mu;\zeta)$ supported for

$$\zeta \geq c > 0,\ d(\theta,\pm\omega') \ll 1,\ d(-\omega',\omega) \ll 1 \tag{2.2.26}$$

satisfying for any α,β,β' in $\mathbb{N}^{d-1}$, $j,j',\ell',\gamma,N \in \mathbb{N}$

$$\Big|\partial_\theta^\alpha\partial_\omega^\beta\partial_{\omega'}^{\beta'}\partial_\nu^j\partial_\mu^{j'}\partial_{\rho'}^{\ell'}\partial_\zeta^\gamma S^{\pm}_{-\frac{d-1}{2}}\Big| \leq C\mu^{-|\alpha|-|\beta|-j-j'}\langle\rho'\rangle^{-\ell'}\langle\zeta\rangle^{-\frac{d-1}{2}-\gamma}\Big\langle\frac{d(\theta,\mp\omega)}{\mu}\Big\rangle^{-N}. \tag{2.2.27}$$

- *The following remainders*

$$e^{-i\rho\rho'\theta\cdot\omega}\mu^{d-1}R(\theta,\omega,\omega',\rho',r',\nu,\mu;\rho\rho'\mu) \tag{2.2.28}$$

where $R(\theta,\omega,\omega',\rho',r',\nu,\mu;\zeta)$ is smooth in $(\theta,\omega,\omega',\rho',\nu,\mu;\zeta)$ and satisfies for any α,β,β' in $\mathbb{N}^{d-1}$, j,j',ℓ',γ,N in $\mathbb{N}$

$$|\partial_\theta^\alpha\partial_\omega^\beta\partial_{\omega'}^{\beta'}\partial_\nu^j\partial_\mu^{j'}\partial_{\rho'}^{\ell'}\partial_\zeta^\gamma R| \leq C\mu^{-|\alpha|-|\beta|-j-j'}\langle\rho'\rangle^{-\ell'}\langle\zeta\rangle^{-N}\mathbb{1}_{|r-\rho'|\ll\rho'\mu}. \tag{2.2.29}$$

Proof: We set

$$\lambda = \rho\rho' \geq c > 0. \tag{2.2.30}$$

One has $\lambda\mu^2 \geq c\rho r'\langle r'\rangle^{-1+2\delta}$ as $\rho' \sim r'$ and by (2.2.17). Since we have seen that $\rho \geq c$, $r' \sim r \geq c$ and since $\delta \geq 0$, we have $\lambda\mu^2 \geq c > 0$. If the support of χ_0 in (2.2.24) is small enough, we have that $|\theta' - \omega'| \ll 1$ on the support of the integrand. Moreover, because of the χ cut-off, we have also $|\theta' + \omega| \ll \mu \leq 1$. This implies $|\omega + \omega'| \ll 1$ on the support i.e. the last condition (2.2.26) holds. With notation (2.2.23) for ν, we define a function

$$F_+(\theta,\theta',\omega,\omega',\nu,\mu;\rho',r) = \chi_0\Big(\frac{\rho'\theta' - r'\omega'}{(1+\rho'^2+r'^2)^{\frac{1}{2}}}\Big)\chi\Big(\frac{-\nu\omega+\theta'}{\mu}\Big)\chi_0(\theta'-\theta) \tag{2.2.31}$$

and

$$F_0(\theta,\theta',\omega,\omega',\nu,\mu;\rho',r) = \chi_0\Big(\frac{\rho'\theta' - r'\omega'}{(1+\rho'^2+r'^2)^{\frac{1}{2}}}\Big)\chi\Big(\frac{-\nu\omega+\theta'}{\mu}\Big)[1-\chi_0(\theta'-\theta)-\chi_0(\theta'+\theta)]. \tag{2.2.32}$$

One may write (2.2.24) as $J_+ + J_0 + J_-$ where

$$
\begin{aligned}
J_+ &= \int e^{i\lambda\theta\cdot\theta'} F_+(\theta,\theta',-\omega,\omega',\nu,\mu;\rho',r')\,d\theta' \\
J_- &= \int e^{-i\lambda\theta\cdot\theta'} F_+(\theta,\theta',\omega,-\omega',\nu,\mu;\rho',r')\,d\theta' \\
J_0 &= \int e^{i\lambda\theta\cdot\theta'} F_0(\theta,\theta',-\omega,\omega',\nu,\mu;\rho',r')\,d\theta'.
\end{aligned} \tag{2.2.33}
$$

We notice that in J_+ the integrand is supported for $d(\theta,\theta')+d(\omega',\theta') \ll 1$ so that the assumptions of Corollary A.1.3 are satisfied, with ω replaced by $-\omega$, (ρ',r') in (2.2.31) being extra parameters. Notice that a $\partial_{\rho'}$ or $\partial_{r'}$ derivative of F_+ makes gain $\rho'^{-1} \sim r'^{-1}$. Consequently, J_+ is an integral of the form (A.1.21), that may be written under the form (A.1.22) in terms of a symbol satisfying (A.1.23) (up to the replacement of ω by $-\omega$). This gives (2.2.27) with sign $+$, the extra estimates of $\partial_{\rho'}$ derivatives following from the analogous properties satisfied by F_+. Since J_- may be obtained from J_+ by conjugation and replacing $(-\omega,\omega')$ by $(\omega,-\omega')$, we obtain as well the contribution to (2.2.25) with sign $-$.
It remains to study J_0. Since on the support of F_0, $|\theta'-\theta| \geq c > 0$ and $|\theta'+\theta| \geq c > 0$, one may apply to J_0 Corollary A.1.6, which gives expression (2.2.28) with estimates (2.2.29), the last cut-off in the right hand side of this inequality coming from the χ factor in F_0 and the definition (2.2.23) of ν. □

To prove Proposition 2.2.6, we shall plug the expression of (2.2.24) obtained in Lemma 2.2.7 inside (2.2.16).

Proof of Proposition 2.2.6: We shall study the contributions to (2.2.16) given successively by (2.2.25) and (2.2.28).

- **Contribution of (2.2.25) to (2.2.16)**

We shall treat explicitly only the contribution of the term with sign $+$ of (2.2.25) to (2.2.16). We shall write $S_{-\frac{d-1}{2}}$ for $S^+_{-\frac{d-1}{2}}$. We replace in (2.2.16) the $d\theta'$ integral (2.2.24) by (2.2.25) with sign $+$, recalling that $\mu = \mu(\rho') = \langle\rho'\rangle^{-\frac{1}{2}+\delta}$. We obtain

$$
\begin{aligned}
&\frac{\varphi(2^{-k}r)}{(2\pi)^d}\int_{(\mathbb{R}_+^*)^2\times\mathbb{S}^{d-1}} e^{i\rho(\rho'-r')}\mu(\rho')^{d-1}J(\omega,\omega',\rho,\rho',r,r') \\
&\qquad\qquad \times \mathbb{1}_{\rho'<r}\kappa\Big(\frac{r}{\rho'}\Big)(\rho\rho')^{d-1}(1-m)(\rho)\,d\rho d\rho'
\end{aligned} \tag{2.2.34}
$$

with

$$
J(\omega,\omega',\rho,\rho',r,r') = \int e^{i\rho r'(1-\omega'\cdot\theta)} S_{-\frac{d-1}{2}}(\theta,\omega,\omega',\rho',r',\nu,\mu;\zeta)\,d\theta \tag{2.2.35}
$$

where μ stands for $\mu(\rho')$ and ζ for $\rho r'\mu(\rho')^2 \sim \rho\rho'\mu(\rho')^2$. We may apply to this $d\theta$ integral again Corollary A.1.3, where $\lambda = \rho r' \sim \rho\rho'$, with the argument $(\theta,\theta',\omega,\omega')$ of F in (A.1.21) replaced here by $(\omega',\theta,-\omega,\omega')$ and where we consider the case of sign $-$ in the phase of (A.1.21). Then conditions (2.2.26) (with sign $+$) imply that the support assumption in the statement of Corollary A.1.3 is satisfied. The bounds (A.1.20) are satisfied because of (2.2.27). Moreover, we have the extra bounds for the ∂_ζ derivatives in (2.2.27). We may thus write (2.2.35) using (A.1.22) with sign $-$ as

$$
\mu(\rho')^{d-1}\Sigma(\omega,\omega';\rho',\nu,\mu,\zeta,\zeta')
$$

where $\mu = \mu(\rho')$, $\zeta = \zeta' = \rho r'\mu(\rho')^2$ and Σ is supported by (2.2.26) for $d(-\omega',\omega) \ll 1$ and satisfies combining (2.2.27) and (A.1.23) estimates

$$\begin{aligned}|\partial_\omega^\alpha \partial_{\omega'}^{\alpha'} \partial_{\rho'}^{\ell'} \partial_\nu^j \partial_\mu^{j'} \partial_\zeta^\gamma \partial_{\zeta'}^{\gamma'} \Sigma(\omega,\omega';\rho',\nu,\mu,\zeta,\zeta')| \\ \leq C\mu^{-|\alpha|-|\alpha'|-j-j'} \langle\rho'\rangle^{-\ell'} \langle\zeta\rangle^{-\frac{d-1}{2}-|\gamma|} \langle\zeta'\rangle^{-\frac{d-1}{2}-|\gamma'|} \Big\langle \frac{d(\omega',-\omega)}{\mu}\Big\rangle^{-N}.\end{aligned}$$

Since $r' \sim \rho' \sim r$ on the support of J, we have thus, using (2.2.23)

$$\begin{aligned}|\partial_\omega^\alpha \partial_{\omega'}^{\alpha'} \partial_\rho^\ell \partial_{\rho'}^{\ell'} J(\omega,\omega',\rho,\rho',r,r')| \\ \leq C\mu(r')^{d-1-|\alpha|-|\alpha'|} \langle\rho\rangle^{-\ell} \langle\rho'\rangle^{-\ell'} \langle \rho r'\mu(r')^2\rangle^{-(d-1)} \Big\langle \frac{d(\omega',-\omega)}{\mu(r')}\Big\rangle^{-N}.\end{aligned} \tag{2.2.36}$$

We make in (2.2.34) one $\partial_{\rho'}$-integration by parts. We get on the one hand the boundary term

$$-i\frac{\varphi(2^{-k}r)}{(2\pi)^d} \int e^{i\rho(r-r')} \mu(r')^{d-1} (\rho r)^{d-1} J(\omega,\omega',\rho,r,r,r')(1-m)(\rho)\frac{d\rho}{\rho} \tag{2.2.37}$$

and according to (2.2.36) another term of the form (2.2.34) but with an extra $\rho^{-1}\rho'^{-1}$ factor under the integral. Performing one more integration by parts, we get yet another term of the form (2.2.37) (with an extra $(\rho\rho')^{-1}$ factor) and another contribution of the form (2.2.34), with an extra $(\rho\rho')^{-2}$ factor. We have thus to study on the one hand (2.2.37) and on the other hand (2.2.34) with $(\rho\rho')^{d-1}$ replaced by $(\rho\rho')^{d-3}$.

- Study of (2.2.37): Performing in (2.2.37) one integration by parts in ∂_ρ, we gain a factor ρ^{-1} and a factor $(r-r')^{-1}$. Iterating this twice, we conclude using (2.2.36) and the fact that $r \sim r'$ that (2.2.37) may be written as $\Omega(\omega,\omega',r,r')$, for a function Ω satisfying (2.2.19).
- Study of (2.2.34) with $(\rho\rho')^{d-1}$ replaced by $(\rho\rho')^{d-3}$: By (2.2.36),we see that the integrand is $O(\rho^{-2}\rho'^{-2})$. Moreover, performing two ∂_ρ-integrations by parts, one may gain a factor $\langle r'-\rho'\rangle^{-2}$. One has thus to estimate $\int_{\rho'\sim r\sim r'\geq c} \rho^{-2}\rho'^{-2}\langle r'-\rho'\rangle^{-2}\, d\rho d\rho'$, which brings a bound of the form (2.2.19) (without logarithmic term). Moreover, Ω is supported for $d(\omega,-\omega') \ll 1$ since we have seen that a similar property holds for Σ.

• Contributions of (2.2.28) to (2.2.16)

We plug the contribution of (2.2.28) to (2.2.24) inside (2.2.16). We obtain

$$\frac{\varphi(2^{-k}r)}{(2\pi)^d} \int e^{-i\rho\theta\cdot(\rho'\omega+r'\omega')} G(\theta,\omega,\omega',r,r',\rho,\rho')\rho^{d-1}\, d\rho d\theta d\rho' \tag{2.2.38}$$

where

$$\begin{aligned}G(\theta,\omega,\omega',r,r',\rho,\rho') = \mu(\rho')^{d-1} R\Big(\theta,\omega,\omega',\rho',r',\frac{r}{\rho'},\mu;\rho\rho'\mu\Big) \\ \times \mathbb{1}_{\rho'<r}\kappa\Big(\frac{r}{\rho'}\Big)(1-m)(\rho)\mathbb{1}_{|r-\rho'|\ll\rho'\mu}\rho'^{d-1}\end{aligned} \tag{2.2.39}$$

with still $\mu(\rho') = \langle\rho'\rangle^{-\frac{1}{2}+\delta}$. By (2.2.29), for any N

$$G = O\big(\mu(\rho')^{d-1}\langle\rho\rho'\mu(\rho')\rangle^{-N}\rho'^{d-1}\big) = O\big(\rho^{-N}\rho'^{-N\left(\frac{1}{2}+\delta\right)+d-1}\big). \tag{2.2.40}$$

If one takes a ∂_ρ derivative of (2.2.39), one gains a factor $\langle\rho\rangle^{-1}$ by (2.2.29). If one takes a ∂_θ derivative, one loses $\mu(\rho')^{-1}$, that is largely compensated by the rapid decay in (2.2.40). If one sets $z=\rho\theta$, one thus sees that by integrations by parts in z, one may rewrite (2.2.38) under the form

$$\varphi(2^{-k}r)\int K(\omega,\omega',r,r',\rho')\,d\rho' \tag{2.2.41}$$

where K is a function satisfying

$$|K(\omega,\omega',r,r',\rho')|\le C\langle\rho'\omega+r'\omega'\rangle^{-N_0}\rho'^{-N_0}\mathbb{1}_{\rho'\ge c}\mathbb{1}_{|r-\rho'|\ll\rho'\langle\rho'\rangle^{-\frac{1}{2}+\delta}} \tag{2.2.42}$$

for a fixed large enough N_0 and a small $\delta'>0$. If we write $\rho'\omega+r'\omega'=r\omega+r'\omega'+(\rho'-r)\omega$, it follows from the last cut-off in (2.2.42) (where $\delta\le\frac{1}{2}$) that (2.2.41) is $O(\langle r\omega+r'\omega'\rangle^{-d-1})$ if N_0 has been taken large enough. This gives (2.2.21). □

Proof of Proposition 2.2.4: We have to prove that operator $\mathrm{Op}(b''_k)$ is bounded on L^2 uniformly in k, when b''_k is given in (2.2.14). We have decomposed $b''_k=b''_{k,1}+b''_{k,2}$ and shown in Lemma 2.2.5 that $\mathrm{Op}(b''_{k,1})$ is bounded on L^2 uniformly in k. Moreover, $b''_{k,2}$ is given by (2.2.16) and may be written by Proposition 2.2.6 as $\varphi(2^{-k}r)[\Omega(\omega,\omega',r,r')+\Omega^R(\omega,\omega',r,r')]$ with Ω,Ω^R satisfying bounds (2.2.19) and (2.2.21) respectively. Returning to variables $x=r\omega$, $\eta=r'\omega'$, this means that the Ω^R term is a symbol $c(x,\eta)$ satisfying $|c(x,\eta)|\le C\langle x+\eta\rangle^{-d-1}$ uniformly in k. The associated operator is trivially bounded on L^2. On the other hand, we have to consider the operator

$$u\to\int e^{irr'\omega\cdot\omega'}\varphi(2^{-k}r)\Omega(\omega,\omega',r,r')r'^{d-1}\hat{u}(r'\omega')\,d\omega'dr'. \tag{2.2.43}$$

Set $f(r',\omega')=r'^{\frac{d-1}{2}}\hat{u}(r'\omega')$, so that f is in $L^2(dr'd\omega')$. We have thus to prove a bound

$$\Big\|\int(rr')^{\frac{d-1}{2}}\Big\|\int e^{irr'\omega\cdot\omega'}\Omega(\omega,\omega',r,r')f(r',\omega')\,d\omega'\Big\|_{L^2(d\omega)}dr'\Big\|_{L^2(dr)}\le C\|f\|_{L^2(dr'd\omega')}. \tag{2.2.44}$$

We may apply Proposition A.1.7 to the $d\omega'$ integral in view of estimates (2.2.19) and since $D_\omega D_{\omega'}(\omega\cdot\omega')$ is non degenerate on the support of the integrand, as this one is contained in $d(-\omega',\omega)\ll1$. We thus obtain that the $\mathcal{L}(L^2(d\omega))$ norm of the angular operator in (2.2.44) compensates the $(rr')^{\frac{d-1}{2}}$ factor outside the $L^2(d\omega)$ norm, by (A.1.42), so that, by the last factor in (2.2.19), we reduce (2.2.44) to the estimate

$$\Big\|\int|\log|r-r'||\langle r-r'\rangle^{-2}\|f(r',\cdot)\|_{L^2(d\omega')}\,dr'\Big\|_{L^2(dr)}\le C\|f\|_{L^2(dr'd\omega')}$$

which holds by Schur's lemma. □

3 Proof of the main theorems

This section is devoted to the proof of the main theorems of subsection 1.1. To show for instance (1.1.5), (1.1.6), we express the left hand side of these equalities as some explicit phase integrals. Because of the $\mathcal{L}(L^2)$ bounds obtained in the preceding section, we may replace the general u_0 function by a function belonging to any dense subspace of L^2. This allows us to study those phase integrals when the amplitudes in the integrand are smooth and compactly supported outside a neighborhood of zero. One is thus reduced to the study of the limit of these quantities when the time parameter goes to infinity. In subsection 3.1, we shall compute such limits for some auxiliary expressions, using essentially the stationary phase formula. These technical results will be used in the following subsections in order to complete the proofs of the main theorems, expressing the phase integrals giving (1.1.5), (1.1.6) or (1.1.9) from these auxiliary ones.

