Adaptive Rejection of Narrow-band Disturbances in the Presence of Plant Uncertainties - A Dual Youla-Kucera Approach

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Abstract

The stability of adaptive disturbance rejection schemes using Youla-Kucera (YK) parametrization and the internal model principle (IMP) in the presence of plant model uncertainties is investigated. The problem is approached by using the dual Youla Kucera parametrization for the description of the plant model uncertainties. The known disturbance case is discussed first, emphasizing the need of over parametrization of the Youla Kucera filter used for control in order to both solving the IMP and the stability problems. Then this solution is extended for the case of unknown disturbances leading to the use of a parameter adaptation algorithm with projection. A stability analysis of the adaptive scheme is provided. Simulation results on relevant examples and experimental evaluation on an active noise attenuation system illustrate the possibilities of this approach for handling significant plant-model mismatch.

Key words: Adaptive disturbance rejection, Youla-Kucera parameterization, Adaptive control, Active noise control, Internal model principle

1 Introduction

In the last twenty years, the issue of adaptive rejection of unknown disturbances has received a significant interest in the field of control. The basic assumption is that the disturbances are the result of a white noise or of a Dirac impulse passed through the "model of the disturbance" and therefore, using the "internal model principle" (IMP) they can be asymptotically rejected. However the parameters of these models are unknown and can be time-varying and therefore an adaptive approach has to be considered¹. In this context however, it is assumed that the plant model is known (often obtained by system identification from experimental data) and remains almost constant. This type of adaptive problem has been termed "adaptive regulation". Most of the proposed algorithms are based on the Youla-Kucera (YK) parametrization (or Q-parametrization) [2,20], that al-

lows for the tuning of a compensator within the class of all-stabilizing controllers. The basic diagram of such a system is shown in Fig 1 [11][13]. R_0 and S_0 define a polynomial central stabilizing controller with pre-specified fixed parts H_R and H_S ($R_0 = R'_0 H_R$, $S_0 = S'_0 H_S$). An observer of the disturbance d is used. The output of the observer w is processed through the Youla-Kucera filter Q . The Q filter will be adapted in real time in order to cancel asymptotically the disturbance d .

A successful adaptive rejection of disturbances resulting from an unstable exogenous system is mentioned in [22]. In the field of active noise control, first references seem to be [3], [4] and a more recent reference is [16]. In the field of active vibration control a first reference seems to be [11]. In all these papers, the model of the system to be controlled is assumed to be perfectly known (models obtained by system identification). Under this assumption these algorithms do not require a specific stability condition, except that the closed-loop including the central controller must be stable. However in many situations, the plant model is subject to changes during operation and therefore the plant model uncertainties have to be taken into account.

In order to address the issue of plant model uncertain-

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¹ For known disturbances subject to small variations of their characteristics a robust approach can be considered [1], [16].

Fig. 1. Youla-Kucera direct adaptive feedback regulation scheme.

ties, Kinney and Callafon proposed an adaptive scheme called REACT [9] that aims at rejecting unknown disturbances acting on an uncertain system. The uncertainties are represented by means of the dual Youla-Kucera parameterization [2] that had been previously used in the context of identification (see for example [21]). In the REACT scheme, conditions about stability of a robust controller are given. However no stability analysis is provided for the add-on adaptive scheme. In [7] the problem of adaptive rejection of unknown periodic disturbances in the presence of plant model uncertainties is discussed in the context of a different controller parametrization and the uncertainties are unstructured. Stability conditions are provided.

The various applications reported in the literature, particularly in the fields of active vibration and noise control (see [14], [15]), indicate that very reliable discrete linear models can be obtained by experimental system identification techniques (very small differences between simulations and real experiments) and that changes of the physical configuration and of various operational conditions lead to models of the same structure (with different orders and parameters). The work reported in this paper has been motivated by significant variations of the plant model which may occur in some active noise and vibration attenuation applications.

There are various way to express the model uncertainties. However in this context it seems reasonable to use the dual Youla Kucera parametrization as a way to express the relationship between the nominal model used for design and the true model of the system.

The main objectives of the paper are:

- to develop a Youla-Kucera controller parametrization for the case of plant uncertainties described by dual YK parametrization,
- to develop adaptation algorithms for the case of plant model uncertainties described by the dual YK

parametrization,

- to provide conditions assuring the stability of the adaptive regulation scheme,
- to illustrate the methodology by simulations on relevant examples and by experimental results (an active noise control system).

The main ingredient for solving this problem is to augment the size of the (finite impulse response) Q filter beyond the minimum value resulting from the application of the internal model principle (IMP) when there are no plant uncertainties. In this context, as a consequence of plant uncertainties, the closed loop poles are no more invariant with respect to the values of the (finite impulse response) Youla-Kucera filter parameters (as it will be shown in section 3.2) and they may become unstable. Therefore stability conditions related to the Q-filter parameters and dual YK filter parameters will have to be fulfilled. Furthermore in the adaptive operation since the size of the Q-filter is larger than the minimal one, persistent excitation arguments can no more be used for stability analysis and projection has to be included in the parameter adaptation algorithm. While the idea of augmenting the size of the Q filter to take into account plant-model mismatch has been probably first suggested in [23], pp. 48-49, to our knowledge this approach has never been worked in detail to establish a clear procedure and to provide stability conditions. Preliminary results have been presented in [24].

The paper is organized as follows: Section 2 gives the description of the basic control structure. The development of the adaptive control strategy is presented in Section 3. Stability of the resulting scheme is analyzed in Section 4. The case of open loop stable plant models is discussed in Section 5. Simulation results on relevant examples are presented in Section 6. Experimental results on an active noise control system subject to important plant model variations are presented in Section 7.

