



HAL
open science

Relaxed solutions for incompressible inviscid flows: A variational and gravitational approximation to the initial value problem

Yann Brenier, Iván Moyano

► **To cite this version:**

Yann Brenier, Iván Moyano. Relaxed solutions for incompressible inviscid flows: A variational and gravitational approximation to the initial value problem. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 2022, 380 (2219), 10.1098/rsta.2021.0078 . hal-03212634

HAL Id: hal-03212634

<https://hal.science/hal-03212634>

Submitted on 29 Apr 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RELAXED SOLUTIONS FOR INCOMPRESSIBLE INVISCID FLOWS: A VARIATIONAL AND GRAVITATIONAL APPROXIMATION TO THE INITIAL VALUE PROBLEM

YANN BRENIER AND IVÁN MOYANO

ABSTRACT

Following Arnold’s geometric interpretation, the Euler equations of an incompressible fluid moving in a domain D are known to be the optimality equation of the minimizing geodesic problem along the group of orientation and volume preserving diffeomorphisms of D . This problem admits a well-established convex relaxation which generates a set of “relaxed”, “multi-stream”, version of the Euler equations. However, it is unclear that such relaxed equations are appropriate for the initial value problem and the theory of turbulence, due to their lack of well-posedness for most initial data. As an attempt to get a more relevant set of relaxed Euler equations, we address the multi-stream pressure-less gravitational Euler-Poisson system as an approximate model, for which we show that the initial value problem can be stated as a concave maximization problem from which we can at least recover a large class of smooth solutions for short enough times.

INTRODUCTION

In [5], the first author introduced the following “relaxed”, “multi-stream”, version of the Euler equations for an incompressible homogeneous fluid:

$$(0.1) \quad \partial_t c + \nabla \cdot q = 0, \quad c = c(t, x, a) \geq 0, \quad q = q(t, x, a) \in \mathbb{R}^d,$$

$$(0.2) \quad \int_a c(t, x, a) = 1,$$

$$(0.3) \quad E = -\nabla\psi(t, x) \in \mathbb{R}^d, \quad q = cv, \quad v = v(t, x, a) = \nabla\theta(t, x, a) \in \mathbb{R}^d,$$

$$(0.4) \quad \partial_t(cv) + \nabla \cdot (cv \otimes v) = cE,$$

where $t \in [0, T]$, $x \in D \subset \mathbb{R}^d$, $a \in \mathcal{A}$ and ∇ denotes the nabla operator

$$\nabla = \left(\frac{\partial}{\partial x^i} \right)_{i=1, \dots, d} \quad \text{on } \mathbb{R}^d.$$

The final time $T > 0$ is fixed and, for simplicity, we assume the space domain D to be the periodic cube $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. The space of labels \mathcal{A} is taken as a compact metric space equipped with a Borel probability measure μ (typically, the counting measure when \mathcal{A} is discrete and finite, or the Lebesgue measure when $\mathcal{A} = D$) and the integration in $a \in \mathcal{A}$ in formula (0.2) is performed according to μ . This corresponds to the description of an incompressible fluid viewed as a superposition of different streams, labelled by a , moving across each other with their own concentration $c = c(t, x, a) \geq 0$, potential velocity

$v = v(t, x, a) = \nabla\theta(t, x, a) \in \mathbb{R}^d$ and momentum $q(t, x, a) = c(t, x, a)v(t, x, a) \in \mathbb{R}^d$ fields, while they are driven by a *common* potential acceleration field $E = -\nabla p(t, x)$, independent of a , which maintains the equal occupation of volumes through (0.2). When \mathcal{A} is taken to be D itself, μ being the Lebesgue measure, these equations turn out to be the correct “relaxation” of the Euler equations when solving the minimizing geodesic problem (MGP) in the framework of their geometric interpretation, going back to Arnold [2], in terms of the geodesic flow over the group $SDiff(D)$ of all orientation and volume-preserving diffeomorphisms of D . Let us give a short explanation of this statement. The MGP can definitely be uniquely solved, in the classical setting of diffeomorphisms, in a small Sobolev neighborhood of the identity map, as shown by Ebin and Marsden [12], but not in the large as shown by Shnirelman [19], at least as $d \geq 3$. Indeed, minimizing sequences may develop small scale oscillations and admit no limit in the classical setting. Therefore, one has to complete $SDiff(D)$ by the set of “doubly stochastic measures” $DS(D)$ (also called “polymorphisms” as in [18]), i.e. the set all Borel measures c over $D \times D$ that project to the Lebesgue measure on each copy of D , each diffeomorphism $a \in D \rightarrow X(a) \in D$ of $SDiff(D)$ generating a corresponding c through $c(x, a) = \delta(x - X(a))$ or, more precisely,

$$\int_{D \times D} f(x, a)c(dx da) = \int_D f(X(a), a)da, \quad \forall f \in C(D \times D).$$

