ELLIOTT-HALBERSTAM CONJECTURE AND VALUES TAKEN BY THE LARGEST PRIME FACTOR OF SHIFTED PRIMES

Jie Wu

ELLIOTT-HALBERSTAM CONJECTURE AND VALUES TAKEN BY THE LARGEST PRIME FACTOR OF SHIFTED PRIMES

JIE WU

Abstract. Denote by $\mathbb{P}$ the set of all primes and by $P^+(n)$ the largest prime factor of integer $n \geqslant 1$ with the convention $P^+(1) = 1$. For each $\eta > 1$, let $c = c(\eta) > 1$ be some constant depending on η and

$$\mathcal{P}_{a,c,\eta} := \{p \in \mathbb{P} \, : \, p = P^+(q-a) \text{ for some prime } q \text{ with } p^\eta < q \leqslant c(\eta)p^\eta\}.$$

In this paper, under the Elliott-Halberstam conjecture we prove, for $y \to \infty$,

$$\pi_{a,c,\eta}(x) := |(1,x] \cap \mathcal{P}_{a,c,\eta}| \sim \pi(x) \quad \text{or} \quad \pi_{a,c,\eta}(x) \gg_{a,\eta} \pi(x)$$

according to values of η. These complement for some results of Banks-Shparlinski [1], of Wu [12] and of Chen-Wu [2].

1. Introduction

Denote by $\mathbb{P}$ the set of all prime numbers and by $P^+(n)$ the largest prime factor of the positive integer $n \geqslant 1$ with the convention $P^+(1) = 1$. Banks & Shparlinski [1] proposed to estimate the number of primes p that occur as the largest prime factor of a shifted prime $q - a$ when $q \in \mathbb{P}$ lies in a certain interval determined by p. This question has applications in theoretical computer science and has been considered by Vishnoi [10].

Let $\mathbb{Z}^*$ be the set of non-zero integers. For $a \in \mathbb{Z}^*$, $c > 1$ and $\eta > 0$, we put

$$\mathcal{P}_{a,c,\eta} := \{r \in \mathbb{P} \, : \, r = P^+(q-a) \text{ for some prime } q \text{ with } r^\eta < q \leqslant cr^\eta\}$$

and

$$\pi_{a,c,\eta}(y) := |\{r \leqslant y \, : \, r \in \mathcal{P}_{a,c,\eta}\}|, \qquad \pi(y) := |\{r \leqslant y \, : \, r \in \mathbb{P}\}|.$$

Banks & Shparlinski [1, Theorem 1.1] proved that for each $\eta \in (\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})$, there exists a constant $c = c(\eta) > 1$ such that the asymptotic formula

$$\pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta}\bigg(\frac{y}{(\log y)^A}\bigg) \qquad (y \to \infty) \tag{1.1}$$

holds for every fixed non-zero integer $a \in \mathbb{Z}^*$ and any constant $A > 1$. Moreover for $2 \leqslant \eta < 1 + \frac{3}{4}\sqrt{2} \approx 2.0606$, this estimate holds for any constant $c > 1$. Very recently, Wu [12] extended Banks-Shparlinski's interval $(\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})$ to $(\frac{32}{17}, \eta_0)$, where $\eta_0 \approx 2.142$ is the unique solution of the equation $\eta - 1 - 4\eta\log(\eta - 1) = 0$ in $(1, \infty)$. Banks & Shparlinski [1, page 144] also remarked that the asymptotic formula (1.1) holds for $\eta \in (1, \frac{32}{17}]$ if we assume the Elliott-Halberstam conjecture (see $\text{EH}_{\text{prime}}[\varepsilon]$ below). Subsequently, Chen & Wu

Date: July 2, 2019.
2010 *Mathematics Subject Classification.* 11N05, 11N25, 11N36.
Key words and phrases. Shifted prime, Friable integer, Sieve.

[2] further extended the domain of η at the price proportion positive instead of density 1. More precisely, they proved that

$$\pi_{a,c,\eta}(y) \geqslant \big(\log \sqrt[4]{2}\big)\frac{\eta-1}{\eta}\bigg(1 - 4\log(\eta-1) - \delta\frac{\log(\eta-1)^8}{c-1}\bigg)\pi(y), \tag{1.2}$$

where $\delta = \delta(c,\eta)$ is sufficiently small positive number. Clearly, (1.2) implies

$$\pi_{a,c,\eta}(y) \gg \pi(y)$$

provided $\eta < 1 + \sqrt[4]{\mathrm{e}}$. This is complement for the results of Banks-Shparlinski and of Wu mentioned above. It seems rather natural to pose the following question.

Question 1. *Is the asymptotic formula* (1.1) *true for all* $\eta > 1$ *?*

In this paper, we shall try to answer this question under the well-known Elliott-Halberstam conjecture. Firstly we state two versions of this conjecture for prime numbers.

