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Essential conformal structures in Riemannian and Lorentzian geometry

Charles Frances *

Abstract. This paper is devoted to pseudo-Riemannian essential structures, namely the pseudo-Riemannian manifolds, whose conformal group of transformations can not be reduced to a group of isometries. A celebrated result of M.Obata and J.Ferrand states that for Riemannian manifolds, the only essential structures are the standard sphere, and the Euclidean space. We discuss what becomes this result in higher signature, focusing on the Lorentzian case.

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1. Introduction

The starting point of what has been called *Lichnerowicz's conjecture*, is the very simple and naïve question: “*given a Riemannian manifold, is the group of angle-preserving transformations bigger than the group of distance-preserving ones?*” At first glance, the natural answer should be almost always affirmative. Indeed, the data of a Riemannian metric g on a manifold M , seems to be stronger than a simple “angle-structure”, most commonly called *conformal structure*, i.e the data of a whole family of metrics $[g] = \{e^\sigma g \mid \sigma \in C^\infty(M)\}$.

As an illustration, one thinks at once to a similarity $x \mapsto \lambda A.x + T$ of $\mathbf{R}^n$, with $|\lambda| \neq 1$, $A \in O(n)$. Such a transformation is conformal for the flat metric on $\mathbf{R}^n$. It fixes a unique point $x_0 \in \mathbf{R}$, at which its differential is λId . Since $u \mapsto \lambda u$ can't preserve any scalar product, we get that $x \mapsto \lambda Ax + T$ can't preserve any smooth Riemannian metric in the conformal class of the flat metric.

This example motivates the following:

Definition 1.1. Let (M, g) be a smooth Riemannian manifold. Let f be a conformal transformation of (M, g) , i.e there exists a smooth function $\sigma_f : M \rightarrow \mathbf{R}$, such that $f^*g = e^{2\sigma_f}g$. The transformation is said to be *essential* if f does not preserve any metric in the conformal class $[g]$ of the metric g .

More generally, a subgroup G of conformal transformations of (M, g) is said to be essential if it does not preserve any metric in the conformal class.

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One says also that the structure (M, g) itself is essential when its group of conformal transformations is essential.

By the stereographic projection, the similarities $x \mapsto \lambda x + T$ can be extended smoothly to transformations of the sphere $\mathbf{S}^n$ fixing the “point at infinity”. It turns out that these extensions act as conformal transformations for the round metric g_{can} , of constant curvature $+1$, on $\mathbf{S}^n$. Exactly by the same argument as above, such transformations are essential. Another way to see quickly that the conformal group of $(\mathbf{S}^n, g_{can})$ is essential, is to notice that this conformal group, the *Möbius group* $PO(1, n+1)$, is not compact. On the other hand, by Ascoli’s theorem, the group of isometries of a compact Riemannian manifold has to be compact.

Let us now try to determine the conformal group of a flat torus $\mathbf{T}^n = \mathbf{R}^n / \Gamma$, where $\Gamma = \mathbb{Z}\tau_1 \oplus \dots \oplus \mathbb{Z}\tau_n$ is a lattice. We endow this torus with the metric $\bar{g}_{flat}$ induced by the flat metric on $\mathbf{R}^n$. Any conformal transformation $\bar{f}$ lifts to a conformal transformation f of $(\mathbf{R}^n, g_{eucl})$. Thus f is of the form $x \mapsto \lambda A.x + T$, with $\lambda \in \mathbf{R}^*$ and $A \in O(n)$. Let us suppose that $|\lambda| \neq 1$. We can then assume $|\lambda| > 1$. If U is a small open subset, such that the covering map $\pi : \mathbf{R}^n \rightarrow \mathbf{T}^n$ is injective on U . Then π has to be injective on every $f^k(U)$, $k \in \mathbb{N}$. On the other hand, $\lim_{k \rightarrow +\infty} Vol(f^k(U)) = +\infty$, where Vol is the euclidean volume form on $\mathbf{R}^n$. But π can not be injective on an open subset with a volume strictly greater than $Vol(\tau_1, \dots, \tau_n)$, yielding a contradiction. We infer that $|\lambda| = 1$, and that $\bar{f}$ is an isometry of $(\mathbf{T}^n, \bar{g}_{flat})$. Thus, the conformal group of $(\mathbf{T}^n, \bar{g}_{flat})$ is exactly the group of isometries. One says in this case that *the conformal group is inessential*.

Looking for more examples, we could determine the conformal group of other Riemannian manifolds. For example that of the hyperbolic space $\mathbf{H}^n$, or of $\mathbf{RP}^n$. Everytime, we get that this group is inessential, and reduces to the group of isometries.

So, starting with the feeling that essential Riemannian manifolds should be quite numerous, we still have only two examples of such essential structures! This lack of examples led to the:

Lichnerowicz’s Conjecture: *The only Riemannian manifolds of dimension at least two having an essential conformal group, are, up to conformal diffeomorphism, the standard sphere $(\mathbf{S}^n, g_{can})$ and the Euclidean space $(\mathbf{R}^n, g_{eucl})$.*

Several partial results toward the conjecture, were made during the sixties by [4], [30], [28], [31], among others.

Almost simultaneously, but with different approaches, J.Ferrand and M.Obata proved in [12] and [32] that the conjecture was true in the compact case. Finally, in 1996, J.Ferrand answered definitively to the original question of Lichnerowicz, proving:

Theorem 1.2. [13] *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. If the group of conformal transformations of (M, g) is essential, then (M, g) is conformally diffeomorphic to:*

- (i) $(\mathbf{S}^n, g_{can})$ if M is compact.
- (ii) $(\mathbb{R}^n, g_{can})$ if M is not compact.

This theorem has been proved independently by R.Schoen in [34] (see also [18]).

Ferrand's result is often presented as a nice example of the following general principle. Generically, rigid geometric structures have a trivial group of automorphisms (even if we just consider the local group of automorphism). So, when the group of automorphism is nontrivial, and even "big", then the geometric structure has to be very peculiar. Of course, we must precise what we mean by "big". When we are looking at compact manifold, a big group of automorphisms is for example a non compact one. To understand why Ferrand's theorem illustrates this principle, we have to precise that Riemannian conformal structures (and more generally pseudo-Riemannian ones) are rigid geometric structure. Indeed, such a structure defines naturally a parallelism on a subbundle $\mathcal{B}^2(M) \subset \mathcal{R}^2(M)$, of the bundle $\mathcal{R}^2(M)$ of 2-frames of M (details can be found in [23]). Any local conformal transformation acts on an open subset of $\mathcal{B}^2(M)$ preserving this parallelism. We thus see that a conformal transformation whose 2-jet at a point is the 2-jet of the identical transformation, will fix a point of $\mathcal{B}^2(M)$. Since the parallelism is preserved, this means that the transformation induces the identical transformation on $\mathcal{B}^2(M)$, hence the transformation is itself the identical transformation of M . We thus see that any conformal transformation is completely determined by its 2-jet at a point of M , so that the dimension of the Lie algebra of infinitesimal conformal transformations is finite. This is a manifestation of the rigidity of conformal structures.

Now, how can we interpret the condition of essentiality, as a criteria for the conformal group to be big.

Let us recall that the action of a group G , by homeomorphisms of a manifold M , is said to be *proper* if for every compact subset $K \subset M$, the set:

$$G_K = \{g \in G \mid g(K) \cap K \neq \emptyset\}$$

has compact closure in $Homeo(M)$ (where $Homeo(M)$, the group of homeomorphisms of M is endowed with the compact-open topology). In particular, when the manifold M is compact, the action of G is proper if and only if G is compact. Nonproperness can be thought as the weakest condition of non triviality, for the dynamics of a group action on a manifold.

A key point is the following theorem of Alekseevski, which, in the Riemannian framework, makes the link between essentiality and dynamics of the conformal group:

Theorem 1.3. [2] *Let (M, g) be a Riemannian manifold. The conformal group of (M, g) is essential if and only if its action on M is not proper.*

One part of this theorem is clear: by Ascoli's theorem, the isometry group of a Riemannian manifold acts properly.

When the manifold (M, g) is compact, the converse is quite easy to prove. Assume that the conformal group H acts properly, i.e is compact. Then, if $d\mu$ is

the bi-invariant Haar measure on H , the metric $\bar{g} = \int_{h \in H} h^* g d\mu(h)$ is a smooth Riemannian metric of $[g]$, left invariant by the group H . Thus, the conformal group is inessential.

For noncompact manifolds, the proof is more technical.

In the light of Alekseevski's result, Theorem 1.2 is a remarkable example of geometrico-dynamical rigidity. Here, nonproperness, a very weak assumption on the dynamics of the conformal group, implies very strong consequences on the geometry of the manifold: the only possible geometries turn out to be, up to conformal diffeomorphism, $(\mathbf{S}^n, g_{can})$ and $(\mathbf{R}^n, g_{eucl})$.

Our aim in this article, is to study to what extent Theorem 1.2 generalizes (or does not) to more general frameworks. In the next section, we will discuss what such generalizations could be. In section 3, we recall the main arguments to prove Theorem 1.2. Then, in the next sections, we focus on Lorentzian conformal geometry, trying to understand the meaning of essentiality in this case.

2. Generalizations of Lichnerowicz's conjecture

2.1. Conjecture in the pseudo-Riemannian framework. The definition 1.1 of an essential structure carries in an obvious way to general pseudo-Riemannian manifolds (recall that a (smooth) pseudo-Riemannian metric g on a manifold M is a (smooth) field of nondegenerate quadratic forms of constant signature (p, q) on TM). It is thus natural to ask whether Ferrand's theorem also generalizes in some way, to any signature.

Let us point out the first difficulty occurring when we pass from Riemannian conformal geometry to general signature (p, q) . While in Riemannian signature, we saw, thanks to Theorem 1.3, that essentiality is equivalent to nonproperness of the action of the conformal group, this equivalence is no longer true in higher signature. It is still true that the properness of the action of the conformal group implies inessentiality, but the converse is false, as shows the following example. Endow the space $\mathbf{R}^n$ with the Lorentzian metric $g_{mink} = -(dx_1)^2 + (dx_2)^2 + \dots + (dx_n)^2$. If $O(1, n-1)$ is the group of linear transformations preserving the quadratic form $-x_1^2 + x_2^2 + \dots + x_n^2$, then the conformal group of $(\mathbf{R}^n, g_{mink})$ is the group generated by homotheties, translations, and elements of $O(1, n-1)$. We look at the torus $\mathbf{R}^n/\mathbb{Z}^n$, endowed with the induced metric $\bar{g}_{mink}$. By a proof analogous to that made in the introduction for a Riemannian flat torus, it is not hard to check that the conformal group of $(\mathbf{T}^n, g_{mink})$ is exactly the group of isometries of $(\mathbf{T}^n, \bar{g}_{mink})$, so that the structure is inessential. Nevertheless, by a Theorem of Borel and Harisch-Chandra, the subgroup $O_{\mathbb{Z}}(1, n-1) = O(1, n-1) \cap SL(n, \mathbb{Z})$ is a lattice in $O(1, n-1)$. In particular, it is noncompact, and since it normalizes $\mathbb{Z}^n$, it induces an isometric action on $(\mathbf{T}^n, \bar{g}_{mink})$. Thus the action of $Isom(\mathbf{T}^n, \bar{g}_{mink})$, and hence of $Conf(\mathbf{T}^n, \bar{g}_{mink})$, is not proper, while the structure is inessential.