This completion process can be justified in a static way, as in [18], or, even better, in a dynamical way as in [20]. Then, as shown in [5, 1], the MGP can be “relaxed” as a convex minimization problem over $DS(D)$. The optimality equations turn out to be the “multi-stream” equations (0.1,0.2,0.3,0.4), in the special case when \mathcal{A} is taken as D . Nevertheless, the “relaxed” MGP, for which time-boundary data $c(0, x, a)$ and $c(T, x, a)$ are provided at both $t = 0$ and $t = T$, without any information on the velocity field, differs very much from the initial value problem (IVP), when both $c(0, x, a)$ and $v(0, x, a)$ are prescribed at the initial time $t = 0$, without any information on the final time T . Therefore, it is not clear that the “relaxed Euler equations” (0.1,0.2,0.3,0.4) are relevant as a good relaxation of the Euler equations for the IVP. Notice they can also be written in “Vlasov form”

$$(0.5) \quad \partial_t f(t, x, \xi) + \nabla_x \cdot (\xi f(t, x, \xi)) + \nabla_\xi \cdot (E(t, x) f(t, x, \xi)) = 0, \quad (t, x, \xi) \in [0, T] \times D \times \mathbb{R}^d,$$

$$(0.6) \quad \int_{\xi \in \mathbb{R}^d} f(t, x, \xi) = 1, \quad E = -\nabla\psi(t, x) \in \mathbb{R}^d,$$

where

$$(0.7) \quad f(t, x, \xi) = \int_{a \in \mathcal{A}} c(t, x, a) \delta(\xi - v(t, x, a)).$$

These equations can be seen as a “kinetic formulation” of the Euler equations. Observe that, after integration in $\xi \in \mathbb{R}^d$, we get for the “macroscopic” quantities

$$V(t, x) = \int_{\xi \in \mathbb{R}^d} \xi f(t, x, \xi), \quad M(t, x) = \int_{\xi \in \mathbb{R}^d} \xi \otimes \xi f(t, x, \xi),$$

the (incomplete) set of equations

$$\nabla \cdot V = 0, \quad \partial_t V + \nabla \cdot M + \nabla\psi = 0,$$

together with inequality $M \geq V \otimes V$, in the sense of symmetric matrices. In other words, (V, M) is a *subsolution* to the Euler equations, in the De Lellis-Székelyhidi framework of “convex integration” [10]. It is known that such subsolutions, at least when they are strict, i.e. $M > V \otimes V$, can be approximated by standard weak solutions to the Euler equations thanks to “convex integration” tools [10]. (We also refer to [5, 8] for a discussion about the connections between such a Vlasov-type formulation, the concept of “sharp measure-valued solutions” to the Euler equations and the older concept of “measure-valued solutions”.)

It is important to notice that equations (0.5,0.6) can also be thought as the highly singular limit of the well-known gravitational Vlasov-Poisson system,

$$(0.8) \quad \partial_t f(t, x, \xi) + \nabla_x \cdot (\xi f(t, x, \xi)) + \nabla_\xi \cdot (E(t, x) f(t, x, \xi)) = 0, \quad (t, x, \xi) \in [0, T] \times D \times \mathbb{R}^d,$$

$$(0.9) \quad 1 - \epsilon \nabla \cdot E(t, x) = \int_{\xi \in \mathbb{R}^d} f(t, x, \xi), \quad E = -\nabla \psi(t, x) \in \mathbb{R}^d,$$

as $\epsilon \downarrow 0$. In sharp contrast with the Vlasov-Poisson system, for which the IVP is well-posed, the limit equations are not well-posed in the usual sense [16], although there is a set of initial conditions, defined by a suitable “Penrose-Rayleigh” type condition around which the IVP is presumably well-posed [6]. As a matter of fact, the study of the limiting process is very delicate and is an active field of research, for which we refer, as a very recent reference, to [4]. The Vlasov-Poisson system itself admits a “multi-stream formulation” which has been used for a while, for instance by Grenier in [15] and recently by Baradat [4] (at least in the framework of Plasma Physics for which ϵ is negative). The corresponding “multi-stream” (pressure-less) Euler-Poisson system (MEPS) reads

$$(0.10) \quad \partial_t c + \nabla \cdot q = 0, \quad c = c(t, x, a) \geq 0, \quad q = q(t, x, a) \in \mathbb{R}^d,$$

$$(0.11) \quad 1 - \epsilon \nabla \cdot E(t, x) = \int_a c(t, x, a),$$

$$(0.12) \quad E = -\nabla \psi(t, x) \in \mathbb{R}^d, \quad q = cv, \quad v = v(t, x, a) = \nabla \theta(t, x, a) \in \mathbb{R}^d,$$

$$(0.13) \quad \partial_t (cv) + \nabla \cdot (cv \otimes v) = cE.$$

This system is (formally) energy conservative:

$$(0.14) \quad \frac{d}{dt} \left(\int_{(x,a) \in D \times \mathcal{A}} c(t, x, a) \frac{|v(t, x, a)|^2}{2} - \int_{x \in D} \epsilon \frac{|\nabla \psi(t, x)|^2}{2} \right) = 0.$$

Let us now sketch a physical interpretation of these equations. The MEPS describes the dynamics of a self-gravitating pressure-less multi-stream fluid, moving according to the standard Newtonian model of gravitation.