2 Closed-loop Structure

The nominal plant model used for controller design (*design model*)² is denoted $G_o(q^{-1})$ and is described by:

$$G_o(q^{-1}) = \frac{q^{-d_o} B_o(q^{-1})}{A_o(q^{-1})} \quad (1)$$

² In active vibration and noise control the plant is called "secondary path" or "compensation path"

with:

$$\begin{aligned} d_0 &= \text{the plant pure time delay in} \\ &\quad \text{number of sampling periods;} \\ A_o &= 1 + a_1 q^{-1} + \dots + a_{n_A} q^{-n_A} ; \\ B_o &= b_1 q^{-1} + \dots + b_{n_B} q^{-n_B} = q^{-1} B^* ; \\ B_o^* &= b_1 + \dots + b_{n_B} q^{-n_B+1} ; \end{aligned}$$

where $A(q^{-1})$, $B(q^{-1})$, $B^*(q^{-1})$ are polynomials in the delay operator q^{-1} and n_{A_0} , n_{B_0} and $n_B - 1$ represent their orders³.

A stabilizing controller called "central controller" $C_o(q^{-1})$ is designed based on the nominal model:

$$C_o(q^{-1}) = \frac{R_o(q^{-1})}{S_o(q^{-1})} = \frac{R'_o(q^{-1})H_R(q^{-1})}{S'_o(q^{-1})H_S(q^{-1})} \quad (2)$$

where $H_r(q^{-1})$ and $H_s(q^{-1})$ are some fixed parts which may be imposed during the controller synthesis⁴. $C_o(q^{-1})$ is causal and $S_o(q^{-1})$ is a monic polynomial. The closed loop poles of the nominal feedback system are defined by the polynomial:

$$P_o(q^{-1}) = A_o(q^{-1})S_o(q^{-1}) + q^{-d_o} B_o(q^{-1})R_o(q^{-1}) \quad (3)$$

which has (by design) all its roots strictly inside the unit circle. The real plant model denoted $G(q^{-1})$ has the same structure as $G_0(q^{-1})$,

$$G(q^{-1}) = \frac{q^{-d} B(q^{-1})}{A(q^{-1})} \quad (4)$$

but the order of the various polynomials (n_A, n_B), the delay d and the parameters are unknown.

Remark: A system representation using coprime factorisation [2,9,8] can also be used.⁵

The following hypothesis is made throughout the paper: **H1**- The real plant model G is also stabilized by the central controller C_0 .

Therefore the polynomial:

$$P(q^{-1}) = A(q^{-1})S_o(q^{-1}) + q^{-d} B(q^{-1})R_o(q^{-1}) \quad (5)$$

³ The complex variable z^{-1} will be used for characterizing the system's behavior in the frequency domain and the delay operator q^{-1} will be used for describing the system's behavior in the time domain.

⁴ Typical examples are: opening the loop at the Nyquist frequency, including the internal model of known disturbance, opening the loop at 0 Hz in active vibration and noise control, see [10].

⁵ While this representation is without doubt advantageous for the MIMO systems, it does not seem to offer in our opinion an advantage with respect to the transfer function representation in the SISO case.

which defines the poles of the closed loop when G_0 is replaced by G has its roots strictly inside the unit circle. In other terms we will consider the class of system G satisfying the hypothesis H1.⁶

A class of systems satisfying this hypothesis can be defined by using the Vinnicombe distance (sufficient condition). One has the following result [25], [10]:

Lemma 1 : The controller C_o which stabilizes the model G_o will also stabilize the model G if:

$$\delta_\nu(G_o, G) \leq b(C_o, G_o) \quad (6)$$

where $\delta_\nu(\cdot)$ is the Vinnicombe distance and $b(\cdot)$ is the generalized Vinnicombe stability margin.

3 Development of the algorithm

As already indicated in the Introduction, applications in the fields of active vibration and noise control (see [14], [15]) clearly emphasize that model representation of the form (1) and (4) are very reliable and the changes of physical configurations and operation conditions lead to models of the same structure (with different orders and parameters). In this context it seems reasonable to use the dual Youla Kucera parametrization as a way to express the relationship between $G(q^{-1})$ and $G_o(q^{-1})$. Such an approach for describing model uncertainty has been already used. See for example [8] and [2].

In order to develop the adaptive scheme for handling both unknown disturbances and plant model uncertainties, we will consider first the "nominal" case without plant model uncertainties and assuming that the model of the disturbance is known. This will allow to set up the structure of the control law. Then the model uncertainty (assumed temporarily to be known) will be added. This will emphasize the need for an overparametrized Q filter in order to simultaneously fulfill the IMP conditions and the stability conditions related to the size of the uncertainty. In the next step the adaptive control algorithm will be set up and analyzed.

