TEMPORAL PROXIMITIES. SELF-SIMILAR, TEMPORALLY CLOSE SHAPES

Muhammad Shangol, James F Peters

► To cite this version:

Muhammad Shangol, James F Peters. TEMPORAL PROXIMITIES. SELF-SIMILAR, TEMPORALLY CLOSE SHAPES. 2021. hal-03199089

HAL Id: hal-03199089

<https://hal.science/hal-03199089>

Preprint submitted on 15 Apr 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

TEMPORAL PROXIMITIES. SELF-SIMILAR, TEMPORALLY CLOSE SHAPES

MUHAMMAD SHANGOL HAIDER AND JAMES F. PETERS

ABSTRACT. This article introduces temporal proximity spaces as a framework to observe surface shapes as well as geometric shapes that change over time. A **surface shape** has a boundary and a non-empty interior, which is approximated by a **geometric shape** that lies within the boundary of a surface shape recorded in a video frame. Temporally close shapes (briefly, $\delta_{\Delta t}$ shapes) persist over the some temporal interval. Those surface shapes that appear withing the same video frame are strongly self-similar as well as temporally close. The rate of change of self-similar shapes $\text{sh}E, \text{sh}E'$ is represented by $\nabla(\text{sh}E), \nabla(\text{sh}E')$ pairs. Temporally close shapes that appear during the same temporal interval may or may not be spatially or descriptively close to each other. Persistent as well as spatially close shapes share belong to the same era and also overlap along their boundaries or withing their interiors. Overlapping appearances of shapes such as vortexes occur within temporal CW (Closure-finite Weak) spaces (briefly, tCW spaces), which are an extension of the CW spaces introduced by J.H.C. Whitehead during the 1940s. Because of their simplicity, tCW space provide a workable setting for the study of $\delta_{\Delta t}$ surface as well as geometric shapes in sequences of video frames.

CONTENTS

1. Introduction	1
2. Preliminaries	2
3. Temporal Proximities	4
4. Temporal proximities between shapes in video frame sequences	5
5. Temporally Close Shapes Across Segmented Video Frames	9
Appendix A. Cell Complexes	12
Appendix B. Shape Complexes	13
Appendix C. Planar Vortexes	14
Appendix D. Čech Proximity Spaces	15
References	16

1. INTRODUCTION

The study of nearness or apartness of shapes began a consideration of spatial closeness [24, 3, 22, 9, 10, 2] and later focused on descriptive as well as spatial closeness [14, 21, 13, 15, 16, 20, 1]. Both forms of closeness are unified here with the

2010 *Mathematics Subject Classification.* 54E05 (proximity).54C56,55P55,57N25 (shape theory).

Key words and phrases. Descriptive closeness,Geometric shape,Planar vortex,Self-similar,Spatial closeness,Temporal CW space,Temporal proximity,Temporally near vortexes.

introduction temporally close shapes, which can either be spatially or descriptively close. A **shape** is a bounded region that has a nonvoid interior. Two forms of shapes are considered here, namely, surface shapes and geometric shape. A **surface shape** is a bounded physical region with a nonempty interior recorded in a video frame. A **geometric shape** in the Euclidean plane is a bounded cell complex with a nonempty interior. Cell complexes live in what are known as CW spaces. Briefly, a CW space is a Hausdorff space (distinct points reside in distinct neighborhoods) that has two main properties, namely, containment (the closure of a set of cells in the space is also in the space) and intersection (the intersection of any pair of cell complexes in the space is also in the space). In a planar CW topology, there 3 types of cells in the formation of any cell complex, namely, vertex (0-cell), edge (1-cell), and filled triangle (2-cell). In a planar, closure-finite weak (CW) topological space, a **cell complex** is collection of cells attached to each other. Incarnation of both forms of shapes appear in triangulated video frames in computer vision [18], which provide an ideal hunting ground spatial, descriptive as well as temporal proximities.

Geometric shapes in triangulated video frames offer a straightforward means of approximating surface shapes that appear either within the same video frame or which appear in separate frames. In this work, the focus is on geometric shapes in the form of vortexes that lie with the borders of a triangulated surfaces shapes. Surface shapes that appear within the same video frame exhibit strong self-affinities, temporally, as well as possible self-affinities descriptively.

Just take an example of video, in which an object seems to be moving from one point to another within a video frame. If we consider only one video frame, then we will perceive the motion as stationary. But if we will play these static video frames at some definite rate and in a proper sequence, then our mind itself will develop the sense of apparent motion happening in the video. While watching video, our mind unconsciously matches the points of the spatial domain (i.e., coordinates of the character in each video frame) with nearby points in the temporal domain. As a result of this phenomena in which our mind unconsciously connects points in spatial domain and look for nearby descriptively close points in the time domain, we perceive the motion of objects in video.