Conjecture 1 (Elliott–Halberstam). *Let* $a \in \mathbb{Z}^*$ *and* $\varepsilon \in (0,1)$ *be fixed constants.*

(i) *For any* $A > 0$*, the inequality*

$$\sum_{\substack{q\leqslant x^{1-\varepsilon}\\ (a,q)=1}} \bigg|\sum_{\substack{p\leqslant x\\ p\equiv a (\mathrm{mod}\, q)}} 1 - \frac{\pi(x)}{\varphi(q)}\bigg| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A} \tag{$\mathrm{EH}_{\mathrm{prime}}[\varepsilon]$}$$

holds uniformly for all $x \geqslant 3$*, where the letter* p *always denotes prime numbers,* $\varphi(q)$ *is the Euler function and the implied constant depends on* A, a *and* ε*.*

(ii) *Let* $\kappa_1(m)$ *and* $\kappa_2(m)$ *be the characteristic functions of the odd integers and of even integers, respectively. Then for any* $A > 0$*, we have*

$$\sum_{\substack{q\leqslant x^{1-\varepsilon}\\ (a,q)=1}} \bigg|\sum_{\substack{mp\leqslant x\\ mp\equiv a (\mathrm{mod}\, q)}} \kappa_i(m) - \frac{1}{\varphi(q)}\sum_{\substack{mp\leqslant x\\ (mp,q)=1}} \kappa_i(m)\bigg| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A} \tag{$\mathrm{EH}^*_{\mathrm{prime}}[\varepsilon]$}$$

uniformly for all $x \geqslant 3$*, where the letter* p *always denotes prime numbers and the implied constant depends on* A *and* a*.*

Remark 1. According to the classical Bombieri-Vinogradov theorem and Proposition 2.2 of Wu [12], the Elliott-Halberstam conjectures $\mathrm{EH}_{\mathrm{prime}}[\varepsilon]$ and $\mathrm{EH}^*_{\mathrm{prime}}[\varepsilon]$ hold for all $\varepsilon \in (\frac{1}{2}, 1)$.

Secondly we also need a version of this conjecture for friable numbers.

Conjecture 2 (Elliott–Halberstam). *Let* $a \in \mathbb{Z}^*$ *and* $\varepsilon \in (0,1)$ *be fixed constants. For any* $A > 0$*, we have*

$$\sum_{\substack{q\leqslant x^{1-\varepsilon}\\ (a,q)=1}} \bigg|\sum_{\substack{n\leqslant x\\ n\equiv a (\mathrm{mod}\, q),\, P^+(n)\leqslant y}} 1 - \frac{1}{\varphi(q)}\sum_{\substack{n\leqslant x\\ (n,q)=1,\, P^+(n)\leqslant y}} 1\bigg| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A} \tag{$\mathrm{EH}_{\mathrm{friable}}[\varepsilon]$}$$

uniformly in $x \geqslant y \geqslant 2$*.*

Remark 2. According to Wolke's work [11] (see also [4, Theorem 6]), $\mathrm{EH}_{\mathrm{friable}}[\varepsilon]$ holds unconditionally for all $\varepsilon \in (\frac{1}{2}, 1)$.

Our results are as follows.

Theorem 1. *Let $a \in \mathbb{Z}^*$ and $c > 1$ be fixed constants.*

(i) *Let $\eta \in (1, \frac{32}{17}]$ and assume the Elliott-Halberstam conjecture* $\mathrm{EH}_{\mathrm{prime}}[\varepsilon]$ *with $\varepsilon = 1 - 1/\eta > 0$. Then for any $A > 1$ we have*

$$\pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta}\bigg(\frac{y}{(\log y)^A}\bigg), \tag{1.3}$$

as $y \to \infty$.

(ii) *Let $\eta_1 \approx 2.3303$ be the unique positive zero of the equation $\eta - 1 - 2\eta \log(\eta - 1) = 0$ in $(1, \infty)$. For each $\eta \in [\eta_0, \eta_1)$, there is a sufficiently small positive number $\varepsilon = \varepsilon(\eta)$ such that assuming the Elliott-Halberstam conjecture* $\mathrm{EH}_{\mathrm{prime}}[\varepsilon]$*, for any $A > 1$ we have*

$$\pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta}\bigg(\frac{y}{(\log y)^A}\bigg), \tag{1.4}$$

as $y \to \infty$.

Theorem 2. *Let $a \in \mathbb{Z}^*$ and $c > 1$ be fixed constants. For every $\eta \in [2, \infty)$, there is a sufficiently small positive number $\varepsilon = \varepsilon(\eta)$ such that under the Elliott-Halberstam conjecture* $\mathrm{EH}_{\mathrm{friable}}[\varepsilon]$*, we have*

$$\pi_{a,c,\eta}(y) \geqslant \big(\log \sqrt[4]{2}\big)\frac{\eta - 1}{\eta}\pi(y)\bigg\{1 + O_{a,c,\eta}\bigg(\frac{1}{\sqrt[3]{\log y}} + \varepsilon\bigg)\bigg\}, \tag{1.5}$$

as $y \to \infty$.

Remark 3. (a) For comparaison, we have

$$1 + \tfrac{3}{4}\sqrt{2} \approx 2.060, \quad \eta_0 \approx 2.142, \quad \eta_1 \approx 2.330, \quad 1 + \sqrt[4]{\mathrm{e}} \approx 2.284, \quad 1 + \sqrt{\mathrm{e}} \approx 2.648.$$

(b) The first assertion of Theorem 1 is due to Banks & Shparlinski [1, page 144]. Here we give a proof for convenience of the reader. By combining Theorem 1 with results of [12], we see that the asymptotic formula (1.1) holds for $1 < \eta < \eta_1$ under the Elliott-Halberstam conjecture.