Hopping a direct generalization of Theorem 1.2, we could ask:

Question 2.1. *Let $(p, q) \in \mathbb{N}^2$ be two integers. Are there, up to conformal diffeomorphism, only a finite number of pseudo-Riemannian manifolds (M, g) of signature (p, q) , for which the conformal group is essential?*

It turns out that with this degree of generality, there is no hope to get a positive answer to this question.

Given a basis $(e_1, \dots, e_n)$ of $\mathbf{R}^n$, we look at the one-parameter group of transformations: $\psi^t : (x_1, \dots, x_n) \mapsto (e^{2t}x_1, e^tx_2, \dots, e^tx_{n-1}, x_n)$. This group acts as conformal transformations for a lot of Lorentzian metrics on $\mathbf{R}^n$. In fact, Alekseevski proved in [3] that ψ^t acts as an homothetic flow for every metric of the form $g = 2dx_1dx_n + \sum_{i=2}^{n-1}(dx_i)^2 + \sum_{i,j=2}^{n-1}\lambda_{ij}y_i dy_j dx_n + (Q(x_n)(x_2, \dots, x_{n-1}) + b(x_n)x_1)(dx_n)^2$, for λ_{ij} , b smooth functions, and $Q(x_n)$ a smooth family of quadratic forms in the variables $x_2, \dots, x_{n-1}$. This means that for such a metric g , we have $(\psi^t)^*g = e^tg$. This hold in particular at the fixed points of ψ^t , so that ψ^t can not act by isometries of a Lorentz metric. All the structures $(\mathbf{R}^n, g)$, with g as above are thus essential.

Other examples of essential conformal flows of pseudo-Riemannian manifolds, preserving an infinite dimensional space of conformal structures were constructed in [25], [26]. The basic idea here is to consider a flow ϕ^t acting conformally on a Lorentz manifold (M, g) , and having singularities at which the differential is the identity. These flows are then essential, because one knows that a Lorentzian isometry fixing a point, and whose differential is the identity at this point, has to be the identical transformation (the manifold M under consideration is connected). Now, “far from the singularity”, the flow ϕ^t considered in [25], [26] acts properly. There is a piece of transversal $\Sigma \subset M$, such that for every $x \in \Sigma$, the orbit $\phi^t.x$ leaves every compact set of M in a finite time. Thus, choosing $U \subset \Sigma$, an open subset of Σ , perturbing g at the points of U , without changing anything on $\Sigma \setminus U$, and pushing the modified metric along the flow, will yield another conformal structure $[g']$ on M which is preserved by ϕ^t . Since the perturbation on U is arbitrary, we get a huge class of different Lorentzian conformal structures on M , which are preserved by ϕ^t , and for which ϕ^t is an essential subgroup of conformal transformations. Similar constructions by “perturbations” are done in [15].

The process described above uses the noncompactness of the manifold, to get pieces of the manifold where the action of the flow is proper. Such constructions break down on compact manifolds. It is thus quite likely that compact essential structures are more unusual than noncompact ones. So, we could reformulate the previous question:

Question 2.2. *Let $(p, q) \in \mathbb{N}^2$ be two integers. Are there, up to conformal diffeomorphism, only a finite number of compact pseudo-Riemannian manifolds (M, g) of signature (p, q) , for which the conformal group is essential?*

Even with the extra compactness assumption, we will see that no positive answer to Question 2.2 is to expect. In section 6, following [15], we will construct quite a wide class of different Lorentzian conformal structures on compact manifolds, which are all essential. Even if these constructions are achieved in the Lorentzian framework, it is likely that they generalize to any signature (except the Riemannian

one, of course). The structures constructed in section 6 will all be globally distinct, but locally, they are all conformally modelled on open subsets of Minkowski's space (the space $\mathbf{R}^n$ endowed with the Lorentzian metric $-(dx_1)^2 + (dx_2)^2 + \dots + (dx_n)^2$). This still leaves open the following:

Generalized Lichnerowicz's Conjecture: *Let (M, g) be a compact pseudo-Riemannian manifold with an essential group of conformal transformations. Then (M, g) is conformally flat, i.e every $x \in M$ has a neighbourhood U_x which is conformally equivalent to an open subset of $\mathbf{R}^n$, with the conformal structure induced by $-x_1^2 - \dots - x_p^2 + x_{p+1}^2 + \dots + x_n^2$.*

This conjecture is stated in [10], p.96. Notice that the compactness assumption can not be removed if we want the conjecture to be true. Examples of [25], [26] exhibit conformal flows on noncompact manifolds, which are essential for nonconformally flat pseudo-Riemannian structures. Also, in [33], M.N.Podolsenov gives examples of Lorentzian metrics on $\mathbf{R}^n$, which are homogeneous, and for which the flow ψ^t above is conformal (and automatically essential). We see that even under the strong assumption of homogeneity, essentiality does not imply local rigidity if we remove the compactness assumption.

2.1.1. Strong essentiality. Until now, we were not very precise on the regularity required for the conformal structures we consider. Ferrand's theorem 1.2 requires a regularity C^2 on the metric. For a C^k pseudo-Riemannian metric g , one usually defines the conformal class as $[g] = \{e^\sigma g \mid \sigma \in C^k(M)\}$. But in fact, keeping g of class C^k , we could enlarge the conformal class, considering $[g]_{C^0} = \{e^\sigma g \mid \sigma \in C^0(M)\}$ (and in the same way $[g]_{L^\infty}$ etc...). This leads to the following:

Definition 2.3. A pseudo-Riemannian (M, g) is said to be *strongly essential*, if its group of conformal transformations does not preserve any metric in $[g]_{C^0}$.

Notice that the assumption, for a conformal structure, of being strongly essential, is weaker than the simple notion of essentiality. These two notions could be distinct, even if we do not have examples of pseudo-Riemannian structures which are essential, without being strongly essential. At least, since smooth invariant objects can be difficult to build for dynamical systems, the assumption of strong essentiality in generalized Lichnerowicz's conjecture, could make it a little bit simpler to handle.

2.2. Lichnerowicz's conjecture for parabolic geometries. We address now another question, which put generalized Lichnerowicz's conjecture in a wider framework.

2.2.1. Cartan geometries. A way to see pseudo-Riemannian and conformal pseudo-Riemannian structures as rigid geometric structures, is to present them as what is called *Cartan geometries*. We won't give a lot of details about Cartan geometries in this section, but we refer to [35], which is a very good reference.

Let us consider G a Lie group, and P a closed subgroup of G . A Cartan geometry on a manifold M is, roughly speaking, a geometric structure on M which is infinitesimally modelled on the homogeneous space G/P . So, Cartan geometries are curved generalizations of Klein's geometries modelled on G/P , namely manifolds which are locally modelled on G/P . Formally, a Cartan geometry on a manifold M , modelled on the homogeneous space $\mathbf{X} = G/P$, is the data of:

- (i) a principal P -bundle $B \rightarrow M$ over M .
- (ii) a 1-form ω on B , with values in the Lie algebra $\mathfrak{g}$, called *Cartan connection*, and satisfying the following conditions:
 - At every point $p \in B$, ω_p is an isomorphism between $T_p B$ and $\mathfrak{g}$.
 - if $X^\dagger$ is a vector field of B , coming from the action by right multiplication of some one-parameter subgroup $t \mapsto \exp_G(tX)$ of P , then $\omega(X^\dagger) = X$.
 - For every $a \in P$, $R_a^* \omega = \text{Ad}(a^{-1})\omega$ (R_a standing for the right action of a on B).

A Cartan geometry on a manifold M will be denoted by (M, B, ω) .

To each Cartan geometry, is associated a 2-form $\Omega = \omega + \frac{1}{2}[\omega, \omega]$, the curvature of ω , whose vanishing characterizes the manifolds M which are locally modelled on the homogeneous space $\mathbf{X} = G/P$. Such Cartan geometries are called *flat*.

Pseudo-Riemannian manifolds (M, g) are examples of Cartan geometries. The model space is $\mathbf{X} = SO(p, q) \ltimes \mathbf{R}^n / SO(p, q)$. We can choose for the fiber bundle B , the bundle of orthonormal frames on M . The metric g defines in a unique way a Levi-Civita connection, which can be reinterpreted as a Cartan connection ω_g on B , with values in $\mathfrak{so}(p, q) \oplus \mathbf{R}^n$. Each isometry of (M, g) acts on B , and leaves the connection ω_g invariant.

For conformal pseudo-Riemannian structures $(M, [g])$ of signature (p, q) , the model space is $\mathbf{X} = SO(p+1, q+1)/P$, where P is the stabilizer, in $SO(p+1, q+1)$, of an isotropic line for the quadratic form $-x_1^2 - \dots - x_{p+1}^2 + x_{p+2}^2 + \dots + x_{n+2}^2$. The group P is isomorphic to the semi-direct product $(\mathbf{R} \times SO(p, q)) \ltimes \mathbf{R}^n$. In particular, in the Riemannian case $p = 0$, $\mathbf{X}$ is just the sphere $\mathbf{S}^n$ seen as the homogeneous space $SO(1, n+1)/P$, i.e we see $\mathbf{S}^n$ endowed with the conformal class of the round metric.

The fact that a conformal structure $(M, [g])$ determines in a canonical way a Cartan geometry (M, B, ω) is not at all an obvious fact (see for example [23]). Nevertheless, when the dimension of the manifold M is at least three, the data of the conformal class $[g]$ defines on a subbundle B of the bundle of 2-jets of frames, a Cartan connexion ω with values in $\mathfrak{so}(p+1, q+1)$. This connection is flat if and only if the manifold is locally conformally equivalent to open subsets of the space $\mathbf{R}^n$, endowed with $-(dx_1)^2 - \dots - (dx_p)^2 + (dx_{p+1})^2 + \dots + (dx_n)^2$.

2.2.2. Parabolic geometries versus reductive geometries. Among all types of Cartan geometries, we can isolate two large and interesting families.

- The first family is constituted by *reductive geometries*. These are the Cartan geometries (M, B, ω) modelled on $\mathbf{X} = G/P$, such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}$, where $\mathfrak{p}$

is the Lie subalgebra of the subgroup P , and $\mathfrak{n}$ is $Ad(P)$ -invariant. For example, pseudo-Riemannian metrics give rise to reductive Cartan geometries, since $\mathfrak{g} = \mathfrak{so}(p, q) \oplus \mathbf{R}^n$ in this case, and $\mathbf{R}^n$ is $Ad(SO(p, q))$ -invariant. For general reductive geometries, it is possible to define a notion of covariant derivative (see [35], Chap. 5), so that these geometries behave quite closely to pseudo-Riemannian ones. Other examples of reductive geometries are, for example affine structures.

- The second family is constituted by *parabolic geometries*. A good exposition of these geometries is given in [8]. They correspond to Cartan geometries (M, B, ω) modelled on $\mathbf{X} = G/P$, where G is a simple Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with a k -grading, namely $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \dots \oplus \mathfrak{g}_{+k}$ ($[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$), and the Lie algebra of P is $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \dots \oplus \mathfrak{g}_{+k}$. In this case, there is no natural $Ad(P)$ -invariant complement to $\mathfrak{p}$, what makes this kind of geometries more difficult to handle.

Pseudo-Riemannian conformal structures $(M, [g])$ (in dimension ≥ 3) are examples of parabolic geometries. Indeed, in this case, we saw that $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$, and there is a 1-grading of $\mathfrak{so}(p+1, q+1)$, namely $\mathfrak{so}(p+1, q+1) = \mathfrak{n}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}^+$, with $\mathfrak{n}^-$ and $\mathfrak{n}^+$ n -dimensional abelian subalgebras, and $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{so}(p, q)$. The Lie subalgebra corresponding to the group P is $\mathfrak{g}_0 \oplus \mathfrak{n}^+$.