3.1 Some technical lemmas

In order to consider simultaneously the case of the wave and the Schrödinger equation, we introduce $p : \mathbb{R} \to \mathbb{R}$, a smooth function satisfying both conditions:

$$p'(\rho) > 0 \text{ for any } \rho > 0 \tag{3.1.1}$$

$$\text{either } (p(\rho) \equiv \rho) \text{ or } (p \text{ is strictly convex}). \tag{3.1.2}$$

The case of the Schrödinger equation corresponds to $p(\rho) = \frac{\rho^2}{2}$, but the computations being the same for any p satisfying the second condition (3.1.2), we formulate our results in these more general framework.
Let

$$\begin{aligned} F : [1, +\infty[\times]0, +\infty[^3 \times \mathbb{R}^2 &\to \mathbb{C} \\ (t, \rho, \rho', r, \zeta, \zeta') &\to F(t, \rho, \rho'; \zeta, \zeta') \end{aligned} \tag{3.1.3}$$

be a smooth function of $(\rho, \rho', r, \zeta, \zeta')$ satisfying the following conditions:
(i) There are $C > 0$, $\delta' \in [0, 1[$ if $p'' \equiv 0$, $\delta' \in]\frac{1}{2}, 1[$ if p is strictly convex, such that

$$\operatorname{Supp}(F) \subset \{(t, \rho, \rho', r, \zeta, \zeta'); C^{-1} \leq \rho, \rho' \leq C, |\zeta| + |\zeta'| \leq Ct^{\delta'}, C^{-1} \leq \frac{r}{t} \leq C\}. \tag{3.1.4}$$

(ii) For any $j, k, k', \gamma, \gamma'$ in $\mathbb{N}$

$$|\partial_r^j \partial_\rho^k \partial_{\rho'}^{k'} \partial_\zeta^\gamma \partial_{\zeta'}^{\gamma'} F(t, \rho, \rho', r, \zeta, \zeta')| \leq Ct^{-\delta'(\gamma+\gamma'+j)}. \tag{3.1.5}$$

(iii) There is a smooth compactly supported function defined on $]0, +\infty[^2$, $(\rho, \rho') \to F_0(\rho, \rho')$ such that, for any $(\rho, \rho', r, \zeta, \zeta')$, one has the following limits:
• In the case p strictly convex

$$\lim_{t\to+\infty} F(t, \rho, \rho', r\sqrt{t} + tp'(\rho'); \zeta\sqrt{t}, \zeta'\sqrt{t}) = F_0(\rho, \rho'). \tag{3.1.6}$$

• In the case $p(\rho) \equiv \rho$

$$\lim_{t\to+\infty} \partial_\rho^k \partial_{\rho'}^{k'} F(t, \rho, \rho', r + t; \zeta, \zeta') = \partial_\rho^k \partial_{\rho'}^{k'} F_0(\rho, \rho') \text{ for } k + k' \leq 2. \tag{3.1.7}$$

Let $\epsilon, \epsilon', \sigma, \sigma'$ be signs in $\{-, +\}$ and for F as above, define

$$\begin{aligned} I(t, \epsilon, \epsilon', \sigma, \sigma'; F) = \int e^{i[r(\epsilon\rho+\epsilon'\rho')-t(\sigma p(\rho)+\sigma' p(\rho'))]} F\big(t, \rho, \rho', r; r - \epsilon\sigma tp'(\rho), r - \epsilon'\sigma' tp'(\rho')\big) \\ \times \mathbb{1}_{r>|t|\max(p'(\rho),p'(\rho'))}\, d\rho d\rho' dr. \end{aligned} \tag{3.1.8}$$

We shall prove the following proposition, giving the limit of (3.1.8) when t goes to $+\infty$ for some choice of signs. The case of other choices will be treated in a further proposition.

Proposition 3.1.1 *Assume (3.1.1), (3.1.2) and conditions (i) to (iii) above. Then the following limits hold for (3.1.8):*
• *If $\epsilon\sigma = -1$ or $\epsilon'\sigma' = -1$, one has in both cases (3.1.2)*

$$\lim_{t\to+\infty} I(t, \epsilon, \epsilon', \sigma, \sigma'; F) = 0. \tag{3.1.9}$$

• *If $\epsilon = \epsilon' = \sigma = \sigma'$ and if we are in the second case in (3.1.2) (p strictly convex) , then (3.1.9) holds again.*

• *If $\epsilon = \epsilon' = \sigma = \sigma'$ and if we are in the first case in (3.1.2) ($p(\rho) \equiv \rho$), then*

$$\lim_{t\to+\infty} I(t,\epsilon,\epsilon',\sigma,\sigma';F) = \Big\langle i\big(\epsilon(\rho+\rho')+i0\big)^{-1}, F_0(\rho,\rho')\Big\rangle. \tag{3.1.10}$$

In particular, in both cases (3.1.2), we shall have

$$\lim_{t\to+\infty}[I(t,1,1,1,1;F)+I(t,-1,-1,-1,-1;F)] = 0. \tag{3.1.11}$$

Before starting the proof of the proposition, we compute some intermediary integrals.

Lemma 3.1.2 *Denote for $t>0$*

(3.1.12)

$$I_+(t,\epsilon,\epsilon',\sigma,\sigma';F) = \int e^{i[r(\epsilon\rho+\epsilon'\rho')-t(\sigma p(\rho)+\sigma' p(\rho'))]}F\big(t,\rho,\rho',r;r-\epsilon\sigma t p'(\rho), r-\epsilon'\sigma' t p'(\rho')\big) \times \mathbb{1}_{r>tp'(\rho)}\mathbb{1}_{\rho>\rho'}\,d\rho d\rho' dr$$

and let $I_-(t,\epsilon,\epsilon',\sigma,\sigma';F)$ be the same integral with $\mathbb{1}_{r>tp'(\rho)}\mathbb{1}_{\rho>\rho'}$ replaced by $\mathbb{1}_{r>tp'(\rho')}\mathbb{1}_{\rho'>\rho}$. Then if $\epsilon\sigma=-1$ or $\epsilon'\sigma'=-1$, $I_\pm(t,\epsilon,\epsilon',\sigma,\sigma';F)$ goes to zero when t goes to $+\infty$. The same conclusion holds true if $\epsilon=\epsilon'=\sigma=\sigma'$ and p is strictly convex.
In the case $\epsilon=\epsilon'=\sigma=\sigma'$ and $p(\rho)\equiv\rho$, one has

$$\begin{aligned}\lim_{t\to+\infty} I_+(t,\epsilon,\epsilon',\sigma,\sigma';F) &= \int e^{i\epsilon r(\rho+\rho')}F_0(\rho,\rho')\mathbb{1}_{r>0}\mathbb{1}_{\rho>\rho'}\,d\rho d\rho' dr\\ \lim_{t\to+\infty} I_-(t,\epsilon,\epsilon',\sigma,\sigma';F) &= \int e^{i\epsilon r(\rho+\rho')}F_0(\rho,\rho')\mathbb{1}_{r>0}\mathbb{1}_{\rho'>\rho}\,d\rho d\rho' dr\end{aligned} \tag{3.1.13}$$

where the dr integral in the right hand side should be understood as an oscillatory integral, i.e. converges after making at least two integrations by parts using $\frac{\partial}{\partial\rho}+\frac{\partial}{\partial\rho'}$.

Proof: We study successively the different cases in the conclusion of the statement.
• **Case** $\epsilon\sigma=-1$ **or** $\epsilon'\sigma'=-1$
According to assumption (i) on F, the integrand in (3.1.12) is supported for

$$|r-\epsilon\sigma t p'(\rho)| < Ct^{\delta'},\ |r-\epsilon'\sigma' t p'(\rho')| < Ct^{\delta'}.$$

If $\epsilon\sigma=-1$ or $\epsilon'\sigma'=-1$, the fact that $\inf_K p'>0$ for any compact subset K of $]0,+\infty[$ implies that, for large enough t, the integrand in (3.1.12) vanishes (since $\delta'<1$), whence the conclusion.
• **Case** $\epsilon=\epsilon'=\sigma=\sigma'$
Notice that

$$I_+(t,\epsilon,\epsilon',\sigma,\sigma';F) = I_-(t,\epsilon',\epsilon,\sigma',\sigma;\underline{F}) \tag{3.1.14}$$

if $\underline{F}(t,\rho,\rho',r;\zeta,\zeta') = F(t,\rho',\rho,r;\zeta',\zeta)$, so that it will be sufficient to treat integral I_+, that may be rewritten in the case we are considering as

$$\int e^{i\epsilon[r(\rho+\rho')-t(p(\rho)+p(\rho'))]}F(t,\rho,\rho',r;r-tp'(\rho),r-tp'(\rho'))\mathbb{1}_{r>tp'(\rho)}\mathbb{1}_{\rho-\rho'>0}\,d\rho d\rho' dr. \tag{3.1.15}$$

- Sub-case p strictly convex

  We replace inside (3.1.15) r by $t(r+p'(\rho))$ and then make the further change of variables

$$\rho = t^{-(1-\delta')}\frac{u+v}{2},\ \rho' = t^{-(1-\delta')}\frac{u-v}{2},\ r = t^{-(1-\delta')}s.$$

We rewrite thus (3.1.15) as

$$t^{-2+3\delta'}\int e^{it^{2\delta'-1}\Phi_t(s,u,v)}\mathbb{1}_{s>0}\mathbb{1}_{v>0}G(t,s,u,v)\,dsdudv \tag{3.1.16}$$

with

$$\begin{aligned}\Phi_t(s,u,v) = us + ut^{1-\delta'}p'\Big(t^{-1+\delta'}\frac{u+v}{2}\Big)\\ - t^{2(1-\delta')}\Big[p\Big(t^{-1+\delta'}\frac{u+v}{2}\Big)+p\Big(t^{-1+\delta'}\frac{u-v}{2}\Big)\Big]\end{aligned} \tag{3.1.17}$$

and

$$\begin{aligned}G(t,s,u,v) = \frac{1}{2}F\Big[t,t^{-1+\delta'}\frac{u+v}{2},t^{-1+\delta'}\frac{u-v}{2},t^{\delta'}s+tp'\Big(t^{-1+\delta'}\frac{u+v}{2}\Big);\\ t^{\delta'}s,t^{\delta'}\Big[s+t^{1-\delta'}\Big(p'\Big(t^{-1+\delta'}\frac{u+v}{2}\Big)-p'\Big(t^{-1+\delta'}\frac{u-v}{2}\Big)\Big)\Big]\Big].\end{aligned} \tag{3.1.18}$$

Recall that $F(t,\rho,\rho',r;\zeta,\zeta')$ is supported for ρ,ρ' in a compact subset of $]0,+\infty[$, for $r\le Ct$ and for $|\zeta|+|\zeta'|\le Ct^{\delta'}$. It follows that G is supported for

$$|u|+|v|\le Ct^{1-\delta'},\ s\le C,\ \Big|p'\Big(t^{-1+\delta'}\frac{u+v}{2}\Big)-p'\Big(t^{-1+\delta'}\frac{u-v}{2}\Big)\Big|\le Ct^{-1+\delta'} \tag{3.1.19}$$

and satisfies by (3.1.5)

$$|\partial_u^k\partial_v^{k'}\partial_s^jG(t;s,u,v)|\le C_{k,k',j} \tag{3.1.20}$$

for any k,k',j in $\mathbb{N}$. We notice that, because we are in the case $p''>0$ on $\mathbb{R}$, the last inequality (3.1.19) implies that v, that is positive, is also bounded from above on the support of the integrand in (3.1.16).

Consider first (3.1.16) where we insert under the integral a smooth cut-off for $|u|\ge c>0$. If we make a ∂_s-integration by parts, we get a boundary term, bounded by

$$Ct^{-1+\delta'}\int_{c\le|u|\le Ct^{1-\delta'}}\mathbb{1}_{0<v<c}\frac{du}{|u|}dv \tag{3.1.21}$$

that goes to zero if $t\to+\infty$ as $\delta'<1$. On the other hand, the term integrated in s is also bounded by (3.1.21) since $s\le C$ on the support of integration by (3.1.19) and because of (3.1.20). We are thus reduced to (3.1.16) where G satisfies (3.1.20) and is supported moreover for $|u|\ll 1$ and for v,s in a fixed bounded set. The phase $(s,u)\to\Phi_t(s,u,v)$ has at fixed v a unique critical point given by

$$(u=0,s=-\frac{1}{2}p''(0)v+O(t^{-1+\delta'})),\ t\to+\infty. \tag{3.1.22}$$

Moreover, at any bounded (s,u,v), we have when t goes to $+\infty$

$$\frac{\partial^2\Phi_t}{\partial u\partial s}=1,\ \frac{\partial^2\Phi_t}{\partial s^2}=0,\ \frac{\partial^2\Phi_t}{\partial u^2}=\frac{1}{2}p''(0)+O(t^{-1+\delta'}) \tag{3.1.23}$$

with a small remainder since $\delta'<1$, and with $p''(0)>0$. Consider integral (3.1.16) in $dsdu$. If one cuts-off outside a neighborhood of $s=0$, then the cut-off $\mathbb{1}_{s>0}$ disappears, and one gets for any fixed v a phase integral with a unique critical point given by (3.1.22). As by (3.1.23) this critical point is non degenerate, we get for (3.1.16) an estimate in $t^{-1+\delta'}=o(1)$ since $\delta'<1$.

We may thus assume G cut-off for $s \ll 1$. If then $v \geq c > 0$, (3.1.22) shows that the phase has no critical point in (s, u) on the support of G. We may then perform one integration by parts in s or one integration by parts in u in order to bound again the corresponding contribution to (3.1.16) by $Ct^{-1+\delta'}$. It remains to consider the case when G is supported for $|u| \ll 1$, $s \ll 1$ and $v \ll 1$. As $u \to \Phi_t(0, u, 0)$ has 0 has unique critical point, and as this critical point is non degenerate by (3.1.23), for $s \ll 1$ and $v \ll 1$, $u \to \Phi_t(s, u, v)$ has also a unique critical point at some point $u(s,t)$, and this critical point is non degenerate. Moreover, (3.1.23) implies that at any fixed v, the critical value $s \to \Phi_t(s, u(s, v), v)$ has a unique critical point, located at $s = -\frac{1}{2}p''(0) + O(t^{-1+\delta'})$ by (3.1.22), and that this critical point is non degenerate. Then the du integral in (3.1.16) has a $t^{-(2\delta'-1)/2}$ gain in terms of the large parameter $t^{2\delta'-1}$ (recall that $\delta' > \frac{1}{2}$) and the ds integral gains another such factor, by the stationary phase formula (with boundary at $s = 0$ in case of ds-integration). Since the dv integral is taken on a bounded interval, we conclude that (3.1.16) is $O(t^{-1+\delta'})$ when t goes to infinity. As $\delta' < 1$, we obtain the wanted conclusion that (3.1.16), so (3.1.15), goes to zero if t goes to infinity.

- Sub-case $p(\rho) \equiv \rho$

  We rewrite (3.1.15) with $p(\rho) = \rho$ as

$$\int e^{i\epsilon r(\rho+\rho')} F(t, \rho, \rho', r + t; r, r) 1_{r>0} 1_{\rho>\rho'}\, d\rho d\rho' dr. \tag{3.1.24}$$

  We may perform in (3.1.24) integrations by parts in $\partial_\rho + \partial_{\rho'}$ in order to get a factor $\langle r\rangle^{-2}$ under the integral. It follows from (3.1.7) and bound (3.1.5) that (3.1.24) converges to the right hand side of the first equality (3.1.13). This concludes the proof of the lemma.

□

Proof of Proposition 3.1.1: We notice first that by (3.1.8) and the definition of I_+, I_- (see (3.1.12))

$$I(t, \epsilon, \epsilon', \sigma, \sigma'; F) = I_+(t, \epsilon, \epsilon', \sigma, \sigma'; F) + I_-(t, \epsilon, \epsilon', \sigma, \sigma'; F).$$

Then (3.1.9) follows from the statement concerning cases $\epsilon\sigma = -1$ or $\epsilon'\sigma' = -1$ in Lemma 3.1.2. The second point in Proposition 3.1.1 follows in the same way. Consider next the case $\epsilon = \epsilon' = \sigma = \sigma'$ and $p(\rho) \equiv \rho$. Summing the two expressions in (3.1.13), we get (3.1.10). Finally, (3.1.11) is trivial in the case p strictly convex since it follows from the second point of the proposition. In the case $p(\rho) \equiv \rho$, (3.1.10) shows that the left hand side of (3.1.11) is

$$\Big\langle i(\rho + \rho' + i0)^{-1} + i(-(\rho + \rho') + i0)^{-1}, F_0(\rho, \rho')\Big\rangle = 2\pi\langle\delta(\rho + \rho'), F_0(\rho, \rho')\rangle$$

which vanishes since F_0 is compactly supported in $]0, +\infty[^2$. This concludes the proof. □

We still have to determine the limit of $I(t, \epsilon, \epsilon', \sigma, \sigma'; F)$ in some cases that are not covered by Proposition 3.1.1. We shall do that in the following proposition:

Proposition 3.1.3 *(i) Assume that $p(\rho) \equiv \rho$. Then we have for $\epsilon = \pm$*

$$\lim_{t\to+\infty} I(t, \epsilon, -\epsilon, -\epsilon, \epsilon; F) = 0 \tag{3.1.25}$$

and

$$\lim_{t\to+\infty} [I(t, 1, -1, 1, -1; F) + I(t, -1, 1, -1, 1; F)] = 2\pi \int_0^{+\infty} F_0(\rho, \rho)\, d\rho. \tag{3.1.26}$$

(ii) Assume that p is strictly convex. Then (3.1.25) still holds and (3.1.26) is replaced by

$$\lim_{t\to+\infty}[I(t,1,-1,1,-1;F)+I(t,-1,1,-1,1;F)] = \pi\int_0^{+\infty} F_0(\rho,\rho)\,d\rho. \tag{3.1.27}$$

We first prove a lemma giving expressions of $I_+(t,\epsilon,-\epsilon,\epsilon,-\epsilon;F)$.