3.1 The Nominal Case

Let $\{u(t)\}$ and $\{y(t)\}$ be the input and output sequences of the controlled plant, and $\{d(t)\}$ the sequence of an output disturbance such that $|d(t)| < C \forall t$, with $0 < C < \infty$. In the nominal case, one has

$$y(t) = G_o(q^{-1})u(t) + d(t) \quad (7)$$

⁶ If both G and G_0 are open loop stable the central controller is not theoretically necessary. See Section 5 for details.

$$y(t) = \frac{\Gamma A_o - \Delta R_o}{P_o} \frac{S_o - QH_R H_S q^{-d_o} B_o}{\Gamma + QH_R H_S \Delta} d(t) \quad (19)$$

One can notice that for $G(q^{-1}) \neq G_o(q^{-1})$, a stability condition appears (in addition to (5)):

$$\Gamma(q^{-1}) + Q(q^{-1})H_R(q^{-1})H_S(q^{-1})\Delta(q^{-1}) \quad (20)$$

must have all its roots strictly inside the unit circle¹⁰. In this context, one has the following result for $w(t)$:

Lemma 2 *In case of model uncertainty expressed by the Dual-Youla parameterization as in (17), one has*

$$w(t) = \frac{\Gamma(q^{-1})A_o(q^{-1}) - \Delta(q^{-1})R_o(q^{-1})}{\Gamma(q^{-1}) + Q(q^{-1})H_R(q^{-1})H_S(q^{-1})\Delta(q^{-1})} d(t) \quad (21)$$

Proof: Taking into account equation (10) $w(t) = -q^{-d_o}B_o u(t) + A_o y(t)$ and multiplying both sides by $(S_o - QH_R H_S q^{-d_o} B_o)$ leads to $(S_o - QH_R H_S q^{-d_o} B_o)w(t) = (S_o - QH_R H_S q^{-d_o} B_o)(-B_o u(t) + A_o y(t))$. But according to (13) one has $u(t) = -\frac{R_o + QH_R H_S q^{-d_o} A_o}{S_o - QH_R H_S q^{-d_o} B_o} y(t)$. Introducing this expression in the previous ones, yields $(S_o - QH_R H_S q^{-d_o} B_o)w(t) = q^{-d_o} B_o (R_o + QH_R H_S A_o) y(t) + A_o (S_o - QH_R H_S q^{-d_o} B_o) y(t) = P_o y(t)$. But taking into account (19) one has $P_o y(t) = (\Gamma A_o - \Delta R_o) \frac{S_o - QH_R H_S q^{-d_o} B_o}{\Gamma + QH_R H_S \Delta} d(t)$. Therefore one obtains $(S_o - QH_R H_S q^{-d_o} B_o)w(t) = (\Gamma A_o - \Delta R_o) \frac{S_o - QH_R H_S q^{-d_o} B_o}{\Gamma A_o - \Delta R_o} d(t)$, hence the result $w(t) = \frac{\Gamma A_o - \Delta R_o}{\Gamma + QH_R H_S \Delta} d(t)$. $\square$

As a consequence, in order that $w(t)$ be an observation of $d(t)$ (which is bounded) the polynomial (20) must have all its roots strictly inside the unit circle. These observations can be summarized as follows:

Lemma 3 *For the case of plant uncertainty represented by the dual Youla Kucera parametrization and in the presence of bounded disturbances $d(t)$, $y(t)$, $u(t)$ and $w(t)$ will be bounded provided that the polynomial (20) will have all its roots strictly inside the unit circle.*

3.2.1 Disturbance rejection - known disturbance case

Since $G(q^{-1})$ differs from $G_o(q^{-1})$, the minimal solution of (14) denoted $\bar{Q}(q^{-1})$ does not necessarily leads to the stability of the closed-loop. In such a situation

¹⁰ This condition disappears when returning to the hypotheses of a nominal system with a minimal dimension of the Q-filter. Previous works assuming perfect knowledge of the plant model appear as particular case of the results of the present paper.

one can augment the order of $Q(q^{-1})$. Non-minimal solution to (14) can be parametrized as $Q(q^{-1}) = \bar{Q}(q^{-1}) + V(q^{-1})D_D(q^{-1})$, and (14) becomes

$$\begin{aligned} & \left(S' - Vq^{-d_o} B_o H_R H_S \right) D_d + \\ & H_R H_S q^{-d_o} B_o (\bar{Q} + V D_d) = S_o \end{aligned} \quad (22)$$

where $V(q^{-1})$ is any polynomial (there is no need for it to be monic). One has the following result about the existence of a polynomial $Q(q^{-1})$ with a degree $n_Q < \infty$ guaranteeing the closed-loop stability:

Lemma 4 *The existence of a finite dimensional Q which stabilizes the closed-loop and asymptotically reject the disturbance $d(t)$ is assured (sufficient condition) if:*

- For the frequencies ω_j of the disturbance corresponding to $D_d(e^{-i\omega_j}) = 0$, the following inequality is satisfied:

$$\begin{aligned} & \left| \frac{S_o(e^{-i\omega_j})}{e^{-i\omega_j} B_o(e^{-i\omega_j})} \right| < \\ & \left| \frac{A(e^{-i\omega_j}) S_o(e^{-i\omega_j}) + e^{-i\omega_j d} B(e^{-i\omega_j}) R_o(e^{-i\omega_j})}{e^{-i\omega_j d} B(e^{-i\omega_j}) A_o(e^{-i\omega_j}) - e^{-i\omega_j d_o} A(e^{-i\omega_j}) q^{-d_o} B_o(e^{-i\omega_j})} \right| \end{aligned} \quad (23)$$

- For all other frequencies, the following inequality is satisfied

$$\begin{aligned} & |Q(e^{-i\omega})| < \\ & \left| \frac{A(e^{-i\omega}) S_o(e^{-i\omega}) + e^{-i\omega d} B(e^{-i\omega}) R_o(e^{-i\omega})}{e^{-i\omega d} B(e^{-i\omega}) A_o(e^{-i\omega}) - e^{-i\omega d_o} A(e^{-i\omega}) q^{-d_o} B_o(e^{-i\omega})} \right| \cdots \\ & \cdots \times \left| \frac{1}{H_R(e^{-i\omega}) H_S(e^{-i\omega})} \right| \end{aligned} \quad (24)$$

The proof of this result is given in Appendix.