(c) Theorem 2 improves signaficantively Chen-Wu's result (1.2) in two aspects : result and method. Firstly the proportion in (1.5) increases from $\frac{\log 2}{8}$ to $\frac{\log 2}{4}$ when η runs over $[2, \infty)$, and that in (1.2) tends toward 0 as $\eta \to 1 + \sqrt[4]{\mathrm{e}}$. Secondly our proof of (1.5) needs information of prime numbers in "arithmetical progressions" $\{a + mq\}_{m \text{ friable}}$ with friable indice. For $(a, q) = 1$ and $x \geqslant y \geqslant 2$, define the counting function

$$\pi(x, y; q, a) := \sum_{\substack{p \leqslant x \\ p \equiv a (\bmod q) \\ P^+((p-a)/q) \leqslant y}} 1. \tag{1.6}$$

A systematic study on the asymptotical behaviour has been done by Liu, Wu & Xi [8], recently. We need a theorem of Bombieri-Vinogradov type and an inequality of Brun-Titichmarsh for this new counting function (see Lemmas 2.2–2.3 below).

2. Some preliminary lemmas

In this section, we present three lemmas, which will be useful later.

2.1. **The Rosser-Iwaniec linear sieve.**

The first lemma is due to Iwaniec [6, 7].

Lemma 2.1. *Let $D \geqslant 2$ and let $\mu(n)$ be the Möbius function. Then there are two sequences $\{\lambda_d^{\pm}\}_{d\geqslant 1}$, vanishing for $d > D$ or $\mu(d) = 0$, satisfying $|\lambda_d^{\pm}| \leqslant 1$, such that*

$$\sum_{d|n} \lambda_d^- \leqslant \sum_{d|n} \mu(d) \leqslant \sum_{d|n} \lambda_d^+ \qquad (n \geqslant 1) \tag{2.1}$$

and

$$\sum_{d|P_{\mathcal{P}}(z)} \lambda_d^+ \frac{w(d)}{d} \leqslant \prod_{\substack{p\leqslant z\\ p\in\mathcal{P}}} \bigg(1 - \frac{w(p)}{p}\bigg) \bigg\{F(s) + O\bigg(\frac{\mathrm{e}^{\sqrt{L}-s}}{\sqrt[3]{\log D}}\bigg)\bigg\} \tag{2.2}$$

$$\sum_{d|P_{\mathcal{P}}(z)} \lambda_d^- \frac{w(d)}{d} \geqslant \prod_{\substack{p\leqslant z\\ p\in\mathcal{P}}} \bigg(1 - \frac{w(p)}{p}\bigg) \bigg\{f(s) + O\bigg(\frac{\mathrm{e}^{\sqrt{L}-s}}{\sqrt[3]{\log D}}\bigg)\bigg\} \tag{2.3}$$

for any $z \in [2, D]$, $s = (\log D)/\log z$, set of prime numbers $\mathcal{P}$ and multiplicative function w satisfying

$$0 < w(p) < p \qquad (p \in \mathcal{P}), \tag{2.4}$$

$$\prod_{u<p\leqslant v,\, p\in\mathcal{P}} \bigg(1 - \frac{w(p)}{p}\bigg)^{-1} \leqslant \frac{\log v}{\log u}\bigg(1 + \frac{L}{\log u}\bigg) \quad (2 \leqslant u \leqslant v), \tag{2.5}$$

where $P_{\mathcal{P}}(z) := \prod_{p\leqslant z,\, p\in\mathcal{P}} p$ and the implied O-constants are absolute. Here F, f are defined by the continuous solutions to the system

$$\begin{cases} sF(s) = 2\mathrm{e}^{\gamma} & (1 \leqslant s \leqslant 2)\\ sf(s) = 0 & (0 < s \leqslant 2)\\ (sF(s))' = f(s-1) & (s > 2)\\ (sf(s))' = F(s-1) & (s > 2)\end{cases}$$

where γ is the Euler constant.

2.2. **Bombieri-Vinogradov theorem of for $\pi(x, y; q, a)$.**

The second lemma is a theorem of Bombieri-Vinogradov type for the counting function $\pi(x, y; q, a)$ defined as in (1.6) (see [8, Theorem 2]).

Lemma 2.2. *Let $a \in \mathbb{Z}^*$, $A > 0$ and κ a non-negative arithmetic function. Assuming the Elliott-Halberstam conjecture $\mathrm{EH}_{\mathrm{prime}}[\varepsilon]$, the following estimate*

$$\sum_{\substack{q\leqslant Q\\ (q,a)=1}} \kappa(q) \bigg|\pi(x, y; q, a) - \frac{\pi(x)}{\varphi(q)}\rho\bigg(\frac{\log(x/q)}{\log y}\bigg)\bigg| \ll_{a,A} \frac{x}{(\log x)^A}\sqrt{\sum_{q\leqslant x}\frac{\kappa(q)^2}{q}} + \pi(x)\varepsilon u \sum_{q\leqslant Q}\frac{\kappa(q)}{\varphi(q)}$$

holds uniformly in $x \geqslant 2$, $\exp\big\{(\log x)^{2/5+\varepsilon}\big\} \leqslant y \leqslant x$ and $Q \leqslant \min\{y, \sqrt{x}\}$, where $\rho(u)$ is the Dickman function.

2.3. **Brun-Titichmarsh inequality for** $\pi(x,y;q,a)$**.**

The lemma below is a variant of [8, Theorem 1].