Lemma 3.1.4 *Let θ be in $C_0^\infty(\mathbb{R})$, even, with small enough support, equal to one close to zero.*
(i) If $p(\rho)\equiv\rho$, we have for $\epsilon=\pm 1$

$$I_+(t,\epsilon,-\epsilon,\epsilon,-\epsilon;F) = \int e^{i\epsilon rw}\mathbb{1}_{w>0}\mathbb{1}_{r>0}F(t,\rho'+w,\rho',r+t;r,r)\,drdwd\rho'. \tag{3.1.28}$$

(ii) If p is strictly convex, define

$$g(\rho,\rho') = \int_0^1 (1-\alpha)p''(\rho+\alpha(\rho'-\rho))\,d\alpha = \frac{p(\rho')-p(\rho)-p'(\rho)(\rho'-\rho)}{(\rho'-\rho)^2} \tag{3.1.29}$$

and

$$\tilde{F}(t,\rho,\rho',r;\zeta,\zeta') = F(t,\rho,\rho',tr;t\zeta,t\zeta'). \tag{3.1.30}$$

Then when t goes to $+\infty$, for $\epsilon=\pm$,

$$\begin{aligned} I_+(t,\epsilon,-\epsilon,\epsilon,-\epsilon;F) = t\int & e^{i\epsilon t[rw+g(\rho'+w,\rho')w^2]}\theta(w)\mathbb{1}_{w>0}\mathbb{1}_{r>0} \\ & \times \tilde{F}(t,\rho'+w,\rho',r+p'(\rho');r,r)\,drdwd\rho' + o(1). \end{aligned} \tag{3.1.31}$$

Proof: By (3.1.12), we have

$$\begin{aligned} I_+(t,\epsilon,-\epsilon,\epsilon,-\epsilon;F) = \int & e^{i\epsilon[r(\rho-\rho')-t(p(\rho)-p(\rho'))]} \\ & \times F(t,\rho,\rho',r;r-tp'(\rho),r-tp'(\rho'))\mathbb{1}_{r>tp'(\rho)}\mathbb{1}_{\rho>\rho'}\,d\rho d\rho' dr. \end{aligned} \tag{3.1.32}$$

(i) In the case $p(\rho)\equiv\rho$, (3.1.28) follows from (3.1.32), replacing r by $r+t$ and ρ by $\rho'+w$.
(ii) In the case p strictly convex, the change of variables $r\to r+tp'(\rho)$ transforms the phase in (3.1.32) into $\epsilon[r(\rho-\rho')+tg(\rho,\rho')(\rho-\rho')^2]$ according to (3.1.29), so that we have to study

$$\begin{aligned} t\int e^{i\epsilon t[rw+g(\rho'+w,\rho')w^2]}\tilde{F}\big(t,\rho'+w,\rho',r+p'(\rho'+w);r,r+p'(\rho'+w)-p'(\rho')\big) \\ \times \mathbb{1}_{r>0}\mathbb{1}_{w>0}\,dwd\rho' dr \end{aligned} \tag{3.1.33}$$

using notation (3.1.30). Notice that assumptions (3.1.4) to (3.1.6) on F imply that

$$\mathrm{Supp}\,(\tilde{F}) \subset \{(t,\rho,\rho',r,\zeta,\zeta'); C^{-1}\le\rho,\rho'\le C, |\zeta|+|\zeta'|\le Ct^{\delta'-1}, C^{-1}\le r\le C\} \tag{3.1.34}$$

$$|\partial_r^j\partial_\rho^k\partial_{\rho'}^{k'}\partial_\zeta^\gamma\partial_{\zeta'}^{\gamma'}\tilde{F}| \le Ct^{(1-\delta')[j+\gamma+\gamma']} \tag{3.1.35}$$

$$\lim_{t\to+\infty}\tilde{F}\Big(t,\rho,\rho',\frac{r}{\sqrt{t}}+p'(\rho');\frac{\zeta}{\sqrt{t}},\frac{\zeta'}{\sqrt{t}}\Big) = F_0(\rho,\rho'). \tag{3.1.36}$$

Moreover, by (3.1.29) and the assumption that p is strictly convex we have

$$\Big|\frac{\partial}{\partial w}\big[g(\rho'+w,\rho')w^2\big]\Big| \geq c|w|, \quad \partial_w^\alpha\big[g(\rho'+w,\rho')w^2\big] = O(1) \text{ for } |\alpha|\geq 2 \tag{3.1.37}$$

when ρ' stays in a compact set of $[0,+\infty[$ and w stays bounded.
We shall study (3.1.33) in two steps.

• **Step 1**: Integral cut-off for $w \geq c > 0$

Take θ in $C_0^\infty(\mathbb{R})$, equal to one close to zero and consider

$$I_1 = t\int e^{it\phi(r,w,\rho')}(1-\theta)(w)\tilde{F}\big(t,\rho'+w,\rho',r+p'(\rho'+w);r,r+p'(\rho'+w)-p'(\rho')\big) \times \mathbb{1}_{r>0}\mathbb{1}_{w>0}\,dwd\rho'dr \tag{3.1.38}$$

where

$$\phi(r,w,\rho') = \epsilon(rw+g(\rho'+w,\rho')w^2).$$

If we make one ∂_r-integration by parts in (3.1.38), we get the boundary term

$$i\epsilon\int e^{i\epsilon tg(\rho'+w,\rho')w^2}(1-\theta)(w)\tilde{F}\big(t,\rho'+w,\rho',p'(\rho'+w);0,p'(\rho'+w)-p'(\rho')\big)w^{-1}\mathbb{1}_{w>0}\,dwd\rho' \tag{3.1.39}$$

and a term of the form (3.1.38) where the prefactor t has been replaced by $t^{1-\delta'}$ because of (3.1.35), and where $(1-\theta)(w)$ is replaced by $w^{-1}(1-\theta)(w)$. In this last term, we perform one more integration by parts using $\frac{\partial}{\partial r}$. We get a new boundary term of the form (3.1.39), with an extra $t^{-\delta'}$ factor and w^{-1} replaced by w^{-2}, and a contribution of the form (3.1.38) with the prefactor t replaced by $t^{1-2\delta'}$ and $(1-\theta)(w)$ replaced by $(1-\theta)(w)w^{-2}$. Taking (3.1.34) into account, we see that this last integral is $O(t^{-\delta'})$, so goes to zero when t goes to infinity. The same is true for (3.1.39) multiplied by an extra $t^{-\delta'}$ factor. We thus have just to study the limit of (3.1.39) when t goes to infinity. In this last expression, we notice that by (3.1.34) the integrand is supported for $C^{-1}\leq \rho'+w\leq C$, $C^{-1}\leq\rho'\leq C$ and for $|p'(\rho'+w)-p'(\rho')|\leq Ct^{\delta'-1}$. As $\delta'<1$ and $p''>0$, dominated convergence shows that (3.1.39) goes to zero.
As a conclusion of Step 1, we have shown that (3.1.38) contributes to $o(1)$ in (3.1.31).

• **Step 2**: Integral cut-off for $w \ll 1$

We have to study

$$t\int e^{it\phi(r,w,\rho')}\theta(w)\tilde{F}\big(t,\rho'+w,\rho',r+p'(\rho'+w);r,r+p'(\rho'+w)-p'(\rho')\big)\mathbb{1}_{r>0}\mathbb{1}_{w>0}\,dwd\rho'dr \tag{3.1.40}$$

with θ supported close to zero and we have to show, in order to get (3.1.31), that we may replace the argument of $\tilde{F}$ by $(t,\rho'+w,\rho',r+p'(\rho');r,r)$, up to a remainder contributing to $o(1)$. Write

$$\tilde{F}\big(t,\rho'+w,\rho',r+p'(\rho'+w);r,r+p'(\rho'+w)-p'(\rho')\big) = \tilde{F}\big(t,\rho'+w,\rho',r+p'(\rho');r,r\big) + w\tilde{G}(t,\rho',w,r) \tag{3.1.41}$$

where by (3.1.35), $\tilde{G}$ satisfies

$$|\partial_r^j\partial_{\rho'}^{k'}\partial_w^\ell\tilde{G}| \leq Ct^{(1-\delta')(1+j+k'+\ell)} \tag{3.1.42}$$

and by (3.1.34)

$$\operatorname{Supp}\tilde{G} \subset \{C^{-1} \leq \rho' \leq C, r \leq Ct^{\delta'-1}\}. \tag{3.1.43}$$

Consequently, (3.1.40) is the sum of the first term in the right hand side of (3.1.31) and of a remainder that may be written

$$\int e^{i\epsilon t g(\rho'+w,\rho')w^2}\theta(w)\mathbb{1}_{w>0}M(t,\rho',w)\,dwd\rho' \tag{3.1.44}$$

with

$$\begin{aligned} M(t,\rho',w) &= t\int_0^{+\infty} e^{it\epsilon rw} w\tilde{G}(t,\rho',w,r)\,dr \\ &= i\epsilon\tilde{G}(t,\rho',w,0) + i\epsilon\int_0^{+\infty} e^{it\epsilon w}\partial_r\tilde{G}(t,\rho',w,r)\,dr. \end{aligned} \tag{3.1.45}$$

By (3.1.42) and (3.1.43) we get that

$$|M(t,\rho',w)| \leq Ct^{1-\delta'}. \tag{3.1.46}$$

If we denote by $M_1(t,\rho',w)$ the integral term in the right hand side of (3.1.45), another integration by parts shows that

$$wM_1(t,\rho',w) = -\frac{1}{t}\partial_r\tilde{G}(t,\rho',w,0) - \frac{1}{t}\int_0^{+\infty} e^{it\epsilon rw}\partial_r^2\tilde{G}(t,\rho',w,r)\,dr$$

whence

$$\begin{aligned} \partial_w[wM_1] = -\frac{1}{t}\partial_r\partial_w\tilde{G}(t,\rho',w,0) - i\epsilon\int_0^{+\infty} e^{it\epsilon rw} r\partial_r^2\tilde{G}(t,\rho',w,r)\,dr \\ - \frac{1}{t}\int_0^{+\infty} e^{it\epsilon rw}\partial_r^2\partial_w\tilde{G}(t,\rho',w,r)\,dr \end{aligned}$$

which is by (3.1.42), (3.1.43), $O(t^{3(1-\delta')-1}) + O(t^{1-\delta'}) = O(t^{1-\delta'})$ since $\delta' > \frac{1}{2}$. Together with (3.1.46), this implies

$$|M_1(t,\rho',w)| + |w\partial_w M_1(t,\rho',w)| = O(t^{1-\delta'}). \tag{3.1.47}$$

We plug next decomposition (3.1.45) $M(t,\rho',w) = i\epsilon\tilde{G}(t,\rho',w,0) + M_1$ inside (3.1.44). We get two contributions:

$$\int e^{i\epsilon t g(\rho'+w,\rho')w^2}\theta(w)\mathbb{1}_{w>0}i\epsilon\tilde{G}(t,\rho',w,0)\,dwd\rho' \tag{3.1.48}$$

$$\int e^{i\epsilon t g(\rho'+w,\rho')w^2}\theta(w)\mathbb{1}_{w>0}M_1(t,\rho',w)\,dwd\rho'. \tag{3.1.49}$$

To conclude the proof, we have to show that these two expressions are $o(1)$ when t goes to $+\infty$. If we insert under these integrals a cut-off for $\sqrt{t}w \leq 1$, it follows immediately from (3.1.42), (3.1.43), (3.1.47) that we get a bound in $t^{\frac{1}{2}-\delta'} = o(1)$ since $\delta' > \frac{1}{2}$. On the other hand, if we cut-off for $\sqrt{t}w \geq 1$, we may use that, for θ with small enough support, $w = 0$ is the only critical point on that support, so that making one ∂_w-integration by parts, we get a $O(t^{\frac{1}{2}-\delta'})$ or a $O(t^{1-2\delta'}\log t)$ bound, using (3.1.43), (3.1.47), which is again $o(1)$. This concludes the proof of Step 2 and thus of the lemma. $\Box$

Proof of Proposition 3.1.3: We recall from the definition (3.1.8) of I, the definition (3.1.12) of $I_\pm$ and (3.1.14) that

$$I(t,\epsilon,\epsilon',\sigma,\sigma';F) = I_+(t,\epsilon,\epsilon',\sigma,\sigma';F) + I_+(t,\epsilon',\epsilon,\sigma',\sigma;\underline{F}). \tag{3.1.50}$$

By the case $\epsilon\sigma = -1$ in Lemma 3.1.2, we conclude that (3.1.25) holds true, both in cases (i) and (ii) of the proposition. We are thus left with proving (3.1.26) and (3.1.27).

• **Proof of (3.1.26)**

We plug expression (3.1.28) inside the expression of $I(t,\epsilon,-\epsilon,\epsilon,-\epsilon;F)$ given by (3.1.50). We obtain, using the definition of $\underline{F}$ given after (3.1.14)

$$\begin{aligned} I(t,\epsilon,-\epsilon,\epsilon,-\epsilon;F) = \int e^{i\epsilon rw}\mathbb{1}_{r>0}\big[&F(t,\rho'+w,\rho',r+t;r,r)\mathbb{1}_{w>0}\\ &+F(t,\rho',\rho'-w,r+t;r,r)\mathbb{1}_{w<0}\big]\,drdwd\rho'. \end{aligned} \tag{3.1.51}$$

We define

$$F_1(t,\rho,\rho',r) = F(t,\rho,\rho',r+t;r,r)\mathbb{1}_{r>0} + F(t,\rho,\rho',-r+t;-r,-r)\mathbb{1}_{r<0}. \tag{3.1.52}$$

By assumptions (3.1.4), (3.1.5)

$$\operatorname{Supp} F_1 \subset \{(t,\rho,\rho',r); C^{-1}\le \rho,\rho'\le C, |r|\le Ct^{\delta'}\} \tag{3.1.53}$$

and for any j,k,k' in $\mathbb{N}$ with $0\le j\le 1$

$$|\partial_r^j\partial_\rho^k\partial_{\rho'}^{k'}F_1(t,\rho,\rho',r)| \le Ct^{-j\delta'}. \tag{3.1.54}$$

By (3.1.51), we may write

$$I(t,1,-1,1,-1;F) + I(t,-1,1,-1,1;F) = \int e^{irw}W(t,w,r)\,drdw \tag{3.1.55}$$

with

$$W(t,w,r) = \int \big[F_1(t,\rho'+w,\rho',r)\mathbb{1}_{w>0} + F_1(t,\rho',\rho'-w,r)\mathbb{1}_{w<0}\big]\,d\rho'. \tag{3.1.56}$$

By (3.1.53), (3.1.54), W is a Lipschitz function of r, supported for w in a compact subset of $\mathbb{R}$ and for $|r|\le Ct^{\delta'}$ and satisfies for $0\le j\le 1$, $0\le \ell\le 1$ and any k in $\mathbb{N}$

$$|\partial_r^j\partial_w^\ell(w\partial_w)^kW(t,w,r)| \le Ct^{-j\delta'}. \tag{3.1.57}$$

Moreover by (3.1.7), (3.1.52), (3.1.53)

$$\lim_{t\to+\infty} W(t,0,r) = \int F_0(\rho',\rho')\,d\rho'. \tag{3.1.58}$$

We decompose (3.1.55) as I_1+I_2 with

$$\begin{aligned} I_1 &= \int e^{irw}W(t,0,r)\theta(w)\,dwdr\\ I_2 &= \int e^{irw}[W(t,w,r)-W(t,0,r)]\theta(w)\,dwdr \end{aligned} \tag{3.1.59}$$

where $\theta \in C_0^\infty(\mathbb{R})$ is equal to one on the w-support of W. By (3.1.57), (3.1.58), when t goes to $+\infty$, I_1 converges to the right hand side of (3.1.26). We are thus left with showing that I_2 converges to 0 when t goes to infinity. We write $W(t,w,r) - W(t,0,r) = w\tilde{W}(t,w,r)$ with, because of (3.1.57),

$$|\partial_r^j (w\partial_w)^k \tilde{W}(t,w,r)| \leq Ct^{-j\delta'} \mathbb{1}_{|r|\leq Ct^{\delta'}} \tag{3.1.60}$$

if $0 \leq j \leq 1$, $k \in \mathbb{N}$. By one ∂_r integration by parts in I_2, we have

$$I_2 = i \int e^{irw} \partial_r \tilde{W}(t,w,r)\theta(w)\, dwdr.$$

By (3.1.60) and ∂_w-integrations by parts, we get that $|I_2| \leq Ct^{-\delta'} \int_{|r|\leq ct^{\delta'}} \langle rw\rangle^{-2}\theta(w)\, dwdr$ which goes to zero if t goes to infinity since $\delta' > 0$. This concludes the proof of (3.1.26).