3.3 Adaptive disturbance rejection

In the presence of unknown narrow band disturbances and plant model uncertainties, the polynomials D_d , Γ and Δ are unknown. In this situation one can consider a Q-filter with adjustable parameters:

$$\hat{Q}(q^{-1}, t) = \hat{q}_0^Q(t) + \cdots + \hat{q}_{n_Q}^Q(t) q^{-n_Q} \quad (25)$$

and the objective is to find a parameter adaptation algorithms driving this parameters towards the values assuring asymptotic rejection of the disturbance. We will follow up to certain extent the development procedure described in [11] however including from the beginning the presence of model uncertainties described by the dual Youla Kucera representation and the use of a Q filter of

higher order than the minimal one used for the nominal case (when $G = G_0$).

When using a Q-filter with constant coefficients, the output of the system $y(t)$ in the presence of a disturbance $d(t)$ can be expressed as:

$$y(t) = \frac{\Gamma A_0 - \Delta R_0}{P_0} \frac{(S_0 - q^{-d_o} B_0 \hat{Q} H_R H_S)}{\Gamma + \hat{Q} H_R H_S \Delta} d(t) \quad (26)$$

By combining (26) and (21), one obtains

$$y(t) = \left(\frac{S_o}{P_o} - \hat{Q} \frac{H_R H_S q^{-d_o} B_o}{P_o} \right) w(t) \quad (27)$$

Since the objective is to drive $y(t)$ asymptotically to zero, it is logically to consider this variable as an adaptation error which will be denoted $\varepsilon(t)$. Now replacing in (27) the fixed value of Q by an estimation of $\hat{Q}$ one gets an expression for the a-priori adaptation error $\varepsilon^o(t+1)$:

$$\varepsilon^o(t+1) = \left(\frac{S_o}{P_o} - \hat{Q}(q^{-1}, t) \frac{H_R H_S q^{-d_o} B_o}{P_o} \right) w(t+1) \quad (28)$$

One can define the a-posteriori adaptation as:

$$\varepsilon(t+1) = \left(\frac{S_o}{P_o} - \hat{Q}(q^{-1}, t+1) \frac{H_R H_S q^{-d_o} B_o}{P_o} \right) w(t+1) \quad (29)$$

Taking into account Eq.(14) (resulting from the application of the IMP principle for the case of known disturbances), Eq.(29) can be rewritten as:

$$\varepsilon(t+1) = \left(Q(q^{-1}) - \hat{Q}(q^{-1}, t+1) \right) \frac{q^{-d_o} B_o^* H_R H_S}{P_o} w(t) + v(t+1) \quad (30)$$

where $v(t+1) = \frac{D_p(q^{-1})S'(q^{-1})}{P_o(q^{-1})} w(t+1)$ is a signal which tends towards 0. Set

$$w_2(t) = \frac{q^{-d_o} B_o^* H_R H_S}{P_o} w(t) \quad (31)$$

The a-posteriori adaptation error can be expressed under the form [13]:

$$\varepsilon(t+1) = H(q^{-1}) \left(\theta - \hat{\theta}(t+1) \right)^T \phi(t) \quad (32)$$

where

$$\begin{aligned} \phi^T(t) &= [w_2(t) \cdots w_2(t-n_Q)] \\ \theta^T &= [q_0^Q \cdots q_{n_Q}^Q] \\ \hat{\theta}^T(t) &= [\hat{q}_0^Q(t) \cdots \hat{q}_{n_Q}^Q(t)] \end{aligned}$$

and

$$H(q^{-1}) = 1$$

Taking into account the fact that the order of the polynomial $\hat{Q}$ is higher than the minimal order required by the IMP, the parameter adaptation algorithm proposed in [11] has to be completed with a projection of the estimated parameter vector on a bounded domain in order to prove the stability of the adaptive control scheme (see [13] p.340). The adaptation algorithm to be used is

$$\hat{\theta}(t+1) = \hat{\theta}_p(t) + F(t)\phi(t)\varepsilon(t+1) \quad (33)$$

$$\varepsilon(t+1) = \frac{\varepsilon^o(t+1)}{1 + \phi^T F(t) \phi(t)} \quad (34)$$

$$F(t+1)^{-1} = \lambda_1 F(t)^{-1} + \lambda_2 \phi(t)\phi^T(t) \quad (35)$$

$$0 < \lambda_1 < 1 \quad 0 \leq \lambda_2 < 2, F_0 > 0 \quad (36)$$

where $\hat{\theta}_p(t)$ is the projection of $\hat{\theta}(t)$ which is computed as follows:

$$\begin{aligned} \hat{\theta}'(t) &= F(t)^{-1/2} \hat{\theta}(t) \\ \hat{\theta}'_p(t) &= \hat{\theta}'(t) \quad \text{if } \hat{\theta}'(t+1) \in \mathcal{D}' \\ \hat{\theta}'_p(t) &= \perp \text{ proj of } \hat{\theta}'(t) \text{ on } \mathcal{D}' \quad \text{if } \hat{\theta}'(t) \notin \mathcal{D}' \\ \hat{\theta}_p(t) &= F(t)^{1/2} \hat{\theta}'_p(t) \end{aligned} \quad (37)$$

The projection domain $\mathcal{D}'$ is defined as follows:

$$\hat{\theta} \in \mathcal{D}, \quad \hat{\theta}'(t) = F(t)^{-1/2} \hat{\theta}(t) \in \mathcal{D}' \quad (38)$$

where the projection domain $\mathcal{D}$ is such that:

$$\mathcal{D}: \|\hat{\theta}(t)\|_2^2 < \mathcal{R} < \infty \quad (39)$$

Two particular choices for the adaptation gain $F(t)$ are mainly used in practice in order to assure the alertness of the adaptation with respect to possible variations of the disturbance characteristics [13,11]¹¹:

- *Constant trace*: for a constant ratio $\lambda_1(t)/\lambda_2(t)$, $\lambda_1(t)$ is chosen such that the trace of the adaptation gain matrix $F(t)$ remain constant ($\text{trace} F(t) = \text{trace} F_0$)
- *Constant gain*: $\lambda_1(t) = 1, \lambda_2(t) = 0$ and $F(t) = F_0 = \alpha I; \alpha > 0$.