This article introduces two types of temporal closeness of changing triangulated surface shapes in the form of vortexes that overlap temporally during their appearance.

2. PRELIMINARIES

In this section, we briefly present a framework for $\delta_{\Delta t}$ (temporally-near) cell complexes in tCW spaces.

For our analysis, we will take voxel as basic entity which is defined in [18] as *'Each picture element in a video frame is a 3D object called a voxel with coordinates $v_{(x,y,t)}$, where t (in fractions of a second) is the elapsed time from first frame to the current frame and x,y corresponds to coordinates of any object in video'*. This idea of voxels can be easily extended for real life objects having 3-D coordinates (x, y, z) . Voxel in such case will form a Quaternion of $v_{(x,y,z,t)}$. But for sake of simplicity and compatibility with video processing literature, we will limit our discussions to

FIGURE 1. Temporally near vortices

2-D coordinates.

Voxels $v_1, v_2, \dots, v_n$ in a video will provide us with a nonempty set of points on which we will constitute a topology. We will define our topology as follow,

$$(1) \quad T_r(p) = (v_1, v_2, \dots, v_n \in V \mid \sum_{i=1}^d (v_i - p_i)^3 < r^3)$$

where $p_i \in \mathbb{R}^3$. So, our topology will be a set of all voxels which lie within certain radius of sphere defined by value of r and this collection T of subsets of V satisfies following,

- $\emptyset, V \in T$
- if $\{C_i \mid i \in I\} \subset T$ then $\cup_{i \in I} C_i \in T$
- if $C_i \in T$ for $i = 1, 2, \dots, n$, then $\cap_{i=1}^n C_i \in T$

Therefore, we will define voxel topological space as $\vartheta = (V, T)$. In our ϑ voxel topological space, a nonvoid collection of cell complexes K is a *Closure finite Weak CW space*, provided K is Hausdorff (every pair of distinct cells is contained in disjoint neighbourhoods [11, 5.1, p. 94]) and the collection of cell complexes in K satisfy the Whitehead [25, pp. 315-317], [26, 5, p. 223] CW conditions, namely, the closure of each cell complex is in K and the nonempty intersection of cell complexes is in K . For the details, see App. 5. A CW space K is temporal (briefly, tCW), provided cell complexes appear, disappear and possibly reappear during temporal intervals. In other words, cell complexes in a tCW space change over time, appearing momentarily and eventually disappearing over varying temporal intervals.

The focus here is on cell complexes that are vortices. These are cell complexes that naturally arise, for example, within the boundaries of changing surface shapes in video frames.

Definition 1. A **planar vortex** $\text{vor}E$ is a finite cell complex, which is a collection of path-connected vertices in nested, filled 1-cycles in a CW complex K . A **1-cycle** in $\text{vor}E$ (denoted by $\text{cyc}A$) is a sequence of edges with no end vertex and with a

nonempty interior. A geometric realization of $\text{vor}E$ is denoted by $|\text{vor}E|$ on $|K|$ in the Euclidean plane.

3. TEMPORAL PROXIMITIES

In [8],[4],[5] authors have compared two different theories how a human could perceive time or data from temporal domain. According to strength model, which is based on memory trace, nearness or farness of events depends on trace which persists or diminishes over time. While the inference model suggests that information of near or far events is perceived in comparison of any reference event whose time is known to us. Our model of temporal proximity is more similar to strength model, since we create a trace of structure under observation and then try to find similarity of shape in later frames, we look for that trace in proximity of provided temporal interval.

A nonempty set X equipped with the relation $\delta_{\Delta t}$ is a temporal Čech proximity space (denoted by $(X, \delta_{\Delta t})$), provided temporal form of the Čech axioms in Appendix are satisfied. When write, for example, $\text{vor}E \delta_{\Delta t} \text{vor}E'$ in tCW space K , we mean that $\text{vor}E$ persists over a temporal interval $\Delta t = [t_0, t_{\text{end}}]$ that overlaps with a temporal interval $\Delta t' = [t'_0, t'_{\text{end}}]$ over which a vortex $\text{vor}E'$ persists.

Example 1. In Fig. 1, $\text{vor}E \delta_{\Delta t} \text{vor}E'$ in the temporal CW space K , since vortex $\text{vor}E$ persists over a temporal interval that overlaps the temporal interval over which vortex $\text{vor}E'$ persists.

An interest in overlapping temporal intervals leads to the introduce of the temporal intersection between cell complexes such as vortexes $\text{vor}E, \text{vor}E'$ that overlap in time (denoted by $\text{vor}E \bigcap_{\Delta t} \text{vor}E'$).