• **Proof of (3.1.27)**

We plug expression (3.1.31) in (3.1.50). We obtain that, up to some $o(1)$ remainder, the integral $I(t,\epsilon,-\epsilon,\epsilon,-\epsilon;F)$ is given by

$$\begin{aligned} &t\int e^{i\epsilon t[rw+g(\rho'+w,\rho')w^2]} \tilde{F}_+(t,\rho',w,r)\mathbb{1}_{w>0}\mathbb{1}_{r>0}\, drdwd\rho' \\ &+t\int e^{i\epsilon t[rw-g(\rho'-w,\rho')w^2]} \tilde{F}_-(t,\rho',w,r)\mathbb{1}_{w<0}\mathbb{1}_{r>0}\, drdwd\rho' \end{aligned} \tag{3.1.61}$$

with since θ is even

$$\begin{aligned} \tilde{F}_+(t,\rho',w,r) &= \tilde{F}(t,\rho'+w,\rho',r+p'(\rho');r,r)\theta(w) \\ \tilde{F}_-(t,\rho',w,r) &= \tilde{F}(t,\rho',\rho'-w,r+p'(\rho');r,r)\theta(w). \end{aligned} \tag{3.1.62}$$

By (3.1.35), for any j,k,ℓ in $\mathbb{N}$

$$|\partial_r^j \partial_{\rho'}^k \partial_w^\ell \tilde{F}_\pm(t,\rho',w,r)| \leq Ct^{(1-\delta')(j+k)} \tag{3.1.63}$$

and by (3.1.34)

$$\operatorname{Supp} \tilde{F}_\pm \subset \{(t,\rho',w,r); C^{-1} \leq \rho' \leq C, w \leq C, |r| \leq Ct^{\delta'-1}\}. \tag{3.1.64}$$

Moreover, by (3.1.36)

$$\lim_{t\to+\infty} \tilde{F}_\pm\Big(t,\rho',0,\frac{r}{\sqrt{t}}\Big) = F_0(\rho',\rho'). \tag{3.1.65}$$

Denote by $J_{+,\epsilon}(t)$ (resp. $J_{-,\epsilon}(t)$) the first (resp. second) line in (3.1.61). We decompose

$$J_{\pm,\epsilon}(t) = J'_{\pm,\epsilon}(t) + J''_{\pm,\epsilon}(t) \tag{3.1.66}$$

where $J'_{\pm,\epsilon}(t)$ is obtained inserting a cut-off $\chi_0(\sqrt{t}w)$ under the integral, with $\chi_0 \in C_0^\infty(\mathbb{R})$ equal to one on $[-1,1]$. We get

$$J'_{\pm,\epsilon}(t) = \int e^{i\epsilon rw}\chi_0(w)\Omega_\pm^\epsilon(t,\rho',w)\tilde{F}_\pm\Big(t,\rho',\frac{w}{\sqrt{t}},\frac{r}{\sqrt{t}}\Big)\mathbb{1}_{r>0}\, drdwd\rho' \tag{3.1.67}$$

with

$$\Omega_\pm^\epsilon(t,\rho',w) = e^{\pm i\epsilon g\left(\rho'\pm\frac{w}{\sqrt{t}},\rho'\right)w^2}\mathbb{1}_{\pm w>0}. \tag{3.1.68}$$

If $\Omega^\epsilon_\pm(\infty,\rho',w)$ is the corresponding expression at $t=+\infty$, we have when ρ' stays in a compact set

$$|\Omega^\epsilon_\pm(t,\rho',w)-\Omega^\epsilon_\pm(\infty,\rho',w)|\le C\frac{|w|^3}{\sqrt{t}}.$$

Since the integrand in (3.1.67) is supported for $|w|\le C$, $|r|\le Ct^{\delta'-\frac12}$ according to (3.1.64), we see that if we substitute inside (3.1.67) $\Omega^\epsilon_\pm(\infty,\rho',w)$ for $\Omega^\epsilon_\pm(t,\rho',w)$, the error generated is $O(t^{\delta'-1})=o(1)$ as $\delta'<1$. If we replace next in $\tilde F_\pm$ the argument $\frac{w}{\sqrt{t}}$ by 0, we get an error in

$$\frac{1}{\sqrt{t}}\int_0^1\int e^{i\epsilon rw}w\theta(w)\Omega^\epsilon_\pm(\infty,\rho',w)(\partial_w\tilde F_\pm)\Big(t,\rho',\alpha\frac{w}{\sqrt{t}},\frac{r}{\sqrt{t}}\Big)\mathbb{1}_{r>0}\,drdwd\rho'd\alpha.$$

If one makes one ∂_r-integration by parts and uses (3.1.63), and the fact that the integrand is supported for $|r|\le Ct^{\delta'-\frac12}$, one gets that this error is $O\big(t^{\frac12-\delta'}\big)=o(1)$ since $\delta'>\frac12$. Up to a $o(1)$ remainder, and since $\tilde F_+|_{w=0}=\tilde F_-|_{w=0}$, we may thus express

$$J'_{+,\epsilon}(t)+J'_{-,\epsilon}(t)=\int e^{i\epsilon rw}\chi_0(w)\Omega^\epsilon(\rho',w)\tilde F_+\Big(t,\rho',0,\frac{r}{\sqrt{t}}\Big)\mathbb{1}_{r>0}\,drdwd\rho' \tag{3.1.69}$$

with

$$\begin{aligned}\Omega^\epsilon(\rho',w)&=\Omega^\epsilon_+(\infty,\rho',w)+\Omega^\epsilon_-(\infty,\rho',w)\\&=e^{i\epsilon g(\rho',\rho')w^2}\mathbb{1}_{w>0}+e^{-i\epsilon g(\rho',\rho')w^2}\mathbb{1}_{w<0}.\end{aligned} \tag{3.1.70}$$

Notice that $\Omega^\epsilon\chi_0(w)$ has two ∂_w-derivatives in L^∞, so that we may rewrite (3.1.69) as

$$\int M_\epsilon(\rho',-\epsilon r)\tilde F_+\Big(t,\rho',0,\frac{r}{\sqrt{t}}\Big)\mathbb{1}_{r>0}\,drd\rho' \tag{3.1.71}$$

where $M_\epsilon(\rho',\zeta)$ is the Fourier transform in w of $\chi_0(w)\Omega^\epsilon(\rho',w)$, that satisfies $|M_\epsilon(\rho,\zeta)|=O(\langle\zeta\rangle^{-2})$. By dominated convergence, we deduce from (3.1.69), (3.1.71) and (3.1.65) that

$$\begin{aligned}\lim_{t\to+\infty}[J'_{+,\epsilon}(t)+J'_{-,\epsilon}(t)]&=\iint M_\epsilon(\rho',-\epsilon r)\mathbb{1}_{r>0}\,drF_0(\rho',\rho')\,d\rho'\\&=i\epsilon\int\Big\langle(w+i\epsilon0)^{-1},\Omega^\epsilon(\rho',w)\chi_0(w)\Big\rangle F_0(\rho',\rho')\,d\rho'.\end{aligned} \tag{3.1.72}$$

Let us study next the contributions $J''_{\pm,\epsilon}(t)$ in (3.1.66), obtained inserting in the integrals (3.1.61) the cut-off $(1-\chi_0)(\sqrt{t}w)$. We obtain

$$J''_{\pm,\epsilon}(t)=\int e^{i\epsilon rw}(1-\chi_0)(w)\Omega^\epsilon_\pm(t,\rho',w)\tilde F_\pm\Big(t,\rho',\frac{w}{\sqrt{t}},\frac{r}{\sqrt{t}}\Big)\mathbb{1}_{r>0}\,drdwd\rho'. \tag{3.1.73}$$

We make one integration by parts in r. We obtain the boundary term

$$i\epsilon\int\frac{(1-\chi_0)(w)}{w}\Omega^\epsilon_\pm(t,\rho',w)\tilde F_\pm\Big(t,\rho',\frac{w}{\sqrt{t}},0\Big)\,dwd\rho' \tag{3.1.74}$$

and according to (3.1.63), a term similar to (3.1.73), but with a gain in $t^{\frac12-\delta'}w^{-1}$ inside the integral. Repeating the process, we end up with terms of the form (3.1.73), but with a gain $t^{2\left(\frac12-\delta'\right)}w^{-2}$ inside the integral (that, using (3.1.63), (3.1.64) will provide a $o(1)$ remainder since $\delta'>\frac12$), terms of the form (3.1.74), with a $t^{\frac12-\delta'}w^{-1}$ extra gain under the integral (that are also $o(1)$) and (3.1.74) itself. According to (3.1.68), this may be written as

$$i\epsilon\int\frac{(1-\chi_0)(w)}{w}e^{\pm i\epsilon g\left(\rho'\pm\frac{w}{\sqrt{t}},\rho'\right)w^2}\tilde F_\pm\Big(t,\rho',\frac{w}{\sqrt{t}},0\Big)\mathbb{1}_{\pm w>0}\,dw. \tag{3.1.75}$$

As by (3.1.64), $\frac{w}{\sqrt{t}}$ and ρ' are bounded on the support of the integrand, we have

$$\Big|\partial_w\Big[g\Big(\rho' \pm \frac{w}{\sqrt{t}}, \rho'\Big)w^2\Big]\Big| \geq c|w| \geq c' \tag{3.1.76}$$

(see (3.1.37)), and we may perform a ∂_w-integration by parts to write the sum of the two expressions (3.1.75) corresponding to the two possible signs as

$$-\int_{w>0} e^{i\epsilon g\left(\rho'+\frac{w}{\sqrt{t}},\rho'\right)w^2}\partial_w\Big[\frac{(1-\chi_0)(w)}{w}\frac{\tilde{F}_+\big(t,\rho',\frac{w}{\sqrt{t}},0\big)}{\partial_w\big[g\big(\rho'+\frac{w}{\sqrt{t}},\rho'\big)w^2\big]}\Big]\,dwd\rho'$$
$$+\int_{w<0} e^{-i\epsilon g\left(\rho'-\frac{w}{\sqrt{t}},\rho'\right)w^2}\partial_w\Big[\frac{(1-\chi_0)(w)}{w}\frac{\tilde{F}_+\big(t,\rho',\frac{w}{\sqrt{t}},0\big)}{\partial_w\big[g\big(\rho'-\frac{w}{\sqrt{t}},\rho'\big)w^2\big]}\Big]\,dwd\rho'.$$

By (3.1.63), (3.1.65) at $r = 0$, (3.1.76) and dominated convergence, this goes, when t tends to $+\infty$, to

$$-\iint\Big[e^{i\epsilon g(\rho',\rho')w^2}\mathbb{1}_{w>0} - e^{-i\epsilon g(\rho',\rho')w^2}\mathbb{1}_{w<0}\Big]\partial_w\Big[\frac{(1-\chi_0)(w)}{2w^2}\Big]\,dw\frac{F_0(\rho',\rho')}{g(\rho',\rho')}\,d\rho' \tag{3.1.77}$$

that may be rewritten using notation (3.1.70) as

$$i\epsilon\int\Big\langle\frac{(1-\chi_0)(w)}{w}, \Omega^\epsilon(\rho',w)\Big\rangle F_0(\rho',\rho')\,d\rho' \tag{3.1.78}$$

interpreting the inner bracket as an oscillatory integral in w.

Consequently, we have expressed the limit when t goes to $+\infty$ of $J'_{+,\epsilon}(t) + J'_{-,\epsilon}(t)$ by (3.1.72) and the limit of $J''_{+,\epsilon}(t) + J''_{-,\epsilon}(t)$ by (3.1.78). By (3.1.66), the limit of $J_{+,\epsilon}(t) + J_{-,\epsilon}(t)$ is the sum of these two expressions, namely

$$i\epsilon\int\Big\langle(w+i\epsilon 0)^{-1}, \Omega^\epsilon(\rho',w)\Big\rangle F_0(\rho',\rho')\,d\rho'. \tag{3.1.79}$$

Since by (3.1.61) this is the limit when t goes to $+\infty$ of $I(t,\epsilon,-\epsilon,\epsilon,-\epsilon;F)$, it remains, in order to get (3.1.27), to express the sum of quantities (3.1.79) corresponding to $\epsilon = 1$ and $\epsilon = -1$ as the right hand side of (3.1.27). The definition (3.1.70) of Ω^ϵ may be rewritten as

$$\Omega^\epsilon(\rho',w) = \cos[g(\rho',\rho')w^2] + i\epsilon\mathrm{sgn}(w)\sin[g(\rho',\rho')w^2].$$

Then (3.1.79) is just

$$i\epsilon\int\Big\langle(w+i\epsilon 0)^{-1}, \cos[g(\rho',\rho')w^2]\Big\rangle F_0(\rho',\rho')\,d\rho' - \int\Big[\int\frac{\sin[g(\rho',\rho')w^2]}{|w|}\,dw\Big]F_0(\rho',\rho')\,d\rho'. \tag{3.1.80}$$

Since the last dw integral is just $\int_0^{+\infty}\frac{\sin y}{y}\,dy = \frac{\pi}{2}$, the sum of expressions (3.1.80) for $\epsilon = 1$ and $\epsilon = -1$, which gives (3.1.27), is just

$$i\int\Big\langle(w+i0)^{-1} - (w-i0)^{-1}, \cos\big[g(\rho',\rho')w^2\big]\Big\rangle F_0(\rho',\rho')\,d\rho' - \pi\int F_0(\rho',\rho')\,d\rho'$$
$$= \pi\int F_0(\rho',\rho')\,d\rho'$$

expressing $(w+i0)^{-1} - (w-i0)^{-1}$ as $-2i\pi\delta_0$. This gives the right hand side of (3.1.27) and concludes the proof. □

3.2 Proof of the main theorem

We shall give here the proof of Theorem 1.1.1.

• **Case of the half-wave equation**

Let us write explicitly the truncated energy $E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t)$ defined in (1.1.4) as, using (1.1.2), (1.1.1),

$$
\begin{aligned}
(3.2.1)\quad E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) &= \Big\langle \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}\big)\left[e^{it|D_x|}u_0\right], \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}\big)\left[e^{it|D_x|}u_0\right]\Big\rangle\\
&= \frac{1}{(2\pi)^{2d}}\int e^{ix\cdot(\xi-\eta)+it(|\xi|-|\eta|)}\mathbb{1}_{|x|>|t|}\tilde{\chi}\big(|t|^{-\delta}(|t|-|x|)\big)^2\\
&\qquad\times \chi\Big(|t|^{-\frac{1}{2}-\delta}\Big(x+t\frac{\xi}{|\xi|}\Big)\Big)\chi\Big(|t|^{-\frac{1}{2}-\delta}\Big(x+t\frac{\eta}{|\eta|}\Big)\Big)\hat{u}_0(\xi)\overline{\hat{u}_0(\eta)}\,d\xi d\eta dx.
\end{aligned}
$$

Because of the boundedness property in Theorem 2.1.1, it suffices to prove (1.1.5) when u_0 is taken in a dense subspace of $L^2(\mathbb{R}^d)$. Consequently, we may assume that in (3.2.1), $\hat{u}_0$ is in $C_0^\infty(\mathbb{R}^d-\{0\})$. Setting $x=r\omega$, $\xi=\rho\theta$, $\eta=\rho'\theta'$, we may rewrite (3.2.1) as

$$
\begin{aligned}
(3.2.2)\quad E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,\pm t) &= \frac{1}{(2\pi)^{2d}}\int e^{ir\omega\cdot(\rho\theta-\rho'\theta')\pm it(\rho-\rho')}\\
&\quad\times \tilde{\chi}(t^{-\delta}(t-r))^2\chi\big(t^{-\delta-\frac{1}{2}}(r\omega\pm t\theta)\big)\chi\big(t^{-\delta-\frac{1}{2}}(r\omega\pm t\theta')\big)\\
&\qquad\times \mathbb{1}_{r>t}r^{d-1}(\rho\rho')^{d-1}\hat{u}_0(\rho\theta)\overline{\hat{u}_0(\rho'\theta')}\,d\rho d\rho' dr d\omega d\theta d\theta'
\end{aligned}
$$

for any $t>0$. We apply Corollary A.1.4 to the $d\theta$ (resp. $d\theta'$) integral in (3.2.2) with $\epsilon'=1$ (resp. $\epsilon'=-1$) and $\epsilon=\pm$. We get first a contribution to (3.2.2) coming from the principal term (A.1.25) for both integrals, which is

$$
\begin{aligned}
\frac{1}{(2\pi)^{d+1}}&\int e^{-i\epsilon(r-t)(\rho-\rho')}\tilde{\chi}(t^{-\delta}(t-r))^2\chi\big(t^{-\delta-\frac{1}{2}}(r-t)\big)^2\\
&\qquad\times(\rho\rho')^{\frac{d-1}{2}}\mathbb{1}_{r>t}\hat{u}_0(-\epsilon\rho\omega)\overline{\hat{u}_0(-\epsilon\rho'\omega)}\,drd\omega d\rho d\rho'.
\end{aligned}
$$

We notice that for t large enough, $\tilde{\chi}\big(t^{-\delta}(r-t)\big)\chi\big(t^{-\delta-\frac{1}{2}}(r-t)\big)=\tilde{\chi}\big(t^{-\delta}(r-t)\big)$ since $\chi\equiv 1$ close to zero, so that we may rewrite this contribution as

$$
(3.2.3)\qquad \frac{1}{(2\pi)^{d+1}}\int e^{-i\epsilon(r-t)(\rho-\rho')}\tilde{\chi}(t^{-\delta}(t-r))^2(\rho\rho')^{\frac{d-1}{2}}\mathbb{1}_{r>t}\hat{u}_0(-\epsilon\rho\omega)\overline{\hat{u}_0(-\epsilon\rho'\omega)}\,drd\omega d\rho d\rho'.
$$

We get also a remainder, coming from (A.1.27) for at least one of the $d\theta$ or $d\theta'$ integrals in (3.2.2). This remainder will be given by expressions of the form

$$
(3.2.4)\qquad \int e^{-i\epsilon(r-t)(\rho-\rho')}\tilde{\chi}\big(t^{-\delta}(r-t)\big)^2 r^{d-1}t^{2(d-1)\left(\delta-\frac{1}{2}\right)}S_{-d}(\omega,\rho,\rho',r,t;t^{2\delta-1}r)\,drd\omega d\rho d\rho',
$$

where the factor S_{-d} comes from products of expressions of the form (A.1.27) and (A.1.25), or from two factors (A.1.27), and satisfies in any case bounds

$$
(3.2.5)\qquad |\partial_\zeta^\gamma\partial_r^j\partial_\rho^\ell\partial_{\rho'}^{\ell'}[S_{-d}(\omega,\rho,\rho',r,t;\zeta)]|\leq Ct^{-j\left(\frac{1}{2}+\delta\right)}\langle\zeta\rangle^{-d-\gamma}\leq Ct^{-j\delta}\langle\zeta\rangle^{-d-\gamma}.
$$

Let us define

$$
(3.2.6)\qquad F(t,\rho,\rho',r;\zeta,\zeta')=\tilde{\chi}\big(t^{-\delta}\zeta\big)\tilde{\chi}\big(t^{-\delta}\zeta'\big)\kappa\Big(\frac{r}{t}\Big)(\rho\rho')^{\frac{d-1}{2}}\int\hat{u}_0(\rho\omega)\overline{\hat{u}_0(\rho'\omega)}\,d\omega
$$

$$F_R(t,\rho,\rho',r;\zeta,\zeta') = \tilde{\chi}\big(t^{-\delta}\zeta\big)\tilde{\chi}\big(t^{-\delta}\zeta'\big)\Big(\frac{r}{t}\Big)^{d-1}\kappa\Big(\frac{r}{t}\Big) \times t^{2(d-1)\delta}\int S_{-d}\big(\omega,\rho,\rho',r,t;t^{2\delta-1}(t+\zeta)\big)\,d\omega \tag{3.2.7}$$

where κ is supported close to one and equal to one on a neighborhood of 1.
Notice that F and F_R satisfy the support condition (3.1.4) with $\delta'=\delta$ because $\hat{u}$ is taken in $C_0^\infty(\mathbb{R}^d-\{0\})$.
Estimates (3.1.5) with δ' replaced by δ follow immediately from (3.2.6) in the case of F and from (3.2.5) in the case of F_R. Moreover, (3.1.7) holds for F and F_R with the limits

$$\begin{aligned} &\lim_{t\to+\infty} F(t,\rho,\rho',r+t;\zeta,\zeta') = (\rho\rho')^{\frac{d-1}{2}}\int_{\mathbb{S}^{d-1}}\hat{u}_0(\rho\omega)\overline{\hat{u}_0(\rho'\omega)}\,d\omega\\ &\lim_{t\to+\infty} F_R(t,\rho,\rho',r+t;\zeta,\zeta') = 0\end{aligned} \tag{3.2.8}$$

as $\delta>0$. In addition, by (3.1.8) with $p(\rho)\equiv\rho$, we may rewrite (3.2.3) and (3.2.4) respectively as

$$\frac{1}{(2\pi)^{d+1}}I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F),\ \ I(t,-\epsilon,\epsilon,-\epsilon,\epsilon;F_R). \tag{3.2.9}$$

Thus $E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t)+E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,-t)$ is the sum of (3.2.3) and (3.2.4) for $\epsilon=1$ and $\epsilon=-1$ and thus by (3.2.9) may be written

$$\begin{aligned}&\frac{1}{(2\pi)^{d+1}}\big[I(t,1,-1,1,-1;F)+I(t,-1,1,-1,1;F)\big]\\ &\quad+I(t,1,-1,1,-1;F_R)+I(t,-1,1,-1,1;F_R)\end{aligned} \tag{3.2.10}$$

If we apply (3.1.26) with F_0 given by the right hand side of (3.2.8) for F and F_R respectively, we get

$$\lim_{t\to+\infty}\big[E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t)+E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,-t)\big] = \frac{1}{(2\pi)^d}\int_0^{+\infty}\int_{\mathbb{S}^{d-1}}|\hat{u}_0(\rho\omega)|^2\rho^{d-1}\,d\rho d\omega = \|u_0\|^2_{L^2}$$

i.e. the conclusion (1.1.5).