4 Stability Analysis

The following theorem summarizes the stability analysis of the adaptive scheme

Theorem 1 *Under the assumptions:*

- *Controller C_0 stabilizes unknown plant model G*
- *Disturbance $d(t)$ has the form (8) and is bounded*

¹¹ For other options and connections with recursive least squares and stable gradient algorithm, see [13,11].

- It exists $Q(q^{-1}) \in \mathcal{D}$ with $n_D - 1 \leq n_Q < \infty$ satisfying (14) and assuring that the polynomial (20) has all its roots strictly inside the unit circle.

then one has:

$$\lim_{t \rightarrow \infty} \varepsilon(t+1) = 0 \quad (40)$$

and if in addition:

$$\max_{\hat{\theta} \in \mathcal{D}} \left(\|\hat{Q} - Q\|_1 \right) \left\| \frac{\bar{\Delta}}{\Gamma + \bar{\Delta}Q} \right\|_1 < 1 \quad (41)$$

where $\bar{\Delta} = \Delta H_R H_S$
one has:

$$w(t), u(t), y(t) \text{ are bounded} \quad (42)$$

and:

$$\lim_{t \rightarrow \infty} \varepsilon^o(t+1) = \lim_{t \rightarrow \infty} y(t+1) = 0 \quad (43)$$

Fig. 3. Generation of $w(t)$.

Condition (41) guarantees that $w(t)$ as well as the regressor vector, the output and the input will be bounded. This condition results from the analysis of the equivalent scheme connecting $d(t)$ with $w(t)$ shown in Fig.3 where $\tilde{Q}(t) = \hat{Q}(t) - Q$ (see details in the proof below). In order that $w(t)$ be bounded the feedback loop should be BIBO. Applying the "small gain theorem" condition (41) results.

Remark : For $G = G_0$ and $\bar{\Delta} = 0$, condition (41) is automatically satisfied. Similar situation occurs for the case $n_Q = n_D - 1$ if (20) is satisfied since the persistence of excitation drives $\tilde{Q}(t) = \hat{Q}(t) - Q$ to zero.

Proof: The equation of the a posteriori adaptation error (32) has the form considered in Theorem 3.2 from [13] (pp 104-105) and since $H(q^{-1}) = 1$, one immediately concludes that (40) holds. To go further one has to show that $\phi(t)$ and respectively $w(t)$ are bounded ($\phi(t)$ contains delayed and filtered signals generated by $w(t)$).

In an adaptive context using (21), one can write:

$$(\Gamma + \bar{\Delta}\hat{Q}(t)) w(t) = (\Gamma A_o - \Delta R_o) d(t) \quad (44)$$

or alternatively:

$$(\Gamma + \bar{\Delta}\hat{Q}(t) \pm \bar{\Delta}Q) w(t) = (\Gamma A_o - \Delta R_o) d(t) \quad (45)$$

Set $N = \Gamma A_o - \Delta R_o$, and $\tilde{Q}(t) = \hat{Q}(t) - Q$, then (45) can be written as:

$$w(t) = \frac{N}{\Gamma + \bar{\Delta}Q} d(t) - \tilde{Q}(t) \frac{\bar{\Delta}}{\Gamma + \bar{\Delta}Q} w(t) \quad (46)$$

which corresponds to a feedback system with an external bounded excitation whose output is $w(t)$, shown in Fig. 3. The input to the equivalent feedback system $\bar{d} = \frac{N}{\Gamma + \bar{\Delta}Q} d(t)$ has the property:

$$\|\bar{d}(t)\|_\infty \leq \left\| \frac{N}{\Gamma + \bar{\Delta}Q} \right\|_1 d_m(t) \quad (47)$$

where $d_m(t) = \sup |d(t)| = \|d(t)\|_\infty$ and $\left\| \frac{N}{\Gamma + \bar{\Delta}Q} \right\|_1$ is finite. The output of the feedback path $x(t)$ is given by

$$x(t) = \frac{\bar{\Delta}\tilde{Q}}{\Gamma + \bar{\Delta}Q} w(t) \quad (48)$$

and therefore:

$$\|w(t)\|_\infty \leq \|\bar{d}(t)\|_\infty + \|\tilde{Q}\|_1 \left\| \frac{\bar{\Delta}}{\Gamma + \bar{\Delta}Q} \right\|_1 \|w(t)\|_\infty \quad (49)$$

Applying small gain arguments [5] and taking into account that the equivalent feedforward path has unitary gain one concludes that condition (41) assures the boundedness of $w(t)$. From the definition of $\phi(t)$ given in (31) and (32) one concludes also that $\phi(t)$ is bounded and taking in account (34) one concludes that (43) is true. Since $y(t)$ is bounded in finite time (and goes to zero), $w(t)$ is bounded and Q is bounded one concludes that $u(t)$ is also bounded. $\square$