Appart from pseudo-Riemannian conformal structures, there are a lot of interesting geometric structures, which define canonically a parabolic Cartan geometry: projective structures, CR and quaternionic CR-structures, some models of path geometries (see [8] and references therein for a lot of examples).

Now, let us remark, that a parabolic geometry (M, B, ω) modelled on $\mathbf{X} = G/P$ defines a family of reductive geometries, that we will call *subordinated to* (M, B, ω) . Indeed, in the case of a parabolic geometry, the Lie algebra $\mathfrak{g}_0$ turns out to be reductive (in the sense of Lie algebras). So, $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}_0$, and $\mathfrak{s} = [\mathfrak{g}_0, \mathfrak{g}_0]$ is semi-simple (see [22]). At the Lie group level, P writes as a semi-direct product $(Z \times S) \ltimes N$, where Z is abelian and centralizes S , which is semi-simple. The group N is nilpotent with Lie algebra $\mathfrak{g}_{+1} \oplus \dots \oplus \mathfrak{g}_{+k}$.

The action of $Z \ltimes N$ is proper on B , so that $B_0 = B/(Z \ltimes N)$ is a smooth manifold. In fact, since S normalizes $Z \ltimes N$, we still have a right action of S on B_0 , which makes B_0 a S -principal bundle over M .

Definition 2.4. We call a *unimodular reductive geometry* on M , *subordinated to* (M, B, ω) , the data of a S -equivariant section $\sigma : B_0 \rightarrow B$.

Let us make this definition more explicit. Pick a S -equivariant section $\sigma : B_0 \rightarrow B$, and call $\Sigma = \sigma(B_0)$. The bundle map $\pi : B \rightarrow M$, when restricted to Σ , makes Σ into a S -principal bundle over M . Now, let us denote by $\mathfrak{g}'$ the Lie algebra $\mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{s}$ (where $\mathfrak{s} = [\mathfrak{g}_0, \mathfrak{g}_0]$ is the Lie algebra of S), and by $\rho : \mathfrak{g} \rightarrow \mathfrak{g}'$, the projection onto $\mathfrak{g}'$ relatively to $\mathfrak{z} \oplus \mathfrak{g}_{+1} \oplus \dots \oplus \mathfrak{g}_{+k}$. On Σ , we define $\bar{\omega} = \rho \circ \omega$. We claim that $(M, \Sigma, \bar{\omega})$ is a Cartan geometry over M , modelled on $\mathbf{X}' = S \ltimes N^-/S$. By N^- , we denote the connected Lie subgroup of G , whose Lie algebra is $\mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$. Notice that the Lie algebra of $S \ltimes N^-$ is $\mathfrak{g}'$, so that the

geometry $(M, \Sigma, \bar{\omega})$ is reductive. This reductive geometry is unimodular because since S is semi-simple, $Ad(S)$ acts on $\mathfrak{n}^- = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$ by elements of $SL(\mathfrak{n}^-)$.

Let us illustrate this construction for conformal pseudo-Riemannian structures. In this case, $\mathfrak{g} = \mathfrak{so}(p+1, q+1) = \mathfrak{n}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}^+$, as we already said. The Lie algebra $\mathfrak{g}_0$ writes $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{so}(p, q)$ in this case, and thus $\mathfrak{s} = \mathfrak{so}(p, q)$. At each point $p \in B$, we call $V_p = \omega^{-1}(\mathfrak{p})$ (recall that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}^+$). Then, $D_p\pi$ defines an isomorphism from $T_p B/V_p$ on $T_x M$, where $x\pi(p)$. Also, ω_p induces an isomorphism from $T_p B/V_p$ onto $\mathfrak{g}/\mathfrak{p}$. Hence, $\omega_p \circ (D_p\pi)^{-1}$ defines an isomorphism $i_p : T_x M \rightarrow \mathfrak{g}/\mathfrak{p}$. It is not difficult to check that $i_{p,b} = Ad(b^{-1}) \circ i_p$, for every $b \in B$. The conformal structure at $T_x M$ is just the pullback through i_p of the unique $Ad(P)$ -invariant conformal class of scalar products of signature (p, q) on $\mathfrak{g}/\mathfrak{p}$.

We choose now $\langle, \rangle_0$, an $Ad(S)$ -invariant scalar product of signature (p, q) on $\mathfrak{g}/\mathfrak{p}$. Let $\sigma : B_0 \rightarrow B$ a $SO(p, q)$ -equivariant section. This data defines a metric g_σ on M , in the conformal class $[g]$. Indeed, for each $x \in M$, choose $p \in \Sigma$ over x , and define $g_\sigma(x) = (i_p)^* \langle, \rangle_0$. this does not depend on the choice of $p \in \Sigma$ above x , since $i_{p \cdot s} = Ad(s^{-1})i_p$, for $s \in SO(p, q)$, and $\langle, \rangle_0$ is $SO(p, q)$ -invariant. Hence, a unimodular reductive geometry $(M, \Sigma, \bar{\omega})$ subordinated to the Cartan geometry (M, B, ω) associated to a pseudo-Riemannian conformal structure $(M, [g])$, is just the data of a metric in the conformal class.

Given a Cartan geometry (M, B, ω) , we define $Aut(M, B, \omega)$ as the set of diffeomorphisms $\hat{f}$ of B such that $\hat{f}^*\omega = \omega$. Such a diffeomorphism has to respect the fibers of B , so that it induces a diffeomorphism f of M . The set of such induced diffeomorphisms is called $Aut(M, \omega)$.

Let now (M, B, ω) be a parabolic geometry modelled on $\mathbf{X} = G/P$. We will say that $Aut(M, B, \omega)$ leaves invariant a subordinated unimodular reductive geometry, if there exists a S -invariant section $\sigma : B_0 \rightarrow B$, such that $\hat{f}(\Sigma) = \Sigma$. Notice that in this case, $\hat{f}^*\bar{\omega} = \bar{\omega}$. In the case of a pseudo-Riemannian conformal structure, saying that the group $Aut(M, B, \omega)$ preserves a subordinated unimodular reductive geometry, means that the conformal group preserves a metric in the conformal class, i.e the conformal group is inessential. We can now formulate a:

Lichnerowicz's conjecture for parabolic Cartan geometries: *Let (M, B, ω) be a compact parabolic geometry modelled on $\mathbf{X} = G/P$. Then, either the geometry is flat, or there is a C^0 unimodular reductive geometry, subordinated to (M, B, ω) , which is preserved by $Aut(M, B, \omega)$.*

Schoen's theorem [34] on CR -structures, and more generally the results of [18], are evidences that the conjecture must be true when G is simple of rank one (here, regularity of the connection, a mild restriction on its curvature, is made).

3. Some words about the proof of Theorem 1.2

Let us now explain the main ideas of the proof of Theorem 1.2. In dimension 2, the theorem is a consequence of the uniformization theorem of Riemann surfaces. We will explain the proof when $\dim(M) \geq 3$.

By Alekseevski's theorem, the assumption on the essentiality of the conformal group can be replaced, by the assumption that this group does not act properly. The first, and the most difficult step in the proof, is to show that under this hypothesis, the manifold has to be conformally flat, i.e every point x of M has a neighbourhood U_x conformally equivalent to an open subset V of the Euclidean space $\mathbf{R}^n$.

If we know that (M, g) is conformally flat, then we also know since Kuiper that there is a conformal immersion $\delta : \tilde{M} \rightarrow \mathbf{S}^n$ (where $\tilde{M}$ stands for the universal cover of M), called *devellopping map*, as well as a morphism $\rho : \text{Conf}(\tilde{M}, \tilde{g}) \rightarrow PO(1, n+1)$ satisfying the equivariance relation $\delta \circ \gamma = \rho(\gamma) \circ \delta$. When $\text{Conf}(\tilde{M}, \tilde{g})$ does not act properly, the dynamics of $\rho(\text{Conf}(\tilde{M}, \tilde{g})) \subset PO(1, n+1)$ allows to understand the map δ (see for example [27], [19]), and we get:

Proposition 3.1. *Let (M, g) a Riemannian manifold which is conformally flat. If the conformal group of (M, g) does not act properly on M , then the devellopping map $\delta : \tilde{M} \rightarrow \mathbf{S}^n$ is a diffeomorphism on $\mathbf{S}^n$, or on $\mathbf{S}^n \setminus \{p\}$, for some point $p \in \mathbf{S}^n$.*

Notice that at the begining of the seventies, notions like (G, X) -structures, and tools like devellopping maps were not yet very “popular”, so that this part of the proof in [32], for example, is not really correct.

On a general pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$, the conformal flatness is detected by tensorial conditions.

The *Weyl tensor* is the $(1, 3)$ tensor given by the formula:

$$W = R - \frac{S}{2n(n-1)}g \cdot g - \frac{1}{n-2}\left(Ric - \frac{S}{n}g\right) \cdot g$$

here, R , Ric and S stand for the Riemann, Ricci and Scalar curvature associated to g , and $h \cdot q$ stands for the Kulkarni-Nomizu product of two symetric 2-tensors (see [6] p.47).

In dimension $n \geq 4$, the vanishing of the Weyl tensor is equivalent to the manifold (M, g) to be conformally flat. In dimension 3, the Weyl tensor always vanishes, but another tensor substitutes it.

The *Schouten tensor* on (M, g) is given by:

$$S = \frac{1}{n-2}\left(Ric - \frac{S}{2(n-1)}g\right)$$

Then, one defines the *Cotton tensor* by $C(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$.

In dimension 3, the tensor C vanishes if and only if (M, g) is conformally flat.

Thanks to Proposition 3.1, Theorem 1.2 is proved if one can show that essentiality implies conformal flatness. When we suppose the manifold M compact, there is a trick to do that. Indeed, on a manifold which is not conformally flat, one can build the following nontrivial singular metric $h_g = \|W\|_g g$ (resp. $\|C\|_g^{\frac{2}{3}} g$ when $\dim(M) = 3$). One checks that the conformal group acts by isometries for this singular metric. In fact, this singular metric defines a singular distance $d_h(x, y) = \inf_{\gamma} \int h_g(\gamma', \gamma')$, the infimum being taken over all γ 's joining x to y . If K denotes the closed subset on which the Weyl tensor (resp. the Cotton tensor in dimension 3) vanishes, and if $K_\epsilon = \{x \in M, d_h(x, K) \geq \epsilon\}$, then (K_ϵ, d_h) is, for ϵ sufficiently small, a nonempty, compact, non-singular metric space, left invariant by the conformal group. One then infers that the conformal group of (M, g) is compact, and thus, inessential (see [19] for details).

For a noncompact Riemannian manifold (M, g) , the previous demonstration breaks down, and far more involved tools must be used. This is J.Ferrand, who first gave a correct proof in the noncompact case. For this, she introduced conformal invariants, which allowed her to understand the global dynamical behaviour of sequences of conformal transformations which don't act properly on (M, g) .

Let $H(M)$ (resp. $H_0(M)$) denote the space of continuous functions on M (resp. continuous with compact support), with an L^n -integrable differential distribution. Thanks to the metric g , this later can be seen as a gradient vector field.

Now, if C_1 and C_2 are closed connected sets of M , let $A(C_0, C_1)$ denote the set of functions $u \in H(M)$ such that $u = 0$ on C_0 and $u = 1$ on C_1 . Then one defines $Cap(C_0, C_1) = \inf_{u \in A(C_0, C_1)} \int_M |\nabla u|^n dVol_g$. Notice that $Cap(C_1, C_2)$ is invariant by conformal change of metric $g \rightarrow e^{2\sigma} g$.