Each stream, labelled by some a in \mathcal{A} , admits its own concentration field $c(t, x, a) \geq 0$ and potential velocity field $v(t, x, a) = \nabla \theta(t, x, a) \in \mathbb{R}^d$, while the momentum $c(t, x, a)v(t, x, a)$ is denoted by $q(t, x, a)$. All together, these streams share a common acceleration field $E(t, x) = -\nabla \psi(t, x)$ where the gravitational potential ψ is coupled to them through the Poisson equation

$$(0.15) \quad \epsilon \Delta \psi(t, x) = 1 - \int_a c(t, x, a),$$

with $\epsilon > 0$, ϵ^{-1} being the gravitational constant, properly rescaled. We assume the initial concentration field c_0 to satisfy

$$\int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) = 1,$$

so that the “total mass” is always equal to 1:

$$(0.16) \quad \int_{(x,a) \in D \times \mathcal{A}} c(t, x, a) = 1,$$

because of (0.10) (since D is periodic). This way, in the Poisson equation (0.15), the right-hand side has zero mean over D , where constant 1 induces a background repulsive gravitational potential due to the periodicity of the spatial domain D (as usual in computational Cosmology [14, 9]).

Ultimately, our hope would be to adapt the relaxation technique successfully used for the MGP to address the IVP in the case of the Euler equations, with the goal of getting new “relaxed” equations, that might be of some interest for the theory of turbulence [13], beyond the one we started with, namely (0.1,0.2,0.3,0.4). As a preliminary step, we address in the present paper the case of the more accessible, but nevertheless interesting, gravitational Vlasov-Poisson system, with finite $\epsilon > 0$. In that case, we first explain, in the spirit of [5, 17], how the solutions of the MEPS can be recovered on a given time interval $[0, T]$ by the least action principle through a suitable space-time convex minimization problem, the concentration fields being prescribed at both $t = 0$ and $t = T$ without any information required on the velocity field. Next, we show that, surprisingly enough, through a suitable *augmented Lagrangian* technique, the initial value problem, when both concentration and velocity fields are prescribed at $t = 0$, without any information needed at time $t = T$, can *also* be relaxed as a (dual) concave maximization problem, through Theorem 3.1. We finally explain how, under some smallness condition on T , smooth solutions of the MEPS system can be recovered, through Theorem 4.1. Let us conclude this introduction by saying that the results obtained in the present paper for the multi-stream Euler-Poisson system MEPS for each fixed $\epsilon > 0$ give some hope that a similar analysis could be performed in the more challenging case of the multi-stream Euler equations as $\epsilon \downarrow 0$.

1. THE LEAST ACTION PRINCIPLE FOR THE MEPS

As already established by Loeper in [17] in the case of a single stream, the MEPS can be derived from the *convex* minimization problem in (c, q)

$$(1.17) \quad \inf_{c, q, E} \int_{Q'} \frac{|q|^2}{2c} + \int_Q \frac{\epsilon |E|^2}{2},$$

where

$$(1.18) \quad Q = [0, T] \times D, \quad Q' = Q \times \mathcal{A},$$

as the fields c, q, E are subject to *linear* constraints (0.10,0.11) and c is prescribed at both $t = 0$ and $t = T$. Indeed, using Lagrange multipliers for constraints (0.10,0.11), we

first get the equivalent saddle-point formulation of the minimization problem:

$$(1.19) \quad \inf_{c,q,E} \sup_{\theta,\psi} BT_0(\theta) + \int_{(t,x) \in Q} \psi + \epsilon E \cdot \nabla \psi + \frac{\epsilon |E|^2}{2} + \int_{(t,x,a) \in Q'} \frac{|q|^2}{2c} - (\partial_t \theta + \psi)c - q \cdot \nabla \theta,$$

where

$$(1.20) \quad BT_0(\theta) = \int_{(x,a) \in D \times \mathcal{A}} c_T(x,a)\theta(T,x,a) - c_0(x,a)\theta(0,x,a).$$

Then, differentiating the Lagrangian with respect to c , q and E , we get

$$q = c \nabla \theta, \quad -\frac{|q|^2}{2c^2} - \partial_t \theta - \psi = 0, \quad E = -\nabla \psi,$$

which implies

$$(1.21) \quad \partial_t \theta(t,x,a) + \frac{|\nabla \theta(t,x,a)|^2}{2} + \psi(t,x) = 0, \quad \epsilon \Delta \psi(t,x) = \int_a c(t,x,a) - 1,$$

leading to (0.12) and (0.13) using (0.10).