5 An important particular case

When hypothesis H1 is not satisfied but G and G_0 are asymptotically stable one can suppress the central stabilizing controller by taking $R_o(q^{-1}) = 0$ and $H_S = 1$ and replace H1 by the stability condition upon G and G_0 . All the previous results remain valid but there are some significant simplifications. In this situation, one obtains

$$\begin{aligned} \Gamma(q^{-1}) &= A(q^{-1})S_o(q^{-1}) \\ y(t) &= \frac{A}{A_o} \left(\frac{S_o - QH_R q^{-d_o} B_o}{AS_o + QH_R(q^{-d}BA_o - q^{-d_o}B_oA)} \right) d(t) \end{aligned} \quad (50)$$

and Lemma 4 becomes:

Lemma 5 *In the case where $R_o = 0$ and $H_S = 1$ with G and G_0 asymptotically stable, sufficient conditions for the existence of a finite dimensional Q which stabilizes the closed-loop and ensure the asymptotic rejection of the disturbance are*

- At the frequencies ω_j corresponding to $D(e^{-i\omega_j}) = 0$

$$\left| e^{-i(d-d_o)\omega_j} \frac{B(e^{-i\omega_j})A_o(e^{-i\omega_j})}{B_o(e^{-i\omega_j})A(e^{-i\omega_j})} - 1 \right| < 1 \quad (51)$$

- At all other frequencies the inequality (24) is verified.

Proof: Follows directly from Lemma 4 by taking $R_o = 0$ and $H_S = 1$.

Lemma 5 shows that, if $R_o = 0$, the feasibility of the disturbance rejection depends on the discrepancy between the nominal and uncertain plant. This result can also be interpreted as the condition that $z^{-(d_o-d)} \frac{B_o(z^{-1})}{A_o(z^{-1})} \frac{A(z^{-1})}{B(z^{-1})} - \frac{1}{2}$ is a strictly positive real transfer function for $z = e^{i\omega_j}$ [17], or that the phase of this function is within $]-90^\circ; +90^\circ[$ at least at the frequencies of the disturbance.

6 Simulation results

6.1 Two examples

Let us consider the nominal plant model with $A_o(q^{-1}) = 1 - 1.45q^{-1} + 0.475q^{-2}$ and $B_o(q^{-1}) = 0.5q^{-1} - 0.5q^{-2}$, driven by the central controller $R_o(q^{-1}) = 0.15 - 0.14q^{-1}$, and $S_o(q^{-1}) = 1 - 0.74q^{-1}$ where $H_R = H_S = 1$. In a first configuration, the uncertain plant is given by $A(q^{-1}) = 1 - 2.8635q^{-1} + 2.9811q^{-2} - 1.330q^{-3} + 0.213q^{-4}$, and $B(q^{-1}) = 0.59q^{-1} - 1.3566q^{-2} + 0.9916q^{-3} - 0.2250q^{-4}$, which corresponds to $\Gamma(q^{-1}) = 1 - 1.4q^{-1} + 0.45q^{-2}$, and $\Delta(q^{-1}) = 0.09q^{-1} - 0.09q^{-2}$. The sampling frequency is $Ts = 1s$. The frequency characteristics of the nominal and uncertain models are represented in Figure. 4.

The uncertain plant is disturbed by an harmonic out-

Fig. 4. Frequency characteristics of the nominal and uncertain plant models

put noise having a frequency of $0.0318Hz$ (0.2 rad/s), corresponding to $D_d(q^{-1}) = 1 - 1.96q^{-1} + q^{-2}$. Under this assumption, one can show that the minimal order Q-filter is $\bar{Q}(q^{-1}) = -8.56 + 11q^{-1}$, ($n_Q = 1$) and one verifies easily that $\Gamma + \bar{Q}H_RH_S\Delta$ has not all its zeros

within the unit circle. A simulation displayed in Figure. 5 shows that convergence is not obtained. This leads

Fig. 5. Evolution of the residual error and parameters for example 1: $n_Q = 1$

to use an augmented Q-filter: $Q = \bar{Q} + VD$, and by choosing $V(q^{-1})$ as a scalar we will have $n_Q = 2$. For $V \in [0.85; 8.93]$ the stability and the perfect rejection of the disturbance is guaranteed. Figure. 6 shows that the error signal tends towards zero and stable operation of the adaptive scheme is obtained.

Now, in a second configuration, the uncertain system

Fig. 6. Evolution of the residual error and parameters for example 1: $n_Q = 2$

differs more significantly from the nominal one, with $B(q^{-1}) = 0.65q^{-1} - 1.461q^{-2} + 1.036q^{-3} - 0.225q^{-4}$ and $A(q^{-1}) = 1 - 2.8755q^{-1} + 2.9985q^{-2} - 1.3385q^{-3} + 0.2138q^{-4}$, corresponding to $\Gamma(q^{-1}) = 1 - 1.4q^{-1} + 0.45q^{-2}$ and $\Delta(q^{-1}) = 0.15q^{-1} - 0.15q^{-2}$, the corresponding Bode diagram is displayed in Figure 4. It can be verified that no scalar V assuring both perfect disturbance rejection and stability of the closed-loop exists. Indeed the adaptive system shows instability phenomena in this case with $n_Q = 2$ as shown in Fig. 7. If the Q-filter order is augmented to $n_Q = 3$, one can check that for $V(q^{-1}) = -5 - 8q^{-1}$, the polynomial $\Gamma + \bar{Q}H_RH_S\Delta$ has all its zeros strictly inside the unit circle. Therefore, for $n_Q = 3$, there exists at least one expression of

Fig. 7. Evolution of the residual error and parameters for example 2: $n_Q = 2$

$Q(q^{-1})$ satisfying the internal model principle condition and providing the closed-loop stability at the same time. Fig. 8 shows that, effectively, for $n_Q = 3$ that stable operation of the adaptive scheme is obtained.