- **Case of the Schrödinger equation**

The truncated energy for the Schrödinger equation in (1.1.4) is

$$\begin{aligned} E^{\mathrm{Schr}}_{\chi,\delta}(u_0,t) &= \Big\langle \mathrm{Op}\big(a^{\mathrm{Schr}}_{\chi,\delta}\big)\big(e^{it\frac{|D_x|^2}{2}}u_0\big),\mathrm{Op}\big(a^{\mathrm{Schr}}_{\chi,\delta}\big)\big(e^{it\frac{|D_x|^2}{2}}u_0\big)\Big\rangle\\ &= \frac{1}{(2\pi)^{2d}}\int e^{ix\cdot(\xi-\eta)+i\frac{t}{2}\left(|\xi|^2-|\eta|^2\right)}\mathbb{1}_{|x|>|t|\max(|\xi|,|\eta|)}\\ &\quad\times\chi\bigg(\frac{x+t\xi}{|t||\xi|\langle\sqrt{|t|}|\xi|\rangle^{-\frac12+\delta}}\bigg)\chi\bigg(\frac{x+t\eta}{|t||\eta|\langle\sqrt{|t|}|\eta|\rangle^{-\frac12+\delta}}\bigg)\hat{u}_0(\xi)\overline{\hat{u}_0(\eta)}\,dxd\xi d\eta\end{aligned} \tag{3.2.11}$$

with $\delta\in[0,\frac12[$. Again, by Proposition 2.2.1, we may assume that $\hat{u}_0$ is in $C_0^\infty(\mathbb{R}^d-\{0\})$. Setting $x=r\omega$, $\xi=\rho\theta$, $\eta=\rho'\theta'$, we write for $t>0$,

(3.2.12)
$$\begin{aligned}E^{\mathrm{Schr}}_{\chi,\delta}(u_0,\pm t) &= \frac{1}{(2\pi)^{2d}}\int e^{ir\omega\cdot(\rho\theta-\rho'\theta')\pm it(p(\rho)-p(\rho'))}\chi\bigg(\frac{r\omega\pm t\rho\theta}{t\rho\langle\sqrt{t}\rho\rangle^{-\frac12+\delta}}\bigg)\chi\bigg(\frac{r\omega\pm t\rho'\theta'}{t\rho'\langle\sqrt{t}\rho'\rangle^{-\frac12+\delta}}\bigg)\\ &\quad\times r^{d-1}\mathbb{1}_{r>t\max(\rho,\rho')}(\rho\rho')^{d-1}\hat{u}_0(\rho\theta)\overline{\hat{u}_0(\rho'\theta')}\,drd\omega d\theta d\theta' d\rho d\rho'\end{aligned}$$

where $p(\rho) = \frac{\rho^2}{2}$. We apply Corollary A.1.5 to the $d\theta$ (resp. $d\theta'$) integral with $\epsilon' = 1$ (resp. $\epsilon' = -1$) and $\epsilon = \pm$. We obtain from (A.1.31) for both integrals a main contribution given by

$$(3.2.13)\quad \frac{1}{(2\pi)^{d+1}} \int e^{-i\epsilon\left(r(\rho-\rho')-t(p(\rho)-p(\rho'))\right)} \chi\Big(\frac{r-t\rho}{t\rho\langle\sqrt{t\rho}\rangle^{-\frac{1}{2}+\delta}}\Big)\chi\Big(\frac{r-t\rho'}{t\rho'\langle\sqrt{t\rho'}\rangle^{-\frac{1}{2}+\delta}}\Big) \times \mathbb{1}_{r>t\max(\rho,\rho')}(\rho\rho')^{\frac{d-1}{2}}\hat{u}_0(-\epsilon\rho\omega)\overline{\hat{u}_0(-\epsilon\rho'\omega)}\,dr d\omega d\rho d\rho'$$

and a remainder, coming from the replacement of at least one of the $d\theta$ or $d\theta'$ integrals in (3.2.12) by (A.1.33). If one notices that (A.1.31) may be also written under form (A.1.33) with $S_{-\frac{d+1}{2}}$ replaced by $S_{-\frac{d-1}{2}}$, with $S_{-\frac{d-1}{2}}$ satisfying (A.1.34) with in the right hand side $\langle\zeta\rangle^{-\frac{d+1}{2}-\gamma}$ replaced by $\langle\zeta\rangle^{-\frac{d-1}{2}-\gamma}$, we see that the remainder is of the form

$$(3.2.14)\quad \int e^{-i\epsilon\left(r(\rho-\rho')-t(p(\rho)-p(\rho'))\right)} \chi\Big(\frac{r-t\rho}{t\rho\langle\sqrt{t\rho}\rangle^{-\frac{1}{2}+\delta}}\Big)\chi\Big(\frac{r-t\rho'}{t\rho'\langle\sqrt{t\rho'}\rangle^{-\frac{1}{2}+\delta}}\Big) \times r^{d-1}t^{(d-1)\left(\delta-\frac{1}{2}\right)}S_{-d}\Big(\omega,\rho,\rho',\frac{r}{t\rho}-1,\frac{r}{t\rho'}-1,t;t^{\delta-\frac{1}{2}}r\Big)\mathbb{1}_{r>t\max(\rho,\rho')}\,dr d\omega d\rho d\rho'$$

where S_{-d} satisfies

$$(3.2.15)\quad |\partial_\rho^\ell\partial_{\rho'}^{\ell'}\partial_s^j\partial_{s'}^{j'}\partial_\zeta^\gamma S_{-d}(\omega,\rho,\rho',s,s',t;\zeta)| \leq Ct^{(j+j')\left(\frac{1}{4}-\frac{\delta}{2}\right)}\langle\zeta\rangle^{-d-\gamma}.$$

We define thus

$$(3.2.16)\quad F(t,\rho,\rho',r;\zeta,\zeta') = \chi\Big(\frac{\zeta}{t\rho\langle\sqrt{t\rho}\rangle^{-\frac{1}{2}+\delta}}\Big)\chi\Big(\frac{\zeta'}{t\rho'\langle\sqrt{t\rho'}\rangle^{-\frac{1}{2}+\delta}}\Big)\kappa\Big(\frac{r}{t}\Big) \times (\rho\rho')^{\frac{d-1}{2}}\int_{\mathbb{S}^{d-1}}\hat{u}_0(\rho\omega)\overline{\hat{u}_0(\rho'\omega)}\,d\omega$$

and

$$(3.2.17)\quad F_R(t,\rho,\rho',r;\zeta,\zeta') = \chi\Big(\frac{\zeta}{t\rho\langle\sqrt{t\rho}\rangle^{-\frac{1}{2}+\delta}}\Big)\chi\Big(\frac{\zeta'}{t\rho'\langle\sqrt{t\rho'}\rangle^{-\frac{1}{2}+\delta}}\Big)\Big(\frac{r}{t}\Big)^{d-1}\kappa\Big(\frac{r}{t}\Big) \times t^{(d-1)\left(\delta+\frac{1}{2}\right)}\int_{\mathbb{S}^{d-1}}S_{-d}\Big(\omega,\rho,\rho',\frac{\zeta}{t\rho},\frac{\zeta'}{t\rho'},t;t^{\delta-\frac{1}{2}}r\Big)\,d\omega$$

where $\kappa \in C_0^\infty(]0,+\infty[)$ is equal to one on a large enough compact set. Then the support conditions (3.1.4) are satisfied by F and F_R with $\delta' = \frac{3}{4}+\frac{\delta}{2} \in]\frac{1}{2},1[$ since $0 \leq \delta < \frac{1}{2}$. Inequalities (3.1.5) follow from the expressions of F, F_R and from (3.2.15). Finally, (3.1.6) holds, with

$$(3.2.18)\quad \begin{aligned} &F_0(\rho,\rho') = (\rho\rho')^{\frac{d-1}{2}}\int_{\mathbb{S}^{d-1}}\hat{u}_0(\rho\omega)\overline{\hat{u}_0(\rho'\omega)}\,d\omega \text{ for } F\\ &F_0(\rho,\rho') = 0 \text{ for } F_R. \end{aligned}$$

Then the sum of (3.2.13) and (3.2.14) is given again by the sum of the two quantities (3.2.9) associated to the F, F_R given by (3.2.16), (3.2.17) and thus $E_{\chi,\delta}^{\mathrm{Schr}}(u_0,t) + E_{\chi,\delta}^{\mathrm{Schr}}(u_0,-t)$ may be written as (3.2.10) for these functions. Applying (3.1.27) with the values (3.2.18) of F_0, we obtain conclusion (1.1.6) of Theorem 1.1.1. This concludes the proof.

3.3 The case of the wave equation

We shall prove in this subsection Theorem 1.1.2. We re-express first the energy defined in (1.1.8). If w is the solution of (1.1.7), define

$$\begin{aligned} u = (D_t + |D_x|)w,\ u_0 = -iw_1 + |D_x|w_0 \\ D_t w = \frac{u-\bar{u}}{2}, D_x w = R\Big(\frac{u+\bar{u}}{2}\Big) \end{aligned} \tag{3.3.1}$$

where R is the Riesz transform $R = \frac{D_x}{|D_x|}$.

Proof of Theorem 1.1.2 and of Theorem 1.1.3: By (1.1.7), we have $u = e^{it|D_x|}u_0$ so that (1.1.8) may be written

$$\begin{aligned} E^{\mathrm{W}}_{c,\delta}(w_0,w_1,t) = {} & \frac{1}{4}\big\|\mathbb{1}_{|t|<|x|<|t|+c|t|^\delta}\big(e^{it|D_x|}u_0 - e^{-it|D_x|}\bar{u}_0\big)\big\|^2_{L^2} \\ & + \frac{1}{4}\big\|\mathbb{1}_{|t|<|x|<|t|+c|t|^\delta}\big(e^{it|D_x|}Ru_0 + e^{-it|D_x|}R\bar{u}_0\big)\big\|^2_{L^2}. \end{aligned} \tag{3.3.2}$$

Again, to prove the theorem, we may assume that $\hat{u}_0$ is in $C_0^\infty(\mathbb{R}^d - \{0\})$. Then if $\chi \in C_0^\infty(\mathbb{R}^d)$ is equal to one close to zero, it follows from non stationary phase that, if $\delta > 0$ and N is an arbitrary integer, for $|x| \sim |t|$,

$$\Big|\mathrm{Op}\Big((1-\chi)\Big(x + t\frac{\xi}{|\xi|}\Big)|t|^{-\delta-\frac{1}{2}}\Big)\Big[e^{it|D_x|}u_0\Big]\Big| \le C|t|^{-N}.$$

A similar estimate holds if we cut-off $e^{it|D_x|}u_0$ for $|x| > |t| + c'|t|^\delta$ for any $c' > 0$. Consequently, up to a perturbation going to zero when $|t|$ goes to infinity, we may rewrite $E^{\mathrm{W}}_{c,\delta}(w_0,w_1,t)$ as

$$\begin{aligned} & \frac{1}{4}\Big\langle \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,\cdot)\big)e^{it|D_x|}u_0, \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,\cdot)\big)e^{it|D_x|}u_0\Big\rangle \\ & - \frac{1}{2}\mathrm{Re}\,\Big\langle \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,\cdot)\big)e^{it|D_x|}u_0, \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(-t,\cdot)\big)e^{-it|D_x|}\bar{u}_0\Big\rangle \\ & + \frac{1}{4}\Big\langle \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(-t,\cdot)\big)e^{-it|D_x|}\bar{u}_0, \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(-t,\cdot)\big)e^{-it|D_x|}\bar{u}_0\Big\rangle \\ & + \frac{1}{4}\Big\langle \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,\cdot)\big)e^{it|D_x|}Ru_0, \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,\cdot)\big)e^{it|D_x|}Ru_0\Big\rangle \\ & + \frac{1}{2}\mathrm{Re}\,\Big\langle \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(t,\cdot)\big)e^{it|D_x|}Ru_0, \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(-t,\cdot)\big)e^{-it|D_x|}R\bar{u}_0\Big\rangle \\ & + \frac{1}{4}\Big\langle \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(-t,\cdot)\big)e^{-it|D_x|}R\bar{u}_0, \mathrm{Op}\big(a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(-t,\cdot)\big)e^{-it|D_x|}R\bar{u}_0\Big\rangle \end{aligned} \tag{3.3.3}$$

that is, using notation (1.1.4) and the fact that $R\bar{u}_0 = -\overline{Ru_0}$,

$$\begin{aligned} E^{\mathrm{W}}_{c,\delta}(w_0,w_1,t) = {} & \frac{1}{4}\Big[E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) + E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(\bar{u}_0,-t) + E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(Ru_0,t) + E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(R\bar{u}_0,-t)\Big] \\ & - \frac{1}{2}\mathrm{Re}\,\Big[\tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) + \tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(Ru_0,t)\Big] \end{aligned} \tag{3.3.4}$$

where we define

$$\begin{aligned} \tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) = {} & \frac{1}{(2\pi)^{2d}}\int \mathbb{1}_{|x|>|t|}e^{ix\cdot(\xi-\eta)+it(|\xi|+|\eta|)} \\ & \times \chi\Big(\Big(x + t\frac{\xi}{|\xi|}\Big)|t|^{-\frac{1}{2}-\delta}\Big)\chi\Big(\Big(x - t\frac{\eta}{|\eta|}\Big)|t|^{-\frac{1}{2}-\delta}\Big) \\ & \times \tilde{\chi}\big(|t|^{-\delta}(|t|-|x|)\big)^2\hat{u}_0(\xi)\hat{u}_0(-\eta)\,d\xi d\eta dx. \end{aligned} \tag{3.3.5}$$

For $t > 0$, $\epsilon = \pm$, rewrite

$$\tag{3.3.6} \begin{aligned} \tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0, \epsilon t) = \frac{1}{(2\pi)^{2d}} \int & e^{ir\omega\cdot(\rho\theta-\rho'\theta)+i\epsilon t(\rho+\rho')} \chi\big((r\omega+\epsilon t\theta)t^{-\frac{1}{2}-\delta}\big)\chi\big((r\omega-\epsilon t\theta')t^{-\frac{1}{2}-\delta}\big) \\ & \times \tilde{\chi}\big(t^{-\delta}(t-r)\big)^2 r^{d-1}(\rho\rho')^{d-1}\mathbb{1}_{r>t}\hat{u}_0(\rho\theta)\hat{u}_0(-\rho'\theta')\,dr d\rho d\rho' d\theta d\theta' d\omega. \end{aligned}$$

Let us show:

Lemma 3.3.1 *One has if d is odd*

$$\tag{3.3.7} \lim_{t\to+\infty}\Big[\tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) + \tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,-t)\Big] = 0$$

while if d is even

$$\tag{3.3.8} \begin{aligned} &\lim_{t\to+\infty}\Big[\tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,t) + \tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,-t)\Big] \\ &\quad = \frac{2(-1)^{\frac{d}{2}-1}}{(2\pi)^{d+1}} \int_{]0,+\infty[\times\mathbb{S}^{d-1}} H\big(\rho^{\frac{d-1}{2}}\hat{u}_0(\rho\omega)\big)(\rho')\rho'^{\frac{d-1}{2}}\hat{u}_0(\rho'\omega)\,d\rho' d\omega \end{aligned}$$

where H is the Hankel transform (1.1.11) acting on the function $\rho \to \rho^{\frac{d-1}{2}}\hat{u}_0(\rho\omega)$ at fixed ω.