Lemma 2.3. *Let* $a \in \mathbb{Z}^*$ *and* $c > 1$ *be fixed constants. For any* $\varepsilon > 0$, *we have*

$$\begin{aligned}\pi(cx,y;q,a) - \pi(x,y;q,a) &\leqslant \frac{4(c-1)x}{\varphi(q)\log(x/q)}\rho\Big(\frac{\log(x/q)}{\log y}\Big)\Big\{1 + O_{a,c,\varepsilon}\Big(\frac{1}{\sqrt[3]{\log x}}\Big)\Big\} \\ &\quad + O_{A,a,c}\Big(\frac{x}{q(\log x)^A}\Big)\end{aligned} \tag{2.6}$$

uniformly in $\exp\{\log\log x)^{5/3+\varepsilon}\} \leqslant y \leqslant x$ *and* $1 \leqslant q \leqslant \min\{y, \sqrt{x}\}$ *with* $(a,q) = 1$.

Proof. Denote by $\mathcal{S}$ the quantity on the left-hand side of (2.6). Without loss of generality, we can assume q is even and a is odd. Put $P_{2a}(z) := \prod_{p\leqslant z, p\nmid 2a} p$. By the Möbius inversion, we can write

$$\begin{aligned}\mathcal{S} &= \sum_{\substack{x<a+mq\leqslant cx \\ (mq,a)=(a+mq,P_{2a}(\sqrt{cx}))=1 \\ P^+(m)\leqslant y}} 1 + O(x^\varepsilon) \\ &= \sum_{\substack{x<a+mq\leqslant cx \\ (a,mq)=1,\, P^+(m)\leqslant y}} \sum_{d\mid(a+mq,P_{2a}(\sqrt{cx}))} \mu(d) + O(x^\varepsilon),\end{aligned}$$

Using Lemma 2.1 and switching summations, it follows that

$$\begin{aligned}\mathcal{S} &\leqslant \sum_{\substack{x<a+mq\leqslant cx \\ (a,mq)=1,\, P^+(m)\leqslant y}} \sum_{d\mid(a+mq,P_{2a}(\sqrt{cx}))} \lambda_d^+ + O(x^\varepsilon) \\ &= \sum_{\substack{d\leqslant D \\ d\mid P_{2aq}(\sqrt{cx})}} \lambda_d^+ \sum_{\substack{(x-a)/q<m\leqslant(cx-a)/q \\ m\equiv -a\overline{q}\,(\mathrm{mod}\, d) \\ (a,m)=1,\, P^+(m)\leqslant y}} 1 + O(x^\varepsilon),\end{aligned}$$

where $\{\lambda_d^+\}_{d\geqslant 1}$ is an upper bound sieve of level D as in Lemma 2.1 and $\overline{q}$ is the inverse of q modulo d (i.e. $q\overline{q} \equiv 1\,(\mathrm{mod}\, d)$). To apply the Elliott-Halberstam conjecture $\mathrm{EH}_{\mathrm{friable}}[\varepsilon]$, we would like to remove the restriction $(a,m) = 1$ by Möbius inversion, so that

$$\begin{aligned}\mathcal{S} &\leqslant \sum_{\ell\mid a} \mu(\ell) \sum_{\substack{d\leqslant D \\ d\mid P_{2aq}(\sqrt{cx})}} \lambda_d^+ \sum_{\substack{(x-a)/q<m\leqslant(cx-a)/q \\ m\equiv -a\overline{q}\,(\mathrm{mod}\, d) \\ \ell\mid m,\, P^+(m)\leqslant y}} 1 + O(x^\varepsilon) \\ &= \sum_{\substack{\ell\mid a \\ P^+(\ell)\leqslant y}} \mu(\ell) \sum_{\substack{d\leqslant D \\ d\mid P_{2aq}(\sqrt{cx})}} \lambda_d^+ \sum_{\substack{(x-a)/\ell q<m\leqslant(cx-a)/\ell q \\ m\equiv -a\overline{\ell q}\,(\mathrm{mod}\, d) \\ P^+(m)\leqslant y}} 1 + O(x^\varepsilon).\end{aligned}$$

We are now in a good position to employ the Elliott-Halberstam conjecture $\mathrm{EH}_{\mathrm{friable}}[\varepsilon]$ with $D = (x/q)^{1-\varepsilon}$, getting

$$\mathcal{S} \leqslant \mathcal{S}^+ + O\Big(\frac{x}{q(\log x)^A}\Big), \tag{2.7}$$

where

$$\mathcal{S}^+ := \sum_{\ell | a} \mu(\ell) \sum_{\substack{d \leqslant D \\ d | P_{2aq}(\sqrt{cx})}} \frac{\lambda_d^+}{\varphi(d)} \sum_{\substack{x/\ell q < m \leqslant cx/\ell q \\ (d,m)=1,\, P^+(m) \leqslant y}} 1.$$

and we have used the trivial bound

$$\sum_{\ell | a} \sum_{\substack{d \leqslant D \\ d | P_{2aq}(\sqrt{cx})}} \frac{|\lambda_d^+|}{\varphi(d)} \sum_{\substack{cx/\ell q < m \leqslant (cx-a)/\ell q \\ (d,m)=1,\, P^+(m) \leqslant y}} 1 \ll_a \log x.$$

Here we removed the restriction that $P^+(\ell) \leqslant y$ since we henceforth assume $y > |a|$.