2. SADDLE-POINT FORMULATION OF THE INITIAL VALUE PROBLEM (IVP)

We now address the IVP by *adding* to the Lagrangian the extra term

$$\int_{(t,x,a) \in Q'} -\partial_t A \cdot q - \nabla A \cdot \left(\frac{q \otimes q}{c}\right) - cA \cdot E - \int_{(x,a) \in D \times \mathcal{A}} A(0,x,a) \cdot q_0(x,a),$$

which takes into account the *weak* formulation of (0.13), where $A = A(t,x,a) \in \mathbb{R}^d$ is any test function subject to $A(T,x,a) = 0$. Then, we get a new saddle-point problem

$$(2.22) \quad I(c_0, q_0) = \inf_{c,q,E} \sup_{\theta,A,\psi} BT(\theta, A) + \int_{(t,x) \in Q} \psi + \epsilon E \cdot \nabla \psi + \frac{\epsilon |E|^2}{2} \\ + \int_{(t,x,a) \in Q'} \frac{|q|^2}{2c} - (\partial_t \theta + \psi)c - q \cdot \nabla \theta - \partial_t A \cdot q - \nabla A \cdot \left(\frac{q \otimes q}{c}\right) - cA \cdot E$$

where

$$(2.23) \quad BT(\theta, A) = - \int_{(x,a) \in D \times \mathcal{A}} c_0(x,a)\theta(0,x,a) + q_0(x,a) \cdot A(0,x,a)$$

and test functions θ and A must vanish at time $t = T$. This ‘‘augmented Lagrangian’’ strategy allows us to input initial condition q_0 , together with c_0 , while data c_T is no longer needed. This idea has been already applied by the first author to the Euler equations of incompressible fluids and, also, to the class of first-order systems of conservation laws with a convex entropy in [7] and extended to various interesting models (including ideal MHD) by Vorotnikov in [21].

3. DUAL FORMULATION OF THE IVP FOR THE MEPS

By exchanging the sup and the inf, we obtain the "dual" problem

$$(3.24) \quad J(c_0, q_0) = \sup_{\theta, A, \psi} \inf_{c, q, E} (\cdot).$$

Observe that we have a priori no more than the "weak" duality property

$$I(c_0, q_0) = \inf_{c, q, E} \sup_{\theta, A, \psi} (\cdot) \geq \sup_{\theta, A, \psi} \inf_{c, q, E} (\cdot) = J(c_0, q_0).$$

Indeed, by augmenting the Lagrangian, we have destroyed the convex structure of the original problem, because of the nonlinear term $q \otimes q/c$ and, therefore, a duality gap cannot be excluded. Anyway, the dual problem corresponds to a *concave* maximization problem in (θ, A, ψ) , due to the linearity of weak formulations with respect to their test functions. Let us compute it more explicitly. First, we perform the infimum in q . We see that this infimum is $-\infty$ unless A satisfies the inequality constraint

$$(3.25) \quad (\nabla A + \nabla A^T)(t, x, a) \leq I,$$

pointwise in the sense of symmetric matrices, where I denotes the identity $d \times d$ matrix. In that case, the optimal value of q is obtained as

$$(I - \nabla A - \nabla A^T)q = (\partial_t A + \nabla \theta)c$$

and we find, after minimization in q ,

$$(3.26) \quad J(c_0, q_0) = \sup_{\theta, A, \psi} BT(\theta, A) + \inf_{c \geq 0, E} \int_Q \psi + \epsilon E \cdot \nabla \psi + \frac{\epsilon |E|^2}{2} \\ - \int_{Q'} \left(\frac{1}{2} (I - \nabla A - \nabla A^T)^{-1} (\partial_t A + \nabla \theta)^{\otimes 2} + \partial_t \theta + \psi + E \cdot A \right) c.$$

(Here notation $Mw^{\otimes 2}$ stands for $w \cdot (Mw)$ whenever M is a $d \times d$ matrix and w is a vector in \mathbb{R}^d .) Let us now perform the infimum in $c \geq 0$. We first observe that

$$\inf_{c \geq 0, E} \leq \inf_{c \geq 0, E=0}$$

which is $-\infty$ unless

$$(3.27) \quad -\eta(t, x, a) = \left(\frac{1}{2} (I - \nabla A - \nabla A^T)^{-1} (\partial_t A + \nabla \theta)^{\otimes 2} + \partial_t \theta \right)(t, x, a) + \psi(t, x) \leq 0.$$

Thus, we may now perform the infimum in c , under this second inequality constraint, and immediately obtain

$$(3.28) \quad J(c_0, q_0) = \sup_{\theta, A, \psi} BT(\theta, A) + \inf_E \int_Q \psi + \frac{\epsilon |E|^2}{2} + \epsilon E \cdot \nabla \psi$$

where E is subject to the pointwise inequality

$$E(t, x) \cdot A(t, x, a) \leq \eta(t, x, a).$$

So, we have finally obtained:

Theorem 3.1. *The concave dual maximization problem, proposed to solve the IVP for the MEPS (0.10,0.11,0.12,0.13) with initial condition (c_0, q_0) , reads:*

$$(3.29) \quad J(c_0, q_0) = \sup_{\theta, A, \psi} BT(\theta, A) + \int_{(t,x) \in Q} (\psi(t, x) - \epsilon K_{(A, \eta)(t, x, \cdot)}(\nabla \psi(t, x))) dxdt,$$

where (θ, A) must vanish at $t = T$, A is subject to $(\nabla A + \nabla A^T)(t, x, a) \leq I$,

$$(3.30) \quad \eta(t, x, a) = -\left(\frac{1}{2}(I - \nabla A - \nabla A^T)^{-1}(\partial_t A + \nabla \theta)^{\otimes 2} + \partial_t \theta\right)(t, x, a) - \psi(t, x)$$

must be nonnegative, and we use notations:

$$BT(\theta, A) = - \int_{(x,a) \in D \times \mathcal{A}} c_0(x, a)\theta(0, x, a) + q_0(x, a) \cdot A(0, x, a),$$

$$K_{(A, \eta)(\cdot, \cdot, \cdot)}(B) = - \inf \left\{ \frac{|E|^2}{2} + E \cdot B, \quad E \in \mathbb{R}^d \text{ s. t. } E \cdot A(\cdot, \cdot, a) \leq \eta(\cdot, \cdot, a), \quad \forall a \in \mathcal{A} \right\}.$$

4. A “NO DUALITY-GAP” RESULT

Theorem 4.1. *Let $(c^s > 0, q^s, E^s)$ be a smooth solution on $Q' = [0, T] \times D \times \mathcal{A}$ to the MEPS of form*

$$q^s = c^s v^s, \quad v^s(t, x, a) = \nabla \theta^s(t, x, a), \quad E^s(t, x) = -\nabla \psi^s(t, x).$$

We make two assumptions on such a solution. First,

$$(4.31) \quad (\nabla v^s + (\nabla v^s)^T)(t, x, a) < \frac{I}{T-t},$$

holds true pointwise in Q' , in the sense of symmetric matrices, where I denotes the identity $d \times d$ matrix. Second, the velocity field v^s is “weakly absorbing” in the sense that, for all fixed $(t, x) \in Q$ and for all vector V in \mathbb{R}^d , there is a nonnegative measure λ (depending on t, x and V) on \mathcal{A} such that

$$(4.32) \quad V = \int_{a \in \mathcal{A}} v^s(t, x, a) \lambda(da).$$

Then there is no duality gap between the primal and dual problems, namely

$$I(c_0, q_0) = J(c_0, q_0).$$

Moreover, a solution of the dual problem (3.29) is explicitly given by

$$(4.33) \quad A(t, x, a) = (t - T)v^s(t, x, a), \quad \psi(t, x) = \partial_t((T - t)\psi^s(t, x)),$$

$$\theta(t, x, a) = (t - T)\left(\psi^s(t, x) - \frac{|v^s(t, x, a)|^2}{2}\right).$$

N.B. In these notations, the superscript “s” stands for “solution” and enables us to make a clear and crucial distinction between the solution and the various test functions used in the following calculations.

Remark 4.1. Notice that assumption (4.32) is automatically satisfied as soon as, for each (t, x) the convex hull of the velocity range $\{v^s(t, x, a), a \in \mathcal{A}\}$ contains a small open ball containing 0 in \mathbb{R}^d . Surprisingly enough, in the case of a single stream, i.e. when there is only one label a , condition (4.32) implies $v^s = 0$ and rules out all non trivial solutions!

Proof of Theorem 4.1.

Step 1. Since (c^s, q^s, E^s) is a smooth solution to the MEPS with initial condition (c_0, q_0) , it weakly solves both (0.10) and (0.13). Thus, for all (θ, A) that vanish at $t = T$, we get

$$BT(\theta, A) + \int_{Q'} -\partial_t \theta c^s - q^s \cdot \nabla \theta - \partial_t A \cdot q^s - \nabla A \cdot \left(\frac{q^s \otimes q^s}{c^s} \right) - c^s A \cdot E^s = 0,$$

where we recall that the time-boundary term BT is defined by (2.23). We also have (0.11) which implies, for all ψ :

$$\int_{(t,x) \in Q} \psi(t, x) + \epsilon E^s \cdot \nabla \psi(t, x) - \int_{(t,x,a) \in Q'} \psi(t, x) c^s(t, x, a) = 0.$$

So, the supremum in (θ, A, ψ) of

$$BT(\theta, A) + \int_Q \psi^s + \epsilon E^s \cdot \nabla \psi^s + \frac{\epsilon |E^s|^2}{2} \\ + \int_{Q'} \frac{|q^s|^2}{2c^s} - (\partial_t \theta + \psi) c^s - q^s \cdot \nabla \theta - \partial_t A \cdot q^s - \nabla A \cdot \left(\frac{q^s \otimes q^s}{c^s} \right) - c^s A \cdot E^s,$$

is just equal to

$$\int_Q \frac{\epsilon |E^s|^2}{2} + \int_{Q'} \frac{|q^s|^2}{2c^s},$$

where we recall that $q^s = cv^s$, $v^s = \nabla \theta^s$ and $E^s = -\nabla \psi^s$. By definition (2.22) of the primal problem, this supremum is an upper bound for $I(c_0, q_0)$ and, therefore, for $J(c_0, q_0)$ as well. Thus, at the end of this first step, we already have

$$(4.34) \quad \int_Q \frac{\epsilon |\nabla \psi^s|^2}{2} + \int_{Q'} \frac{c |v^s|^2}{2} \geq I(c_0, q_0) \geq J(c_0, q_0).$$