Proof: As in subsection 3.2, we apply Corollary A.1.4 to the $d\theta$ and $d\theta'$ integrals in (3.3.6), except that now we have to use (A.1.24) with $\epsilon' = 1$ for *both* integrals (up to the change of variables $\theta' \to -\theta'$ in order to reduce the $d\theta'$ integral to form (A.1.24)). In that way, the $e^{i\epsilon\epsilon'\frac{\pi}{4}(d-1)}$ factors of (A.1.25) do not cancel each other, and we get that (3.3.6) is given by a main contribution of the form

$$\tag{3.3.9} \begin{aligned} \frac{1}{(2\pi)^{d+1}} e^{i\epsilon\frac{\pi}{2}(d-1)} \int & e^{-i\epsilon(\rho+\rho')(r-t)}\tilde{\chi}\big(t^{-\delta}(r-t)\big)^2(\rho\rho')^{\frac{d-1}{2}}\mathbb{1}_{r>t} \\ & \times \int_{\mathbb{S}^{d-1}} \hat{u}_0(-\epsilon\rho\omega)\hat{u}_0(-\epsilon\rho'\omega)\,d\omega d\rho d\rho' dr \end{aligned}$$

up to a $o(1)$ remainder. Set

$$\tag{3.3.10} F(t,\rho,\rho',r;\zeta,\zeta') = \kappa\Big(\frac{r}{t}\Big)(\rho\rho')^{d-1}\int_{\mathbb{S}^{d-1}} \hat{u}_0(\rho\omega)\hat{u}_0(\rho'\omega)\,d\omega\tilde{\chi}\big(t^{-\delta}\zeta\big)\big(t^{-\delta}\zeta'\big)$$

for some function $\kappa \in C_0^\infty(]0,+\infty[)$ equal to one on a large enough compact subset. Then assumptions (3.1.4), (3.1.5) are satisfied with $\delta' = \delta$, and (3.1.7) holds with

$$\tag{3.3.11} F_0(\rho,\rho') = (\rho\rho')^{\frac{d-1}{2}}\int_{\mathbb{S}^{d-1}} \hat{u}_0(\rho\omega)\hat{u}_0(\rho'\omega)\,d\omega.$$

Moreover, (3.3.9) may be rewritten according to (3.1.8) as

$$\frac{1}{(2\pi)^{d+1}} e^{i\epsilon\frac{\pi}{2}(d-1)} I(t,-\epsilon,-\epsilon,-\epsilon,-\epsilon;F)$$

so that the quantity to be computed in (3.3.7), (3.3.8), which is nothing but the sum of (3.3.6) (or of (3.3.9)) for $\epsilon = -1$ and $\epsilon = 1$, is just

$$\tag{3.3.12} \frac{1}{(2\pi)^{d+1}}\Big[e^{i\frac{\pi}{2}(d-1)}I(t,-1,-1,-1,-1;F) + e^{-i\frac{\pi}{2}(d-1)}I(t,1,1,1,1;F)\Big]$$

up to a $o(1)$ remainder. If d is odd, (3.1.11) implies that the limit of (3.3.12) vanishes, hence (3.3.7). If d is even, $d = 2\ell$, we use (3.1.10) to write that limit as

$$\frac{1}{(2\pi)^{d+1}} e^{i\pi\ell} \Big\langle -2(\rho+\rho')^{-1}, F_0(\rho,\rho') \Big\rangle$$

with F_0 given by (3.3.11). This gives (3.3.8). $\square$

End of the proof of Theorem 1.1.2: We need to study the limit when t goes to infinity of the sum $E^{\mathrm{W}}_{c,\delta}(w_0, w_1, t) + E^{\mathrm{W}}_{c,\delta}(w_0, w_1, -t)$ i.e. by (3.3.4)

(3.3.13)
$$\begin{aligned}\lim_{t\to+\infty} \frac{1}{4}\Big[\sum_{\epsilon=\pm} E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,\epsilon t) + \sum_{\epsilon=\pm} E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(\bar{u}_0,\epsilon t) + \sum_{\epsilon=\pm} E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(Ru_0,\epsilon t) + \sum_{\epsilon=\pm} E^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(R\bar{u}_0,\epsilon t)\Big] \\ - \lim_{t\to+\infty} \frac{1}{2} \mathrm{Re}\Big[\sum_{\epsilon=\pm} \tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(u_0,\epsilon t) + \sum_{\epsilon=\pm} \tilde{E}^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}(Ru_0,\epsilon t)\Big].\end{aligned}$$

By Theorem 1.1.1, the first limit is equal to $\|u_0\|^2_{L^2} = \|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}$. By (3.3.7), when d is odd, the second limit is zero, so that we obtain (1.1.9).
End of the proof of Theorem 1.1.3: In this case, we have to add to the first contribution $\|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}$ the last line of (3.3.13), which is given, according to (3.3.8), by

$$2\frac{(-1)^{\frac{d}{2}}}{(2\pi)^{d+1}} \mathrm{Re} \int_{\mathbb{S}^{d-1}} H\big(\rho^{\frac{d-1}{2}} \hat{u}_0(\rho\omega)\big)(\rho')\rho'^{\frac{d-1}{2}} \hat{u}_0(\rho'\omega)\, d\omega$$

since the contribution coming from Ru_0 is equal to the one given by u_0. This gives (1.1.10).
To prove the last statement of Theorem 1.1.3, we notice that if w_0 is even and w_1 is odd (resp. w_0 is odd and w_1 is even) then $\hat{u}_0(\eta) = -i\hat{w}_1(\eta) + |\eta|\hat{w}_0(\eta)$ is real valued (resp. purely imaginary). If we set $f(\rho,\omega) = \rho^{\frac{d-1}{2}} \hat{u}_0(\rho\omega)$, the last term in (1.1.10) may be written as a positive multiple of

$$\int_{\mathbb{S}^{d-1}} \langle H(f(\cdot,\omega)), f(\cdot,\omega)\rangle \, d\omega$$

when (1.1.12) holds since then $f = \bar{f}$, and where we denoted by $\langle \cdot,\cdot \rangle$ the $L^2(d\rho')$ scalar product. As H is a positive operator we thus get that (1.1.10) is bounded from below by $\|w_1\|^2_{L^2} + \|\nabla_x w_0\|^2_{L^2}$. Under (1.1.13), the same conclusion holds, since then $\bar{f} = -f$.
Finally, (1.1.14) follows if we show that $t \to E^{\mathrm{W}}_{c,\delta}(w_0, w_1, t)$ is an even function. This is clear as, under condition (1.1.12) (resp. (1.1.13)), the solution w of the wave equation satisfies for any (t,x), $w(-t,-x) = w(t,x)$ (resp. $w(-t,-x) = -w(t,x)$). This concludes the proof. $\square$

A Appendix

A.1 Stationary phase related properties

We shall first state some phase integral estimates in a general framework and write then the corollaries we use in the bulk of the text.
Let $\mathcal{M}$, $\mathcal{N}$ be two boundaryless riemannian manifolds of dimension m and n respectively, with $\mathcal{N}$ compact. Denote by $d_{\mathcal{M}}$ the riemannian distance on $\mathcal{M}$. Let $\Phi : \mathcal{N} \times \mathcal{M} \to \mathbb{R}$ be a smooth map, $(x,y) \to \Phi(x,y)$ and let

(A.1.1)
$$\begin{aligned}\chi :]0,1] \times \mathcal{N} \times \mathcal{M} \times \mathcal{M} \times \mathcal{M} &\to \mathbb{R} \\ (\mu, x, y, z, z') &\to \chi(\mu, x, y, z, z')\end{aligned}$$

be smooth in (x, y, z, z') and satisfy for any $\alpha \in \mathbb{N}^n$, $\beta, \gamma, \gamma' \in \mathbb{N}^m$, $j, N \in \mathbb{N}$, estimates

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \partial_{z'}^{\gamma'} \partial_\mu^j \chi(\mu, x, y, z, z')| \le C\mu^{-|\alpha|-|\beta|-|\gamma|-j} \Big\langle \frac{d_{\mathcal{M}}(y,z)}{\mu} \Big\rangle^{-N}. \tag{A.1.2}$$

Denote by L the projection of the support of χ on the (x, y) variables and assume that for any x in $\mathcal{N}$, $y \to \Phi(x, y)$ has a unique critical point $y(x)$ such that $(x, y(x)) \in L$, and that moreover this critical point is non degenerate. Finally, assume that for some small positive δ'',

$$\operatorname{Supp}\big(\chi(\mu, x, y, z, z')\big) \subset \{(\mu, x, y, z, z'); d_{\mathcal{M}}(y, y(x)) < \delta'' \text{ and } d_{\mathcal{M}}(y, z') < \delta''\}. \tag{A.1.3}$$

For $\lambda \ge 1$, define

$$I(x, z, z'; \lambda, \mu) = \int_{\mathcal{M}} e^{i\lambda\Phi(x,y)} \chi(\mu, x, y, z, z')\, dy \tag{A.1.4}$$

where dy is the riemannian measure on $\mathcal{M}$. Our first result is the following:

Proposition A.1.1 *There is a function*

$$\begin{aligned} S_{-\frac{m}{2}} : \mathcal{N} \times \mathcal{M} \times \mathcal{M} \times]0, 1] \times \mathbb{R} &\to \mathbb{C} \\ (x, z, z', \mu, \zeta) &\to S_{-\frac{m}{2}}(x, z, z', \mu; \zeta) \end{aligned} \tag{A.1.5}$$

smooth in (x, z, z', ζ) and satisfying for any $\alpha \in \mathbb{N}^n$, $\beta, \beta' \in \mathbb{N}^m$, $\gamma \in \mathbb{N}$, j, N in $\mathbb{N}$

$$|\partial_x^\alpha \partial_z^\beta \partial_{z'}^{\beta'} \partial_\mu^j \partial_\zeta^\gamma S_{-\frac{m}{2}}(x, z, z', \mu; \zeta)| \le C\mu^{-|\alpha|-|\beta|-j} \langle\zeta\rangle^{-\frac{m}{2}-\gamma} \Big\langle \frac{d_{\mathcal{M}}(y(x), z)}{\mu} \Big\rangle^{-N}, \tag{A.1.6}$$

supported inside

$$\{(x, z, z', \mu, \zeta); d_{\mathcal{M}}(y(x), z') < 2\delta''\} \tag{A.1.7}$$

such that one may write if $\delta'' > 0$ is small enough and $\lambda\mu^2 \ge c > 0$,

$$I(x, z, z'; \lambda, \mu) = e^{i\lambda\Phi(x,y(x))} \mu^m S_{-\frac{m}{2}}(x, z, z', \mu; \lambda\mu^2). \tag{A.1.8}$$

Moreover, the symbol $S_{-\frac{m}{2}}$ may be decomposed for $\zeta \ge c$ as

$$\begin{aligned} S_{-\frac{m}{2}}(x, z, z', \mu; \zeta) = (2\pi)^{\frac{m}{2}} e^{i\frac{\pi}{4}(p-q)} |\det\big(\operatorname{Hess}(\Phi_y''(x, y(x)))\big)|^{-\frac{1}{2}} \chi(\mu, x, y(x), z, z')\zeta^{-\frac{m}{2}} \\ + S_{-\frac{m}{2}-1}(x, z, z', \mu; \zeta) \end{aligned} \tag{A.1.9}$$

where $\operatorname{Hess}(\Phi_y''(x, y(x)))$ is the Hessian matrix of $y \to \Phi(x, y)$ at point $y(x)$, (p, q) is the signature of that matrix, and where $S_{-\frac{m}{2}-1}$ satisfies (A.1.6) with $\frac{m}{2}$ replaced by $\frac{m}{2}+1$ in the right hand side.
Moreover, if χ depends smoothly on some parameter w, then so does I, and $S_{-\frac{m}{2}}$, $S_{-\frac{m}{2}-1}$ above are also smooth functions of w. Their ∂_w-derivatives of order ℓ may be estimated then by inequalities of the form (A.1.6) with a constant C depending only on the constants obtained in (A.1.2) for the ∂_w-derivatives of order ℓ of χ.

Proof: Replacing in (A.1.5) $\Phi(x, y)$ by $\Phi(x, y) - \Phi(x, y(x))$, we may always assume that the critical value is zero. By compactness of $\mathcal{N}$, we may reduce ourselves to the case of x staying in a small enough open subset U of $\mathcal{N}$ such that for any x in U, $y(x)$ stays in a same chart domain of $\mathcal{M}$. By (A.1.3), for small enough δ'', y stays as well in this chart domain. From now on, we shall thus assume that in (A.1.4), χ is compactly supported for y in a ball of center 0 in $\mathbb{R}^m$ and that $y(x)=0$ for any x in U. Making the change of variables $y=\mu y'$, we rewrite (A.1.4) as

$$I(x,z,z';\lambda,\mu)=\mu^m\int_{\mathbb{R}^m}e^{i\lambda\mu^2\Phi_1(x,y',\mu)}\chi_1(\mu,x,y',z,z')\,dy' \tag{A.1.10}$$

where, since Φ has a non degenerate critical point at $y(x)=0$, Φ_1 satisfies on the domain $|y'|\mu=O(\delta'')$ with $\delta''\ll 1$ properties

$$\begin{aligned}&\partial_{y'}\Phi_1(x,0,\mu)=0,\ \partial^2_{y'}\Phi_1(x,y',\mu)=A(x)+O(\mu|y'|)\\&\partial^{\alpha}_{y'}\Phi_1(x,y',\mu)=O(1),\forall\alpha\in\mathbb{N}^m,|\alpha|\geq 3\end{aligned} \tag{A.1.11}$$

for some non singular matrix $A(x)$, and where χ_1 satisfies because of (A.1.2) estimates

$$|\partial^j_\mu\partial^\alpha_x\partial^\beta_{y'}\partial^\gamma_z\partial^{\gamma'}_{z'}\chi_1(\mu,x,y',z,z')|\leq C\mu^{-|\alpha|-|\gamma|-j}\Big\langle\frac{d_{\mathcal{M}}(\mu y',z)}{\mu}\Big\rangle^{-N} \tag{A.1.12}$$

and

$$\chi_1(\mu,x,0,z,z')=\chi(\mu,x,y(x),z,z')|J(x)| \tag{A.1.13}$$

where $J(x)$ is a jacobian such that

$$|\det\mathrm{Hess}\,(\Phi''_{1,y'}(x,0,\mu))|^{-\frac{1}{2}}|J(x)|=|\det\mathrm{Hess}\,(\Phi''_y(x,y(x)))|^{-\frac{1}{2}}. \tag{A.1.14}$$

Take χ_0 in $C_0^\infty(\mathbb{R}^m)$ with small enough support, equal to one close to zero and decompose in (A.1.10) $I=I_0+I_1$ with

$$I_0(x,z,z';\lambda,\mu)=\mu^m\int e^{i\lambda\mu^2\Phi_1(x,y',\mu)}\chi_1(\mu,x,y',z,z')\chi_0(y')\,dy'. \tag{A.1.15}$$

For δ'' small enough so that (A.1.11) holds, $y'\to\Phi_1(x,y',\mu)$ has a unique critical point in $\operatorname{Supp}\chi_0$ at $y'=0$, that is non degenerate, with zero critical value, so that by the stationary phase formula, the above integral is a symbol of order $-\frac{m}{2}$ in $\lambda\mu^2$ of the form (A.1.5), with $\zeta=\lambda\mu^2$. Moreover, its ∂_x (resp. ∂_z, resp. $\partial_{z'}$)-derivatives are estimated from the same derivatives of χ_1, and from its $\partial_{y'}$-ones, so that (A.1.12) implies that bounds (A.1.6) hold true, using also that

$$\Big\langle\frac{d_{\mathcal{M}}(\mu y',z)}{\mu}\Big\rangle^{-N}\sim\Big\langle\frac{|\mu y'-z|}{\mu}\Big\rangle^{-N}\sim\Big\langle\frac{|z|}{\mu}\Big\rangle^{-N}\sim\Big\langle\frac{d_{\mathcal{M}}(y(x),z)}{\mu}\Big\rangle^{-N}$$

since y' is bounded on the support of χ_0 in (A.1.15). The support condition (A.1.7) follows from (A.1.3). Finally, expansion (A.1.9) is just the first term in the expansion of the stationary phase formula.

We are left with studying the integral I_1, given by expression (A.1.15) where we replace χ_0 by $1-\chi_0$. Since, on the support of the integrand, $\mu|y'|=O(\delta'')$, it follows from (A.1.11) that, if $(1-\chi_0)(y')\neq 0$,

$$|\partial_{y'}\Phi_1(x,y',\mu)|\sim|y'|\geq c>0$$

if δ'' is small enough. We may thus integrate by parts in y' and conclude that

$$|I_1(x,z,z';\lambda,\mu)|\leq C\mu^m\int_{|y'|\geq c}\langle\lambda\mu^2y'\rangle^{-N'}\Big\langle\frac{d_{\mathcal{M}}(\mu y',z)}{\mu}\Big\rangle^{-N'}dy'$$

for any $N'\in\mathbb{N}$. As $\lambda\mu^2\geq c$, we get a bound in

$$\mu^m(\lambda\mu^2)^{-N'}\int_{|y'|\geq c}|y'|^{-N'}\Big[1+\Big|y'-\frac{z}{\mu}\Big|\Big]^{-N'}dy'\leq C\mu^m(\lambda\mu^2)^{-N}\Big\langle\frac{z}{\mu}\Big\rangle^{-N}.$$

As derivatives are estimated similarly, this shows that I_1 contributes to $S_{-\frac{m}{2}-1}$ in (A.1.9). This concludes the proof. $\square$

We study now integrals of the form (A.1.5) when there is no critical point on the domain of integration.

Proposition A.1.2 *Let χ be a function of the form (A.1.1) satisfying (A.1.2) and compactly supported in y. Let $\Phi : \mathcal{M} \times \mathcal{N} \to \mathbb{R}$ be a smooth map such that there is $c > 0$ with for any $(x, y) \in \mathcal{N} \times \mathcal{M}$, $|\nabla_y \Phi(x, y)| \geq c$. Then (A.1.4) may be written*

$$I(x, z, z'; \lambda, \mu) = e^{i\lambda\Phi(x,z)}\mu^m G(x, z, z', \lambda, \mu) \tag{A.1.16}$$

where G is smooth in μ, x, z, z' and satisfies for any $\alpha \in \mathbb{N}^n$, $\beta, \beta' \in \mathbb{N}^m$, $j, N \in \mathbb{N}$

$$|\partial_x^\alpha \partial_z^\beta \partial_{z'}^{\beta'} \partial_\mu^j G(x, z, z', \lambda, \mu)| \leq C\mu^{-|\alpha|-|\beta|-j}\langle \lambda\mu\rangle^{-N}. \tag{A.1.17}$$

Proof: Consider first the case when on the support of χ, one has $d_{\mathcal{M}}(y, z) \geq c' > 0$. Then the last factor in (A.1.2) shows that χ and its derivatives are $O(\mu^N)$ for any N. Since $y \to \Phi(x, y)$ has no critical point uniformly in (x, y), integrations by parts in ∂_y in (A.1.4) show then that a bound in $O(\lambda^{-N}\mu^N)$ holds for any N for I and its derivatives, which implies (A.1.16), (A.1.17). We are thus reduced to the case when χ is supported for $d_{\mathcal{M}}(y, z) \leq c' \ll 1$. As y stays in a compact subset of $\mathcal{M}$, we may thus assume that y, z belong to a same chart domain of $\mathcal{M}$ and set, in local coordinates, $y = z + y'$, with $|y'| \ll 1$. Then (A.1.4) may be rewritten

$$e^{i\lambda\Phi(x,z)} \int e^{i\lambda\Phi_1(x,z,y')}\chi_1(\mu, x, y', z, z')\, dy' \tag{A.1.18}$$

where Φ_1 satisfies

$$|\partial_{y'}\Phi_1(x, z, y')| \sim 1,\ |\partial_x^\alpha \partial_z^\beta \Phi_1(x, z, y')| = O(|y'|),\ y' \to 0 \tag{A.1.19}$$

and where χ_1 is such that, by (A.1.2)

$$|\partial_\mu^j \partial_x^\alpha \partial_{y'}^\beta \partial_z^\gamma \partial_{z'}^{\gamma'} \chi_1| \leq C\mu^{-|\alpha|-|\beta|-|\gamma|-j}\Big\langle \frac{|y'|}{\mu}\Big\rangle^{-N}.$$

If we make $\partial_{y'}$ integrations by parts in (A.1.18), we conclude that this integral is $O(\mu^m\langle\lambda\mu\rangle^{-N})$ for any N. Using (A.1.19), we get as well the estimates (A.1.17) for the derivatives. $\square$

We specialize now the above results to the case $\mathcal{N} = \mathcal{M} = \mathbb{S}^{d-1}$ to obtain the corollaries used in subsection 2.2. We denote by d the distance $d_{\mathcal{M}}$.