Switching summations, it follows that

$$\begin{aligned} \mathcal{S}^+ &= \sum_{\ell | a} \mu(\ell) \sum_{\substack{x/\ell q < m \leqslant cx/\ell q \\ P^+(m) \leqslant y}} \sum_{\substack{d \leqslant D \\ d | P_{2amq}(\sqrt{cx})}} \frac{\lambda_d^+}{\varphi(d)} \\ &= \sum_{\substack{x/q < m \leqslant cx/q \\ (a,m)=1,\, P^+(m) \leqslant y}} \sum_{\substack{d \leqslant D \\ d | P_{2amq}(\sqrt{cx})}} \frac{\lambda_d^+}{\varphi(d)}. \end{aligned}$$

From Lemma 2.1, we deduce

$$\begin{aligned} \mathcal{S}^+ &\leqslant \frac{\log \sqrt{x}}{\log(x/q)} \bigg\{ 2\mathrm{e}^\gamma + O\bigg(\frac{1}{\sqrt[3]{\log x}} \bigg) \bigg\} \sum_{\substack{x/q < m \leqslant cx/q \\ (a,m)=1,\, P^+(m) \leqslant y}} \prod_{\substack{p < \sqrt{x} \\ p \nmid 2amq}} \Big(1 - \frac{1}{p-1} \Big) \\ &\leqslant \frac{\log \sqrt{x}}{\log(x/q)} \bigg\{ 2\mathrm{e}^\gamma + O\bigg(\frac{1}{\sqrt[3]{\log x}} \bigg) \bigg\} \prod_{\substack{p < \sqrt{x} \\ p \nmid 2aq}} \Big(1 - \frac{1}{p-1} \Big) \sum_{\substack{x/q < m \leqslant cx/q \\ (a,m)=1,\, P^+(m) \leqslant y}} H(m), \end{aligned} \tag{2.8}$$

where $H(m)$ is the multiplicative function, defined by

$$H(p^\nu) = \begin{cases} 1 & \text{if } p \mid 2q \text{ or } p > x^{1/2} \\ \frac{p-1}{p-2} & \text{if } p \nmid 2q \text{ and } p \leqslant x^{1/2} \end{cases}$$

for all $\nu \geqslant 1$. According to [8, (3.5)], we have

$$\sum_{\substack{m \in S(x/q,y) \\ (a,m)=1}} H(m) = \Psi\Big(\frac{x}{q}, y \Big) \frac{\varphi(a)}{a} \prod_{\substack{p < \sqrt{x} \\ p \nmid 2aq}} \Big(1 + \frac{1}{p(p-2)} \Big) \bigg\{ 1 + O\bigg(\frac{(\log \log x)^2}{\log y} \bigg) \bigg\},$$

where $S(x,y) := \{n \leqslant x : P^+(n) \leqslant y\}$ and $\Psi(x,y) := |S(x,y)|$. Combining this with (2.8), we find that

$$\mathcal{S}^+ \leqslant \frac{\log x}{\log(x/q)} \bigg\{ 2\mathrm{e}^\gamma + O\bigg(\frac{1}{\sqrt[3]{\log x}} \bigg) \bigg\} \frac{\varphi(a)}{a} \bigg\{ \Psi\Big(\frac{cx}{q}, y \Big) - \Psi\Big(\frac{x}{q}, y \Big) \bigg\} \prod_{\substack{p \leqslant \sqrt{x} \\ p \nmid aq}} \Big(1 - \frac{1}{p} \Big). \tag{2.9}$$

By the Mertens formula, it follows that

$$\prod_{\substack{p\leqslant x^{1/2}\\ p\nmid aq}}\Big(1-\frac{1}{p}\Big)=\frac{aq}{\varphi(a)\varphi(q)}\cdot\frac{2\mathrm{e}^{-\gamma}}{\log x}\bigg\{1+O\bigg(\frac{1}{\log x}\bigg)\bigg\}.$$

On the other hand, according to [5, Theorem 1], we have

$$\Psi(x,y)=x\rho\bigg(\frac{\log x}{\log y}\bigg)\bigg\{1+O\bigg(\frac{\log((\log x)/\log y+1)}{\log y}\bigg)\bigg\}$$

uniformly for $x\geqslant 3$ and $\exp\{(\log\log x)^{5/3+\varepsilon}\}\leqslant y\leqslant x$. Combining these with (2.9) and (2.7), we can get the required inequality (2.6). □

3. Proof of Theorem 2

For each prime $r\in(\frac{1}{2}y,y]$, consider

$$\mathcal{Q}_r(y):=\sum_{\substack{x<q\leqslant cx\\ P^+(q-a)=r}}1. \tag{3.1}$$

Noticing that

$$P^+(q-a)=r\ \Leftrightarrow\ q\equiv a\,(\mathrm{mod}\,r)\ \text{ and }\ P^+(q-a)\leqslant r,$$

we can write

$$\begin{aligned}\sum_{y<r\leqslant 2y}\mathcal{Q}_r(y)&\geqslant\sum_{y<r\leqslant 2y}\sum_{\substack{x<q\leqslant cx\\ q\equiv a\,(\mathrm{mod}\,r),\,P^+(q-a)\leqslant y}}1\\ &=\sum_{y<r\leqslant 2y}\big(\pi(cx,y;r,a)-\pi(x,y;r,a)\big)\\ &=\mathcal{M}+\mathcal{E},\end{aligned}\tag{3.2}$$

where

$$\mathcal{M}:=\sum_{y<r\leqslant 2y}\bigg(\frac{\pi(cx)}{\varphi(r)}\rho\bigg(\frac{\log(cx/r)}{\log y}\bigg)-\frac{\pi(x)}{\varphi(r)}\rho\bigg(\frac{\log(x/r)}{\log y}\bigg)\bigg),$$

$$\mathcal{E}:=\sum_{y<r\leqslant 2y}\big(\mathcal{E}(cx,y;r,a)-\mathcal{E}(x,y;r,a)\big),$$

and

$$\mathcal{E}(x,y;r,a):=\pi(x,y;r,a)-\frac{\pi(x)}{\varphi(r)}\rho\bigg(\frac{\log(x/r)}{\log y}\bigg).$$