Step 2. Let us now move to the dual side. By definition (3.29), we get the lower bound

$$J(c_0, q_0) \geq BT(\theta, A) + \int_Q \psi + \inf_{c,q,E} \int_Q \epsilon E \cdot \nabla \psi + \frac{\epsilon |E|^2}{2} \\ + \int_{Q'} \frac{|q|^2}{2c} - (\partial_t \theta + \psi) c - q \cdot \nabla \theta - \partial_t A \cdot q - \nabla A \cdot \left(\frac{q \otimes q}{c} \right) - c A \cdot E,$$

whatever is our choice of (θ, A, ψ) . Let us make the “ansatz”

$$(4.35) \quad A = (t - T) v^s(t, x), \quad \theta = (t - T) \left(\psi^s(t, x) - \frac{|v^s(t, x, a)|^2}{2} \right),$$

while, the choice of ψ will be made later. So, (A, θ, ψ) being fixed, we may write our lower bound as

$$(4.36) \quad J(c_0, q_0) \geq \mathcal{J} + BT(\theta, A) + \int_Q \psi,$$

where we denote

$$(4.37) \quad \mathcal{J} = \inf_{c,q,E} \int_Q \epsilon E \cdot \nabla \psi + \frac{\epsilon |E|^2}{2}$$

$$+ \int_{Q'} \frac{|q|^2}{2c} - (\partial_t \theta + \psi)c - q \cdot \nabla \theta - \partial_t A \cdot q - \nabla A \cdot \left(\frac{q \otimes q}{c} \right) - cA \cdot E.$$

Let us first focus on the infimum in q in this definition of \mathcal{J} . Since we have assumed $(T - t)(\nabla v^s + (\nabla v^s)^T) < I$, inequality $(I - \nabla A - \nabla A^T) > 0$ holds true in the sense of symmetric matrices and, therefore, the infimum in q is obtained for

$$(I - \nabla A - \nabla A^T)q = (\partial_t A + \nabla \theta)c$$

i.e., setting $q = cv$, as

$$(I - \nabla A - \nabla A^T)v = \partial_t A + \nabla \theta,$$

which reads, by definition of (θ, A) :

$$(I - (t - T)\nabla v^s - (t - T)(\nabla v^s)^T)v = v^s + (t - T)\partial_t v^s + (t - T)\nabla(\psi^s - \frac{|v^s|^2}{2}),$$

or, equivalently,

$$v - v^s = (t - T) \left(\partial_t v^s + (\nabla v^s + (\nabla v^s)^T)v + \nabla(\psi^s - \frac{|v^s|^2}{2}) \right).$$

Observe that this equation admits $v = v^s = \nabla \theta^s$ as an obvious solution. Indeed, we have

$$\partial_t v^s + (\nabla v^s + (\nabla v^s)^T)v^s + \nabla(\psi^s - \frac{|v^s|^2}{2}) = \nabla \left(\partial_t \theta^s + \frac{|\nabla \theta^s|^2}{2} + \psi^s \right) = 0$$

since θ^s and ψ^s solve (1.21). At this stage, we may therefore rewrite \mathcal{J} defined by (4.37) as

$$\begin{aligned} \mathcal{J} &= \inf_{c \geq 0, E} \int_Q \epsilon E \cdot \nabla \psi + \frac{\epsilon |E|^2}{2} \\ &+ \int_{Q'} \left(\frac{|v^s|^2}{2} - \partial_t \theta - \psi - v^s \cdot \nabla \theta - \partial_t A \cdot v^s - \nabla A \cdot (v^s \otimes v^s) - A \cdot E \right) c. \end{aligned}$$

The infimum with respect to $c \geq 0$ immediately leads to the new value of \mathcal{J} :

$$(4.38) \quad \mathcal{J} = \inf \int_Q \epsilon E \cdot \nabla \psi + \frac{\epsilon |E|^2}{2}$$

where E is subject to the pointwise inequality

$$\frac{|v^s|^2}{2} - \partial_t \theta - \psi - v^s \cdot \nabla \theta - \partial_t A \cdot v^s - \nabla A \cdot (v^s \otimes v^s) - A \cdot E \geq 0.$$

Let us recall that θ, A have been fixed according to (4.35), i.e. :

$$A = (t - T)v^s, \quad \theta = (t - T)(\psi^s - \frac{|v^s|^2}{2}).$$

So, our inequality actually reads

$$\begin{aligned} &\frac{|v^s|^2}{2} - \partial_t((t - T)(\psi^s - \frac{|v^s|^2}{2})) - \psi - (t - T)v^s \cdot \nabla(\psi^s - \frac{|v^s|^2}{2}) \\ &- \partial_t((t - T)v^s) \cdot v^s - (t - T) \cdot \nabla v^s \cdot (v^s \otimes v^s) - (t - T)v^s \cdot E \geq 0, \end{aligned}$$

which simply reduces to

$$-\partial_t((t - T)\psi^s) - \psi - (t - T)v^s \cdot (\nabla \psi^s + E) \geq 0.$$

