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ELLIOTT-HALBERSTAM CONJECTURE AND VALUES TAKEN BY THE LARGEST PRIME FACTOR OF SHIFTED PRIMES

JIE WU

ABSTRACT. Denote by $\mathbb{P}$ the set of all primes and by $P^+(n)$ the largest prime factor of integer $n \geq 1$ with the convention $P^+(1) = 1$. For each $\eta > 1$, let $c = c(\eta) > 1$ be some constant depending on η and

$$\mathcal{P}_{a,c,\eta} := \{p \in \mathbb{P} : p = P^+(q - a) \text{ for some prime } q \text{ with } p^\eta < q \leq c(\eta)p^\eta\}.$$

In this paper, under the Elliott-Halberstam conjecture we prove, for $y \rightarrow \infty$,

$$\pi_{a,c,\eta}(x) := |(1, x] \cap \mathcal{P}_{a,c,\eta}| \sim \pi(x) \quad \text{or} \quad \pi_{a,c,\eta}(x) \gg_{a,\eta} \pi(x)$$

according to values of η . These complement for some results of Banks-Shparlinski [1], of Wu [12] and of Chen-Wu [2].

1. INTRODUCTION

Denote by $\mathbb{P}$ the set of all prime numbers and by $P^+(n)$ the largest prime factor of the positive integer $n \geq 1$ with the convention $P^+(1) = 1$. Banks & Shparlinski [1] proposed to estimate the number of primes p that occur as the largest prime factor of a shifted prime $q - a$ when $q \in \mathbb{P}$ lies in a certain interval determined by p . This question has applications in theoretical computer science and has been considered by Vishnoi [10].

Let $\mathbb{Z}^*$ be the set of non-zero integers. For $a \in \mathbb{Z}^*$, $c > 1$ and $\eta > 0$, we put

$$\mathcal{P}_{a,c,\eta} := \{r \in \mathbb{P} : r = P^+(q - a) \text{ for some prime } q \text{ with } r^\eta < q \leq cr^\eta\}$$

and

$$\pi_{a,c,\eta}(y) := |\{r \leq y : r \in \mathcal{P}_{a,c,\eta}\}|, \quad \pi(y) := |\{r \leq y : r \in \mathbb{P}\}|.$$

Banks & Shparlinski [1, Theorem 1.1] proved that for each $\eta \in (\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})$, there exists a constant $c = c(\eta) > 1$ such that the asymptotic formula

$$(1.1) \quad \pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta}\left(\frac{y}{(\log y)^A}\right) \quad (y \rightarrow \infty)$$

holds for every fixed non-zero integer $a \in \mathbb{Z}^*$ and any constant $A > 1$. Moreover for $2 \leq \eta < 1 + \frac{3}{4}\sqrt{2} \approx 2.0606$, this estimate holds for any constant $c > 1$. Very recently, Wu [12] extended Banks-Shparlinski's interval $(\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})$ to $(\frac{32}{17}, \eta_0)$, where $\eta_0 \approx 2.142$ is the unique solution of the equation $\eta - 1 - 4\eta \log(\eta - 1) = 0$ in $(1, \infty)$. Banks & Shparlinski [1, page 144] also remarked that the asymptotic formula (1.1) holds for $\eta \in (1, \frac{32}{17}]$ if we assume the Elliott-Halberstam conjecture (see $\text{EH}_{\text{prime}}[\varepsilon]$ below). Subsequently, Chen & Wu

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[2] further extended the domain of η at the price proportion positive instead of density 1. More precisely, they proved that

$$(1.2) \quad \pi_{a,c,\eta}(y) \geq (\log \sqrt[4]{2}) \frac{\eta - 1}{\eta} \left(1 - 4 \log(\eta - 1) - \delta \frac{\log(\eta - 1)^8}{c - 1} \right) \pi(y),$$

where $\delta = \delta(c, \eta)$ is sufficiently small positive number. Clearly, (1.2) implies

$$\pi_{a,c,\eta}(y) \gg \pi(y)$$

provided $\eta < 1 + \sqrt[4]{e}$. This is complement for the results of Banks-Shparlinski and of Wu mentioned above. It seems rather natural to pose the following question.

Question 1. *Is the asymptotic formula (1.1) true for all $\eta > 1$?*

In this paper, we shall try to answer this question under the well-known Elliott-Halberstam conjecture. Firstly we state two versions of this conjecture for prime numbers.