Since $\eta\geqslant 2$, we have $y=x^{1/\eta}\leqslant x^{1/2}$ and $Q=\min(y,\sqrt{x})=y$. Using Lemma 2.2 with the characteristic function of prime numbers in $(y,2y]$ in place of $\kappa(q)$, we easily derive, under the conjecture of Elliott-Halberstam $\mathrm{EH}_{\mathrm{prime}}[\varepsilon]$, that

$$\begin{aligned}|\mathcal{E}|&\leqslant\sum_{y<r\leqslant 2y}\big(|\mathcal{E}(cx,y;r,a)|+|\mathcal{E}(x,y;r,a)|\big)\\ &\ll_a\frac{x}{(\log x)^3}+\varepsilon\eta\frac{\pi(x)}{\log y}\ll_{a,\eta}\varepsilon\frac{\pi(x)}{\log y}\end{aligned}\tag{3.3}$$

for all $x \geqslant x_0(\varepsilon)$, where we have used the following bound

$$\sum_{y<r\leqslant 2y} \frac{1}{\varphi(r)} = \sum_{y<r\leqslant 2y} \frac{1}{r}\bigg\{1+O\Big(\frac{1}{y}\Big)\bigg\} = \frac{\log 2}{\log y}\bigg\{1+O\Big(\frac{1}{\log y}\Big)\bigg\} \tag{3.4}$$

and the implied constant depends on a, η at most.

According to [9, Corollary III.5.8.3], we have $|\rho'(u)| \ll \rho(u)\log u$ $(u > 1)$. Thus for all $r \in (y, 2y]$, we have

$$\rho\bigg(\frac{\log(x/r)}{\log y}\bigg) = \rho(\eta-1)\bigg\{1+O_\eta\Big(\frac{1}{\log y}\Big)\bigg\}.$$

From this and (3.4), we derive

$$\mathcal{M} = (\log 2)(c-1)\rho(\eta-1)\frac{\pi(x)}{\log y}\bigg\{1+O_\eta\Big(\frac{1}{\log y}\Big)\bigg\}. \tag{3.5}$$

Inserting (3.5) and (3.3) into (3.2), it follows that

$$\sum_{y<r\leqslant 2y} \mathcal{Q}_r(y) \geqslant (\log 2)(c-1)\rho(\eta-1)\frac{\pi(x)}{\log y}\bigg\{1+O_{a,c,\eta}\Big(\frac{1}{\log y}+\varepsilon\Big)\bigg\}. \tag{3.6}$$

On the other hand, the Brun-Titchmarsh inequality (2.6) give us

$$\begin{aligned}\mathcal{Q}_r(y) &\leqslant \frac{4(c-1)x}{\varphi(r)\log(x/r)}\rho(\eta-1)\bigg\{1+O_{a,\eta}\Big(\frac{1}{\sqrt[3]{\log x}}\Big)\bigg\}\\ &\leqslant 4(c-1)\frac{\rho(\eta-1)}{\eta-1}\cdot\frac{x}{y\log y}\bigg\{1+O_{a,\eta}\Big(\frac{1}{\sqrt[3]{\log x}}\Big)\bigg\}\end{aligned}$$

for all primes $r \in (y, 2y]$. This implies that

$$\sum_{y<r\leqslant 2y} \mathcal{Q}_r(y) \leqslant \rho(\eta-1)\frac{4(c-1)}{(\eta-1)}\cdot\frac{x}{y\log y}\bigg\{1+O_{a,\eta}\Big(\frac{1}{\sqrt[3]{\log x}}\Big)\bigg\}\sum_{\substack{y<r\leqslant 2y\\ \mathcal{Q}_r(y)\neq 0}} 1. \tag{3.7}$$

Combining (3.6) and (3.7), it follows that

$$\sum_{\substack{y<r\leqslant 2y\\ \mathcal{Q}_r(y)\neq 0}} 1 \geqslant \big(\log\sqrt[4]{2}\big)\frac{\eta-1}{\eta}\pi(y)\bigg\{1+O_{a,c,\eta}\Big(\frac{1}{\sqrt[3]{\log x}}+\varepsilon\Big)\bigg\}.$$

This completes the proof of Theorem 2.