If $(x, y, z) \in M^3$, $z \neq x$, $z \neq y$, J.Ferrand defines $\nu(x, y, z) = \inf_{C_0, C_1} Cap(C_0, C_1)$, for C_0 a noncompact closed connected set containing z , and C_1 a compact connected set containing x and y .

J.Ferrand then proves that noncompact Riemannian manifolds split into two classes. On the first class, she can define a conformally-invariant distance. So, the conformal group of manifolds of the first class acts properly, and these manifolds are inessential.

For manifolds which are not in this class, the function ν can be extended to $\nu : (M \times M \times \hat{M}) \setminus \Delta \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ (where $\hat{M}$ is the Alexandroff compactification of M , and Δ is the diagonal), and satisfies:

- $\nu(x, y, z) = 0$ if and only if $y = x$ or $z = \infty$.
- when $x \neq y$, $\nu(x, y, z) = +\infty$ if and only if $z = x$ or $z = y$.

Now, let us consider a noncompact essential Riemannian manifold (M, g) . By Alekseevski's theorem, the action of the conformal group of (M, g) is nonproper, and thus we can find a sequence (x_k) of M converging to x_∞ , and a sequence (f_k) of conformal transformations, leaving every compact subset of $Homeo(M)$, and such that $y_k = f_k(x_k)$ converges to $y_\infty \in M$. Now, three cases have to be considered:

- (i) there is a subsequence of (f_k) , also noted (f_k) , and a converging sequence $z_k \rightarrow z_\infty$, $z_\infty \neq x_\infty$, such that $f_k(z_k) \rightarrow w_\infty$, $w_\infty \neq y_\infty$.
- (ii) there is a subsequence of (f_k) , also noted (f_k) , and a converging sequence

$z_k \rightarrow z_\infty$, $z_\infty \neq x_\infty$, such that $f_k(z_k) \rightarrow y_\infty$.

(iii) For any converging sequence $z_k \rightarrow z_\infty$, $z_\infty \neq x_\infty$, $f_k(z_k) \rightarrow \infty$.

In the first case, J.Ferrand proves that (f_k) has a subsequence converging in $Conf(M, g)$, which contradicts the hypothesis on (f_k) .

Now, if we are in the case (ii), we look at a converging sequence $a_k \rightarrow a_\infty$, $a_\infty \in M$, and looking at subsequence if necessary, we suppose that $f_k(a_k)$ tends to $b_\infty \in \hat{M}$. Then, $\nu(x_\infty, z_\infty, a_\infty) > 0$, and $\nu(x_\infty, z_\infty, a_\infty) = \lim_{k \rightarrow +\infty} \nu(x_k, z_k, a_k)$. On the other hand, if $b_\infty \neq y_\infty$, $\nu(x_\infty, z_\infty, a_\infty) = \lim_{k \rightarrow +\infty} \nu(f_k(x_k), f_k(z_k), f_k(a_k)) = \nu(y_\infty, y_\infty, b_\infty) = 0$, by the conformal invariance of ν , yielding a contradiction. We thus must have $b_\infty = y_\infty$. This implies the following dynamical property for (f_k) :

For any open subset $U \subset M$, with compact closure in M , and any $\epsilon > 0$, $f_k(U) \subset B_g(y_\infty, \epsilon)$, the g -ball of center y_∞ and radius ϵ , for k sufficiently big.

But such a dynamical behaviour implies that (M, g) is conformally flat. Indeed, let us suppose that $\dim(M) \geq 4$, and look at the integral $\int_U \|W\|_g^{\frac{n}{2}} dVol_g$. This integral is conformally invariant, so that by the previous assertion, $\int_U \|W\|_g^{\frac{n}{2}} dVol_g \leq \int_{B_g(y_\infty, \epsilon)} \|W\|_g^{\frac{n}{2}} dVol_g$, and this for every $\epsilon > 0$ arbitrary small. This implies $\int_U \|W\|_g^{\frac{n}{2}} dVol_g = 0$, and finally $W = 0$ on U . In dimension 3, one considers $\|C\|$ instead of $\|W\|_g^{\frac{n}{2}}$ to conclude the proof.

The invariant ν allows also to conclude in case (iii). Let us consider any sequence $a_k \rightarrow a_\infty$ in M . Since $f_k(z_k)$ tends to ∞ , we see that $\nu(y_k, a_k, f_k(z_k))$ tends to $\nu(y_\infty, a_\infty, \infty)$, namely 0. But by conformal invariance $\nu(y_k, a_k, f_k(z_k)) = \nu(x_k, f_k^{-1}(a_k), z_k)$. Since any cluster value of $\nu(x_k, f_k^{-1}(a_k), z_k)$ has to be 0, we infer that the only possible cluster values for $(f_k^{-1}(a_k))$ are ∞ and x_∞ . But the set of cluster values of $(f_k^{-1}(a_k))$, over all the convergent sequences (a_k) has to be connected, because M is, and since $f_k^{-1}(y_k) \rightarrow x_\infty$, we infer that for any convergent sequence $a_k \rightarrow a_\infty$, $f_k^{-1}(a_k) \rightarrow x_\infty$. The end of the proof of point (ii), when applied to (f_k^{-1}) , yields the conclusion $W = 0$ on M .

4. Conformal dynamics on compact manifolds

Given a pseudo-Riemannian compact manifold (M, g) and a conformal transformation f , can we describe all the possible dynamical patterns for the dynamics of (f^k) on M ? In the Riemannian case, Theorem 1.2 allows a complete description.

Proposition 4.1. *Let (M, g) be a compact Riemannian manifold, and f a conformal transformation. Then two cases can occur:*

(i) *The sequence (f^k) is contained in a group $Isom(M, e^{2\sigma}g)$, for some $\sigma \in C^\infty(M)$. In this case, for every $x \in M$, the closure of $(f^k(x))$ in M is a torus, on which f acts by translation.*

(ii) *The manifold (M, g) is conformally diffeomorphic to $(\mathbf{S}^n, g_{can})$, and under this identification, (f^k) is a non relatively compact sequence of Moebius transformations.*

To complete the dynamical description, let us recall that Moebius transformations generating a non relatively compact group have a so called *North-South* dynamics:

Lemma 4.2. *Let (f_k) be a sequence of $PO(1, n+1)$ tending to infinity (i.e leaving every compact subset of $PO(1, n+1)$). Then there exist two points p^+ and p^- on $\mathbf{S}^n$, such that considering a subsequence of (f_k) if necessary:*

(i) *for every $x \in \mathbf{S}^n \setminus \{p^-\}$, $\lim_{k \rightarrow +\infty} f_k(x) = p^+$, the convergence being uniform on every compact subset of $\mathbf{S}^n \setminus \{p^-\}$.*

(ii) *for every $x \in \mathbf{S}^n \setminus \{p^+\}$, $\lim_{k \rightarrow +\infty} (f_k)^{-1}(x) = p^-$, the convergence being uniform on every compact subset of $\mathbf{S}^n \setminus \{p^+\}$.*

We see that the dynamics of essential sequences (f^k) (case (ii) of the proposition, described in the lemma) is a North-South dynamics, qualitatively very different from that of isometric sequences (case (i) of the proposition). More interesting, the proof of Theorem 1.2 done by J.Ferrand (and in fact all existing proofs) consists roughly to show that an essential conformal transformation on a Riemannian manifold, must have a north-South dynamics. And the key point, at the end of the proof, is to observe that a North-South dynamics *forces* the Weyl tensor to vanish, and the geometry to be conformally flat (see the end of the previous section).

So, an answer to Lichnerowicz's conjecture for general pseudo-Riemannian signatures, should begin by a good understanding of the dynamics of essential transformations. A central question being:

Question 4.3. *Let (M, g) be a pseudo-Riemannian manifold. What are qualitatively the differences between the dynamics of isometries on M , and the dynamics of essential conformal transformations?*

We are far from having an answer to this question, but at least for the Lorentzian signature, we have some hints of what the answer could be. To understand better the question, our first task is to exhibit and study, quite a lot of compact Lorentzian manifolds, having essential conformal transformations. That is what we are going to do in the two next sections.

5. The conformal model space in Lorentzian geometry

5.1. Geometry of Einstein's universe. Just as the standard sphere is a central geometrical object in conformal Riemannian geometry, there is in the Lorentzian framework a distinguished compact conformal space. This space calls Einstein's universe, and it is so important from the geometrical and dynamical point of view, that we must, at least briefly, describe it. A more detailed description can be found in [16], [15]. For the physical point of view, see [21].

Let $\mathbf{R}^{2,n}$ be the space $\mathbf{R}^{n+2}$, endowed with the quadratic form $q^{2,n}(x) = -2x_0x_{n+1} - 2x_1x_n + x_2^2 + \dots + x_{n-1}^2$. The isotropic cone of $q^{2,n}$ is the subset

of $\mathbf{R}^{2,n}$ on which $q^{2,n}$ vanishes. We call $C^{2,n}$ this isotropic cone, with the origin removed. We will denote by π the projection from $\mathbf{R}^{2,n}$ minus the origin, on $\mathbf{RP}^{n+1}$. The set $\pi(C^{2,n})$ is a smooth hypersurface Σ of $\mathbf{RP}^{n+1}$. This hypersurface turns out to be endowed with a natural Lorentzian conformal structure. Indeed, for any $x \in C^{2,n}$, the restriction of $q^{2,n}$ to the tangent space $T_x C^{2,n}$, that we call $\hat{q}_x^{2,n}$, is degenerated. Its kernel is just the kernel of the tangent map $d_x \pi$. Thus, pushing $\hat{q}_x^{2,n}$ by $d_x \pi$, we get a well defined Lorentzian metric on $T_{\pi(x)} \Sigma$. If $\pi(x) = \pi(y)$ the two Lorentzian metrics on $T_{\pi(x)} \Sigma$ obtained by pushing $\hat{q}_x^{2,n}$ and $\hat{q}_y^{2,n}$ are in the same conformal class. Thus, the form $q^{2,n}$ determines naturally a well defined conformal class of Lorentzian metrics on Σ . In fact, one can check that the manifold Σ is the quotient of $\mathbf{S}^1 \times \mathbf{S}^{n-1}$ by the product of the antipodal maps, and the natural conformal structure is induced on this quotient by the conformal class of the metric $-dt^2 + g_{can}$. The manifold Σ , together with its canonical conformal structure will be called *Einstein's universe*, and denoted by $\mathbf{Ein}_n$.

Notice also that the metric induced on Σ by $-dt^2 + g_{can}$, gives rise to a smooth n -form, called Vol on Σ . If $(X_1, \dots, x_n)$ is a (local) smooth field of orthonormal frames on Σ , then $Vol(X_1, \dots, X_n) = 1$.

5.1.1. Conformal group and Liouville's theorem.. From the very construction of $\mathbf{Ein}_n$, it is clear that the group $PO(2, n)$ acts naturally by conformal transformations on $\mathbf{Ein}_n$. It turns out that $PO(2, n)$ is the full conformal group of $\mathbf{Ein}_n$. Moreover, there is a *Liouville theorem*, asserting that any conformal transformation between connected open subsets of $\mathbf{Ein}_n$ is the restriction of a unique transformation of $PO(2, n)$ (this theorem is proved in [7]).

