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INDUCED SUBGRAPHS OF THE POWER OF A CYCLE

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Abstract. In this article, we show that if G is an induced subgraph of the d -th power of a cycle of length n , and G has minimum degree $d+k$, then G has at least $\frac{d+k}{2d} n$ vertices. This answers a problem of A. Kézdy.

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Let C_n^d be the d -th power of a cycle on n vertices, that is, the graph whose vertices are the integers modulo n , two vertices being joined if, and only if, their difference is less than or equal to d .

During the problem session at the SIAM Conference in San Francisco in June 1988, A. Kézdy [5] proposed the following problem : Find the minimum number of vertices of any induced subgraph G of C_n^d , of minimum degree $d+1$. In particular, he asked to prove that such a graph must have at least $\frac{n}{2}$ vertices.

This question is motivated by the study of the d -girth of a graph. The d -girth of a graph G is defined, according to [6], as the minimum number of vertices of an induced subgraph of G with minimum degree d (note that the 2-girth is nothing but the girth). Paul Erdős originally proposed the problem of determining the largest d -girth of a graph on n vertices and m edges. A. Kézdy and M. Markert [6] considered this problem, and in order to find graphs with a large $(d+1)$ -girth asked to determine the $(d+1)$ -girth of C_n^d . In [6], they gave constructions that show that the $(d+1)$ -girth of C_n^d is at most $\frac{n}{2} [1 + \frac{d+1}{d^2+d-\varepsilon}]$, where $\varepsilon=0$ or 1 whenever $d+1 \equiv 0$ or $1 \pmod{2}$, respectively. They conjectured that this value is exactly the $(d+1)$ -girth of C_n^d and furthermore showed that the $(d+1)$ -girth of C_n^d is at least $n(\sqrt{2}-1)$.

In this article we show that the $(d+1)$ -girth of C_n^d is at least $\frac{d+1}{2d} n$, solving the problem proposed in [5] and proving the above conjecture when $d+1$ is even. In fact, we solve a slightly more general problem by determining the minimum order of an induced subgraph of C_n^d of degree $d+k$ (with $k \geq 1$).

Note that a minimum induced subgraph of minimum degree $d-k$ (with $k \geq 0$) of C_n^d is a complete graph on $d-k+1$ vertices.

Note also that these graphs C_n^d , which are particular circulant graphs, have nice properties. For example, they are the graphs with n vertices and connectivity $2d$ with the smallest possible number of edges (see the book of C. Berge [1]). They also have "super line-connectivity properties" (see the article of F.T. Boesch and J.F. Wang [2]). They are also important in the design of fault-tolerant networks. For example G. Chartrand and S.F. Kapoor [3], J.P. Hayes [4] and W.W. Wong and C.K. Wong [7] proved that they are minimum $2d-2$ hamiltonian graphs (resp. $2d-2$ edge-hamiltonian). That is, they are the graphs with the minimum number of edges among the graphs on n vertices that remain hamiltonian after the deletion of up to $2d-2$ vertices (resp. up to $2d-2$ edges).

Theorem.

Let G be an induced subgraph of C_n^d of minimum degree $d+k$, $1 \leq k \leq d$, then

$$\frac{|V(G)|}{n} \geq \frac{d+k}{2d}.$$

Proof.

Denote by V the set of the vertices of G and by $\bar{V}$ the complement of this set in the interval $[0, n-1]$.

Let $I = [u, v]$ be an interval of cardinality less than or equal to d , with v in V . First observe that if I contains $\bar{n}$ elements of $\bar{V}$, then the interval $[v+1, v+d+1]$ must contain at least $\bar{n}+k$ elements of V to insure that the degree of v in G is at least $d+k$.

Now, we can construct an infinite sequence of consecutive intervals of integers modulo n : $J_1 = [a_1+1, a_2]$, $J_2 = [a_2+1, a_3]$, ..., $J_j = [a_j+1, a_{j+1}]$, ... satisfying the following properties:

- a_j belongs to V , for all j , $j \geq 2$
- $|J_j| \leq d$, for all j , $j \geq 1$
- $n_{j+1} = \bar{n}_j + k$, for all j , $j \geq 1$, where $n_j = |J_j \cap V|$ and $\bar{n}_j = |J_j \cap \bar{V}|$.

Indeed, we can define these intervals recursively as follows :

- $J_1 = [a_1+1, a_2]$ is any interval of length at most d such that a_2 is in V .
- Suppose we have constructed the sequence up to $J_j = [a_j+1, a_{j+1}]$, with a_{j+1} in V . Then, $J_{j+1} = [a_{j+1}+1, a_{j+2}]$, where a_{j+2} is the $\bar{n}_j+k$ -th clockwise neighbor of a_{j+1} . Note that, by the above observation, a_{j+2} is well defined as a_{j+1} has at least $\bar{n}_j$ counterclockwise neighbors in $\bar{V}$. Note also that J_{j+1} has length at most d .

As there exist only a finite number of intervals of length at most d , there exist two positive integers m and p such that $J_m = J_{m+p}$.