Conjecture 1 (Elliott-Halberstam). *Let $a \in \mathbb{Z}^*$ and $\varepsilon \in (0, 1)$ be fixed constants.*

(i) *For any $A > 0$, the inequality*

$$(EH_{\text{prime}}[\varepsilon]) \quad \sum_{\substack{q \leq x^{1-\varepsilon} \\ (a,q)=1}} \left| \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} 1 - \frac{\pi(x)}{\varphi(q)} \right| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A}$$

holds uniformly for all $x \geq 3$, where the letter p always denotes prime numbers, $\varphi(q)$ is the Euler function and the implied constant depends on A, a and ε .

(ii) *Let $\kappa_1(m)$ and $\kappa_2(m)$ be the characteristic functions of the odd integers and of even integers, respectively. Then for any $A > 0$, we have*

$$(EH_{\text{prime}}^*[\varepsilon]) \quad \sum_{\substack{q \leq x^{1-\varepsilon} \\ (a,q)=1}} \left| \sum_{\substack{mp \leq x \\ mp \equiv a \pmod{q}}} \kappa_i(m) - \frac{1}{\varphi(q)} \sum_{\substack{mp \leq x \\ (mp,q)=1}} \kappa_i(m) \right| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A}$$

uniformly for all $x \geq 3$, where the letter p always denotes prime numbers and the implied constant depends on A and a .

Remark 1. According to the classical Bombieri-Vinogradov theorem and Proposition 2.2 of Wu [12], the Elliott-Halberstam conjectures $EH_{\text{prime}}[\varepsilon]$ and $EH_{\text{prime}}^*[\varepsilon]$ hold for all $\varepsilon \in (\frac{1}{2}, 1)$.

Secondly we also need a version of this conjecture for friable numbers.

Conjecture 2 (Elliott-Halberstam). *Let $a \in \mathbb{Z}^*$ and $\varepsilon \in (0, 1)$ be fixed constants. For any $A > 0$, we have*

$$(EH_{\text{friable}}[\varepsilon]) \quad \sum_{\substack{q \leq x^{1-\varepsilon} \\ (a,q)=1}} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}, P^+(n) \leq y}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n,q)=1, P^+(n) \leq y}} 1 \right| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A}$$

uniformly in $x \geq y \geq 2$.

Remark 2. According to Wolke's work [11] (see also [4, Theorem 6]), $EH_{\text{friable}}[\varepsilon]$ holds unconditionally for all $\varepsilon \in (\frac{1}{2}, 1)$.

Our results are as follows.

Theorem 1. *Let $a \in \mathbb{Z}^*$ and $c > 1$ be fixed constants.*

(i) *Let $\eta \in (1, \frac{32}{17}]$ and assume the Elliott-Halberstam conjecture $\text{EH}_{\text{prime}}[\varepsilon]$ with $\varepsilon = 1 - 1/\eta > 0$. Then for any $A > 1$ we have*

$$(1.3) \quad \pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta} \left(\frac{y}{(\log y)^A} \right),$$

as $y \rightarrow \infty$.

(ii) *Let $\eta_1 \approx 2.3303$ be the unique positive zero of the equation $\eta - 1 - 2\eta \log(\eta - 1) = 0$ in $(1, \infty)$. For each $\eta \in [\eta_0, \eta_1)$, there is a sufficiently small positive number $\varepsilon = \varepsilon(\eta)$ such that assuming the Elliott-Halberstam conjecture $\text{EH}_{\text{prime}}[\varepsilon]$, for any $A > 1$ we have*

$$(1.4) \quad \pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta} \left(\frac{y}{(\log y)^A} \right),$$

as $y \rightarrow \infty$.

Theorem 2. *Let $a \in \mathbb{Z}^*$ and $c > 1$ be fixed constants. For every $\eta \in [2, \infty)$, there is a sufficiently small positive number $\varepsilon = \varepsilon(\eta)$ such that under the Elliott-Halberstam conjecture $\text{EH}_{\text{friable}}[\varepsilon]$, we have*

$$(1.5) \quad \pi_{a,c,\eta}(y) \geq (\log \sqrt[4]{2}) \frac{\eta - 1}{\eta} \pi(y) \left\{ 1 + O_{a,c,\eta} \left(\frac{1}{\sqrt[3]{\log y}} + \varepsilon \right) \right\},$$

as $y \rightarrow \infty$.

Remark 3. (a) For comparaisn, we have

$$1 + \frac{3}{4}\sqrt{2} \approx 2.060, \quad \eta_0 \approx 2.142, \quad \eta_1 \approx 2.330, \quad 1 + \sqrt[4]{e} \approx 2.284, \quad 1 + \sqrt{e} \approx 2.648.$$

(b) The first assertion of Theorem 1 is due to Banks & Shparlinski [1, page 144]. Here we give a proof for convenience of the reader. By combining Theorem 1 with results of [12], we see that the asymptotic formula (1.1) holds for $1 < \eta < \eta_1$ under the Elliott-Halberstam conjecture.

