NEW DIVISIBILITY TESTS

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Abstract

In this article, we first made the inventory of some various divisibility tests of 7 known, studied the effectiveness of these tests, demonstrate the new divisibility test of 7 proposed by the young Nigerian Chika Ofili and an algorithm in #Python. Secondary, we proposed and demonstrated a new test of divisibility by 3 which we called in this paper Diouf-Pagdame Test based mainly on the calculation of $d - 2e$ where d and e represents respectively digits of tens and units of any given number. We have also emphasized the educational aspect of these methods.

1. Introduction

The divisibility of a number by 2 or 5 is very simple. It is known that a number N is divisible by 2 if its last digit is an even integer and is divisible by 5 if its last digit is 0 or 5. A number N is divisible by 3 if the sum of its digits is divisible by 3. However, these three tests can be easily explained using modulo arithmetic.

In this paper, we only deal with the case of the divisibility by 7 test and

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briefly discuss the 3-divisibility test. The 7-divisibility test has drawn the interest of some researchers. This gave birth to several techniques of divisibility. However, some tests have been proven more effective than others have, especially when it comes to verify the divisibility by 7 of a large number. In fact, if the divisibility of some numbers seems easy, it is however not the case of the test divisibility by 7. Some tests allow it to be done easily with few lines whereas for others, we need a lot of pages. It is in this thought that the recent discovery of a Nigerian youth Chika Ofili has caused a lot of ink in the newspapers, which had allowed him to win a British prize for the discovery of a new mathematical formula. Before, its discovery there were many existence tests. Among those tests, we have the Toja's test, the property of multiples of 7, the criterion for a large number, the method of utilization of a diagram, the criterion Pascal divisibility, and so on. These new tests of divisibility can be used for educational purpose due to their simplicity and efficiency. Therefore, we think we can start using them earlier in school environment.

In this article, we will list some known tests, demonstrate the divisibility test of 7 proposed by Chika, state and demonstrate a new divisibility test of 3 that we have discovered, compare these tests and determine the most effective of these tests by comparing the execution times of different algorithms in Python of each methods.

2. Historical Background

We can define a divisibility test as an algorithm, which uses the digits of an integer N to determine whether N is divisible by a divisor d . The history of divisibility tests started back from at least 500 C.E. when a divisibility test for 7 was included in the Babylonian Talmud. It was from then that methods which provide divisibility tests for all positive integers have been discovered and rediscovered by different mathematicians such as Joseph-Louis Lagrange [10], and Charles Dodgson (a.k.a. Lewis Carroll, author of Alice In Wonderland) [8]. A remarkable summary of the literature in relation to divisibility tests published prior to 1915 is provided in Leonard Dickson's History of the

Theory of Numbers [3]. Edward Brooks dedicated two chapters to the study of divisibility tests in his 1880 book titled *The Philosophy of Arithmetic* [2]. Recently in (2006), Marc Renault published a wonderful article [11] that provides divisibility tests for all integers between 2 and 102, which includes brief explanations for why the tests work. And lastly, Eric L. McDowell (Berry College) in *Divisibility Tests: A History and User's Guide* presents a modest survey of some of the more recent literature regarding divisibility tests.

3. Some Existing Divisibility Tests of 7

3.1. Property of Multiples of 7

There are already some effective methods that will allow to verify the divisibility of a given number. Among those tests, we can cite property of multiples of 7. The principle is quite simple. This test actually consists of verifying if the difference of the tens digit by the double of the units is a multiple of 7. If that difference is a multiple of 7, then the number to be tested is also a multiple of 7.

3.2. Criterion for a large Number

A second method based only on the fact that 103 is congruent to -1 modulo 7, is to separate this number in 3-digit slices starting from the units and insert alternately - and + between the slices. We carry out the operation thus written and this result is divisible by 7 if and only if the starting number was.

Example Let us take the number 5527579818992. We will separate it into three-digit sections starting from the units:

$$5 \mid 527 \mid 579 \mid 818 \mid 992. \quad (1)$$

We interpose alternately - and +:

$$5 - 527 + 579 - 818 + 992. \quad (2)$$

We carry out the operation thus written:

$$5 - 527 + 579 - 818 + 992 = 231. \quad (3)$$

We see if 231 is divisible using the divisibility lemma by 7 : $23 + 5 \times 1 = 28$ is divisible by 7 so 5527579818992 is.

Proof. Since $1000 \equiv -1 \pmod{7}$ and $1000000 \equiv 1 \pmod{7}$, then for all integer $n > 0$, we have: $(10^3)^{2n} \equiv 1 \pmod{7}$ and $(10^3)^{2n-1} \equiv -1 \pmod{7}$.

Any integer N can be written in the form:

$a \times (10^3)^{2n} + b \times (10^3)^{2n-1} + \dots + x(10^3) + y \times 10^3 + z$. Hence we can directly conclude that $N \equiv 0 \pmod{7}$ if $z - y + x - \dots - b + a \equiv 0 \pmod{7}$.