4. Proof of Theorem 1

As in [12], the letters p, q, r and ℓ are always used to denote prime numbers, and d, m, and n always denote positive integers. In what follows, let $a \in \mathbb{Z}^*$ and $\eta \in (1, \frac{32}{17}] \cup [\eta_0, \eta_1)$. Let δ be a sufficiently small positive constant and let $c > 1$ be a parameter to be chosen later. Let $x_0(A, a, c, \eta, \delta)$ be a large constant depending on A, a, c, η, δ at most. For $x \geqslant x_0(A, a, c, \eta, \delta)$ and $r \in (\frac{1}{2}y, y]$, put $x := r^\eta$. As usual, for $(a, d) = 1$ define

$$\pi(x; d, a) := \sum_{\substack{p\leqslant x\\ p\equiv a (\bmod d)}} 1.$$

4.1. **The case of $\eta \in (1, \frac{32}{17}]$.**

For $\eta > 1$, $c > 1$, $y \geqslant 3$ and $x = y^\eta$, put

$$\mathcal{R}_{\mathrm{b}}'(y) := \left\{ y < r \leqslant 2y \,:\, \left| \pi(x; r, a) - \frac{\pi(x)}{\varphi(r)} \right| \geqslant \delta \frac{\pi(x)}{\varphi(r)} \right\},$$
$$\mathcal{R}_{\mathrm{b}}''(y) := \left\{ y < r \leqslant 2y \,:\, \left| \pi(cx; r, a) - \frac{\pi(cx)}{\varphi(r)} \right| \geqslant \delta \frac{\pi(cx)}{\varphi(r)} \right\}.$$

Noticing that $y = x^{1/\eta} = x^{1-(1-1/\eta)}$, the Elliott-Halberstam conjecture $\mathrm{EH}_{\mathrm{prime}}[\varepsilon]$ with $\varepsilon = 1 - 1/\eta$ allows us to deduce tthat

$$\begin{aligned} \delta \frac{\pi(x)}{y} |\mathcal{R}_{\mathrm{b}}'(y)| &\leqslant \sum_{y<r\leqslant 2y} \left| \pi(x; r, a) - \frac{\pi(x)}{\varphi(r)} \right| \\ &\ll_{A,a,\delta,\eta} \frac{x}{(\log x)^{A+1}}, \end{aligned}$$

which gives immediately

$$|\mathcal{R}_{\mathrm{b}}'(y)| \ll_{A,a,\delta,\eta} \frac{y}{(\log y)^A}. \tag{4.1}$$

Similarly

$$|\mathcal{R}_{\mathrm{b}}''(y)| \ll_{A,a,c,\delta,\eta} \frac{y}{(\log y)^A}. \tag{4.2}$$

Define

$$\mathcal{R}_{\mathrm{g}}'(y) := \left\{ y < r \leqslant 2y \,:\, \pi(x; r, a) \leqslant (1 + \delta) \frac{\pi(x)}{\varphi(r)} \right\},$$
$$\mathcal{R}_{\mathrm{g}}''(y) := \left\{ y < r \leqslant 2y \,:\, \pi(cx; r, a) \geqslant (1 - \delta) \frac{\pi(cx)}{\varphi(r)} \right\},$$

and

$$\mathcal{R}_{\mathrm{g}}(y) := \mathcal{R}_{\mathrm{g}}'(y) \cap \mathcal{R}_{\mathrm{g}}''(y).$$

Clearly

$$\mathcal{R}_{\mathrm{g}}(y) \subset \mathbb{P} \cap (y, 2y] \subset \mathcal{R}_{\mathrm{b}}'(x) \cup \mathcal{R}_{\mathrm{b}}''(x) \cup \mathcal{R}_{\mathrm{g}}(y).$$

Thus the estimations (4.1) and (4.2) imply that

$$|\mathcal{R}_{\mathrm{g}}(y)| = \pi(2y) - \pi(y) + O_{A,a,c,\delta,\eta}\left(\frac{y}{(\log y)^A} \right) \qquad (y \geqslant 2). \tag{4.3}$$

Let $r \in \mathcal{R}_{\mathrm{g}}(y)$ and let $\mathcal{Q}_r(y)$ be defined as in (3.1). When $\eta \in (1, \frac{32}{17}]$, we have $r > y = x^{1/\eta} \geqslant x^{17/32} > (cx)^{1/2}$. Thus the definition of $\mathcal{R}_{\mathrm{g}}(y)$ allows us to write

$$\mathcal{Q}_r(y) = \pi(cx; r, a) - \pi(x; r, a) \geqslant (c - 1 - 3\delta) \frac{\pi(x)}{\varphi(r)} > 0, \tag{4.4}$$

where we have used the inequality $\pi(cx) \geqslant (c - \delta)\pi(x)$ for $x \geqslant x_0(a, c, \delta)$. By the definition of $\mathcal{P}_{a,c,\eta}$ and (4.4), it is easy to see that $\mathcal{R}_{\mathrm{g}}(y) \subset \mathcal{P}_{a,c,\eta} \cap [y, 2y]$. In view of (4.3), we find that

$$\pi_{a,c,\eta}(2y) - \pi_{a,c,\eta}(y) = \pi(2y) - \pi(y) + O_{A,a,c,\eta}\left(\frac{y}{(\log y)^A} \right).$$

This implies the first assertion of Theorem 1, thanks to standard dyadic split.