(c) Theorem 2 improves significantly Chen-Wu's result (1.2) in two aspects : result and method. Firstly the proportion in (1.5) increases from $\frac{\log 2}{8}$ to $\frac{\log 2}{4}$ when η runs over $[2, \infty)$, and that in (1.2) tends toward 0 as $\eta \rightarrow 1 + \sqrt[4]{e}$. Secondly our proof of (1.5) needs information of prime numbers in “arithmetical progressions” $\{a + mq\}_m$ friable with friable indice. For $(a, q) = 1$ and $x \geq y \geq 2$, define the counting function

$$(1.6) \quad \pi(x, y; q, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{q} \\ P^+((p-a)/q) \leq y}} 1.$$

A systematic study on the asymptotical behaviour has been done by Liu, Wu & Xi [8], recently. We need a theorem of Bombieri-Vinogradov type and an inequality of Brun-Titchmarsh for this new counting function (see Lemmas 2.2–2.3 below).

2. SOME PRELIMINARY LEMMAS

In this section, we present three lemmas, which will be useful later.

2.1. The Rosser-Iwaniec linear sieve.

The first lemma is due to Iwaniec [6, 7].

Lemma 2.1. *Let $D \geq 2$ and let $\mu(n)$ be the Möbius function. Then there are two sequences $\{\lambda_d^\pm\}_{d \geq 1}$, vanishing for $d > D$ or $\mu(d) = 0$, satisfying $|\lambda_d^\pm| \leq 1$, such that*

$$(2.1) \quad \sum_{d|n} \lambda_d^- \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda_d^+ \quad (n \geq 1)$$

and

$$(2.2) \quad \sum_{d|P_{\mathcal{P}}(z)} \lambda_d^+ \frac{w(d)}{d} \leq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right) \left\{ F(s) + O\left(\frac{e^{\sqrt{L}-s}}{\sqrt[3]{\log D}}\right) \right\}$$

$$(2.3) \quad \sum_{d|P_{\mathcal{P}}(z)} \lambda_d^- \frac{w(d)}{d} \geq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right) \left\{ f(s) + O\left(\frac{e^{\sqrt{L}-s}}{\sqrt[3]{\log D}}\right) \right\}$$

for any $z \in [2, D]$, $s = (\log D)/\log z$, set of prime numbers $\mathcal{P}$ and multiplicative function w satisfying

$$(2.4) \quad 0 < w(p) < p \quad (p \in \mathcal{P}),$$

$$(2.5) \quad \prod_{u < p \leq v, p \in \mathcal{P}} \left(1 - \frac{w(p)}{p}\right)^{-1} \leq \frac{\log v}{\log u} \left(1 + \frac{L}{\log u}\right) \quad (2 \leq u \leq v),$$

where $P_{\mathcal{P}}(z) := \prod_{p \leq z, p \in \mathcal{P}} p$ and the implied O -constants are absolute. Here F, f are defined by the continuous solutions to the system

$$\begin{cases} sF(s) = 2e^\gamma & (1 \leq s \leq 2) \\ sf(s) = 0 & (0 < s \leq 2) \\ (sF(s))' = f(s-1) & (s > 2) \\ (sf(s))' = F(s-1) & (s > 2) \end{cases}$$

where γ is the Euler constant.

2.2. Bombieri-Vinogradov theorem of for $\pi(x, y; q, a)$.

The second lemma is a theorem of Bombieri-Vinogradov type for the counting function $\pi(x, y; q, a)$ defined as in (1.6) (see [8, Theorem 2]).

Lemma 2.2. *Let $a \in \mathbb{Z}^*$, $A > 0$ and κ a non-negative arithmetic function. Assuming the Elliott-Halberstam conjecture $\text{EH}_{\text{prime}}[\varepsilon]$, the following estimate*

$$\sum_{\substack{q \leq Q \\ (q, a) = 1}} \kappa(q) \left| \pi(x, y; q, a) - \frac{\pi(x)}{\varphi(q)} \rho\left(\frac{\log(x/q)}{\log y}\right) \right| \ll_{a, A} \frac{x}{(\log x)^A} \sqrt{\sum_{q \leq x} \frac{\kappa(q)^2}{q}} + \pi(x) \varepsilon u \sum_{q \leq Q} \frac{\kappa(q)}{\varphi(q)}$$

holds uniformly in $x \geq 2$, $\exp\{(\log x)^{2/5+\varepsilon}\} \leq y \leq x$ and $Q \leq \min\{y, \sqrt{x}\}$, where $\rho(u)$ is the Dickman function.