3.3. The Toja's Method

This third method is similar to the second one based only on the fact that 10^3 is congruent to -1 modulo 7, from which we deduct that if

si

$$x = 100^m b_m + \dots + 100^2 b_2 + 100^1 b_1 + b_0$$

and

$$y = 10^m b_0 - 10^{m-1} b_1 + 10^{m-2} b_2 \dots (-1)^m b_m \quad (4)$$

then

$$10^m x \equiv y \pmod{7}. \quad (5)$$

Therefore, x is divisible by 7 if and only if y is. We can definitely replace by the way every b_i by any integer, which is congruent to modulo 7. The principle is therefore to cut the number x into slices of 2 digits and find the distance between each number of 2 digits and the nearest multiple of 7 (alternately by excess and by default).

Example 2.3. Let us take the number 5527579818992. We will separate it into two-digit sections starting from the units $5 \mid 52 \mid 75 \mid 79 \mid 81 \mid 89 \mid 92$.

- From the right, the nearest multiple of 7 by default is 91: the distance $92 - 91 = 1$;
- For the second pair, the nearest multiple of 7 by excess is 91: the distance $91 - 89 = 2$;
- For the third pair, the nearest multiple of 7 by default is 77: the distance $81 - 77 = 4$;
- For the fourth pair, the distance: $84 - 79 = 5$, and so on.

The starting number is a multiple of 7 if and only if $1 \mid 2 \mid 4 \mid 5 \mid 5 \mid 4 \mid 5$ is a multiple of 7 (The different "remainders" are writtens in reverse order). We also notice that the divisibility by 7 of 1245545 is equal to that of 3136, and of 14, therefore 5527579818992 is divisible by 7.

3.4. Method of utilization of a diagram

This technique is based [2] on the writing of a number in base 10 and on the modulo 7 congruence. The use of a diagram was proposed in 2009 by David Wilson [3, 4]. On a circle, we have all the numbers from 0 to 6, i.e. all possible modulo 7 remainders. We then match with an arrow each remainder r with the rest modulo 7 of $r * 10$.

The diagram is then used as follows: for the integer: $\overline{a_n a_{n-1} \dots a_1 a_0}$ equal to

$$(\dots((a_n * 10 + a_{n-1}) * 10 + a_{n-2}) * 10 + \dots) * 10 + a_0. \quad (6)$$

- We stand on the case 0 and we are moving on the circle of a_n cases. We then obtain the remainder of an modulo 7;
- We then borrow the arrow, which starts from the square where we are and, from the point of arrival of the arrow, we move on the circle of a_{n-1} boxes. We then obtain the remainder of $a_n * 10 + a_{n-1}$ modulo 7;
- We then repeat the process (borrowing an arrow, and then moving on the circle) until a_0 .

The number is divisible by 7 if and only if the arrival box is box 0.

This figure is:

Standard and we move on the boxes according to the procedure stated previously.

Example 2.4. Suppose we want to verify the divisibility by 7 of 4564. Here $a_0 = 4$, $a_1 = 6$, $a_2 = 5$ et $a_3 = 4$.

- We move from 0 to $a_3 = 4$ boxes and we arrive at 4;
- We take the arrow which leads from 4 to 5. We move from $a_2 = 5$ boxes and we arrive at 3;
- We take the arrow which leads from 3 to 2. We move from $a_1 = 6$ boxes and we arrive at 1;
- We take the arrow which leads from 1 to 3. We move from $a_0 = 4$ boxes and we reach 0.

The arrival box being 0, then the number 4564 is a multiple of 7.

3.5. Pascal's Criterion of divisibility

We divide 1 by 7, and we write at each stage of the calculation the successive remains. We obtain: ..., 5, 4, 6, 2, 3 and 1, if we reverse the order.

This number is called Pascal's key.

With the number to be tested, we multiply the number of units by 1, the tens digit by 3, the hundreds by 2, and so on. Finally, we add the results obtained. If the number obtained is a multiple of 7, then the initial number to be tested is a multiple of 7.

Example 2.5. 644 is a multiple of 7, because $4 \times 1 + 4 \times 3 + 6 \times 2 = 28$ which is a multiple of 7. So therefore, 644 is also a multiple of 7.

3.5.1. Proof of Pascal's criterion

If we perform the division of 1 by 7, we obtain 0; 142857 and a sequence of remains in the division: 1, 3, 2, 6, 4, 5, ...

For example, for the first decimal places of 0; 142857, we have

$$7 \times 0 + 1 = 10^0$$

$$7 \times 1 + 3 = 10^1$$

$$7 \times 14 + 2 = 10^2$$

$$7 \times 142 + 6 = 10^3 \text{ etc...}$$

this table is presented as a bridge which links base 10 and base 7.

Let r_n the rest of the division for the n^{th} power of 10 and d_n , the number $1/7$ estimated at n decimal places multiplied by 10^n .

We have then $7 \times d_n + r_n = 10^n$ which is a generalisation of the precedent table.