5.1.2. Lightlike geodesics and lightcones.. The projection on $\mathbf{Ein}_n$ of the intersection of $C^{2,n}$ with linear subspaces of $\mathbf{R}^{2,n}$ will yield various interesting geometrical objects of $\mathbf{Ein}_n$.

For example, the projection on $\mathbf{Ein}_n$ of the intersection of $C^{2,n}$ with null 2-planes of $\mathbf{R}^{2,n}$ (resp. degenerate hyperplanes of $\mathbf{R}^{2,n}$) are called *lightlike geodesics* (resp. *lightcones*) of $\mathbf{Ein}_n$. The lightlike geodesics are smooth circles. The lightcones are the sets of lightlike geodesics passing through a same point p , *the vertex of the lightcone*. If $p \in \mathbf{Ein}_n$, we will call $C(p)$ the lightcone with vertex p . A lightcone $C(p)$ has always a singularity at its vertex p (this singularity is locally that of a lightcone in Minkowski's space, as we will see later), but $C(p) \setminus \{p\}$ is smooth, diffeomorphic to $\mathbf{R} \times \mathbf{S}^{n-2}$.

5.1.3. Complementary of a lightlike geodesic. Let $\Delta \subset \mathbf{Ein}_n$ be a lightlike geodesic. We call Ω_Δ the complementary of Δ in $\mathbf{Ein}_n$. Since this kind of open subsets will play an important role in the following, we recall some of their main geometrical properties. Open sets like Ω_Δ admit a natural foliation by degenerate hypersurfaces, and this foliation $\mathcal{H}_\Delta$ is preserved by the whole conformal group of Ω_Δ . This foliation can be described as follows: given a point $p \in \Delta$, we consider the lightcone $C(p)$ with vertex p . Since Δ is a lightlike geodesic, we have $\Delta \subset C(p)$. Now, the intersection of $C(p)$ with Ω_Δ is a degenerate hypersurface of Ω_Δ ,

diffeomorphic to $\mathbf{R}^{n-1}$. We call it $\mathcal{H}(p)$. If $p \neq p'$, $C(p)$ and $C(p')$ only intersect along Δ , so that $\mathcal{H}(p) \cap \mathcal{H}(p') = \emptyset$. We thus get a foliation $\mathcal{H}_\Delta$ whose leaves are the $\mathcal{H}(p)$'s, for $p \in \Delta$. We also get a smooth fibration $\rho_\Delta : \Omega_\Delta \rightarrow \Delta$ defined as follows: for every $x \in \Omega_\Delta$, $\rho_\Delta(x)$ is the unique $p \in \Delta$ such that $x \in \mathcal{H}(p)$.

5.1.4. Complement of a lightcone: stereographic projections. Let us identify Minkowski's space $\mathbf{R}^{1,n-1}$ with the subspace of $\mathbf{R}^{2,n}$ spanned by $e_1, \dots, e_n$, and let us denote by $\langle, \rangle$ the restriction of $q^{2,n}$ to $\text{Span}(e_1, \dots, e_n)$. We define:

$$\bar{s} : \mathbf{R}^{1,n-1} \rightarrow C^{2,n}$$

$$x \mapsto \langle x, x \rangle e_0 + 2x + e_{n+1}$$

The map $s = \pi \circ \bar{s}$ is a conformal embedding of $\mathbf{R}^{1,n-1}$ into $\mathbf{Ein}_n$, and is called *stereographic projection*. The image $s(\mathbf{R}^{1,n-1})$ is the complement in $\mathbf{Ein}_n$ of the lightlike cone with vertex $p_\infty = \pi(e_0)$. This cone is called *cone at infinity* and denoted by C_∞ . To understand better the way $\mathbf{R}^{1,n-1}$ compactifies, the following lemma is useful (a proof is given in [14] p.53):

Lemma 5.1. *After identifying $\mathbf{R}^{1,n-1}$ $\mathbf{Ein}_n \setminus C_\infty$ thanks to the stereographic projection s , one has:*

- (i) *Let u be a timelike or a spacelike vector, and $a + \mathbf{R}.u$ an affine straightline of $\mathbf{R}^{1,n-1}$. Then $\lim_{t \rightarrow \pm\infty} (a + tu) = p_\infty$.*
- (ii) *To each lightlike direction u of $\mathbf{R}^{1,n-1}$ is associated a unique lightlike geodesic $\Delta_u \subset C_\infty$, such that the leaves $\mathcal{H}(p)$, $p \in \Delta \setminus \{p_\infty\}$, are the image through the stereographic projection of the affine hyperplanes $a + u^\perp$.*
 - *For any $a \in \mathbf{R}^{1,n-1}$, $\lim_{t \rightarrow \pm\infty} (a + tu) = \rho_{\Delta_u}(a)$.*
 - *Two straightlines $a + \mathbf{R}.u$ and $b + \mathbf{R}.u$ "hit" C_∞ at the same point of $\Delta \setminus \{p_\infty\}$ if and only if they belong to a same degenerate affine hyperplane of $\mathbf{R}^{1,n-1}$.*

To summarize the last part of the lemma, let us say that any lightlike affine straightline of Minkowski's space (still identified with $\mathbf{Ein}_n \setminus C_\infty$), compactifies in $\mathbf{Ein}_n$ as a lightlike geodesics. Two lightlike affine straightlines are parallel if and only if they hit $C_\infty \setminus \{p_\infty\}$ at points which are on a same lightlike geodesic of C_∞ . In particular, two lightlike straightlines meet at infinity if and only if they are parallel and in a same lightlike hyperplane.

Given a lightlike geodesic $\Delta \subset \mathbf{Ein}_n$, the lemma helps for understanding the foliation $\mathcal{H}_\Delta$. Indeed, choose a point $p_\infty \in \Delta$. The "Minkowski component" $\mathbf{Ein}_n \setminus C(p_\infty)$ is included in Ω_Δ . Then the lemma says that, after identifying $\mathbf{Ein}_n \setminus C(p_\infty)$ to $\mathbf{R}^{1,n-1}$ thanks to a stereographic projection, the restriction of $\mathcal{H}_\Delta$ to $\mathbf{Ein}_n \setminus C(p_\infty)$ is just a foliation by parallel lightlike hyperplanes.

Remark 5.2. The previous construction generalizes to any signature. If $\mathbf{R}^{p+1,q+1}$ is the space $\mathbf{R}^{n+2}$ endowed with $-2x_0x_{n+1} - 2x_1x_n \dots - 2x_px_{n-p+1} + x_{p+1}^2 + \dots + x_{n+2}^2$, the projection of the isotropic cone on $\mathbf{RP}^{n+1}$ is a smooth manifold, endowed with a conformal structure of signature (p, q) , and with conformal group $PO(p+1, q+1)$. This space is called $\mathbf{Ein}^{p,q}$ (in Lorentzian signature, we write $\mathbf{Ein}_n$ instead of $\mathbf{Ein}^{1,n-1}$). Notice that $\mathbf{Ein}^{p,q}$ is finitely covered by the product $\mathbf{S}^p \times \mathbf{S}^q$, endowed with the product metric $-g_{\mathbf{S}^p}^{can} \times g_{\mathbf{S}^q}^{can}$. The space $\mathbf{Ein}^{p,q}$ is the conformal compactification of $\mathbf{R}^{p,q}$. In particular, it is conformally flat, and turns out to

be the universal model for conformally flat manifolds (i.e the universal cover of every conformally flat manifold of signature (p, q) admits a conformal immersion in $\mathbf{Ein}^{p,q}$).

5.2. Examples of essential dynamics on Einstein's universe.

We are going to show that Einstein's universe has a lot of essential (and in fact strongly essential) conformal transformations. In the following, we fix $p_\infty \in \mathbf{Ein}_n$, $C_\infty = C(p_\infty)$, and a stereographic projection identifying $\mathbf{Ein}_n \setminus C_\infty$ with $\mathbf{R}^{1,n-1}$. By this way, any conformal transformation of $\mathbf{R}^{1,n-1}$ can be seen as a conformal transformation of $\mathbf{Ein}_n \setminus C_\infty$, and by Liouville's theorem, extends in an unique way to a conformal transformation of $\mathbf{Ein}_n$, i.e an element of $PO(2, n)$. So we will always, without further precision, see conformal transformations of $\mathbf{R}^{1,n-1}$ as elements of $PO(2, n)$ fixing p_∞ .

5.2.1. Dynamics of translations . Using Lemma 5.1, it is not difficult to understand the dynamics of translations of $\mathbf{R}^{1,n-1}$, when extended to the whole $\mathbf{Ein}_n$.

Lemma 5.3. - let T be a translation of $\mathbf{R}^{1,n-1}$ of vector $u = (u_1, \dots, u_n)$, then as an element of $O(2, n)$,

$$T = \begin{pmatrix} 1 & \langle T, e_1 \rangle & \langle T, e_2 \rangle & \dots & \langle T, e_n \rangle & \langle T, T \rangle \\ & 1 & 0 & \dots & 0 & 2u_1 \\ & & \ddots & & \vdots & \vdots \\ & & & \ddots & 0 & \vdots \\ & & & & 1 & 2u_n \\ & & & & & 1 \end{pmatrix}$$

- For any translation T , the differential of T at p_∞ is the identity.

- let T be a timelike translation. Then T has p_∞ as unique fixed point in $\mathbf{Ein}_n$. For every $x \in \mathbf{Ein}_n \setminus \{p_\infty\}$, $\lim_{n \rightarrow \pm\infty} T^n \cdot x = p_\infty$. Moreover, for any open subset $U \subset \mathbf{Ein}_n \setminus C_\infty$, with compact closure in $\mathbf{Ein}_n \setminus C_\infty$, $\lim_{n \rightarrow \pm\infty} T^n \cdot U = p_\infty$ (the convergence is to be understood with respect to the Hausdorff topology), and $\lim_{n \rightarrow \pm\infty} \text{Vol}(T^n \cdot U) = 0$.

- let T be a spacelike translation. Then the fixed points of T are the points of a lightcone of codimension one in C_∞ . For any $x \in \mathbf{Ein}_n$ wich is not fixed by T , $\lim_{n \rightarrow \pm\infty} T^n \cdot x = p_\infty$. Moreover, for any open subset $U \subset \mathbf{Ein}_n \setminus C_\infty$, with compact closure in $\mathbf{Ein}_n \setminus C_\infty$, $\lim_{n \rightarrow \pm\infty} T^n \cdot U = p_\infty$, and $\lim_{n \rightarrow \pm\infty} \text{Vol}(T^n \cdot U) = 0$.

- let T be a lightlike translation of vector $u \in \mathbf{R}^{1,n-1}$. Then the fixed points of T are exactly the points of Δ_u . For any compact subset $K \subset \mathbf{Ein}_n \setminus \Delta_u$, $\lim_{n \rightarrow \pm\infty} T^n \cdot K = \rho_{\Delta_u}(K)$. Moreover, if U is an open subset of $\mathbf{Ein}_n \setminus \Delta_u$, with compact closure in $\mathbf{Ein}_n \setminus \Delta_u$, then $\lim_{n \rightarrow \pm\infty} \text{Vol}(T^n \cdot U) = 0$.

The fact that the differential of a translation at p_∞ is the identity, proves that translations are essential conformal transformations of $\mathbf{Ein}_n$. Indeed, an isometry of a C^1 Lorentzian connected manifold (or more generally pseudo-Riemannian manifold), fixing a point, and with differential the identity at this point, has to be

the identical transformation. This is just because at a fixed point, an isometry is conjugated to its differential by the exponential map.