2.3. Brun-Titchmarsh inequality for $\pi(x, y; q, a)$.

The lemma below is a variant of [8, Theorem 1].

Lemma 2.3. *Let $a \in \mathbb{Z}^*$ and $c > 1$ be fixed constants. For any $\varepsilon > 0$, we have*

$$(2.6) \quad \begin{aligned} \pi(cx, y; q, a) - \pi(x, y; q, a) &\leq \frac{4(c-1)x}{\varphi(q) \log(x/q)} \rho\left(\frac{\log(x/q)}{\log y}\right) \left\{ 1 + O_{a,c,\varepsilon}\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \\ &\quad + O_{A,a,c}\left(\frac{x}{q(\log x)^A}\right) \end{aligned}$$

uniformly in $\exp\{\log \log x\}^{5/3+\varepsilon} \leq y \leq x$ and $1 \leq q \leq \min\{y, \sqrt{x}\}$ with $(a, q) = 1$.

Proof. Denote by $\mathcal{S}$ the quantity on the left-hand side of (2.6). Without loss of generality, we can assume q is even and a is odd. Put $P_{2a}(z) := \prod_{p \leq z, p \nmid 2a} p$. By the Möbius inversion, we can write

$$\begin{aligned} \mathcal{S} &= \sum_{\substack{x < a+mq \leq cx \\ (mq, a) = (a+mq, P_{2a}(\sqrt{cx})) = 1 \\ P^+(m) \leq y}} 1 + O(x^\varepsilon) \\ &= \sum_{\substack{x < a+mq \leq cx \\ (a, mq) = 1, P^+(m) \leq y}} \sum_{d | (a+mq, P_{2a}(\sqrt{cx}))} \mu(d) + O(x^\varepsilon), \end{aligned}$$

Using Lemma 2.1 and switching summations, it follows that

$$\begin{aligned} \mathcal{S} &\leq \sum_{\substack{x < a+mq \leq cx \\ (a, mq) = 1, P^+(m) \leq y}} \sum_{d | (a+mq, P_{2a}(\sqrt{cx}))} \lambda_d^+ + O(x^\varepsilon) \\ &= \sum_{\substack{d \leq D \\ d | P_{2aq}(\sqrt{cx})}} \lambda_d^+ \sum_{\substack{(x-a)/q < m \leq (cx-a)/q \\ m \equiv -a\bar{q} \pmod{d} \\ (a, m) = 1, P^+(m) \leq y}} 1 + O(x^\varepsilon), \end{aligned}$$

where $\{\lambda_d^+\}_{d \geq 1}$ is an upper bound sieve of level D as in Lemma 2.1 and $\bar{q}$ is the inverse of q modulo d (i.e. $q\bar{q} \equiv 1 \pmod{d}$). To apply the Elliott-Halberstam conjecture $\text{EH}_{\text{friable}}[\varepsilon]$, we would like to remove the restriction $(a, m) = 1$ by Möbius inversion, so that

$$\begin{aligned} \mathcal{S} &\leq \sum_{\ell | a} \mu(\ell) \sum_{\substack{d \leq D \\ d | P_{2aq}(\sqrt{cx})}} \lambda_d^+ \sum_{\substack{(x-a)/q < m \leq (cx-a)/q \\ m \equiv -a\bar{q} \pmod{d} \\ \ell | m, P^+(m) \leq y}} 1 + O(x^\varepsilon) \\ &= \sum_{\substack{\ell | a \\ P^+(\ell) \leq y}} \mu(\ell) \sum_{\substack{d \leq D \\ d | P_{2aq}(\sqrt{cx})}} \lambda_d^+ \sum_{\substack{(x-a)/\ell q < m \leq (cx-a)/\ell q \\ m \equiv -a\bar{\ell}q \pmod{d} \\ P^+(m) \leq y}} 1 + O(x^\varepsilon). \end{aligned}$$