Let N a number of n digits such that $N = N_n \times 10^n + N_{n-1} \times 10^{n-1} + \dots + N_1 \times 10^1 + N_0$ with $N_i \in [0, 9]$ and $N_i \in \mathbb{N}$.

By applying the calculation suggested by the criterion, we obtain

$N_0(7 \times 0 + 1) + N_1(7 \times 3) + \dots + N_n(7 \times d_n + r_n) = N_0 + N_1(7 \times 1) + N_1 \times 3 + \dots + 7N_nd_n + N_nr_n$. The first terms of each parenthesis being multiples of

7, we can ignore them. If $N = N_n \times 10^n + N_{n-1} \times 10^{n-1} + \dots + N_1 \times 10^1 + N_0$ is a multiple of 7, then $\sum N_i(7d_i + r_i)$ (which by hypothesis is equal to the previous expression must be divisible by 7 for little $\sum m_i r_i$ so are.

4. New Divisibility Tests

Statement of Chika's divisibility test

Let d, e be non-negative integers, $e \leq 9$ and $m = 10d + e$. Thus $7|m \Leftrightarrow 7|d + 5e$.

Here we give a proposal of a new theorem.

Theorem (Diouf-Pagdame). *Let d, e be non-negative integers, $e \leq 9$ and $m = 10d + e$. Then $3|m \Leftrightarrow 3|d - 2e$.*

5. Proof of Theorems

We will prove the two theorems together. Let m be the number to be tested. m is written in the form of $m = 10d + e$ where e is the last digit and d is the remainder of the number. m can also be written in the form $m = 10d + e - 3(d + 5e) + 7(d - 2e)$. With m in this form, we can remark that, $m = 3(d + 5e) + 7(d - 2e)$ is divisible by 3, since $(3, 7) = 1$, if and only if $3|(d - 2e)$. Similarly $m = 3(d + 5e) + 7(d - 2e)$ is divisible by 7 if and only if $7|(d + 5e) = c \cdot c(d + 5e)$ is known as "Chika's number".

This ends the proof.

6. Main Results and Observations

6.1. Main results

In this section, we present the execution time of the different programs on ASUS CORE i5 8th Gen. For a sufficiently given large number, we evaluate the Python code of the various tests listed in this article, and we note the execution time in a summary tables. We have chosen two numbers, one

of which is divisible by 7 and the other is not. This in order to verify the effectiveness of the various tests. These numbers are

$N_1 = 92345677890981132537784986527836547577827648576535125361$
23456789098760774 and

$N_2 = 34526780209364787384690124356272134567890272134563767471$
23456789723427237.

Let's called

Method 1: Property of multiple of 7;

Method 2: Criterion for a large number;

Method 3: Toja's method;

Method 4: Method of utilization of diagram and

Method 5: Pascal's method.

Below are the tables of the execution times of the different methods.

Table 1. For $N_1 = 923456778909811325377849865278365475778276485$
7653512536123456789098760774

Methods	Method 1	Method 2	Method 3
Time of executions	0:024769759000264457	0:024846616999866455	0:024533451000024797

Method 4	Method 5
0:022657221000145	0:020596608999994714

Table 2. For $N_2 = 34526780209364787384690124356272134567890272$
134563767471234567897234272345

Methods	Method 1	Method 2	Method 3
Time of executions	0:022463982999852306	0:023879767999915202	0:02267362900010994

Method 4	Method 5
0:023988764000023366	0:017082971000036196

6.2. Remarks

By the help of Python programs on the different tests, we can compare

them by using the execution time as a parameter of comparison. Indeed, we randomly chose a number large enough (of the order of 60 digits) to test the programs. As the tables above indicate, the differences appear to be negligible. however small it is, makes all the difference between the different methods. We have kept several digits after the comma to be as precise as possible. The difference in execution time between Method 5 (the most efficient) with the lowest execution time and Method 2 (the least efficient) with the longest time is 0:004250008 and 0:006796797 from Table 1 and Table 2 respectively. In view of these results, taking the execution time as a comparison parameter, method 5 seems to be the most efficient and method 2 is the least efficient among those methods which have been studied.

7. Conclusion

Finally, we will conclude that these methods are all very efficient. However, some are more efficient than others in their simplicity. Simple in their understanding and execution. According to the execution times, Pascal's criterion is the most efficient among the tests of divisibility by 7. Pascal's criterion is most efficient in view of the results obtained in the section 6 but is little too heavy. It is based on interesting property which emerges in the passage from base 7 to decimal base. It uses a sequential digit key. But its inconvenience side is usefulness. It is advantageous when we deal with several numbers.

Hence, we should highlight many similarities between the tests. Between the method of property of multiples of 7 and Diouf-Pagdame test, we can notice the use of the number $d - 2u$ in the two cases. The methods "criterion of large number" and the "Toja's method" have in common the separation by slices from the units. The first by the slice of 3 and the second by the slice of 2.

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