At this stage, a natural choice for ψ (which has not been made so far) is

$$(4.39) \quad \psi = -\partial_t((t-T)\psi^s)$$

so that now our inequality just reads

$$v^s \cdot (\nabla\psi^s + E) \geq 0.$$

So we are finally left with minimizing in E

$$\int_Q \epsilon E \cdot \nabla\psi + \frac{\epsilon|E|^2}{2}$$

where E is subject to

$$v^s(t, x, a) \cdot (\nabla\psi^s + E)(t, x) \geq 0, \quad \forall (t, x, a) \in Q'.$$

According to assumption (4.32), which means that the velocity field v^s is “weakly absorbing”, $E = -\nabla\psi^s$ turns out to be the only possible choice for E ! Indeed, once (t, x) is fixed, let us set $V = -(\nabla\psi + E)(t, x)$, which must satisfy

$$v^s(t, x, a) \cdot V \leq 0, \quad \forall a \in \mathcal{A}.$$

By (4.32), we can write

$$V = \int_{a \in \mathcal{A}} v^s(t, x, a) \lambda(da)$$

for some nonnegative measure λ (that, of course, depends on t, x and V) and, therefore,

$$|V|^2 = \int_{a \in \mathcal{A}} V \cdot v^s(t, x, a) \lambda(da) \leq 0$$

which implies $V = 0$, i.e. $E(t, x) = -\nabla\psi^s(t, x)$. So, the value of \mathcal{J} is just

$$\mathcal{J} = \int_Q -\epsilon \nabla\psi^s \cdot \nabla\psi + \frac{\epsilon |\nabla\psi^s|^2}{2}.$$

Since $\psi = -\partial_t((t-T)\psi^s)$, we have

$$\begin{aligned} \int_Q -\nabla\psi^s \cdot \nabla\psi &= \int_Q \nabla\psi^s \cdot \partial_t((t-T)\nabla\psi^s) = \int_Q |\nabla\psi^s|^2 + (t-T)\partial_t\left(\frac{|\nabla\psi^s|^2}{2}\right) \\ &= \int_Q \frac{|\nabla\psi^s|^2}{2} + T \int_{x \in D} \frac{|\nabla\psi^s(0, x)|^2}{2} \end{aligned}$$

so that

$$(4.40) \quad \mathcal{J} = \int_Q \epsilon |\nabla\psi^s|^2 + T \int_{x \in D} \frac{\epsilon |\nabla\psi^s(0, x)|^2}{2}$$

After the computation of \mathcal{J} we have just performed, we still need an evaluation of both $BT(\theta, A)$ and $\int_Q \psi$ in our lower bound (4.36). Concerning the first one, we have, by definition (2.23),

$$\begin{aligned} BT(\theta, A) &= - \int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) \theta(0, x, a) + q_0(x, a) \cdot A(0, x, a) \\ &= T \int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) \left(\psi^s(0, x) - \frac{|v^s(0, x, a)|^2}{2} \right) + q_0(x, a) \cdot v^s(0, x, a) \end{aligned}$$

(since θ and A are given by (4.35))

$$= T \int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) (\psi^s(0, x) + \frac{|v^s(0, x, a)|^2}{2})$$

(since $c_0 v^s(0, \cdot) = q_0$). Because of (1.21), we have $1 + \epsilon \Delta \psi^s = \int_a c$ and, therefore,

$$\int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) \psi^s(0, x) = \int_{x \in D} (1 + \epsilon \Delta \psi^s(0, x)) \psi^s(0, x) = \int_{x \in D} \psi^s(0, x) - \epsilon |\nabla \psi^s(0, x)|^2.$$

Thus

$$BT(\theta, A) = T \int_{x \in D} \psi^s(0, x) - \epsilon |\nabla \psi^s(0, x)|^2 + T \int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) \frac{|v^s(0, x, a)|^2}{2}.$$

Let us now move to the second term $\int_Q \psi$. From (4.35)), we get

$$\int_Q \psi = \int_Q -\partial_t((t - T)\psi^s) = -T \int_{x \in D} \psi^s(0, x).$$

So, we have found

$$BT(\theta, A) + \int_Q \psi = -T \int_{x \in D} \epsilon |\nabla \psi^s(0, x)|^2 + T \int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) \frac{|v^s(0, x, a)|^2}{2}.$$

Using the value of \mathcal{J} given by (4.40), we see that our lower bound (4.36) now reads

$$(4.41) \quad J(c_0, q_0) \geq T \int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) \frac{|v^s(0, x, a)|^2}{2} + \int_Q \epsilon |\nabla \psi^s|^2 - T \int_{x \in D} \frac{\epsilon |\nabla \psi^s(0, x)|^2}{2}$$

Final step. Collecting the results of Step 1 and Step 2, namely (4.34) and (4.41), we have finally obtained

$$\begin{aligned} 0 \leq I(c_0, q_0) - J(c_0, q_0) &\leq -T \int_{(x,a) \in D \times \mathcal{A}} c_0(x, a) \frac{|v^s(0, x, a)|^2}{2} + T \int_{x \in D} \frac{\epsilon |\nabla \psi^s(0, x)|^2}{2} \\ &\quad - \int_Q \frac{\epsilon |\nabla \psi^s|^2}{2} + \int_{Q'} \frac{c |v^s|^2}{2}. \end{aligned}$$

As a matter of fact, the right-hand side is exactly zero because of the conservation of energy (0.14). So there is no duality gap and the proof of Theorem 4.1 is now complete.