We are now in a good position to employ the Elliott-Halberstam conjecture $\text{EH}_{\text{friable}}[\varepsilon]$ with $D = (x/q)^{1-\varepsilon}$, getting

$$(2.7) \quad \mathcal{S} \leq \mathcal{S}^+ + O\left(\frac{x}{q(\log x)^A}\right),$$

where

$$\mathcal{S}^+ := \sum_{\ell|a} \mu(\ell) \sum_{\substack{d \leq D \\ d|P_{2aq}(\sqrt{cx})}} \frac{\lambda_d^+}{\varphi(d)} \sum_{\substack{x/\ell q < m \leq cx/\ell q \\ (d,m)=1, P^+(m) \leq y}} 1.$$

and we have used the trivial bound

$$\sum_{\ell|a} \sum_{\substack{d \leq D \\ d|P_{2aq}(\sqrt{cx})}} \frac{|\lambda_d^+|}{\varphi(d)} \sum_{\substack{cx/\ell q < m \leq (cx-a)/\ell q \\ (d,m)=1, P^+(m) \leq y}} 1 \ll_a \log x.$$

Here we removed the restriction that $P^+(\ell) \leq y$ since we henceforth assume $y > |a|$.

Switching summations, it follows that

$$\begin{aligned} \mathcal{S}^+ &= \sum_{\ell|a} \mu(\ell) \sum_{\substack{x/\ell q < m \leq cx/\ell q \\ P^+(m) \leq y}} \sum_{\substack{d \leq D \\ d|P_{2amq}(\sqrt{cx})}} \frac{\lambda_d^+}{\varphi(d)} \\ &= \sum_{\substack{x/q < m \leq cx/q \\ (a,m)=1, P^+(m) \leq y}} \sum_{\substack{d \leq D \\ d|P_{2amq}(\sqrt{cx})}} \frac{\lambda_d^+}{\varphi(d)}. \end{aligned}$$

From Lemma 2.1, we deduce

$$\begin{aligned} \mathcal{S}^+ &\leq \frac{\log \sqrt{x}}{\log(x/q)} \left\{ 2e^\gamma + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \sum_{\substack{x/q < m \leq cx/q \\ (a,m)=1, P^+(m) \leq y}} \prod_{\substack{p < \sqrt{x} \\ p \nmid 2amq}} \left(1 - \frac{1}{p-1}\right) \\ (2.8) \quad &\leq \frac{\log \sqrt{x}}{\log(x/q)} \left\{ 2e^\gamma + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \prod_{\substack{p < \sqrt{x} \\ p \nmid 2aq}} \left(1 - \frac{1}{p-1}\right) \sum_{\substack{x/q < m \leq cx/q \\ (a,m)=1, P^+(m) \leq y}} H(m), \end{aligned}$$

where $H(m)$ is the multiplicative function, defined by

$$H(p^\nu) = \begin{cases} 1 & \text{if } p \mid 2q \text{ or } p > x^{1/2} \\ \frac{p-1}{p-2} & \text{if } p \nmid 2q \text{ and } p \leq x^{1/2} \end{cases}$$

for all $\nu \geq 1$. According to [8, (3.5)], we have

$$\sum_{\substack{m \in S(x/q, y) \\ (a,m)=1}} H(m) = \Psi\left(\frac{x}{q}, y\right) \frac{\varphi(a)}{a} \prod_{\substack{p < \sqrt{x} \\ p \nmid 2aq}} \left(1 + \frac{1}{p(p-2)}\right) \left\{ 1 + O\left(\frac{(\log \log x)^2}{\log y}\right) \right\},$$

where $S(x, y) := \{n \leq x : P^+(n) \leq y\}$ and $\Psi(x, y) := |S(x, y)|$. Combining this with (2.8), we find that

$$(2.9) \quad \mathcal{S}^+ \leq \frac{\log x}{\log(x/q)} \left\{ 2e^\gamma + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \frac{\varphi(a)}{a} \left\{ \Psi\left(\frac{cx}{q}, y\right) - \Psi\left(\frac{x}{q}, y\right) \right\} \prod_{\substack{p \leq \sqrt{x} \\ p \nmid aq}} \left(1 - \frac{1}{p}\right).$$

By the Mertens formula, it follows that

$$\prod_{\substack{p \leq x^{1/2} \\ p \nmid aq}} \left(1 - \frac{1}{p}\right) = \frac{aq}{\varphi(a)\varphi(q)} \cdot \frac{2e^{-\gamma}}{\log x} \left\{1 + O\left(\frac{1}{\log x}\right)\right\}.$$

On the other hand, according to [5, Theorem 1], we have

$$\Psi(x, y) = x\rho\left(\frac{\log x}{\log y}\right) \left\{1 + O\left(\frac{\log((\log x)/\log y + 1)}{\log y}\right)\right\}$$

uniformly for $x \geq 3$ and $\exp\{(\log \log x)^{5/3+\varepsilon}\} \leq y \leq x$. Combining these with (2.9) and (2.7), we can get the required inequality (2.6). $\square$