Corollary A.1.3 *Let*

$$F : (\theta, \theta', \omega, \omega', \nu, \mu) \to F(\theta, \theta', \omega, \omega', \nu, \mu)$$

be a function defined on $(\mathbb{S}^{d-1})^4 \times [0, 1] \times]0, 1]$ with values in $\mathbb{C}$, smooth in all variables, satisfying for any $\alpha, \alpha', \beta, \beta'$ in $\mathbb{N}^{d-1}$, j, j' in $\mathbb{N}$ the bound

$$|\partial_\theta^\alpha \partial_{\theta'}^{\alpha'} \partial_\omega^\beta \partial_{\omega'}^{\beta'} \partial_\nu^j \partial_\mu^{j'} F(\theta, \theta', \omega, \omega', \nu, \mu)| \leq C\mu^{-|\alpha'|-|\beta|-j-j'}\Big\langle \frac{d(\theta', \omega)}{\mu}\Big\rangle^{-N} \tag{A.1.20}$$

and supported for $d(\theta, \theta') + d(\theta', \omega') < \delta''$. Then if $\delta'' > 0$ is small enough, the integral

$$\int_{\mathbb{S}^{d-1}} e^{\pm i\lambda\theta\cdot\theta'} F(\theta, \theta', \omega, \omega', \nu, \mu)\, d\theta' \tag{A.1.21}$$

may be written, when $\lambda\mu^2 \geq c > 0$, under the form

$$e^{\pm i\lambda}\mu^{d-1}S_{-\frac{d-1}{2}}(\theta,\omega,\omega',\nu,\mu;\lambda\mu^2) \tag{A.1.22}$$

where the symbol $S_{-\frac{d-1}{2}}$ satisfies for any α,β,β' in $\mathbb{N}^{d-1}$, j,j',γ in $\mathbb{N}$

$$|\partial_\theta^\alpha\partial_\omega^\beta\partial_{\omega'}^{\beta'}\partial_\nu^j\partial_\mu^{j'}\partial_\zeta^\gamma S_{-\frac{d-1}{2}}(\theta,\omega,\omega',\nu,\mu;\zeta)| \leq C\mu^{-|\alpha|-|\beta|-j-j'}\Big\langle\frac{d(\theta,\omega)}{\mu}\Big\rangle^{-N}\langle\zeta\rangle^{-\frac{d-1}{2}-\gamma}. \tag{A.1.23}$$

Moreover, $S_{-\frac{d-1}{2}}$ is supported for $d(\theta,\omega') < 2\delta''$. In addition, if F depends smoothly on some extra parameter w, so does $S_{-\frac{d-1}{2}}$, with ∂_w^ℓ-derivatives of (A.1.23) estimated from the ∂_w^ℓ-derivatives of F.

Proof: Consider for instance the case of sign $+$ in the phase of (A.1.21). We apply Proposition A.1.1 with $x=\theta$, $y=\theta'$, $z=\omega$, $z'=\omega'$ and some extra implicit variable ν. The phase is then $\theta'\to\theta\cdot\theta'$, and because of the support condition on F, $\theta'=\theta$ is the only critical point that has to be considered. This critical point is non degenerated and the critical value is equal to 1. Consequently (A.1.22) is just (A.1.8) and (A.1.6) provides (A.1.23), the fact that we get also estimates for ∂_ν derivatives following from the last statement in Proposition A.1.1. The case of sign $-$ in the phase of (A.1.21) follows by conjugation. □

In the preceding corollary, we did not make use of the expansion of Proposition A.1.1. We write a second corollary with the notation that we use in section 3.

Corollary A.1.4 *Let χ be in $C_0^\infty(\mathbb{R})$, χ equal to one close to zero, with small enough support and denote also by $\chi(x)$ the radial function $\chi(|x|)$ for any x in $\mathbb{R}^d$. Let f be in $C_0^\infty(\mathbb{R}^d-\{0\})$. Let $\epsilon,\epsilon'\in\{-,+\}$, $\delta\in]0,\frac{1}{2}]$, $\rho>0$, $r>0$, $t\geq 1$. Then the integral*

$$\int_{\mathbb{S}^{d-1}} e^{i\epsilon' r\rho\omega\cdot\theta}\chi\big(t^{-\frac{1}{2}-\delta}(r\omega+\epsilon t\theta)\big)f(\rho\theta)\,d\theta \tag{A.1.24}$$

may be written as the sum of a principal term

$$e^{-i\epsilon\epsilon' r\rho}(2\pi)^{\frac{d-1}{2}}e^{i\epsilon\epsilon'\frac{\pi}{4}(d-1)}(r\rho)^{-\frac{d-1}{2}}\chi\big(t^{-\frac{1}{2}-\delta}(r-t)\big)f(-\epsilon\rho\omega) \tag{A.1.25}$$

and a remainder, supported for

$$C^{-1}\leq\rho\leq C,\ |r-t|\ll t^{\frac{1}{2}+\delta} \tag{A.1.26}$$

for some $C>0$, that may be written as

$$e^{-i\epsilon\epsilon' r\rho}S_{-\frac{d+1}{2}}\big(\omega,\rho,r,t;t^{2\delta-1}r\rho\big)t^{(d-1)\left(\delta-\frac{1}{2}\right)} \tag{A.1.27}$$

where for any $\alpha\in\mathbb{N}^{d-1}$, ℓ,j,γ in $\mathbb{N}$

$$|\partial_\omega^\alpha\partial_\rho^\ell\partial_r^j\partial_\zeta^\gamma S_{-\frac{d+1}{2}}(\omega,\rho,r,t;\zeta)| \leq Ct^{\left(\frac{1}{2}-\delta\right)|\alpha|-\left(\frac{1}{2}+\delta\right)j}\langle\zeta\rangle^{-\frac{d+1}{2}-\gamma}. \tag{A.1.28}$$

Proof: We make in (A.1.24) the change of notation $r=tr'$ so that we have to study

$$\int_{\mathbb{S}^{d-1}} e^{i\epsilon' tr'\rho\omega\cdot\theta}\chi\Big(\frac{r'\omega+\epsilon\theta}{\mu}\Big)f(\rho\theta)\,d\theta \tag{A.1.29}$$

where $\mu = t^{\delta-\frac{1}{2}} \in]0,1]$. The integrand is supported for $|r'-1| \ll \mu$, $|\omega+\epsilon\theta| \ll \mu$ and for ρ in a compact subset of $]0,+\infty[$. We set $\lambda = r\rho = tr'\rho$. We have $\lambda\mu^2 \geq c > 0$ by the preceding conditions and the fact that $\delta \geq 0$. We apply Proposition A.1.1 to (A.1.29) with $x=\omega, y=\theta, z=-\epsilon\omega$, no z' variable, and smooth dependence in the extra parameter r' of χ in (A.1.5), with uniform estimates when one takes $\mu\partial_{r'}$-derivatives. The unique critical point in the support of the integrand is $\theta(\omega)=-\epsilon\omega$. The support assumption (A.1.3) is satisfied if $\operatorname{Supp}\chi$ is small enough, as well as estimate (A.1.2) by the preceding remarks. The Hessian of $\theta \to \epsilon'\omega\cdot\theta$ at $\theta(\omega)$ is of signature $(d-1,0)$ if $\epsilon\epsilon'=1$ and $(0,d-1)$ if $\epsilon\epsilon'=-1$, and its determinant has absolute value 1. One thus gets from (A.1.9) the first term in the asymptotic expansion (A.1.25) returning to the r variable.
The remainder is given by (A.1.27) according to (A.1.9), the estimates (A.1.28) following from (A.1.6) and from the fact that the coefficient of the exponential in (A.1.29) admits uniform estimates when we take $\mu\partial_{r'} = t^{\frac{1}{2}+\delta}\partial_r$ derivatives. This concludes the proof. $\square$

We state an analogous corollary, used in section 3 to study the integrals associated to the Schrödinger operator.

Corollary A.1.5 *Let χ be a cut-off as in the preceding corollary, f in $C_0^\infty(\mathbb{R}^d-\{0\})$. Let ϵ,ϵ' in $\{-,+\}$, $\delta\in[0,\frac{1}{2}]$. Then if $\operatorname{Supp}\chi$ is small enough, $\rho>0$, $r>0$, $t\geq 1$, the integral*

$$\int_{\mathbb{S}^{d-1}} e^{i\epsilon' r\rho\omega\cdot\theta}\chi\Big(\frac{r\omega+\epsilon t\rho\theta}{t\rho\langle\sqrt{t}\rho\rangle^{-\frac{1}{2}+\delta}}\Big)f(\rho\theta)\,d\theta \tag{A.1.30}$$

may be written as the sum of a principal part

$$e^{-i\epsilon\epsilon' r\rho}(2\pi)^{\frac{d-1}{2}}e^{i\epsilon\epsilon'\frac{\pi}{4}(d-1)}(r\rho)^{-\frac{d-1}{2}}\chi\Big(\frac{r-t\rho}{t\rho\langle\sqrt{t}\rho\rangle^{-\frac{1}{2}+\delta}}\Big)f(-\epsilon\rho\omega) \tag{A.1.31}$$

and a remainder, supported for

$$C^{-1}\leq\rho\leq C,\ |r-t\rho|\ll t\rho\langle\sqrt{t}\rho\rangle^{-\frac{1}{2}+\delta} \tag{A.1.32}$$

that may be written as

$$e^{-i\epsilon\epsilon' r\rho}S_{-\frac{d+1}{2}}\Big(\omega,\rho,\frac{r}{t\rho}-1,t;t^{\delta-\frac{1}{2}}r\rho\Big)t^{\frac{d-1}{2}\left(\delta-\frac{1}{2}\right)} \tag{A.1.33}$$

where for any $\alpha\in\mathbb{N}^d$, ℓ,j,γ in $\mathbb{N}$

$$|\partial_\omega^\alpha\partial_\rho^\ell\partial_{r'}^j\partial_\zeta^\gamma S_{-\frac{d+1}{2}}(\omega,\rho,r',t;\zeta)|\leq Ct^{\left(\frac{1}{4}-\frac{\delta}{2}\right)(|\alpha|+j)}\langle\zeta\rangle^{-\frac{d+1}{2}-\gamma}. \tag{A.1.34}$$

In particular, by (A.1.32), $S_{-\frac{d+1}{2}}(\omega,\rho,r',t;\zeta)$ is supported for $|r'|\ll\langle\sqrt{t}\rho\rangle^{-\frac{1}{2}+\delta}$.

Proof: We set $r=t\rho(r'+1)$. We rewrite (A.1.30) as

$$\int e^{i\epsilon' t\rho^2(1+r')\omega\cdot\theta}\chi\Big(\frac{r'\omega+(\omega+\epsilon\theta)}{\langle\sqrt{t}\rho\rangle^{-\frac{1}{2}+\delta}}\Big)f(\rho\theta)\,d\theta. \tag{A.1.35}$$

The integrand in (A.1.30) is supported for $|r-t\rho|\ll t\rho\langle\sqrt{t}\rho\rangle^{-\frac{1}{2}+\delta}$, and for ρ in a compact subset of $]0,+\infty[$, so that since (A.1.32) holds trivially for the support of (A.1.31), it does also for the support of the remainder. Set $\mu = t^{-\frac{1}{4}+\frac{\delta}{2}}$. Then $\mu \in]0,1]$ since $\delta \leq \frac{1}{2}$ and if $\lambda = t(1+r')\rho^2$, we get $\lambda \geq c > 0$ and $\lambda\mu^2 \geq c > 0$ as r' is small. We define

$$F(x,y,z,\mu;\rho,r') = \chi\Bigg(\frac{r'x+x-y}{\Big\langle \mu^{-\left(\frac{1}{2}-\delta\right)^{-1}}\rho\Big\rangle^{-\frac{1}{2}+\delta}}\Bigg)\tilde{\chi}\Big(\frac{x-y}{\mu}\Big)\tilde{\chi}\Big(\frac{y-z}{\mu}\Big)f(\rho y)$$

where $\tilde{\chi} \in C_0^\infty(\mathbb{R}^d)$ is equal to one on a large enough neighborhood of $\operatorname{Supp}\chi$. Then integral (A.1.35) may be rewritten as

$$\int e^{-i\epsilon\epsilon'\lambda(-\epsilon\omega)\cdot\theta}F(-\epsilon\omega,\theta,-\epsilon\omega,\mu;\rho,r')\,d\theta \tag{A.1.36}$$

since, as by (A.1.32) $|r'| \ll \mu$, the first cut-off in the definition of F imposes that on the integrand of (A.1.36) $|\omega+\epsilon\theta| \ll \mu$, which in turns implies that the $\tilde{\chi}$ cut-offs are equal to one at their argument. Moreover

$$|\partial_x^\alpha\partial_y^{\alpha'}\partial_z^\beta\partial_\mu^p\partial_\rho^\ell\partial_{r'}^j F(x,y,z,\mu;\rho,r')| \leq C\mu^{-|\alpha|-|\alpha'|-|\beta|-p-j}\Big\langle\frac{d(y,z)}{\mu}\Big\rangle^{-N} \tag{A.1.37}$$

for any $\alpha,\alpha',\beta,p,\ell,j,N$. We may apply to (A.1.36) Proposition A.1.1, where in (A.1.2) we have no z' variable and we insert the extra parameters (ρ,r'), allowing a μ^{-1} loss for every $\partial_{r'}$-derivative. We thus get for (A.1.36) a decomposition of the form (A.1.8)-(A.1.9) with m replaced by $\frac{d-1}{2}$ and with a remainder in (A.1.9) given by (A.1.33) with $S_{-\frac{d+1}{2}}$ satisfying estimates given by (A.1.6) (with $m = \frac{d+1}{2}$). This gives (A.1.33) taking into account the extra parameters (ρ,r'), as at the end of the statement of Proposition A.1.1. This concludes the proof. □

We state next the analogous of Corollary A.1.3 when the support of F does not contain any critical point.

Corollary A.1.6 *Let F be a function satisfying (A.1.20). Assume moreover that there is $c > 0$ such that the projection on the (θ,θ') space of the support of F is contained inside the set $\{\min[d(\theta,\theta'),d(\theta,-\theta')] \geq c\}$. Then integral (A.1.21) may be written as*

$$e^{\pm i\lambda\theta\cdot\omega}\mu^{d-1}R(\theta,\omega,\omega',\nu,\mu;\lambda\mu) \tag{A.1.38}$$

where R satisfies for any α,β,β' in $\mathbb{N}^{d-1}$, j,j',γ,N in $\mathbb{N}$

$$|\partial_\theta^\alpha\partial_\omega^\beta\partial_{\omega'}^{\beta'}\partial_\nu^j\partial_\mu^{j'}\partial_\zeta^\gamma R(\theta,\omega,\omega',\nu,\mu;\zeta)| \leq C\mu^{-|\alpha|-|\beta|-j-j'}\langle\zeta\rangle^{-N}. \tag{A.1.39}$$

Proof: One has just to apply Proposition A.1.2 with $x=\theta, y=\theta', z=\omega, z'=\omega'$. □

To finish this subsection, we prove a variant of the well known Hörmander estimate of the $\mathcal{L}(L^2)$ norm of an operator given by a phase integral with non degenerate mixed Hessian.

Proposition A.1.7 *Let $\mathcal{M}$ be riemannian manifold of dimension d, $\Phi : \mathcal{M}\times\mathcal{M} \to \mathbb{R}$ a smooth function. Let $m : \mathcal{M}\times\mathcal{M}\times]0,1] \to \mathbb{C}$ be a function $(x,y,\mu) \to m(x,y,\mu)$, smooth in (x,y), supported in a compact subset $K\times K$ (independent of μ) of $\mathcal{M}\times\mathcal{M}$. Assume that m satisfies for any α,β in $\mathbb{N}^d$, $N \in \mathbb{N}$ bounds of the form*

$$|\partial_x^\alpha\partial_y^\beta m(x,y,\mu)| \leq C_{\alpha,\beta,N}A\mu^{-|\alpha|-|\beta|}\Big\langle\frac{d_{\mathcal{M}}(x,y)}{\mu}\Big\rangle^{-N} \tag{A.1.40}$$

with a constant A independent of α,β,N and $C_{\alpha,\beta,N}$ depending on α,β,N. Assume that at any point (x,y) of $K\times K$, $D_xD_y\Phi(x,y)$ is a non degenerate bilinear form. For u in $C_0^\infty(\mathcal{M})$, supported in K, define if $\lambda\geq 1$

$$T_\lambda u(x)=\int e^{i\lambda\Phi(x,y)}m(x,y,\mu)u(y)\,dy \tag{A.1.41}$$

where dy denotes the riemannian measure. Then there is $C>0$ such that

$$\|T_\lambda\|_{\mathcal{L}(L^2)}\leq CA\lambda^{-\frac{d}{2}} \tag{A.1.42}$$

uniformly in $\mu\in]0,1]$.