Fig. 8. Evolution of the residual error and parameters for example 2: $n_Q = 3$

7 Active noise control experiments

7.1 Test-bench description

The detailed scheme of the control test bench is shown in Fig. 9 and the views of the two implementations which will be considered subsequently are shown in Fig. 10.

The speaker used as the source of disturbances is labeled as 1, the control speaker is 2 and finally, at pipe's open end, the microphone that measures the system's output (residual noise) is denoted as 3. The transfer function between the disturbance's speaker and the microphone (1→3) is named *Primary Path*, while the transfer function between the control speaker and the microphone (2→3) is named *Secondary Path*. $y(t)$ is the system's output (residual noise measurement), $u(t)$ is the control signal and $p(t)$ is the disturbance. The

Fig. 9. Duct active noise control test bench diagram.

Fig. 10. Duct active noise control test bench (Photo): configuration G_o (top), configuration G (bottom)

Secondary Path is the *plant* considered throughout the paper. Both primary and secondary paths have a double differentiator behavior, since as input we have the voice coil displacement, and as output the air acoustic pressure. The sampling frequency is $2500Hz$.

The secondary path models of the two configurations have been identified from experimental data using the methodology described in [18]. Fig. 11 shows the frequency characteristics of the identified models. There are important differences between the two models. These characteristics present multiple resonances (low damped complex poles)¹² and anti-resonances (low damped complex zeros). The orders of the two models are summarized in Table 1.

Model	n_A	n_B	d
Secondary path G_o	38	32	8
Secondary path G	27	27	6

Table 1

Orders of the identified models.

¹² The lowest damping is around 0.01.

Fig. 11. Bode diagrams of the nominal and uncertain plants

7.2 Control design

The adaptive controller is based on the simplified algorithm of Section 5, where $R_o = 0$ and $H_S = 1$. According to lemma 5, inequality (23) can be expressed in this case by $|Q| < |M| |S_o/H_R|$ where $M(\omega) = \left| \frac{A(e^{-i\omega})}{e^{-i\omega d} B(e^{-i\omega}) A_o(e^{-i\omega}) - e^{-i\omega d_o} A(e^{-i\omega}) B_o(e^{-i\omega})} \right|$. $M(\omega)$ is displayed in Fig. 12 (top), in order to calibrate the transfer function $\frac{S_o(z^{-1})}{H_R(z^{-1})}$. $M(\omega)$ has local minima at

Fig. 12. Magnitude of $M(e^{-i\omega})$ (top) and Magnitude of $S_o(e^{-i\omega})/H_R(e^{-i\omega})$ (bottom).

80, 244, 400, 686, 828, 1110, and 1200 Hz. The inequality (24) suggests to design $\frac{S_o(z^{-1})}{H_R(z^{-1})}$ as series of band-pass filters with a strong amplification at these frequencies. Additionally, H_R incorporates zeros at -1 and $+1$, in order to open the loop at 0 Hz and $0.5/Ts$ Hz. The magnitude

of $\frac{S_o(z^{-1})}{H_R(z^{-1})}$ is presented in Fig. 12 (bottom). Furthermore, the magnitude $\left| e^{-i(d-d_o)\omega_j} \frac{B(e^{-i\omega_j}) A_o(e^{-i\omega_j})}{B_o(e^{-i\omega_j}) A(e^{-i\omega_j})} - 1 \right|$ is displayed in Fig. 13, showing that the sufficient condition (51) is satisfied in the ranges $[102 \text{ Hz}; 229 \text{ Hz}]$ and $[252 \text{ Hz}; 295 \text{ Hz}]$. This is a sufficient condition for defining the regions where rejection of disturbances can be achieved (in these regions the system has enough gain for achieving rejection of disturbances-see Fig. 11).

Fig. 13. Magnitude of $e^{-i(d-d_o)\omega_j} \frac{B(e^{-i\omega_j}) A_o(e^{-i\omega_j})}{B_o(e^{-i\omega_j}) A(e^{-i\omega_j})} - 1$

7.3 Simulation results

The objective is to show that despite the fact that the model of the secondary path used for design is different from the identified model of the current implementation, one can obtain performance similar to the case where the model of the secondary path is known, by augmenting the size of the adjustable Youla Kucera Filter.

Two tonal disturbances with variable frequency will be used. Their frequencies are very close, generating a very strong interference phenomenon (hard to attenuate in practice). The first couple of frequencies are 160 Hz and 160.5 Hz. At $t=50$ s these frequencies will be switched to 150 Hz and 150.2 Hz and then at $t=100$ s one switches back to 160 Hz and 160.5 Hz. Figure 14a shows the disturbances. Tests have been conducted using plant model G in order to determine what is the dimension of the Q filter assuring the stability of the system. A Q filter with 14 adjustable parameters is necessary (instead of 4 parameters for the nominal case). It was found that 16 parameters assure much better transients. Further increase of the order of the Q filter did not provide significant improvement in performance¹³. For the simulations, the system operates in open loop for the first 5 s. Figure 14b shows the closed-loop behavior on the system G with a Q filter having 16 adjustable parameters. For comparison the same Q filter complexity is used for the nominal plant model and the results are shown in Figure 14c. One can see that the performance of the two systems are very close. Figure 14d shows the results for

¹³ It was observed that augmenting the size of the Q filter leads to a reduction of $|Q(e^{-i\omega})|$ in the regions concerned by condition (24)

the nominal plant model when using a Q filter of minimal complexity (4 parameters).