Moreover, the fact that translations are volume-collapsing on open subsets of Σ , for the volume form Vol , proves that they can't preserve any L^∞ metric in the conformal class of a smooth Lorentzian metric on Σ . Therefore, translations are strongly essential conformal transformations of $\mathbf{Ein}_n$.

5.2.2. Dynamics of an homothety. Let h_λ be an homothetic transformation of $\mathbf{R}^{1,n-1}$ of ratio λ ; $h_\lambda : x \mapsto \lambda x$. We suppose $|\lambda| < 1$. We denote by p^+ the point of $\mathbf{Ein}_n$ corresponding to the origin in Minkowski's space, and set $p^- = p_\infty$. We call C^+ and C^- the lightcones associated to p^+ and p^- respectively. Then:

Lemma 5.4. *The fixed points of h_λ are p^+ , p^- and the points of $C^+ \cap C^-$ (a codimension 2 Riemannian sphere in $\mathbf{Ein}_n$).*

- If $x \in \Omega^+ = \mathbf{Ein}_n \setminus C^-$, then $\lim_{n \rightarrow +\infty} (h_\lambda)^n \cdot x = p^+$.
- If $x \in \Omega^- = \mathbf{Ein}_n \setminus C^+$, then $\lim_{n \rightarrow -\infty} (h_\lambda)^n \cdot x = p^-$.
- If U is an open subset of Ω^+ (resp. Ω^-) with compact closure in Ω^+ (resp. in Ω^-), then $\lim_{n \rightarrow +\infty} Vol(h_\lambda^n(U)) = 0$, (resp. $\lim_{n \rightarrow -\infty} Vol(h_\lambda^n(U)) = 0$).

Since the differential of h_λ at p^+ is λId , it is clear that h_λ can't preserve any Lorentzian metric on Σ (of any regularity), what shows that h_λ is a strongly essential transformation.

5.2.3. A last example of essential dynamics. We consider now the transformations introduced in section 2, namely $\psi^\lambda : (x_1, \dots, x_n) \mapsto (e^{2\lambda}x_1, e^\lambda x_2, \dots, e^\lambda x_{n-1}, x_n)$. It is quite simple to check that as an element of $O(2, n)$, ψ^t writes as:

$$\psi^\lambda = \begin{pmatrix} e^\lambda & & & & \\ & e^{-\lambda} & & & \\ & & I_{n-2} & & \\ & & & e^\lambda & \\ & & & & e^{-\lambda} \end{pmatrix}$$

We suppose that $\lambda < 0$, and we call Δ^+ the lightlike geodesic, compactification of the straightline $\mathbf{R}.e_n$ in $\mathbf{Ein}_n$. We also call $\Delta^- = \Delta_{e_1}$. Since e_n is not colinear to e_1 , Δ^+ and Δ^- are two disjoint lightlike geodesics of $\mathbf{Ein}_n$. Let us denote $\mathcal{H}^+(p)$, $p \in \Delta^+$ (resp. $\mathcal{H}^-(p)$, $p \in \Delta^-$), the leaves of the natural foliation on Ω_{Δ^+} (resp. of Ω_{Δ^-}) introduced in section 5.1. Now, from basic linear algebra on $\mathbf{R}^{2,n}$, we see that if a lightlike geodesic of $\mathbf{Ein}_n$ is not contained in a lightcone, it intersect this lightcone at exactly one point. It follows that each leaf $\mathcal{H}^+(p)$ meets Δ^- exactly once (resp. each leaf $\mathcal{H}^-(p)$ meets Δ^+ exactly once). So, there is a natural projection $\pi^- : \Omega_{\Delta^+} \rightarrow \Delta^-$ (resp. $\pi^+ : \Omega_{\Delta^-} \rightarrow \Delta^+$), which at each point of a leaf $\mathcal{H}^+(p)$ associates the intersection of Δ^- with $\mathcal{H}^+(p)$. We then have:

Lemma 5.5. *The fixed points of the transformation ψ^λ are the points of $\Delta^+ \cup \Delta^-$.*

- For any open subset $U \subset \Omega_{\Delta^-}$ with compact closure in Ω_{Δ^-} , then $\lim_{n \rightarrow +\infty} \psi^{n\lambda}.U = \pi^+(U)$, and $\lim_{n \rightarrow +\infty} Vol(\psi^{n\lambda}.U) = 0$.

For any open subset $U \subset \Omega_{\Delta^+}$ with compact closure in Ω_{Δ^+} , then $\lim_{n \rightarrow -\infty} \psi^{n\lambda}.U = \pi^-(U)$, and $\lim_{n \rightarrow -\infty} \text{Vol}(\psi^{n\lambda}.U) = 0$

Once again, the volume-collapsing properties of $(\psi^{n\lambda})_{n \in \mathbb{N}}$ show that ψ^λ can't preserve any L^∞ metric in the conformal class of any smooth Lorentzian metric on Σ . It follows that the transformations ψ^λ are strongly essential.

5.3. Remarks. If we look at the dynamical patterns described above, we see that the dynamics of elements of $PO(2, n)$ is a little bit more complicated than dynamics of Moebius elements on $\mathbf{S}^n$. Nevertheless, all these dynamics have a rough common pattern. There are attracting sets (these sets are points, or lightlike geodesics), which attract points of a dense open subset. Moreover, the volume form Vol is collapsed on this dense open subset under the iterates of the essential transformation. We will insist on this point later on.

6. More complicated examples of compact essential Lorentzian manifolds

6.1. Schottky groups on Einstein's universe. A subgroup $\hat{\Gamma} \subset SL(2, \mathbb{R})$, generated by g elements $\hat{\gamma}_1, \dots, \hat{\gamma}_g$ ($g \geq 2$) is called a *Schottky group*, when there exist $2g$ pairwise disjoint half-discs of $\mathbf{H}^2$, denoted by $D_1^+, \dots, D_g^+, D_1^-, \dots, D_g^-$, such that for every $i \in \{1, \dots, g\}$: $\gamma_i(\mathbf{H}^2 \setminus D_i^-) = \overline{D_i^+}$. By an half-disc of $\mathbf{H}^2$, we mean a connected component of the complementary of a geodesic.

The interested reader will find more details on Schottky groups in [29], for instance. For what follows, we will just precise that a Schottky group is always a free discrete subgroup of $SL(2, \mathbb{R})$. When it acts on $\partial\mathbf{H}^2 \simeq \mathbf{S}^1$, a Schottky group has a closed invariant subset $\Lambda_{\hat{\Gamma}}$, homeomorphic to a Cantor set, on which its action is minimal. The action of $\hat{\Gamma}$ on $\Omega_{\hat{\Gamma}} = \mathbf{S}^1 \setminus \Lambda_{\hat{\Gamma}}$ is proper discontinuous, and the quotient $\hat{\Gamma} \backslash \Omega_{\hat{\Gamma}}$ is a finite union of circles.

We consider $\mathbf{R}^{2,n}$, endowed with $q^{2,n}(x) = -2x_0x_{n+1} - 2x_1x_n + x_2^2 + \dots + x_{n-1}^2$, and call $\mathbf{T}_0$ the projection on $\mathbf{Ein}_n$ of the subspace spanned by (e_0, e_1, e_n, e_{n+1}) . Since this subspace has signature $(2, 2)$, $\mathbf{T}_0$ is a sub-Einstein's universe of dimension 2. Conformally, it is just the product $\mathbf{S}^1 \times \mathbf{S}^1$ with the conformal class of the metric $dx dy$. On $\mathbf{T}_0$ there are two foliations by lightlike geodesics. The first, called $\mathcal{F}_1$ has $\{x\} \times \mathbf{S}^1$ for leaves, and the leaves of the second, $\mathcal{F}_2$, are the $\mathbf{S}^1 \times \{y\}$.

We now introduce two representations ρ_R and ρ_L of $SL(2, \mathbb{R})$ in $O(2, n)$ defined in the following way:

For every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbb{R})$,

$$\rho_L(A) = \begin{pmatrix} A & & \\ & I_{n-2} & \\ & & A \end{pmatrix}$$

and

$$\rho_R(A) = \begin{pmatrix} aI_2 & & bI_2 \\ & I_{n-2} & \\ cI_2 & & dI_2 \end{pmatrix}$$

Notice that $\rho_L(A)$ (resp. $\rho_R(A)$) preserves $\mathbf{T}_0 = \mathbf{S}^1 \times \mathbf{S}^1$, and acts projectively by A on the left factor (resp. the right factor) and trivially on the other. In particular, $\rho_L(A)$ (resp. $\rho_R(A)$) leaves every lightlike geodesic of $\mathcal{F}_2$ (resp. of $\mathcal{F}_1$) invariant.

Let us now consider a Schottky group $\hat{\Gamma}$ in $SL(2, \mathbb{R})$, generated by $\hat{s}_1, \dots, \hat{s}_g$. We call Γ the group $\rho_L(\hat{\Gamma})$, and set $s_i = \rho_L(\hat{s}_i)$. If we set $\Lambda_\Gamma = \Lambda_{\hat{\Gamma}} \times \mathbf{S}^1 \subset \mathbf{T}_0$, we get that Λ_Γ is a closed invariant subset for the action of Γ on $\mathbf{Ein}_n$. Moreover, we proved in [16], [15].

Theorem 6.1. - *The action of Γ is proper on $\Omega_\Gamma = \mathbf{Ein}_n \setminus \Lambda_\Gamma$.*

- *The quotient manifold $\Gamma \backslash \Omega_\Gamma$ is compact, diffeomorphic to $\mathbf{S}^1 \times (\mathbf{S}^1 \times \mathbf{S}^{n-2})^{(g-1)\sharp}$, and inherits from $\mathbf{Ein}_n$ a Lorentzian conformal structure. Here $(\mathbf{S}^1 \times \mathbf{S}^{n-2})^{(g-1)\sharp}$ stands for the connected sum of $(g-1)$ copies of $\mathbf{S}^1 \times \mathbf{S}^{n-2}$.*

- *The conformal group of this structure is induced by $O(n-2) \times \rho_R(SL(2, \mathbb{R}))$, and is strongly essential.*

We will explain in the following section why the structures constructed in this way are essential. For the moment, let us just do some remarks on Theorem 6.1. This theorem tells us that the answer to question 2.2 is negative, and there is no hope to have, in the Lorentzian framework, such a strong statement as Ferrand-Obata theorem for conformal Riemannian geometry. Indeed, assumption of essentiality is no more sufficient to fix the topology of the manifold, even in the compact case. Moreover, starting from two Schottky groups $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ in $SL(2, \mathbb{R})$, with the same number g of generators, but which are not conjugated in $SL(2, \mathbb{R})$, we get two groups $\Gamma_1 = \rho_L(\hat{\Gamma}_1)$ and $\Gamma_2 = \rho_L(\hat{\Gamma}_2)$ which are not conjugated in $O(2, n)$. This gives non conformally equivalent Lorentzian structures on $\mathbf{S}^1 \times (\mathbf{S}^1 \times \mathbf{S}^{n-2})^{(g-1)\sharp}$, which are both essential. So, even when one fixes the topology (here, for example, $\mathbf{S}^1 \times (\mathbf{S}^1 \times \mathbf{S}^{n-2})^{(g-1)\sharp}$), there still can be a non trivial moduli space of conformal structures which are essential.

6.2. More complicated essential dynamics. We keep the notations of the previous section: $\hat{\Gamma}$ is a Schottky group of $SL(2, \mathbb{R})$ with g generators, and $\Gamma = \rho_L(\hat{\Gamma})$. We call $M_\Gamma = \Gamma \backslash \Omega_\Gamma$.