Now, let $I_1 = J_m$, $I_2 = J_{m+1}$, ..., $I_p = J_{m+p-1}$. This sequence of intervals clearly satisfies the following properties :

- a) for all i , $1 \leq i \leq p$, $|I_i| \leq d$.
- b) if $n_i = |I_i \cap V|$ and $\bar{n}_i = |I_i \cap \bar{V}|$, then for all i such that $2 \leq i \leq p$, $n_i = \bar{n}_{i-1} + k$ and $n_1 = \bar{n}_p + k$.
- c) all the integers modulo n appear in the same number of intervals.

$$\text{From b) and c), we deduce that } \frac{|V|}{n} = \frac{\sum_{i=1}^p n_i}{\sum_{i=1}^p (n_i + \bar{n}_i)}$$

Let $N = \sum_{i=1}^p n_i$, then $\sum_{i=1}^p \bar{n}_i = N - kp$. Therefore $\frac{|V|}{n} = \frac{N}{2N - kp} = \frac{1}{2 - \frac{kp}{N}}$.

However, for all i , $1 \leq i \leq p-1$, $n_i + n_{i+1} = n_i + \bar{n}_i + k \leq d+k$ by a), and similarly $n_p + n_1 \leq d+k$. Therefore $N \leq \frac{p(d+k)}{2}$ and $\frac{|V|}{n} = \frac{1}{2 - \frac{kp}{N}} \geq \frac{1}{2 - \frac{2k}{d+k}} = \frac{d+k}{2d}$.

Remark 1

If $d+k$ is even, then the bound is sharp. Indeed the subgraph obtained by alternatively taking $\frac{d+k}{2}$ vertices in V and $\frac{d-k}{2}$ vertices in $\bar{V}$ has minimum degree $d+k$. In fact this subgraph is regular of degree $d+k$, and is isomorphic to a power of a cycle.

Remark 2

If $d+k$ is odd, then the bound is not sharp.

For instance, if $d=2$ and $k=1$, then the smallest subgraph of minimum degree 3 of C_n^2 has $\frac{4}{5}n$ vertices. The extremal graph (already given in [6]) is obtained, when 5 divides n , by alternatively taking 4 vertices in V and one in $\bar{V}$. To prove that this subgraph is the smallest one, let us show that any vertex in $\bar{V}$ is followed by at least 4 vertices in V . First recall that if a vertex is in V , then at least 3 of its 4 neighbors are also in V . Then note that there can not be two or more consecutive vertices in $\bar{V}$, as the first vertex in V after the sequence of vertices in $\bar{V}$ would have at most two neighbors in V . Let $\bar{v}$ be a vertex in $\bar{V}$. By the preceding remark, $v+1$ is in V and its two neighbors $v+2$ and $v+3$ are also in V . Now, as $v+2$ has already $\bar{v}$ as neighbor in $\bar{V}$, $v+4$ is also in V .

More generally, one can prove that if $k=d-1$, then $\frac{|V(G)|}{n} \geq \frac{2d}{2d+1}$. An extremal example is obtained by alternatively taking $2d$ vertices in V and one in $\bar{V}$.

If $d=4$ and $k=1$, we can prove that the smallest subgraph of C_n^4 of minimum degree 5 has $\frac{12}{19}n$ vertices. The extremal example, when 19 divides n , corresponds to repetitions of the sequence $\bar{v} v v \bar{v} v v v \bar{v} v \bar{v} v \bar{v} v v v \bar{v} v v \bar{v}$, where v denotes a vertex in V and $\bar{v}$ a vertex in $\bar{V}$. This example is already given in [6], where it is proved to be unique.

If $d=5$ and $k=2$, we can prove that the smallest subgraph of C_n^5 , with minimum degree 7, has $\frac{5}{7}n$ vertices. An extremal graph corresponds to repetitions of the sequence $\bar{v} v v v \bar{v} v v v v \bar{v} v v v \bar{v}$.

If $d=6$ and $k=1$, A. Kézdy and M. Markert proved that the smallest subgraph of C_n^6 , with minimum degree 7, has $\frac{24}{41}$ vertices. They also showed that there are two extremal sequences:

$\bar{v} \bar{v} \bar{v} v v v \bar{v} \bar{v} v v v v \bar{v} \bar{v} v \bar{v} v v \bar{v} \bar{v} v v v v \bar{v} \bar{v} v v \bar{v} \bar{v} v v v v \bar{v} \bar{v} v v v$

and $\bar{v} \bar{v} v \bar{v} v v \bar{v} v v \bar{v} v v \bar{v} v \bar{v} v v \bar{v} v \bar{v} v v v \bar{v} v \bar{v} v \bar{v} v \bar{v} v v v \bar{v} v \bar{v} v v$

In view of these results, we offer the following conjecture which contains the conjecture of A. Kézdy and M. Markert [6], when $k=1$. They verified it for $d=2,4,6$, and 8 and $k=1$.

Conjecture.

The minimum number of vertices of an induced subgraph of C_n^d , with odd minimum degree $d+k$, is $\frac{n d (d+3-k)}{2(d^2-(k-2)d-k)}$.

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