Definition 2. Let $\text{vor}E, \text{vor}E'$ be vortexes in a temporal CW space K . Also let $\bigcap_{\text{vor}E} \Delta t$ be a temporal interval over which $\text{vor}E$ appears and persists and let $\bigcap_{\text{vor}E'} \Delta t'$ be a temporal interval over which $\text{vor}E'$ appears and persists. Then

$$\text{vor}E \bigcap_{\Delta t} \text{vor}E' = \left\{ \text{instant } t : t \in \bigcap_{\text{vor}E} \Delta t \cap \bigcap_{\text{vor}E'} \Delta t' \right\}.$$

Example 2. In Fig. 1, $\text{vor}E \bigcap_{\Delta t} \text{vor}E'$, since

$$\bigcap_{\text{vor}E} \Delta t \cap \bigcap_{\text{vor}E'} \Delta t' \neq \emptyset,$$

since the pair of vortexes persist over the interval $[t', t_{\text{end}}]$.

Temporal Čech Axioms

- (tP.0): All nonempty subsets in X are temporally far from the empty set, i.e., $A \not\delta_{\Delta t} \emptyset$ for all $A \subseteq X$.
- (tP.1): $A \delta_{\Delta t} B \Rightarrow B \delta_{\Delta t} A$.
- (tP.2): $A \bigcap_{\Delta t} B \neq \emptyset \Leftrightarrow A \delta_{\Delta t} B$.
- (tP.3): $A \delta_{\Delta t} (B \cup C) \Rightarrow A \delta_{\Delta t} B$ or $A \delta_{\Delta t} C$.

Lemma 1. Unlike the Čech Axiom P.2, the temporal proximity Axiom tP.2 is an equivalence between overlapping temporal intervals and the $\delta_{\Delta t}$ proximity of subsets in the space $(X, \delta_{\Delta t})$.

Proof. $\Rightarrow$: Assuming that temporal intervals for $A, B \in 2^X$ overlap (i.e., $A \cap_{\Delta t} B \neq \emptyset$), then by definition, $A \delta_{\Delta t} B$. $\Rightarrow$: $A \delta_{\Delta t} B$ means that A and B persist over the same temporal interval Δt . Hence, $A \cap_{\Delta t} B \neq \emptyset$. $\square$

Theorem 1. $(K, \delta_{\Delta t})$ be a *tCW proximity space* and let vortexes $vorE, vorE' \in 2^K$, i.e., vortexes $vorE, vorE'$ are sub-complexes in the collection of cell complexes in space K . Then $vorE \cap_{\Delta t} vorE' \neq \emptyset \Leftrightarrow vorE \delta_{\Delta t} vorE'$.

Proof. Immediate from Lemma 1. $\square$

FIGURE 2. Different Instance of Bouncing Ball Video

4. TEMPORAL PROXIMITIES BETWEEN SHAPES IN VIDEO FRAME SEQUENCES

This section introduces the application of temporal proximity in terms of the similarities between surface shapes recorded in video frames over short periods of time. First we consider a ball bouncing across the screen between different frames of video. Following images show different frames of video, in which we can see how the ball is bouncing as video frames pass by. In the beginning of this video, the ball starts appearing. During video, the ball strikes and bounces back from surface/ground multiple times. While the ball strikes and bounces back from the surface/ground, it shows slight deformation in structure, like a plastic or rubber ball. By the end of the video, the ball disappears on the other side of the video frame.

A convenient way to show the position of any object in 2D frames at different times is by use of 3D spatial temporal plot. Therefore, to show the position of the ball in each video frame, we identified the centroid of the ball and plotted its position in 2D Cartesian plane which was extended to 3D by adding a temporal domain of video frames. Figure (3.1) shows the required spatial temporal plot for bouncing ball video under processing.

After that we detected corners, next we combined these corners with centroids found earlier to get a set of seed points. We then generated mesh of Delaunay Triangulation using those set of seed points. Then we identified the vertex in the mesh which had most triangles connected with it, to get Maximal Nucleus Cluster (MNC). Using the barycentre of the maximal nucleus cluster, we created a vortex cycle inside the ball. Another vortex cycle was formed on the edges of the ball. These two cycles were connected using bridge segments (shown in white colored

FIGURE 3. Position of Ball and Optical Vortex Nerve

line segments) as shown in figure (3.2). Those two vortex cycles which are interconnected with bridge segments lays the foundation for most basic shape referred as Optical Vortex Nerve, whose properties we will be comparing between different frames to determine descriptive and temporal proximity. Figure (3.2) shows the formation of vortex cycles and their interconnection using bridge segments on one frame of bouncing ball video. More details on optical vortex nerve, maximal nucleus cluster, Delaunay mesh etc can be studied in [18]