Let us consider the two flows $\phi^t = \rho_R(\hat{\phi}^t)$ et $\psi^t = \rho_R(\hat{\psi}^t)$ where $\hat{\phi}^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ et $\hat{\psi}^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Since ϕ^t and ψ^t both centralize Γ , and leave Ω_Γ invariant, they induce two conformal flows $\bar{\phi}^t$ and $\bar{\psi}^t$ on M_Γ . We are going to show that $\bar{\phi}^t$ and $\bar{\psi}^t$ are two strongly essential flows on M_Γ .

6.2.1. Dynamics of the flow $\bar{\phi}^t$ on M_Γ . As a flow of $O(2, n)$, ϕ^t writes as:

$$\phi^t = \begin{pmatrix} 1 & t & & & \\ 0 & 1 & & & \\ & & I_{n-2} & & \\ & & & 1 & t \\ & & & 0 & 1 \end{pmatrix}$$

By the matrix expression given in Lemma 5.3, we recognize here a “lightlike translation flow”, as already studied in section 5.2.

Let us recall the dynamical properties of ϕ^t on $\mathbf{Ein}_n$. The flow ϕ^t fixes all the points of a lightlike geodesic $\Delta_0 \subset \mathbf{T}_0$ in $\mathcal{F}_2$ (and Δ_0 is exactly the set of fixed points of ϕ^t). Any lightlike geodesic Δ of $\mathbf{Ein}_n$, passing through a point $p \in \Delta_0$, is preserved by ϕ^t . If such a Δ is different from Δ_0 , ϕ^t acts on Δ as a parabolic transformation (i.e the action is conjugated to that of $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ on

$\mathbf{RP}^1$). Now, let us consider the projection $\pi_\Gamma : \Omega_\Gamma \rightarrow M_\Gamma$. Since Δ_0 is in $\mathcal{F}_2$, it is transverse to Λ_Γ , and thus $\Delta_0 \cap \Omega_\Gamma$ is a nonempty Γ -invariant closed subset of Ω_Γ . We get that $\pi_\Gamma(\Delta_0 \cap \Omega_\Gamma)$ is a finite union of closed lightlike geodesics $\bar{\Delta}_1, \dots, \bar{\Delta}_s$. To see that $\bar{\phi}^t$ is strongly essential, let us pick a point $x_0 \in \Delta_0 \cap \Omega_\Gamma$, and an open neighbourhood $U \subset \Omega_\Gamma$ of x_0 , on which π_Γ is injective. Let V be an open subset of $U \setminus \Delta_0$ with compact closure in $U \setminus \Delta_0$. Then $\lim_{t \rightarrow \pm\infty} \phi^t.V = \rho_{\Delta_0}(V)$ and $\lim_{t \rightarrow \pm\infty} \text{Vol}(\phi^t.V) = 0$. In particular, there is a T_0 such that for $t > T_0$, $\phi^t.V \subset U$. Let us call $\bar{U} = \pi_\Gamma(U)$ (resp. $\bar{V} = \pi_\Gamma(V)$), and let us define a smooth volume form on $\bar{U}$ by $\bar{V}ol = (\pi_\Gamma^{-1})^* \text{Vol}$. Then for $t > T_0$, $\bar{\phi}^t.\bar{V} \subset \bar{U}$, and $\lim_{t \rightarrow +\infty} \bar{V}ol(\bar{\phi}^t.\bar{V}) = 0$. This proves that $\bar{\phi}^t$ is strongly essential.

To have more intuition on how an essential flow of Lorentzian transformations behaves, we now describe more precisely the dynamics of $\bar{\phi}^t$ on M_Γ .

Since we defined Γ to be $\rho_L(\hat{\Gamma})$, the action of Γ on Δ_0 is conjugated to that of $\hat{\Gamma}$ on $\mathbf{S}^1 = \partial\mathbf{H}^2$. Thus $\Lambda_\Gamma \cap \Delta_0$ is homeomorphic to the Cantor set $\Lambda_{\hat{\Gamma}}$. The complementary of this Cantor set in Δ_0 is a family $\mathcal{I}$ of connected components. Since we supposed that $\Gamma \backslash (\Delta_0 \cap \Omega_\Gamma)$ is a union of s closed lightlike geodesics, this means that the action of Γ on the family $\mathcal{I}$ has exactly s orbits. For each $I \in \mathcal{I}$, we define $\Omega_I = \bigcup_{x \in I} (C(x) \setminus \Delta_0)$. This is an open subset of Ω_Γ . In fact each Ω_I is the set of $x \in (\Omega_\Gamma \setminus I)$ such that $\lim_{t \rightarrow +\infty} \phi^t.x = \lim_{t \rightarrow -\infty} \phi^t.x \in I$. The quotient $\Gamma \backslash (\bigcup_{I \in \mathcal{I}} \Omega_I)$ is a finite union $\bar{\Omega}_1, \dots, \bar{\Omega}_s$ of open subsets of M_Γ . For each $j \in \{1, \dots, s\}$, one has the following dynamical characterization of $\bar{\Omega}_j$ as $\bar{\Omega}_j = \{x \in M_\Gamma \setminus \bar{\Delta}_j \mid \lim_{t \rightarrow +\infty} \bar{\phi}^t.x = \lim_{t \rightarrow -\infty} \bar{\phi}^t.x \in \bar{\Delta}_j\}$.

Thus $\bar{\Omega}_1 \cup \dots \cup \bar{\Omega}_s$ is a dense open subset of M_Γ where the dynamics of $\bar{\phi}^t$ is easy to understand.

It remains to understand how the complement of $\bar{\Omega}_1 \cup \dots \cup \bar{\Omega}_s$ in M_Γ looks like. The complement of $\bigcup_{I \in \mathcal{I}} \Omega_I$ in Ω_Γ is $K = \bigcup_{x \in (\Lambda_\Gamma \cap \Delta_0)} (C(x) \cap \Omega_\Gamma)$. This is a closed subset of Ω_Γ and $K \cap (\Omega_\Gamma \setminus \Delta_0)$ is a lamination by lightlike hypersurfaces, transversally modelled on a Cantor set.

Looking at the quotient $\bar{K} = \Gamma \backslash K$, we get that $\bar{K}$ is the complement of $\bar{\Omega}_1 \cup \dots \cup \bar{\Omega}_s$ in M_Γ . The set $\bar{K}$ contains $\bar{\Delta}_1 \cup \dots \cup \bar{\Delta}_s$, and $\bar{K} \setminus \bar{\Delta}_1 \cup \dots \cup \bar{\Delta}_s$ is a lamination $\mathcal{L}$ by lightlike hypersurfaces, transversally modelled on a Cantor set. By the minimality of the action of $\hat{\Gamma}$ on $\Lambda_{\hat{\Gamma}}$, we get that each leaf of $\mathcal{L}$ is dense in K .

6.2.2. Dynamics of the flow $\bar{\psi}^t$ on M_Γ . In $O(2, n)$, the flow ψ^t writes as:

$$\psi^t = \begin{pmatrix} e^t & & & & \\ & e^{-t} & & & \\ & & I_{n-2} & & \\ & & & e^t & \\ & & & & e^{-t} \end{pmatrix}$$

This is the third flow that we studied in section 5.2. We already studied its dynamics on $\mathbf{Ein}_n$ in Lemma 5.5, and we keep the notations of this lemma. The lightlike geodesics Δ^+ and Δ^- are both fixed individually by the group Γ . Thus, $\Delta^+ \cap \Omega_\Gamma$ and $\Delta^- \cap \Omega_\Gamma$ both project in M_Γ on a finite union of closed lightlike geodesics, $\bar{\Delta}_1^+, \dots, \bar{\Delta}_s^+$, and $\bar{\Delta}_1^-, \dots, \bar{\Delta}_s^-$.

The $\Delta^+ \cap \Omega_\Gamma$ (resp. $\Delta^- \cap \Omega_\Gamma$) writes as a union of infinitely many connected components $U_{I \in \mathcal{I}} I$ (resp. $U_{J \in \mathcal{J}} J$). For every $I \in \mathcal{I}$ (resp. every $J \in \mathcal{J}$), we define $\Omega_I^- = \bigcup_{x \in I} (C(x) \setminus \Delta^+)$ (resp. $\Omega_J^+ = \bigcup_{x \in J} (C(x) \setminus \Delta^-)$). We observe that for every $I \in \mathcal{I}$, $\Omega_I^- \subset \Omega_{\Delta^+}$, and that $\pi^-(\Omega_I^-)$ is a connected component of $\Delta^- \cap \Omega_\Gamma$, just obtained from I by “sliding along the leaves of $\mathcal{F}_1$ ”. In particular, if I and I' are in the same Γ -orbit, the same will be true for $\pi^-(\Omega_I^-)$ and $\pi^-(\Omega_{I'}^-)$. Reindexing if necessary the $\bar{\Delta}_j^-$'s, we will now suppose that if I project on $\bar{\Delta}_j^+$, then $\pi^-(\Omega_I^-)$ project on $\bar{\Delta}_j^-$.

Now, we get that the quotient $\Gamma \backslash \bigcup_{I \in \mathcal{I}} \Omega_I^-$ is a finite union of open subsets $\bar{\Omega}_1^- \cup \dots \cup \bar{\Omega}_s^-$. Each $\bar{\Omega}_j^-$ contains $\bar{\Delta}_j^-$, and π^- induces a smooth fibration $\bar{\pi}_j^- : \bar{\Omega}_j^- \rightarrow \bar{\Delta}_j^-$, whose fiber are smooth lightlike hypersurfaces, and such that for any $x \in \bar{\Omega}_j^-$, $\lim_{t \rightarrow -\infty} \bar{\psi}^t . x = \bar{\pi}_j^-(x)$.

Looking at $\Gamma \backslash \bigcup_{J \in \mathcal{J}} \Omega_J^+$, we get a finite union $\bar{\Omega}_1^+ \cup \dots \cup \bar{\Omega}_s^+$ of open subsets. Each $\bar{\Omega}_j^+$ contains $\bar{\Delta}_j^+$, and there is a smooth fibration $\bar{\pi}_j^+ : \bar{\Omega}_j^+ \rightarrow \bar{\Delta}_j^+$, whose fibers are smooth lightlike hypersurfaces, such that for every $x \in \bar{\Omega}_j^+$, $\lim_{t \rightarrow +\infty} \bar{\psi}^t . x = \bar{\pi}_j^+(x)$.

It remains to describe what are the boundaries of the $\bar{\Omega}_j^\pm$. As in the previous example, let us introduce $K^+ = \bigcup_{x \in (\Lambda_\Gamma \cap \Delta^-)} (C(x) \cap \Omega_\Gamma)$, and $K^- = \bigcup_{x \in (\Lambda_\Gamma \cap \Delta^+)} (C(x) \cap \Omega_\Gamma)$. Then $\bar{K}^+ = \pi_\Gamma(K^+)$ and $\bar{K}^- = \pi_\Gamma(K^-)$ are two closed subsets of M_Γ , containing $\bar{\Delta}_1^- \cup \dots \cup \bar{\Delta}_s^-$ and $\bar{\Delta}_1^+ \cup \dots \cup \bar{\Delta}_s^+$ respectively. The sets $K^+ \setminus \{\bar{\Delta}_1^- \cup \dots \cup \bar{\Delta}_s^-\}$ and $K^- \setminus \{\bar{\Delta}_1^+ \cup \dots \cup \bar{\Delta}_s^+\}$ are two laminations $\mathcal{L}^+$ and $\mathcal{L}^-$ by smooth lightlike hypersurfaces, transversally modelled on a Cantor set. Each leaf of $\mathcal{L}^+$ (resp. of $\mathcal{L}^-$) is dense in K^+ (resp. in K^-). Finally, the closure of each $\bar{\Omega}_j^+$ in M_Γ is K^+ , and the closure of each $\bar{\Omega}_j^-$ in M_Γ is K^- .