Acknowledgment. This work originated from a visit of the first author at the Department of Pure Mathematics and Mathematical Statistics (DPMMS), University of Cambridge, in May 2018, supported by the grant MAFRAN. A part of the work was performed while he was visiting the Hausdorff Research Institute for Mathematics, Bonn, in 2019, during the program: “Interfaces and Instabilities in Fluid Dynamics” co-organized by L. Székelyhidi Jr. and G. Weiss.

REFERENCES

- [1] L. Ambrosio, A. Figalli, *Geodesics in the space of measure-preserving maps and plans*, Archive for Rational Mechanics and Analysis 194 (2009) 421-462.
- [2] V. I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Institut Fourier 16 (1966) 319-361.
- [3] V.I. Arnold, B. Khesin, *Topological methods in hydrodynamics*, *Applied Mathematical Sciences, 125*, Springer-Verlag 1998.
- [4] A. Baradat, *Nonlinear instability in Vlasov type equations around rough velocity profiles*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire 37 (2020) 489-547.
- [5] Y. Brenier, *Minimal geodesics on groups of volume-preserving maps*, Commun. Pure Appl. Math 52 (1999) 411-452.
- [6] Y. Brenier, *Homogeneous hydrostatic flows with convex velocity profiles*, Nonlinearity 12 (1999) 495-512.
- [7] Y. Brenier, *The initial value problem for the Euler equations of incompressible fluids viewed as a concave maximization problem*, Comm. Math. Phys. 364 (2018) 579-605.
- [8] Y. Brenier, C. De Lellis, L. Székelyhidi, László, Jr. *Weak-strong uniqueness for measure-valued solutions*, Comm. Math. Phys. 305 (2011) 351-361.
- [9] Y. Brenier, U. Frisch, M. Hénon, G. Loeper, S. Matarrese, Mohayaee, Sobolevskii, *Reconstruction of the early universe as a convex optimization problem*, Mon. Not. R. Astron. Soc. 2002.
- [10] C. De Lellis, L. Székelyhidi Jr, *On turbulence and geometry: from Nash to Onsager*, Notices Amer. Math. Soc. 66 (2019) 677-685.
- [11] D. Ebin, *The motion of slightly compressible fluids viewed as a motion with strong constraining force*, Ann. of Math. (2) 105 (1977) 141-200.
- [12] D. Ebin, J. Marsden, *Groups of diffeomorphisms and the notion of an incompressible fluid*, Ann. of Math. 92 (1970) 102-163.
- [13] U. Frisch, *Turbulence. The legacy of A. N. Kolmogorov*, Cambridge University Press, 1995.
- [14] U. Frisch, S. Matarrese, R. Mohayaee, A. Sobolevski, *A reconstruction of the initial conditions of the Universe by optimal mass transportation*, Nature 417 (2002) 260-262.
- [15] E. Grenier, *Oscillations in quasineutral plasmas*, Communications in PDEs 21(1996) 363-394.
- [16] D. Han-Kwan, T. Nguyen, *Ill-posedness of the hydrostatic Euler and singular Vlasov equations*, Archive for Rational Mechanics and Analysis 221(2016) 1317-1344.
- [17] G. Loeper, *The reconstruction problem for the Euler-Poisson system in cosmology*, Arch. Ration. Mech. Anal. 179 (2006) 153-216.
- [18] Y. Neretin, *Categories of bistochastic measures and representations of some infinite-dimensional groups*, Sb. 183 (1992), no. 2, 52-76.
- [19] A. Shnirelman, *On the geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid*, Math. Sbornik USSR 56 (1987) 79-105.
- [20] A. I. Shnirelman, *Generalized fluid flows, their approximation and applications*, Geom. Funct. Anal. 4 (1994) 586-620.
- [21] D. Vorotnikov, *Partial differential equations with quadratic nonlinearities viewed as matrix-valued optimal ballistic transport problems*, arXiv:1905.06059.

Y.B.: CNRS, DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS, ECOLE NORMALE SUPÉRIEURE, UNIVERSITÉ PSL, 45 RUE D'ULM 75005 PARIS,

I.M.: LABORATOIRE JEAN ALEXANDRE DIEUDONNÉ, UNIVERSITÉ CÔTE-D'AZUR, PARC VALROSE 06108 NICE CEDEX 02.