3. PROOF OF THEOREM 2

For each prime $r \in (\frac{1}{2}y, y]$, consider

$$(3.1) \quad \mathcal{Q}_r(y) := \sum_{\substack{x < q \leq cx \\ P^+(q-a)=r}} 1.$$

Noticing that

$$P^+(q-a) = r \Leftrightarrow q \equiv a \pmod{r} \text{ and } P^+(q-a) \leq r,$$

we can write

$$(3.2) \quad \begin{aligned} \sum_{y < r \leq 2y} \mathcal{Q}_r(y) &\geq \sum_{y < r \leq 2y} \sum_{\substack{x < q \leq cx \\ q \equiv a \pmod{r}, P^+(q-a) \leq y}} 1 \\ &= \sum_{y < r \leq 2y} (\pi(cx, y; r, a) - \pi(x, y; r, a)) \\ &= \mathcal{M} + \mathcal{E}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{M} &:= \sum_{y < r \leq 2y} \left(\frac{\pi(cx)}{\varphi(r)} \rho\left(\frac{\log(cx/r)}{\log y}\right) - \frac{\pi(x)}{\varphi(r)} \rho\left(\frac{\log(x/r)}{\log y}\right) \right), \\ \mathcal{E} &:= \sum_{y < r \leq 2y} (\mathcal{E}(cx, y; r, a) - \mathcal{E}(x, y; r, a)), \end{aligned}$$

and

$$\mathcal{E}(x, y; r, a) := \pi(x, y; r, a) - \frac{\pi(x)}{\varphi(r)} \rho\left(\frac{\log(x/r)}{\log y}\right).$$

Since $\eta \geq 2$, we have $y = x^{1/\eta} \leq x^{1/2}$ and $Q = \min(y, \sqrt{x}) = y$. Using Lemma 2.2 with the characteristic function of prime numbers in $(y, 2y]$ in place of $\kappa(q)$, we easily derive, under the conjecture of Elliott-Halberstam $\text{EH}_{\text{prime}}[\varepsilon]$, that

$$(3.3) \quad \begin{aligned} |\mathcal{E}| &\leq \sum_{y < r \leq 2y} (|\mathcal{E}(cx, y; r, a)| + |\mathcal{E}(x, y; r, a)|) \\ &\ll_a \frac{x}{(\log x)^3} + \varepsilon \eta \frac{\pi(x)}{\log y} \ll_{a, \eta} \varepsilon \frac{\pi(x)}{\log y} \end{aligned}$$

for all $x \geq x_0(\varepsilon)$, where we have used the following bound

$$(3.4) \quad \sum_{y < r \leq 2y} \frac{1}{\varphi(r)} = \sum_{y < r \leq 2y} \frac{1}{r} \left\{ 1 + O\left(\frac{1}{y}\right) \right\} = \frac{\log 2}{\log y} \left\{ 1 + O\left(\frac{1}{\log y}\right) \right\}$$

and the implied constant depends on a, η at most.

According to [9, Corollary III.5.8.3], we have $|\rho'(u)| \ll \rho(u) \log u$ ($u > 1$). Thus for all $r \in (y, 2y]$, we have

$$\rho\left(\frac{\log(x/r)}{\log y}\right) = \rho(\eta - 1) \left\{ 1 + O_\eta\left(\frac{1}{\log y}\right) \right\}.$$

From this and (3.4), we derive

$$(3.5) \quad \mathcal{M} = (\log 2)(c - 1)\rho(\eta - 1) \frac{\pi(x)}{\log y} \left\{ 1 + O_\eta\left(\frac{1}{\log y}\right) \right\}.$$

Inserting (3.5) and (3.3) into (3.2), it follows that

$$(3.6) \quad \sum_{y < r \leq 2y} \mathcal{Q}_r(y) \geq (\log 2)(c - 1)\rho(\eta - 1) \frac{\pi(x)}{\log y} \left\{ 1 + O_{a,c,\eta}\left(\frac{1}{\log y} + \varepsilon\right) \right\}.$$

On the other hand, the Brun-Titchmarsh inequality (2.6) give us

$$\begin{aligned} \mathcal{Q}_r(y) &\leq \frac{4(c - 1)x}{\varphi(r) \log(x/r)} \rho(\eta - 1) \left\{ 1 + O_{a,\eta}\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \\ &\leq 4(c - 1) \frac{\rho(\eta - 1)}{\eta - 1} \cdot \frac{x}{y \log y} \left\{ 1 + O_{a,\eta}\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \end{aligned}$$

for all primes $r \in (y, 2y]$. This implies that

$$(3.7) \quad \sum_{y < r \leq 2y} \mathcal{Q}_r(y) \leq \rho(\eta - 1) \frac{4(c - 1)}{(\eta - 1)} \cdot \frac{x}{y \log y} \left\{ 1 + O_{a,\eta}\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \sum_{\substack{y < r \leq 2y \\ \mathcal{Q}_r(y) \neq 0}} 1.$$