To analyze the appearance or disappearance of any specific optical vortex nerve, we selected two descriptive parameters to observe which are the number of bridge segments and area. Area is computed as the sum of binary image pixels of optical vortex nerve. To visualize the behavior of area and number of bridge segments of optical vortex nerve, we generated a 3D scatter plot between time/number of video frames, area and number of bridge segments. This provided us with a set of points as shown in figure (4.1). Looking directly at this scatter plot does not provide us with any meaningful information but we can observe some trends or behaviors if we observe things for number of bridge segments versus time as shown in figure (4.2) and area of optical vortex nerve versus time as shown in figure (4.3). Figure (4.4) shows a plot of points between the number of bridge segments and area of optical vortex cycle in which we can clearly identify three regions of clusters highlighted by blue, green and yellow colored boxes. These three cluster regions are identifying the appearance or disappearance of three different shapes of optical vortex nerves that existed between the whole video. In the beginning, we mentioned about three different positions of video. First one was the appearance of the ball in the beginning and disappearance of the ball at the end of the video. Second one being when the ball is normally moving in air. Last one was, when the ball strikes the ground or surface and due to the plastic/rubber type nature of the material of the ball it undergoes a bit of deformation of shape. So, these three regions are easily distinguished using the figure (4.4). But an important question yet remains to be answered which is timings of appearance or disappearance of each region shape in video and how temporal proximity is present between each shape.

On the basis of figure (4.4), we determined some threshold values to distinguish the three regions and using relation between temporal and spatial domains of points in figure (4.1) we generated a persistence diagram as shown in figure (5). At each video frame there exists an event which is part of one of the three distinct classes

FIGURE 4. 3D Scatter Plot for No of Bridge Segments VS Optical Vortex Nerve Area VS Time

FIGURE 5. Persistence Diagram for Temporal Proximity

of optical vortex nerves indicated by yellow, blue and green colored bars. Those events which coincides temporally with other events in the same class forms temporal proximity leading to generation of different sets like set A, set B, set C etc. Although the events in set C, set D, set E, set F, set G belong to the same descriptive proximity class or region of cluster (as indicated by green color) but events in those sets neither coincides nor overlap with events in other sets, so they are temporally far from each other. Similar conclusions can be easily drawn for set B and set H. In video under processing, shapes or events in set A appear and last temporally longest. Set B and set C are coinciding with each other in temporal domain but they are not descriptively near each other, so this information of temporal relation between set B and set C is comparatively less important. Similar relations can be

made between other sets also.

FIGURE 6. Different Instances of Automobile Motion

Another real-life example is about automobiles motion all around us. In our video under process, multiple events happen which include appearance of first car in frame as shown in figure (6.1). After some time, second car makes its appearance while first car is still in frame. In figure (6.2), at this instance two events are occurring simultaneously within same video frame in temporal domain. First car disappears after few frames, later, after few frames in which only second car is present, third car shows up while second car is till in video frame. Again, for few frames, two simultaneous events are happening in temporal domain as shown in figure (6.4). Few frames from set of events mentioned earlier are provided in figure (6)

FIGURE 7. Different Instance of Optical Vortex Nerve

Again, to observe the appearance or disappearance of any shape at any instance of time, we will observe the persistence of optical vortex nerve as demonstrated in

FIGURE 8. Scatter Plot of Time VS Bridge Segment VS Area of Cycle

figure (7).

In the scatter plot between time, number of bridge segments and area of cycle for optical vortex nerve, green points are showing appearance of first object in frame while red points are signifying second object which is present in parallel with temporal proximity of first object. In plot between bridge segments and area of cycle, on right side there are multiple long persistent lines which are showing appearance of same optical vortex nerve for comparatively longer period.

Figure (9) shows position of optical vortex nerve in each video frame for cars present in video. In set-A from frame number 200 to 230 two events are temporally overlapping with each other. Two optical vortex nerves for two cars shown in this set demonstrate temporal proximity of two different cars. Similarly, in set-B from frame number 250 to 280 again there are two events are temporally overlapping with each other demonstrating temporal proximity between another set of cars.

FIGURE 9. Position of Temporally Near Optical Vortex Nerve

5. TEMPORALLY CLOSE SHAPES ACROSS SEGMENTED VIDEO FRAMES

This section introduces a fine-grained approach to detect temporal proximities between shapes in terms of segmented video frames. To accomplish this, we divided each frame in eight equal frame segments along the horizontal axis which records

the number of shapes in each frame. Then we determined the appearance and disappearance of our foreground objects in each frame segment against time domain. As a result, we obtained a persistence diagram shown in figure (10) illustrating occurrence of foreground objects in different frame segments. Considering our definition for temporal proximity, we can easily observe simultaneous occurrences of multiple objects at different instances of time or video frame depicting strong temporal proximity between those foreground objects.