Fig. 14. Error Signals from top to bottom a): Open-loop, b): uncertain plant with $n_Q = 16$, c): nominal plant $n_Q = 16$, d): nominal plant $n_Q = 4$

7.4 Experimental results

In this subsection, real time results obtained on configuration G using for the design the model of the secondary path of the configuration G_0 (see Fig. 10)¹⁴. Fig 15 shows the time response of the residual noise for a tonal disturbance at 180 Hz¹⁵. The system operates in open loop for the first 5 s. The transient is very short and the global steady state attenuation is 88.41 dB. Fig. 16 shows the Power Spectral Density of the residual noise in open and in closed loop (steady state)¹⁶. The tonal disturbance is attenuated for about 110 dB. One can see the presence of an harmonic of the disturbance in the residual noise in open loop indicating a small nonlinear behavior of the real system. However, the control algorithm attenuates this harmonic. The Q-filter has 16 parameters.

Fig. 17 shows the behavior of the system for the interfer-

Fig. 15. Evolution of the residual noise for a 180 Hz sinusoidal disturbance (16 parameters).

Fig. 16. Power Spectral density of the residual noise in open loop and in closed loop for a 180 Hz sinusoidal disturbance (16 parameters) .

ence phenomenon considered in the simulation subsection. Two tonal disturbances at 160 Hz and 160.5 Hz are applied. At $t=50$ s their frequencies are switched to 150 Hz and 150.2 Hz and at $t=100$ their frequencies return

¹⁴ Gabriel Buche, GIPSA-LAB has collaborated to the experiments.

¹⁵ For all the experiments, a constant trace adaptation gain has been used. The magnitude of the trace was $30 \times \text{nb.of parameters}$.

¹⁶ PSD evaluated on an horizon of 3s

to the initial values. The system operates in open loop for the first 5 s. The Q filter has 16 parameters. One notes an excellent attenuation of the interference phenomenon like in the simulations. Evolution of the parameters is shown in Fig. 18.

Fig. 19 shows the time response of the residual noise

Fig. 17. Time response of the residual noise for step changes in the frequencies of an interference phenomenon (16 parameters).

Fig. 18. Evolution of the parameters for step changes in the frequencies of an interference phenomenon (16 parameters) .

for a sequence of steps changes on the frequencies of two tonal disturbances. The initial frequencies are 80 Hz and 170 Hz. At $t = 25$ s one switches to 70 Hz and 160 Hz. At $t = 50$ s one returns to 80 Hz and 170 Hz and at $t = 75$, one switches to 90 Hz and 180 Hz. In order to assure simultaneously the asymptotic rejection of the two simultaneous disturbances and the stability of the system over the range 70 Hz to 190 Hz, it was necessary to increase significantly the order of the Q filter (to 60 parameters). One observes that the results are slightly less good when approaching the border of the stability region (however > 60 dB attenuation in steady state). Fig. 20 shows the evolution of the parameters of the Q filter.

8 Concluding remarks

The paper has demonstrated through the use of the dual Youla Kucera plant parametrization, that in the pres-

Fig. 19. Evolution of the residual noise for step changes in the frequency of two simultaneous sinusoidal disturbances around 80 Hz and 170 Hz.

Fig. 20. Evolution of the parameters for step changes in the frequency of two simultaneous sinusoidal disturbances around 80 Hz and 170 Hz (60 parameters).

ence of model uncertainties, adaptive rejection of unknown narrow band disturbances requires to use in most cases an over parametrized Youla Kucera compensator in order to both satisfy the internal model principle and to assure the stability of the whole scheme. Clear conditions for over parametrization have been provided and an appropriate adaptive algorithm have been developed and analyzed. Relevant simulations and experimental results obtained on an active noise control system have illustrated the potentiality of the approach proposed.

9 Appendix A: Proof of Lemma 4

According to (14), the expression of Γ given in (17), and since $A_o S_o + q^{-d_o} q^{-d_o} B_o R_o$ has all its roots strictly inside the unit circle, a necessary and sufficient stability condition for the closed-loop is that the polynomial $AS_o + q^{-d} BR_o + QH_R H_S (q^{-d} BA_o - q^{-d_o} q^{-d_o} B_o A)$ has all its roots inside the unit circle. This condition is equivalent to the stability of a closed-loop including two systems W_1 and W_2 , where W_2 is in negative feedback with W_1 , and where $W_1 = QH_S H_R$ and $W_2 = \frac{\Delta}{\Gamma} = \frac{q^{-d} BA_o - Aq^{-d_o} B_o}{AS_o + q^{-d} BR_o}$ (Fig. 21). From the small gain theorem, a sufficient stability condition for this closed-loop is that $\|Q(z^{-1})H_R(z^{-1})H_S(z^{-1})\|_\infty < \|\frac{z^{-d} B(z^{-1})A_o(z^{-1}) - A(z^{-1})z^{-d_o} B_o(z^{-1})}{A(z^{-1})S_o(z^{-1}) + z^{-d} B(z^{-1})R_o(z^{-1})}\|_\infty$. Consequently a sufficient stability condition for the closed-loop is (24). Now at the frequencies ω_j , equation (14) must be

Fig. 21. Connection of the Q filter with the dual Youla-Kucera filter

satisfied. Since one has $D_d(e^{-i\omega_j}) = 0$, it results that $Q(e^{-i\omega_j}) = \frac{S_o(e^{-i\omega_j})}{H_R(e^{-i\omega_j})H_S(e^{-i\omega_j})e^{-i\omega_j d_o}e^{-d_o i\omega_j}B_o(e^{-i\omega_j})}$, by combining with (24), one obtains condition (23).

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