Proof: We notice first that if m is supported for $d_{\mathcal{M}}(x,y)\geq c>0$ then (A.1.40) implies a uniform bound $|\partial_x^\alpha\partial_y^\beta m(x,y,\mu)|\leq C_{\alpha,\beta}A$ so that the result follows from the usual theorem (see Hörmander [10] Theorem 1.1 or Sogge [12] Theorem 2.1.1). We may thus assume m supported for $d_{\mathcal{M}}(x,y)\ll 1$. By compactness of K, we may moreover reduce to the case when x,y stay in a same local chart and are thus reduced to an integral of the form (A.1.41), with m supported in a neighborhood of 0 in $\mathbb{R}^d\times\mathbb{R}^d$. We write

$$\|T_\lambda u\|_{L^2}^2=\int K_\lambda(y,z)u(y)\overline{u(z)}\,dydz$$

for any u in $\mathcal{S}(\mathbb{R}^d)$, with

$$K_\lambda(y,z)=\int e^{i\lambda[\Phi(x,y)-\Phi(x,z)]}m(x,y,\mu)\overline{m(x,z,\mu)}\,dx. \tag{A.1.43}$$

It is enough to show that there is $C>0$ such that for any $\mu\in]0,1]$, $\lambda\geq 1$

$$\begin{aligned}\mu^d\sup_y\int|K_\lambda(\mu y,\mu z)|\,dz&\leq CA\lambda^{-d}\\ \mu^d\sup_z\int|K_\lambda(\mu y,\mu z)|\,dy&\leq CA\lambda^{-d}\end{aligned} \tag{A.1.44}$$

by Schur's lemma. But one may write

$$\mu^dK_\lambda(\mu y,\mu z)=\mu^{2d}\int e^{i\lambda[\Phi(\mu x,\mu y)-\Phi(\mu x,\mu z)]}M_\mu(x,y,z)\,dx \tag{A.1.45}$$

where by (A.1.40), M_μ satisfies for any $\alpha\in\mathbb{N}^d$, $N\in\mathbb{N}$,

$$|\partial_x^\alpha M_\mu(x,y,z)|\leq C_\alpha A\langle x-y\rangle^{-N}\langle x-z\rangle^{-N} \tag{A.1.46}$$

and is supported for $\mu|x|\ll 1$, $\mu|y|\ll 1$, $\mu|z|\ll 1$. Moreover, if we set

$$\begin{aligned}\tilde\Phi_\mu(x,y,z)&=\Phi(\mu x,\mu y)-\Phi(\mu x,\mu z)\\&=\mu(y-z)\int_0^1(D_2\Phi)(\mu x,\mu(\alpha y+(1-\alpha)z)\,d\alpha\end{aligned}$$

it follows from the assumption on Φ that

$$|D_x\tilde\Phi_\mu(x,y,z)|\sim\mu^2|y-z|$$

for (x,y,z) in the support of the integrand. Making in (A.1.45) integrations by parts in x, we bound this quantity by

$$CA\mu^{2d}\langle\lambda\mu^2(y-z)\rangle^{-d-1}$$

from which (A.1.44) follows. This concludes the proof. □

A.2 Non boundedness of a sharp half-space cut-off

In the heuristics of subsection 1.2, we indicated that one could not use a sharp cut-off along a half-plane because such an operator is not L^2 bounded. Let us prove this statement. Consider $\chi \in C_0^\infty(\mathbb{R}^d)$ a radial function equal to one close to zero, and let us replace the symbol $a^{\mathrm{HW}}_{\chi,\tilde{\chi},\delta}$ of (1.1.2) by

$$\tilde{a}^{\mathrm{HW}}_{\chi,\delta}(t,x,\xi) = \chi\Big(\Big(x - t\frac{\xi}{|\xi|}\Big)|t|^{-\frac{1}{2}-\delta}\Big)\mathbb{1}_{\frac{x}{t}\cdot\frac{\xi}{|\xi|}>1} \tag{A.2.1}$$

with $\delta \in]0, \frac{1}{2}[$.

Proposition A.2.1 *For any large enough t, the operator $\mathrm{Op}\big(\tilde{a}^{\mathrm{HW}}_{\chi,\delta}\big)$ is not L^2 bounded.*

Since we want to prove non L^2-boundedness, we may as well replace x by tx i.e. consider instead of (A.2.1), for $t \geq 1$

$$\tilde{a}^{\mathrm{HW}}_{\chi,\delta}(t,x,\xi) = \chi\Big(t^{\frac{1}{2}-\delta}\Big(x - \frac{\xi}{|\xi|}\Big)\Big)\mathbb{1}_{x\cdot\frac{\xi}{|\xi|}>1}.$$

From now on, t is fixed, and the constants may depend on t. Moreover, it suffices to prove non L^2-boundedness on the space of L^2 radial functions. If u is radial, and if we write $\hat{u}(\xi) = |\xi|^{\frac{d-1}{2}}v(|\xi|)$ for some function v in $L^2([0,+\infty[, d\rho)$, the operator we have to study is thus $v \to Bv$ where

$$Bv(r) = \int_0^{+\infty} e^{i\rho r}a(t,r,\rho)v(\rho)\,d\rho \tag{A.2.2}$$

with

$$a(t,r,\rho) = \int_{\mathbb{S}^{d-1}} e^{i\rho r(\omega\cdot\theta-1)}(r\rho)^{\frac{d-1}{2}}\mathbb{1}_{r\omega\cdot\theta>1}\chi\big(t^{\frac{1}{2}-\delta}(r\omega-\theta)\big)\,d\theta \tag{A.2.3}$$

which is independent of ω since χ is radial, and where we ignored some multiplicative constant. We have to show that B is not bounded from $L^2([0,+\infty[, d\rho)$ to $L^2([0,+\infty[, dr)$. We shall actually show that B is not bounded on $L^2([1,+\infty[, d\rho)$, so that we may assume that a is also cut-off for $\rho \geq c > 0$. Moreover, we shall use (A.2.3) taking $\omega = e_1$, first vector of the canonical basis.

Lemma A.2.2 *For $r > 1$, r close enough to 1, define $u = \sqrt{1-\frac{1}{r}}$ and set $\tilde{a}(t,u,\rho) = a(t,(1-u^2)^{-1},\rho)$ for $u > 0$ small enough. Then when $\rho \geq c > 0$, one may decompose*

$$\tilde{a}(t,u,\rho) = \tilde{a}_0(t,u,\rho) + \tilde{a}_1(t,u,\rho) + R(t,u,\rho u^2) \tag{A.2.4}$$

where, for large enough t, $\tilde{a}_0$ satisfies for any α, β in $\mathbb{N}$

$$|\partial_u^\alpha\partial_\rho^\beta\tilde{a}_0(t,u,\rho)| \leq C\langle\rho\rangle^{-\beta+\frac{\alpha}{2}}, \tag{A.2.5}$$

$\tilde{a}_1$ may be written as

$$\tilde{a}_1(t,u,\rho) = e^{i\rho(1-r)}S(t,u,\rho u^2) \tag{A.2.6}$$

and where S, R satisfy for any α, β integers, and a non vanishing continuous function $A(u)$

$$\begin{aligned} &|\partial_u^\alpha\partial_\zeta^\beta S(t,u,\zeta)| \leq C\langle\zeta\rangle^{\frac{d-3}{2}-\beta} \\ &S(t,u,\zeta) = A(u)\chi\big(t^{\frac{1}{2}-\delta}(r-1)\big)\zeta^{\frac{d-3}{2}} + O(\zeta^{\frac{d-5}{2}}),\ \zeta \to +\infty \\ &|R(t,u,\zeta)| \leq C\langle\zeta\rangle^{-1}. \end{aligned} \tag{A.2.7}$$

Proof: By (A.2.3), since $\delta < \frac{1}{2}$, one has $|r-1| \ll 1$ on the support of a if t is large enough. Moreover $r > 1$ on that support. Passing in local coordinates for θ close to $\omega = e_1$ in the integral, one may rewrite (A.2.3) as

$$\tag{A.2.8} \tilde{a}(t,u,\rho) = \int_{\mathbb{R}^{d-1}} e^{i\rho(1-u^2)^{-1}\left[\sqrt{1-y'^2}-1\right]} \mathbb{1}_{1-\sqrt{1-y'^2}<u^2} \rho^{\frac{d-1}{2}} M(y',u)\, dy'$$

where M is C^∞ on $\mathbb{R}^{r-1}\times\mathbb{R}$, supported for (y',u) close to $(0,0)$ (and depends on t). We notice that in the regime $\rho\mu^2 \le 1$, (A.2.8) may be written as a contribution to R in (A.2.4). We shall thus assume $\rho\mu^2 \ge 1$ from now on. We split

$$\tag{A.2.9} M(y',u) = M(y',u)\chi_0\Big(\frac{y'}{u}\Big) + M(y',u)\chi_1\Big(\frac{y'}{u}\Big)$$

with $\chi_0 \in C_0^\infty(\mathbb{R}^{d-1})$ equal to one close to zero, with small enough support, and $\chi_1 = 1-\chi_0$. We denote the corresponding decomposition of $\tilde{a}$ as $\tilde{a}_0 + \tilde{a}_1^1$. In $\tilde{a}_0$, if χ_0 has been taken with small enough support, we may forget the sharp cut-off in (A.2.8) and notice then that if we set $\lambda = \rho(1-u^2)^{-1}$, $\mu = u$, then (A.2.8) with a factor $\chi_0(y'/u)$ under the integral is of the form (A.1.4) with y replaced by y', $z = z' = 0$ and no parameter x. By Proposition A.1.1, it follows that $\tilde{a}_0(t,u,\rho)$ may be written as

$$\tag{A.2.10} (\rho u^2)^{\frac{d-1}{2}} S_{-\frac{d-1}{2}}\big(t,u,\rho u^2(1-u^2)^{-1}\big)$$

since $\rho u^2 \ge 1$, for S satisfying

$$\tag{A.2.11} |\partial_u^\alpha \partial_\zeta^\beta S_{-\frac{d-1}{2}}(t,u,\zeta)| \le C u^{-\alpha} \langle\zeta\rangle^{-\frac{d-1}{2}-\beta}.$$

It follows that (A.2.5) holds.

Consider now $\tilde{a}_1^1$, given by the substitution of the last term in (A.2.9) to M in (A.2.8). If we make the change of variables $y' = \sigma u\varphi$, with φ in $\mathbb{S}^{d-2}$, $\sigma > 0$, we obtain

(A.2.12)

$$\tilde{a}_1^1(t,u,\rho) = \int_{]0,+\infty[\times\mathbb{S}^{d-2}} e^{i\rho u^2(1-u^2)^{-1}\Phi(u,\sigma)} \mathbb{1}_{\Phi(u,\sigma)>-1} \rho^{\frac{d-1}{2}} M(\sigma u\varphi,u)\chi_1(\sigma\varphi) u^{d-1}\sigma^{d-2}\, d\sigma d\varphi$$

where

$$\Phi(u,\sigma) = u^{-2}\big[\sqrt{1-\sigma^2u^2}-1\big]$$

is smooth and has $\sigma = 0$ as its unique critical point in σ. Moreover, in the integrand the factor $M(\sigma u\varphi,u)\chi_1(\sigma\varphi)u^{d-1}\sigma^{d-2}$ is smooth, supported for $|u| \ll 1$, $\sigma|u| \ll 1$ and $\sigma \ge c_1 > 0$, and each ∂_σ derivative of it gains σ^{-1}. In addition, $\frac{\partial\Phi}{\partial\sigma}(u,\sigma) \le -c\sigma \le -cc_1 < 0$ on the support of the integrand. Since $\sigma|u| \ll 1$, the sharp cut-off $\mathbb{1}_{\Phi(u,\sigma)>-1}$ implies that the integrand is supported for σ bounded. We may thus insert under the integral in (A.2.12) a cut-off for $\sigma \le C_1$. If we make a further change of variables $v = -\Phi(u,\sigma)$ for any fixed u, we thus reduce (A.2.12) to

$$\tag{A.2.13} \int_{]0,+\infty[\times\mathbb{S}^{d-2}} e^{i\rho u^2(1-u^2)^{-1}v} \mathbb{1}_{v<1} \rho^{\frac{d-1}{2}} N(v,\varphi,u) u^{d-1}\, dv d\varphi$$

where N is smooth and supported for v in a fixed compact subset of $]0,+\infty[$ and for $|u| \ll 1$. As $\rho u^2 \ge 1$, if we make integrations by parts in v in (A.2.13), each of them gains a factor $(\rho u^2)^{-1}$. Moreover, the boundary terms generated in the process are of the form

$$\tag{A.2.14} e^{i\rho u^2(1-u^2)^{-1}} (\rho u^2)^{\frac{d-1}{2}-\ell-1} N_\ell(u),\ \ \ell \in \mathbb{N}$$

where N_ℓ is in $C_0^\infty([0,+\infty[)$ and N_0 is given by

$$N_0(u) = -i(1-u^2)\int N(1,\varphi,u)\,d\varphi. \tag{A.2.15}$$

Notice that $v=1$ corresponds in (A.2.12) to $\Phi(u,\sigma)=-1$ i.e. the boundary term (A.2.15) is equal, up to a non zero factor coming from the change of variables, to the function $\chi\big(t^{\frac12-\delta}(r\omega-\theta)\big)$ restricted to $r\omega\cdot\theta=1$ and integrated in θ on this hypersurface of $\mathbb{S}^{d-1}$. Since χ is radial, $\chi\big(t^{\frac12-\delta}(r\omega-\theta)\big)$ restricted to $r\omega\cdot\theta=1$ is just $\chi\big(t^{\frac12-\delta}(r-1)\big)$ and N_0 is a non zero multiple of that function. Moreover, the phase in (A.2.14) is the one in (A.2.3) restricted to $\omega\cdot\theta=\frac1r$ i.e. $\rho(1-r)$. Consequently, we have written $\tilde a_1^1$ as

$$\tilde a_1^1 = e^{i\rho(r-1)}S(u,\rho u^2) + R(u,\rho u^2) \tag{A.2.16}$$

where the first term comes from (A.2.14) with $\ell=0,\dots,L$ with L chosen so that $\frac{d-1}{2}-L\le 0$, and R is provided by an expression of the form (A.2.13) where one has performed at least $L+1$ ∂_v-integrations by parts, that have gained a factor $(\rho u^2)^{-L-1}$. It follows that R satisfies the last estimate (A.2.7), while S obeys the first two lines of (A.2.7). This concludes the proof of the lemma. $\square$

Proof of Proposition A.2.1: We have to show that operator B given by (A.2.2) is not bounded from $L^2([1,+\infty[,d\rho)$ to $L^2([1,+\infty[,dr)$ (since a is supported for $r\ge1$). It is enough to show that there is $\alpha>0$ small such that $\mathbb{1}_{[1+\alpha,1+2\alpha]}\circ B$ is not L^2 bounded. This operator may be written as a sum $B_1+B_2+B_3$ with

$$B_1v(r) = \mathbb{1}_{[1+\alpha,1+2\alpha]}(r)\int_1^{+\infty} e^{i\rho r}\tilde a_0\big(t,\sqrt{1-r^{-1}},\rho\big)v(\rho)\,d\rho \tag{A.2.17}$$

$$B_2v(r) = \mathbb{1}_{[1+\alpha,1+2\alpha]}(r)\int_1^{+\infty} S\Big(t,\sqrt{1-r^{-1}},\rho\Big(1-\frac1r\Big)\Big)e^{i\rho}v(\rho)\,d\rho \tag{A.2.18}$$

$$B_3v(r) = \mathbb{1}_{[1+\alpha,1+2\alpha]}(r)\int_1^{+\infty} e^{i\rho r}R\Big(t,\sqrt{1-r^{-1}},\rho\Big(1-\frac1r\Big)\Big)v(\rho)\,d\rho \tag{A.2.19}$$

according to (A.2.2), (A.2.4), (A.2.6). On the domain $r\ge1+\alpha>1$, inequalities (A.2.5) show that $\tilde a_0$ is a symbol in the Hörmander class $S^0_{1,\frac12}$, so that B_1 is bounded on L^2. As r stays in a compact set, it follows from the last estimate (A.2.7) that $\|B_3v\|_{L^2(dr)}\le C\|v\|_{L^2(d\rho)}$. It remains to prove that if $\alpha>0$ has been fixed small enough, B_2 is not bounded on L^2. We may replace $e^{i\rho}v(\rho)$ by $v(\rho)$ in order to check that. Using the expansion of S in (A.2.7), we may write

$$\begin{aligned} B_2v(r) = \mathbb{1}_{[1+\alpha,1+2\alpha]}(r)\Big[A\Big(\sqrt{1-\frac1r}\Big)\chi\big(t^{\frac12-\delta}(r-1)\big)\Big(1-\frac1r\Big)^{\frac{d-3}{2}}\int_1^{+\infty}\rho^{\frac{d-3}{2}}v(\rho)\,d\rho\\ +O\Big(\int_1^{+\infty}\rho^{\frac{d-5}{2}}|v(\rho)|\,d\rho\Big)\Big].\end{aligned} \tag{A.2.20}$$

When $d>3$, take in (A.2.20), $v=v_k=\mathbb{1}_{[k,k+1]}(\rho)$, $k\in\mathbb{N}^*$. Then for α close enough to 0, positive, (A.2.20) shows that $\|B_2v_k\|_{L^2(dr)}$ goes to infinity if k goes to infinity while $\|v_k\|_{L^2(d\rho)}=1$. This is the wanted conclusion in that case. When $2\le d\le3$, the last term in (A.2.20) has $L^2\big(\mathbb{1}_{[1+\alpha,1+2\alpha]}(r)dr\big)$ norm obviously bounded by $C\|v\|_{L^2}$, so that it is enough to check that

$$v\to\mathbb{1}_{[1+\alpha,1+2\alpha]}(r)\int_1^{+\infty}\rho^{\frac{d-3}{2}}v(\rho)\,d\rho$$

is not L^2 bounded, which is obvious. This concludes the proof. $\square$

References

[1] R. Côte, C. E. Kenig, A. Lawrie, and W. Schlag. Characterization of large energy solutions of the equivariant wave map problem. I. *Am. J. Math.*, 137(1):139–207, 2015.

[2] R. Côte, C. E. Kenig, A. Lawrie, and W. Schlag. Characterization of large energy solutions of the equivariant wave map problem. II. *Am. J. Math.*, 137(1):209–250, 2015.

[3] R. Côte, C. E. Kenig, A. Lawrie, and W. Schlag. Profiles for the radial focusing $4d$ energy-critical wave equation. *Commun. Math. Phys.*, 357(3):943–1008, 2018.

[4] R. Côte, C. E. Kenig, and W. Schlag. Energy partition for the linear radial wave equation. *Math. Ann.*, 358(3-4):573–607, 2014.

[5] R. Côte and C. Laurent. Concentration close to the cone for linear waves. Preprint, `hal-03347492`, Sept. 2021.

[6] M. Dimassi and J. Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268. Cambridge: Cambridge University Press, 1999.

[7] T. Duyckaerts, C. Kenig, and F. Merle. Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation. *J. Eur. Math. Soc. (JEMS)*, 13(3):533–599, 2011.

[8] T. Duyckaerts, C. Kenig, and F. Merle. Universality of the blow-up profile for small type II blow-up solutions of the energy-critical wave equation: the nonradial case. *J. Eur. Math. Soc. (JEMS)*, 14(5):1389–1454, 2012.

[9] T. Duyckaerts, C. Kenig, and F. Merle. Decay estimates for nonradiative solutions of the energy-critical focusing wave equation. *J. Geom. Anal.*, 31(7):7036–7074, 2021.

[10] L. Hörmander. Oscillatory integrals and multipliers on FL^p. *Ark. Mat.*, 11:1–11, 1973.

[11] N. Lerner. Integrating the Wigner distribution on subsets of the phase space, A survey. Preprint, `hal-03145518`, Feb. 2021.

[12] C. D. Sogge. *Fourier integrals in classical analysis. 2nd edition*, volume 210. Cambridge: Cambridge University Press, 2nd edition edition, 2017.

[13] M. Zworski. *Semiclassical analysis*, volume 138. Providence, RI: American Mathematical Society (AMS), 2012.