FIGURE 10. Persistence Barcode for Temporal Proximity

Next, we wanted to know more about the existence of different kinds of shape in each video frame. Therefore, we observed the descriptive proximity via physical properties (area and perimeter) of observed foreground shapes in the video frames and produced a scatter plot between area of shape and perimeter of shape as illustrated in figure (11). All the points in the plot between two observed physical properties shown in figure (11) can be distinctively divided into two clusters represented by blue and orange boxes. These two boxes correspond to two different classes of shapes present in the whole video. Points of cluster in blue boxes are representing instances of shape when it is full in frame while points in orange box are showing those moments of objects when it is either appearing in frame or disappearing from frame.

Both of the last two mentioned plots provide us information about temporal proximity of shapes and descriptive proximity of shapes individually but they do not allow us to decide both temporal and descriptive proximity simultaneously. To solve this problem, each video frame shape shE is defined by a continuous mapping $a\ shE : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by ∇shE in terms of the rate of change of shE boundary $bdyE$, shE interior $intE$ and shE disposition at any instant in time t , namely,

$$(2) \quad \nabla_{shE} = \frac{\partial shE}{\partial bdy} + \frac{\partial shE}{\partial int} + \frac{\partial shE}{\partial t}$$

A persistence barcode that records the value of ∇ShE is illustrated in figure (13). This form of barcode makes it possible to determine temporal and descriptive

FIGURE 11. Descriptive Proximity: Plot Between Area and Perimeter of Shapes

proximities between different shapes at the same time.

FIGURE 12. ∇_{shE} Values in Ghrist Pictograph Barcode

From the ∇ persistence barcode in figure (12), from frame 255 to 265 there are two persistent shapes shE and shE as shown in figure (13), which are in different frame segments but at the same time they are co existing in same temporal domain. So, according to our definition, both shapes due to their simultaneous occurrence in time domain shE and shE lie in strong temporal proximity of each other. This provides us with descriptively close video frame shapes with comparable rates of change, where

$$\Phi(shE) = \nabla shE.$$

Then we have

th = selected threshold.

shE, shE' = selected pair of shapes

$\Phi(shE) = \nabla shE$,

$shE \delta_{\Phi} shE'$, provided $|\nabla shE - \nabla shE'| < th$.

FIGURE 13. Strong temporal proximity

Consecutive frames or events happening in videos will always remain temporally close to each other but for video processing we are interested in temporal closeness of any particular shape. Therefore, the framework of temporal proximity helps in determining quantitatively for how long each shape persists over time and after how much time that shape reappeared in a series of consecutive events. If the time interval between disappearance and reappearance of shape under observation is less than certain threshold then we say those events are in a temporal proximity $\delta_{\Delta t}$ of each other. A strong temporal proximity $\delta_{\Delta t strong}$ happens when the shapes under observation appear concurrently in time domain (the shapes may be spatially and descriptively far from each other). Depending on the setting of an experiment, one can choose to observe different parameters such as ∇shE to characterize the temporal closeness between distinct shapes that appear on either in the same frame or on different frames in a video.

APPENDIX A. CELL COMPLEXES

A planar Whitehead cell complex K [18] (usually called a CW complex) is a collection of n -dimensional minimal cells $e_{\alpha}^n, n \in \{0, 1, 2\}$, i.e.,

$$K = \{e_{\alpha}^n \subset \mathbb{R}^2 : n \in \{0, 1, 2\}\}.$$

in the Euclidean plane π .

Definition 3. A cell subcomplex $shE := \{e_{\alpha}^n\} \in 2^K$ (**shape complex**) is a **closed subcomplex**, provided the subcomplex includes both a nonempty interior (denoted by $int(e_{\alpha}^n)$) and its boundary (denoted by $bdy(e_{\alpha}^n)$). In effect, shE is closed, provided

$$shE = int(shE) \cup bdy(shE) \text{ (Closed subcomplex).}$$

Let 2^π be the collection of all subsets in the Euclidean plane π . In the plane, a Whitehead **Closure-finite Weak (CW)** cell complex $K \in 2^\pi$ has two properties, namely,

C: A cell complex K is **closure-finite**, provided each cell $e_\alpha^n \in K$ is contained in a finite subcomplex of K . In addition, each cell $e_\alpha^n \in K$ has a finite number of immediate faces. One cell e_α^n is an **immediate face** of another cell e_α^m , provided $e_\alpha^n \cap e_\alpha^m \neq \emptyset$ [23] (also called a **common face**).

W: The plane π has a **weak topology** induced by cell complex K , *i.e.*, a subset $S \in 2^\pi$ is closed, if and only if $S \cap e_\alpha^n$ is also closed in e_α^n for each n, α [23, 5.3, p. 65].

A collection $K \in 2^\pi$ is called a **CW complex**, provided it has the closure-finite property and π has the weak topology property induced by K .