4.2. **The case of $\eta \in [\eta_0, \eta_1)$.**

In this case, for every prime $r \in \mathcal{R}_{\rm g}(y)$, we can write

$$\begin{aligned} \mathcal{Q}_r(y) &= \pi(cx; r, a) - \pi(x; r, a) - \mathscr{Q}_r(y) \\ &\geqslant (c - 1 - 3\delta)\frac{\pi(x)}{\varphi(r)} - \mathscr{Q}_r(y). \end{aligned} \tag{4.5}$$

for $x \geqslant x_0(a, c, \delta)$, where

$$\mathscr{Q}_r(y) := \sum_{\substack{x<q\leqslant cx \\ q\equiv a(\bmod r),\, P(q-a)>r}} 1. \tag{4.6}$$

Similar to [12, Proposition 2.1] *, we can prove

$$\mathscr{Q}_r(y) \leqslant (c - 1 + 2\delta)\frac{2\eta\log(\eta-1)}{\eta-1}\cdot\frac{\pi(y)}{\varphi(r)}\bigg\{1 + O_{a,c,\delta,\eta,\varepsilon}\bigg(\frac{1}{\sqrt[3]{\log r}}\bigg)\bigg\} \tag{4.7}$$

for $y \geqslant 3$, $r \in (y, 2y]$ and $\eta \geqslant 2$.

Inserting (4.7) into (4.4) and taking $c = 1 + 2\sqrt{\delta}$, we can find that

$$\begin{aligned} \mathcal{Q}_r(y) &\geqslant 2\frac{\sqrt{\delta}-\delta}{\eta-1}\bigg(\eta - 1 - 2\eta\log(\eta-1)\cdot\frac{1+\sqrt{\delta}}{1-\sqrt{\delta}}\bigg)\frac{\pi(y)}{\varphi(r)} \\ &= \big\{G(\eta) + O\big(\sqrt{\delta}\big)\big\}2\sqrt{\delta}\frac{1-\sqrt{\delta}}{\eta-1}\cdot\frac{\pi(y)}{\varphi(r)}, \end{aligned}$$

where

$$G(\eta) := \eta - 1 - 2\eta\log(\eta-1). \tag{4.8}$$

It is easy to see that $G(\eta)$ is decreasing on $[2, \infty)$ and $G(2) = 1$. Therefore there is a unique real number $\eta_1 \in (2, \infty)$ such that $G(\eta_1) = 0$ and for $\eta \in [2, \eta_1)$ we have the inequality

$$\mathcal{Q}_r(y) \gg_{A,a,\delta,\eta} \frac{\pi(y)}{\varphi(r)} \tag{4.9}$$

for $y \geqslant y_0(A, a, \delta, \eta)$. As before, (4.9) allows us to deduce that $\mathcal{R}_{\rm g}(y) \subseteq \mathcal{P}_{a,c,\eta} \cap (y, 2y]$. Combining this with (4.3) leads to

$$\pi_{a,c,\eta}(2y) - \pi_{a,c,\eta}(y) = \pi(2y) - \pi(y) + O_{A,a,c,\delta,\eta}\bigg(\frac{y}{(\log y)^A}\bigg).$$

This implies the required asymptotic formula (1.4).

Acknowledgements. This work is supported in part by Scientific Research Innovation Team Project Affiliated to Yangtze Normal University (No. 2016XJTD01).

*The proof is identical and the only difference is that we can take $z = (y/q)^{(1-\varepsilon)/2}$ and $D = z^2$ thanks to the Elliott-Halberstam conjecture $\mathrm{EH}^*_{\rm prime}[\varepsilon]$, instead of $z = (y/q)^{1/4}/(\log y)^B$ and $D = z^2$

References

[1] W. Banks & Igor E. Shparlinski, *On values taken by the largest prime factor of shifted primes*, J. Aust. Math. Soc. **82** (2007), 133–147.

[2] B. Chen & J. Wu, *On values taken by the largest prime factor of shifted primes (II)*, International J. Number Theory, **15** (2019), no. 5, 935–944.

[3] B. Feng & J. Wu, *On the density of shifted primes with large prime factors*, Science China Mathematics, **61** (2018), no. 1, 83–94.

[4] É. Fouvry & G. Tenenbaum, *Entiers sans grand facteur premier en progressions arithmétiques*, Proc. London Math. Soc. **63** (1991) 449–494.

[5] A. Hildebrand, *On the number of positive integers $\leqslant x$ and free of prime factors $> y$*, J. Number Theory, **22** (1986), 289–307.

[6] H. Iwaniec, *Rosser's sieve*, Acta Arith., **36** (1980), 171–202.

[7] H. Iwaniec, *A new form of the error term in the linear sieve*, Acta Arith., **37** (1980), 307–320.

[8] Jianya Liu, Jie Wu & Ping Xi, *Primes in arithmetic progressions with friable indices*, Science China Mathematics, to appear.

[9] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Translated from the second French edition (1995) by C. B. Thomas, Cambridge Studies in Advanced Mathematics **46**, Cambridge University Press, Cambridge, 1995. xvi+448 pp.

[10] N. K. Vishnoi, *Theoretical aspects of randomization in computation*, Ph. D. Thesis, Georgia Inst. of Tchnogy, 2004. (`http://smartech.gatech.edu:8282/dspace/handle/1853/5049`)

[11] D. Wolke, *Über die mittlere Verteilung der Werte zahlentheoretischer Funktionen auf Restklassen. II*, *Math. Ann.* **204** (1973), 145–153.

[12] J. Wu, *On values taken by the largest prime factor of shifted primes*, J. Aust. Math. Soc., to appear. `doi:10.1017/S144678871800023X`.

School of Mathematics and Statistics, Yangtze Normal University, Fuling, Chongqing 408100, China

Current address: CNRS LAMA 8050, Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est Créteil, 94010 Créteil Cedex, France

E-mail address: `jie.wu@math.cnrs.fr`