Combining (3.6) and (3.7), it follows that

$$\sum_{\substack{y < r \leq 2y \\ \mathcal{Q}_r(y) \neq 0}} 1 \geq (\log \sqrt[4]{2}) \frac{\eta - 1}{\eta} \pi(y) \left\{ 1 + O_{a,c,\eta}\left(\frac{1}{\sqrt[3]{\log x}} + \varepsilon\right) \right\}.$$

This completes the proof of Theorem 2.

4. PROOF OF THEOREM 1

As in [12], the letters p, q, r and ℓ are always used to denote prime numbers, and d, m , and n always denote positive integers. In what follows, let $a \in \mathbb{Z}^*$ and $\eta \in (1, \frac{32}{17}] \cup [\eta_0, \eta_1)$. Let δ be a sufficiently small positive constant and let $c > 1$ be a parameter to be chosen later. Let $x_0(A, a, c, \eta, \delta)$ be a large constant depending on A, a, c, η, δ at most. For $x \geq x_0(A, a, c, \eta, \delta)$ and $r \in (\frac{1}{2}y, y]$, put $x := r^\eta$. As usual, for $(a, d) = 1$ define

$$\pi(x; d, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{d}}} 1.$$

4.1. The case of $\eta \in (1, \frac{32}{17}]$.

For $\eta > 1$, $c > 1$, $y \geq 3$ and $x = y^\eta$, put

$$\begin{aligned}\mathcal{R}'_b(y) &:= \left\{ y < r \leq 2y : \left| \pi(x; r, a) - \frac{\pi(x)}{\varphi(r)} \right| \geq \delta \frac{\pi(x)}{\varphi(r)} \right\}, \\ \mathcal{R}''_b(y) &:= \left\{ y < r \leq 2y : \left| \pi(cx; r, a) - \frac{\pi(cx)}{\varphi(r)} \right| \geq \delta \frac{\pi(cx)}{\varphi(r)} \right\}.\end{aligned}$$

Noticing that $y = x^{1/\eta} = x^{1-(1-1/\eta)}$, the Elliott-Halberstam conjecture $\text{EH}_{\text{prime}}[\varepsilon]$ with $\varepsilon = 1 - 1/\eta$ allows us to deduce that

$$\begin{aligned}\delta \frac{\pi(x)}{y} |\mathcal{R}'_b(y)| &\leq \sum_{y < r \leq 2y} \left| \pi(x; r, a) - \frac{\pi(x)}{\varphi(r)} \right| \\ &\ll_{A,a,\delta,\eta} \frac{x}{(\log x)^{A+1}},\end{aligned}$$

which gives immediately

$$(4.1) \quad |\mathcal{R}'_b(y)| \ll_{A,a,\delta,\eta} \frac{y}{(\log y)^A}.$$

Similarly

$$(4.2) \quad |\mathcal{R}''_b(y)| \ll_{A,a,c,\delta,\eta} \frac{y}{(\log y)^A}.$$

Define

$$\begin{aligned}\mathcal{R}'_g(y) &:= \left\{ y < r \leq 2y : \pi(x; r, a) \leq (1 + \delta) \frac{\pi(x)}{\varphi(r)} \right\}, \\ \mathcal{R}''_g(y) &:= \left\{ y < r \leq 2y : \pi(cx; r, a) \geq (1 - \delta) \frac{\pi(cx)}{\varphi(r)} \right\},\end{aligned}$$

and

$$\mathcal{R}_g(y) := \mathcal{R}'_g(y) \cap \mathcal{R}''_g(y).$$

Clearly

$$\mathcal{R}_g(y) \subset \mathbb{P} \cap (y, 2y] \subset \mathcal{R}'_b(x) \cup \mathcal{R}''_b(x) \cup \mathcal{R}_g(y).$$

Thus the estimations (4.1) and (4.2) imply that

$$(4.3) \quad |\mathcal{R}_g(y)| = \pi(2y) - \pi(y) + O_{A,a,c,\delta,\eta} \left(\frac{y}{(\log y)^A} \right) \quad (y \geq 2).$$