Minimal cell planar complexes are given in Table 1.

TABLE 1. Minimal Planar Cell Complexes

Minimal Complex	Cell $e^n : n \in \{0, 1, 2\}$	Planar Geometry	Interior
e^0	Vertex	nonempty
e^1	Edge	line segment w/o end points
e^2	Filled triangle	nonempty triangle interior w/o edges

Remark 1. *Closure finite cell complexes with weak topology (briefly, CW complexes) were introduced by J.C.H. Whitehead [25], later formalized in [26]. In this work¹, a cell complex K (or complex) [26, 4, p. 221] is a Hausdorff space (union of disjoint open cells e, e^n, e_i^n) such that the closure of an n cell $e^n \in K$ (denoted by $cl(e^n)$) is the image of a map $f : \sigma^n \rightarrow cl(e^n)$, where σ^n is a fixed n -simplex and where the boundary $bdy(e^n)$ (otherwise known as the **contour** of a complex) is defined by*

Complex contour $\rightarrow$ closure $cl(e^n)$ minus $Int(e^n)$ interior

$$bdy(e^n) = \overbrace{f(bdy(e^n))} = cl(e^n) - int(e^n).$$

Notice that a subcomplex $X \subset K$ has the weak topology, since X is the union of a finite number intersections $X \cap cl(e)$ for single cells $e \in K$ [26, 5, p. 223]. From a geometric perspective, a cell complex is a triangulation of the CW space K [25, p. 246].

APPENDIX B. SHAPE COMPLEXES

A shape complex has two basic parts, namely, contour and interior, introduced in [17].

¹Here, we use $cl(e^n)$ (closure of a cell) and $bdy(e^n)$ (contour of a cell) used in this paper, instead of Whitehead's e^n and $\partial(e^n)$.

Definition 4. A **planar shape contour** of a shape complex shE (denoted by $bdy(shE)$) is a simple, closed curve with no self-loops.

Each shape complex shE has a nonempty interior that excludes all points on the shape contour.

Definition 5. **Shape complex interior.**

Set of points bounded by shE contour

$$\overbrace{int(shE) = shE \setminus bdyE.}$$

The fundamental parts of every shape complex are gathered together in the closure of a shape complex, definite using the Hausdorff distance [6] (see, also, [7, 23, p. 128]) between all points in a CW complex K and a shape shE .

Definition 6. Closure of a planar shape shE (denoted by $cl(shE)$) in a CW space K is defined by

Hausdorff distance $D(x, shE) = \inf \{\|x - p\| : p \in shE\}$.

$$\overbrace{cl(shE) = \inf \{x \in X : D(x, shE) = 0\}.}$$

In other words, we have the closure of a planar shape shE is a finite bounded region of the Euclidean plane such that

$cl(shE)$ includes its contour & its interior

$$\overbrace{cl(shE) = bdy(shE) \cup Int(shE).}$$

APPENDIX C. PLANAR VORTEXES

This section briefly looks at planar vortex structures in planar CW spaces. For simplicity, we consider only 2 cycle vortexes containing a pair of nested 1-cycles that intersect or attached to each other via at least one bridge edge.

Definition 7. *Planar 2 Cycle Vortex* [19].

Let $cycA, cycB$ be a collection of path-connected vertexes on nested filled 1-cycles (with $cycB$ in the interior of $cycA$) defined on a finite, bounded, planar region in a CW space K . A planar 2 cycle vortex $vorE$ is defined by

$cl(cycB)$ is contained (nested) in the interior of $cl(cycA)$.

$$vorE = \overbrace{\{cl(cycA) : cl(cycB) \subset int(cl(cycA))\}}.$$

A vortex containing adjacent non-intersecting cycles has a bridge edge attached to vertexes on the cycles.

Definition 8. A **vortex bridge edge** is an edge attached to vertexes on a pair of non-intersecting, filled 1-cycles.

Remark 2. From Def. 7, the cycles in a 2 cycle vortex can either have nonempty intersection (see, e.g., $cycA' \cap cycB' \neq \emptyset$ in $|vorE'|$ in Fig. 14.2) or there is a bridge edge between the cycles (see, e.g., $pq \in |vorE|$ in Fig. 14.1). In effect, every pair of vertexes in a 2 cycle vortex is path-connected.

14.1: Vortex $|\text{vor } E|$ with non-intersecting 1-cycles $\text{cyc}A, \text{cyc}B$

14.2: Vortex $|\text{vor } E'|$ with intersecting 1-cycles $\text{cyc}A', \text{cyc}B'$

FIGURE 14. Sample planar 2-cycle vortices

Remark 3. The structure of a 2 cycle vortex extends to a vortex with $k > 2$ nested filled 1-cycles, provided adjacent pairs of cycles $\text{cyc}A, \text{cyc}A'$ in a k -cycle vortex either intersect or there is a bridge edge attached between $\text{cyc}A, \text{cyc}A'$.

APPENDIX D. ČECH PROXIMITY SPACES

This section briefly introduces Čech proximity spaces, paving the way for temporal proximity spaces.

The simplest form of proximity relation (denoted by δ) on a nonempty set was introduced by E.Čech [24]. A nonempty set X equipped with the relation δ is a Čech proximity space (denoted by (X, δ)), provided the following axioms are satisfied.

Čech Axioms

- (P.0): All nonempty subsets in X are far from the empty set, i.e., $A \not\delta \emptyset$ for all $A \subseteq X$.
- (P.1): $A \delta B \Rightarrow B \delta A$.
- (P.2): $A \cap B \neq \emptyset \Rightarrow A \delta B$.
- (P.3): $A \delta (B \cup C) \Rightarrow A \delta B$ or $A \delta C$.

Given that a nonempty set E has $k \geq 1$ features such as Fermi energy E_{Fe} , cardinality E_{card} , a description $\Phi(E)$ of E is a feature vector, i.e., $\Phi(E) = (E_{Fe}, E_{card})$. Nonempty sets A, B with overlapping descriptions are descriptively proximal (denoted by $A \delta_\Phi B$). The descriptive intersection of nonempty subsets in $A \cup B$ (denoted by $A \cap_\Phi B$) is defined by

$$A \cap_\Phi B = \overbrace{\{x \in A \cup B : \Phi(x) \in \Phi(A) \cap \Phi(B)\}}^{\text{i.e., Descriptions } \Phi(A) \text{ \& } \Phi(B) \text{ overlap}}.$$

Let 2^X denote the collection of all subsets in a nonvoid set X . A nonempty set X equipped with the relation δ_Φ with non-void subsets $A, B, C \in 2^X$ is a Čech proximity space, provided the Čech axioms are satisfied.

Lemma 2. *Let K be a CW space, $2^{Vor(K)} \int K$, a collection of planar vortexes equipped with the proximity δ and let $vorA, vorB \in Vor(K)$. Then $vorA \cap vorB \neq \emptyset$ implies $vorA \delta vorB$.*

Proof. Immediate from Axiom **P.2**. □

Let (X, δ_1) and (Y, δ_2) be two Čech proximity spaces. Then a map $f : (X, \delta_1) \rightarrow (Y, \delta_2)$ is *proximally continuous*, provided $A \delta_1 B$ implies $f(A) \delta_2 f(B)$, i.e., if A, B are close in δ_1 proximity space X , then $f(A), f(B)$ are close in the δ_2 proximity space Y [12, 1.4]. In general, a proximal continuous map preserves the nearness of pairs of sets in the pre-image space X for the corresponding pairs of set in the image space Y [11, 1.7, p. 16].

Theorem 2. *Let K, K' be a pair of CW spaces, equipped with the proximities δ_1, δ_2 , respectively and let $f : (K, \delta_1) \rightarrow (K', \delta_2)$ be a proximally continuous map. If vortexes $vorA, vorB \in 2^K$ are δ_1 close in space K , then $f(vorA), f(vorB)$ are close in the δ_2 proximity space L'*

Proof. Given $vorA \delta_1 vorB$ in space K , then by definition of a proximally continuous map, $f(vorA) \delta_2 f(vorB)$ in space K' . □

Example 3. *Let be vortexes $vorA, vorB \in 2^K$, each with same number of cycles, have overlapping cycles. In that case, $vorA \cap vorB \neq \emptyset$. by Axiom **P.2** $vorA \delta_1 vorB$ in space K . Let $f : (K, \delta_1) \rightarrow (K', \delta_2)$ be a proximally continuous, defined for any vortex $vorA$ by the identify map.*

$$f(vorA \in K) = vorA \text{ in } K'.$$

In that case, let $vorE, vorE' \in K$, then

$$f(vorE) = vorA \in K'. f(vorE') = vorE'. vorE \delta_1 vorB \Rightarrow f(vorE) \delta_2 f(vorE').$$

In other words, if vortexes $vorE, vorE' \in K$ are close, then $f(vorE), f(vorE') \in K'$ are close.