Let $r \in \mathcal{R}_g(y)$ and let $\mathcal{Q}_r(y)$ be defined as in (3.1). When $\eta \in (1, \frac{32}{17}]$, we have $r > y = x^{1/\eta} \geq x^{17/32} > (cx)^{1/2}$. Thus the definition of $\mathcal{R}_g(y)$ allows us to write

$$(4.4) \quad \mathcal{Q}_r(y) = \pi(cx; r, a) - \pi(x; r, a) \geq (c - 1 - 3\delta) \frac{\pi(x)}{\varphi(r)} > 0,$$

where we have used the inequality $\pi(cx) \geq (c - \delta)\pi(x)$ for $x \geq x_0(a, c, \delta)$. By the definition of $\mathcal{P}_{a,c,\eta}$ and (4.4), it is easy to see that $\mathcal{R}_g(y) \subset \mathcal{P}_{a,c,\eta} \cap [y, 2y]$. In view of (4.3), we find that

$$\pi_{a,c,\eta}(2y) - \pi_{a,c,\eta}(y) = \pi(2y) - \pi(y) + O_{A,a,c,\eta} \left(\frac{y}{(\log y)^A} \right).$$

This implies the first assertion of Theorem 1, thanks to standard dyadic split.

4.2. The case of $\eta \in [\eta_0, \eta_1)$.

In this case, for every prime $r \in \mathcal{R}_g(y)$, we can write

$$(4.5) \quad \begin{aligned} \mathcal{Q}_r(y) &= \pi(cx; r, a) - \pi(x; r, a) - \mathcal{Q}_r(y) \\ &\geq (c - 1 - 3\delta) \frac{\pi(x)}{\varphi(r)} - \mathcal{Q}_r(y). \end{aligned}$$

for $x \geq x_0(a, c, \delta)$, where

$$(4.6) \quad \mathcal{Q}_r(y) := \sum_{\substack{x < q \leq cx \\ q \equiv a \pmod{r}, P(q-a) > r}} 1.$$

Similar to [12, Proposition 2.1] *, we can prove

$$(4.7) \quad \mathcal{Q}_r(y) \leq (c - 1 + 2\delta) \frac{2\eta \log(\eta - 1)}{\eta - 1} \cdot \frac{\pi(y)}{\varphi(r)} \left\{ 1 + O_{a,c,\delta,\eta,\varepsilon} \left(\frac{1}{\sqrt[3]{\log r}} \right) \right\}$$

for $y \geq 3$, $r \in (y, 2y]$ and $\eta \geq 2$.

Inserting (4.7) into (4.4) and taking $c = 1 + 2\sqrt{\delta}$, we can find that

$$\begin{aligned} \mathcal{Q}_r(y) &\geq 2 \frac{\sqrt{\delta} - \delta}{\eta - 1} \left(\eta - 1 - 2\eta \log(\eta - 1) \cdot \frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}} \right) \frac{\pi(y)}{\varphi(r)} \\ &= \{G(\eta) + O(\sqrt{\delta})\} 2\sqrt{\delta} \frac{1 - \sqrt{\delta}}{\eta - 1} \cdot \frac{\pi(y)}{\varphi(r)}, \end{aligned}$$

where

$$(4.8) \quad G(\eta) := \eta - 1 - 2\eta \log(\eta - 1).$$

It is easy to see that $G(\eta)$ is decreasing on $[2, \infty)$ and $G(2) = 1$. Therefore there is a unique real number $\eta_1 \in (2, \infty)$ such that $G(\eta_1) = 0$ and for $\eta \in [2, \eta_1)$ we have the inequality

$$(4.9) \quad \mathcal{Q}_r(y) \gg_{A,a,\delta,\eta} \frac{\pi(y)}{\varphi(r)}$$

for $y \geq y_0(A, a, \delta, \eta)$. As before, (4.9) allows us to deduce that $\mathcal{R}_g(y) \subseteq \mathcal{P}_{a,c,\eta} \cap (y, 2y]$. Combining this with (4.3) leads to

$$\pi_{a,c,\eta}(2y) - \pi_{a,c,\eta}(y) = \pi(2y) - \pi(y) + O_{A,a,c,\delta,\eta} \left(\frac{y}{(\log y)^A} \right).$$

This implies the required asymptotic formula (1.4).

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The proof is identical and the only difference is that we can take $z = (y/q)^{(1-\varepsilon)/2}$ and $D = z^2$ thanks to the Elliott-Halberstam conjecture $\text{EH}_{\text{prime}}^[\varepsilon]$, instead of $z = (y/q)^{1/4}/(\log y)^B$ and $D = z^2$

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