REFERENCES

1. A. Di Concilio, C. Guadagni, J.F. Peters, and S. Ramanna, *Descriptive proximities. properties and interplay between classical proximities and overlap*, Math. Comput. Sci. **12** (2018), no. 1, 91–106, MR3767897, Zbl 06972895.
2. A. Di Concilio, *Proximity: A powerful tool in extension theory, functions spaces, hyperspaces, boolean algebras and point-free geometry*, Beyond Topology, AMS Contemporary Mathematics 486 (F. Mynard and E. Pearl, eds.), Amer. Math. Soc., 2009, pp. 89–114.
3. V.A. Efremovič, *The geometry of proximity I (in Russian)*, Mat. Sb. (N.S.) **31(73)** (1952), no. 1, 189–200.
4. William Friedman, *About time: Inventing the fourth dimension.*, The MIT press, 1990.
5. William J Friedman, *Memory for the time of past events.*, Psychological bulletin **113** (1993), no. 1, 44.
6. F. Hausdorff, *Grundzüge der mengenlehre*, Veit and Company, Leipzig, 1914, viii + 476 pp.
7. ———, *Set theory*, trans. by j.r. aumann, AMS Chelsea Publishing, Providence, RI, 1957, 352 pp.
8. Robin Le Poidevin, *The experience and perception of time*, (2000).
9. S. Naimpally, *Near and far. A centennial tribute to Frigyes Riesz*, Siberian Electronic Mathematical Reports **2** (2009), 144–153.
10. S.A. Naimpally, *Proximity spaces*, Cambridge University Press, Cambridge, UK, 1970, x+128 pp., ISBN 978-0-521-09183-1.
11. S.A. Naimpally and J.F. Peters, *Topology with applications. Topological spaces via near and far*, World Scientific, Singapore, 2013, xv + 277 pp, Amer. Math. Soc. MR3075111.

12. S.A. Naimpally and B.D. Warrack, *Proximity spaces*, Cambridge Tract in Mathematics No. 59, Cambridge University Press, Cambridge, UK, 1970, x+128 pp., Paperback (2008), MR0278261.
13. J.F. Peters, *Near sets. General theory about nearness of sets*, Applied Math. Sci. **1** (2007), no. 53, 2609–2629.
14. ———, *Near sets. Special theory about nearness of objects*, Fundamenta Informaticae **75** (2007), 407–433, MR2293708.
15. ———, *Local near sets: Pattern discovery in proximity spaces*, Math. in Comp. Sci. **7** (2013), no. 1, 87–106, DOI 10.1007/s11786-013-0143-z, MR3043920, ZBL06156991.
16. ———, *Proximal relator spaces*, Filomat **30** (2016), no. 2, 469–472, MR3593041.
17. ———, *Proximal planar shapes. correspondence between triangulated shapes and nerve complexes*, Bulletin of the Allahabad Mathematical Society **33** (2018), 113–137, MR3793556, Zbl 06937935, Review by D. Leseberg (Berlin).
18. ———, *Computational geometry, topology and physics of digital images with applications. Shape complexes, optical vortex nerves and proximities*, Springer Nature, Cham, Switzerland, 2020, xxv+440 pp., ISBN 978-3-030-22191-1/hbk; 978-3-030-22192-8/ebook, Zbl07098311.
19. ———, *Ribbon complexes & their approximate descriptive proximities. Ribbon & vortex nerves, betti numbers and planar divisions*, Bull. Allahabad Math. Soc. **35** (2020), 1–13, in press, preprint: <https://arxiv.org/abs/1911.09014> [math.GT].
20. J.F. Peters and E. İnan M.A. Öztürk, *Spatial and descriptive isometries in proximity spaces*, Gen. Math. Notes **21** (2014), no. 2, 125–134.
21. J.F. Peters and S.A. Naimpally, *Applications of near sets*, Notices of the Amer. Math. Soc. **59** (2012), no. 4, 536–542, DOI: <http://dx.doi.org/10.1090/noti817>, MR2951956.
22. Ju. M. Smirnov, *On proximity spaces*, Math. Sb. (N.S.) **31** (1952), no. 73, 543–574, English translation: Amer. Math. Soc. Trans. Ser. 2, 38, 1964, 5–35.
23. R.M. Switzer, *Algebraic topology – homology and homotopy*, Springer, Berlin, 2002, xii+526 pp., Zbl 1003.55002.
24. E. Čech, *Topological spaces*, John Wiley & Sons Ltd., London, 1966, fr seminar, Brno, 1936–1939; rev. ed. Z. Frolik, M. Katětov.
25. J.H.C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proceedings of the London Math. Soc. **45** (1939), 243–327.
26. ———, *Combinatorial homotopy. I*, Bulletin of the American Mathematical Society **55** (1949), no. 3, 213–245, Part 1.

COMPUTATIONAL INTELLIGENCE LABORATORY, UNIVERSITY OF MANITOBA, WPG, MB, R3T 5V6, CANADA

Email address: haiderms@myumanitoba.ca

COMPUTATIONAL INTELLIGENCE LABORATORY, UNIVERSITY OF MANITOBA, WPG, MB, R3T 5V6, CANADA AND DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, ADIYAMAN UNIVERSITY, 02040 ADIYAMAN, TURKEY.

Email address: james.peters3@umanitoba.ca