6.2.3. Interpretation of the previous examples in dimension 3. Let us now say a little bit more about the previous examples when the construction is performed on $\mathbf{Ein}_3$. In this case, $\mathbf{Ein}_3 \setminus \mathbf{T}_0$ carries an action of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ by $\rho_L(SL(2, \mathbb{R})) \times \rho_L(SL(2, \mathbb{R}))$. This allow to identify $\mathbf{Ein}_3 \setminus \mathbf{T}_0$ with $SL(2, \mathbb{R})$, endowed with a Lorentzian conformal structure invariant by the action of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ by left and right multiplications. Now $\mathbf{Ein}_3 \setminus \mathbf{T}_0$ is a dense open subset of Ω_Γ , and project to a dense open subset $N \subset M_\Gamma$. The manifold N is $\Gamma \backslash SL(2, \mathbb{R})$, and can thus be identified with a two-fold cover of $T^1(\Gamma \backslash \mathbf{H}^2)$, the unit tangent bundle of the non-compact hyperbolic surface $\Gamma \backslash \mathbf{H}^2$. The Killing form of $SL(2, \mathbb{R})$ induces on $N = \Gamma \backslash SL(2, \mathbb{R})$ a Lorentzian metric of constant curvature -1 , which is preserved by the right action of $SL(2, \mathbb{R})$. Our manifold M_Γ can thus be understood as a conformal compactification of N . this compactification is made thanks to a finite union $\Gamma \backslash (\mathbf{T}_0 \cap \Omega_\Gamma) = T_1 \cup \dots \cup T_s$ of Lorentzian tori.

Now, how can we interpret the flows $\bar{\phi}^t$ and $\bar{\psi}^t$? On $\Gamma \backslash SL(2, \mathbb{R})$, the right multiplication by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$) can be seen as the action of the horocyclic flow (resp. geodesic flow) on $T^1(\Gamma \backslash \mathbf{H}^2)$ (once again, up to a two-fold cover). So, the flows $\bar{\phi}^t$ and $\bar{\psi}^t$ can be seen as the extension to M_Γ of the horocyclic and geodesic flows on N . Notice that the action of $\bar{\phi}^t$ and $\bar{\psi}^t$ is inessential on N (as we saw, those flows preserve a Lorentzian metric with constant curvature in the conformal class), but become essential when extended to the conformal compactification M_Γ .

7. Essential versus isometric dynamics

Now that we have a sample of examples of essential conformal transformations on compact Lorentz manifolds, we can try to guess what could be the answer to Question 4.3, and isolate what are the dynamical properties which distinguish essential actions from inessential ones. A useful notion will be:

Definition 7.1. Let M be a compact manifold, x_0 in M , and (f_k) a sequence of homeomorphisms of M . Let us denote by $\Lambda_{f_k}(x_0)$ the set of cluster points of $f_k(x_0)$. Then the sequence (f_k) is said to be equicontinuous at x_0 if for every sequence x_k tending to x_0 , the set of cluster points of $f_k(x_k)$ is also $\Lambda_{f_k}(x_0)$.

An homeomorphism f of M is said to be equicontinuous at x_0 if the sequence (f^k) is equicontinuous at x_0 .

Stated briefly, a transformation f is equicontinuous at x_0 if the following phenomena does not occur: $f^k(x_0)$ tends to x_∞ whereas $f^k(x_k)$ tends to x'_∞ ($x'_\infty \neq x_\infty$), for a sequence x_k tending to x_∞ .

The dynamical study of Lorentzian isometries on a compact manifold gave rise to a great amount of works: [1], [36], [38], [10], [9], [24] among others. One of the basic properties of Lorentzian isometric dynamics is:

Theorem 7.2. *Let (M, g) be a compact Lorentz manifold. Let f be an isometry of (M, g) , such that $(f^k)_{k \in \mathbb{Z}}$ does not have compact closure in $\text{Isom}(M, g)$. Then f is nowhere equicontinuous on M .*

On the contrary, the dynamical behaviour of the essential transformations we met until now offered a quite different picture. For such an essential conformal transformation f on M , there always existed:

- two finite families of closed subsets $F_1^+, \dots, F_s^+$ and $F_1^-, \dots, F_s^-$ (this two families being sometimes the same), playing the role of attracting and repelling sets. In the example we had, these sets were finite union of points, or finite union of closed lightlike geodesics.

- a family of open subsets $(\Omega_{ij})_{i,j \in \{1, \dots, s\}}$, endowed with continuous projections $\pi_{ij}^+ : \Omega_{ij} \rightarrow F_i^+$ and $\pi_{ij}^- : \Omega_{ij} \rightarrow F_j^-$, and such that $\bigcup_{i,j \in \{1, \dots, s\}} \Omega_{ij}$ is a dense open set of M .

The dynamical behaviour of f^k on Ω_{ij} was described by the fact that for any compact subset $K \subset \Omega_{ij}$, $\lim_{k \rightarrow +\infty} f^k(K) = \pi_{ij}^+(K)$, and $\lim_{k \rightarrow -\infty} f^k(K) = \pi_{ij}^-(K)$. In particular, in all the examples we met, f was equicontinuous on $\bigcup_{i,j \in \{1, \dots, s\}} \Omega_{ij}$.

This dynamical pattern could be a general picture for essential transformations, and it would distinguish them from inessential ones. Let us formulate the following dynamical conjecture:

Conjecture 7.3. *Let (M, g) be a compact Lorentz manifold. Let f be an essential conformal transformation of (M, g) . Then f is equicontinuous on a dense open subset of M .*

7.1. Stable conformal dynamics, and its consequences on the geometry. We would like now to explain why Conjecture 7.3 is linked to generalized Lichnerowicz conjecture (at least for the Lorentzian signature). We are going to see that when an essential conformal transformation f on a compact Lorentzian manifold (M, g) is equicontinuous on a dense open subset, then it imposes some constraints on the geometry of this open subset. We will see, in the following section, that sometimes, these constraints forces the dense open subset (and hence the whole M) to be conformally flat. Most of the ideas presented here are at the basis of [17].

7.1.1. Stable conformal transformations. Equicontinuity implies properties on the differential maps Df^k , which are more tractable technically.

We consider (f_k) , a sequence of conformal diffeomorphisms of a Lorentz manifold (M, g) . We suppose that there is $x_0 \in M$ such that $x_k = f_k(x_0)$ has a limit point $x_\infty \in M$. We then choose smooth frame fields $x \mapsto (E_1(x), E_2(x), \dots, E_n(x))$ and $y \mapsto (F_1(y), F_2(y), \dots, F_n(y))$ in neighbourhoods of x_0 and x_∞ respectively. We suppose moreover that $(E_1(x), \dots, E_n(x))$ and $(F_1(y), \dots, F_n(y))$ satisfy $g_x(E_1(x), E_2(x)) = 1$ (resp. $g_y(F_1(y), F_2(y)) = 1$) and $g_x(E_i(x), E_i(x)) = 1, i \geq 3$ (resp. $g_y(F_i(y), F_i(y)) = 1, i \geq 3$), all the other products being zero. The differential $D_{x_0} f_k$, when we read

it in the frames $(E_1(x_0), \dots, E_n(x_0))$ and $(F_1(f_k(x_0)), \dots, F_n(f_k(x_0)))$ yields a matrix $M_k(x_0)$ in $\mathbf{R} \times O(1, n-1)$. The projection on the $\mathbf{R}$ -factor is just the square root of the conformal distortion, namely $e^{\sigma_k(x_0)}$, if $f^*g = e^{2\sigma_k}g$.

We now use the Cartan decomposition $O(1, n-1) = KAK$, where K is the maximal compact subgroup of $O(1, n-1)$, namely $K = O(1) \times O(n-1)$, and A is a maximal abelian subgroup in $O(1, n-1)$. We perform a Cartan decomposition of the sequence $M_k(x_0)$, so that $M_k(x_0)$ writes as a product $L_1^{(k)}(x_0)D_k(x_0)L_2^{(k)}(x_0)$. The two matrix $L_1^{(k)}(x_0)$ and $L_2^{(k)}(x_0)$ are in K and $D_k(x_0)$ is a diagonal matrix of the form

$$e^{\sigma_k(x_0)} \begin{pmatrix} e^{\lambda_k(x_0)} & & & & \\ & e^{-\lambda_k(x_0)} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

with $\lambda_k(x_0) \geq 0$.

In what follows, we will use the notation $\delta_k^+(x_0) = \sigma_k(x_0) + \lambda_k(x_0)$ and $\delta_k^-(x_0) = \sigma_k(x_0) - \lambda_k(x_0)$. Remark that one always has $\delta_k^- \leq \sigma_k \leq \delta_k^+$.

Now, given a sequence (f_k) such that $f_k(x_0) \rightarrow x_\infty$, we will say that this sequence *is simple* if:

(i) $e^{\lambda_k(x_0)}$, $e^{\sigma_k(x_0)}$, $e^{\delta_k^+(x_0)}$ and $e^{\delta_k^-(x_0)}$ all have a limit in $\mathbf{R} \cup \{+\infty\}$ when $k \rightarrow +\infty$.

(ii) The two sequences $L_1^{(k)}(x_0)$ and $L_2^{(k)}(x_0)$ converge in K .

Every sequence (f_k) admits a simple subsequence.

In [36], A.Zeghib did the dynamical study of sequences of isometries of a compact manifold, and introduced the following notion of stability:

Definition 7.4. Let (f_k) be a simple sequence of conformal transformations of (M, g) , such that $f_k(x_0) \rightarrow x_\infty$. The stable space at x_0 for the sequence (f_k) is defined as the subspace $H_{x_0}^< = \{u \in T_{x_0}M \mid \exists(u_k) \subset T_{x_0}M, u_k \rightarrow u, \text{ and } D_{x_0}f_k(u_k) \text{ is bounded}\}$. We also define the strongly stable space at x_0 as $H_{x_0}^{<<} = \{u \in T_{x_0}M \mid \exists(u_k) \subset T_{x_0}M, u_k \rightarrow u, \text{ and } D_{x_0}(f_k)(u_k) \rightarrow 0 \in T_{x_\infty}M\}$.

The sequence (f_k) is said to be stable at x_0 if $H_{x_0}^< = T_{x_0}M$, and strongly stable when $H_{x_0}^{<<} = T_{x_0}M$.

The key lemma, making the link between equicontinuity and stability is:

Lemma 7.5. Let (M, g) be a compact Lorentzian manifold, and f a conformal transformation of (M, g) . If (f^k) is equicontinuous at x_0 , then for any sequence (n_k) such that $f^{n_k}(x_0)$ converges and (f^{n_k}) is simple, then (f^{n_k}) is stable at x_0 .

The lemma uses the fact that conformal transformations preserve a distinguished class of projective parameters on lightlike conformal geodesics, see [17].

7.1.2. Influence on the Weyl and Cotton tensors. The link between dynamics and geometry is made clearer by the following propositions of [17].

Proposition 7.6. *Let (f_k) be a sequence of conformal transformations of a three dimensional Lorentz manifold (M, g) . We suppose that $\lim_{k \rightarrow +\infty} f_k(x_0) = x_\infty$ for some $x_\infty \in M$. We suppose also that (f_k) is stable at x_0 , and $D_{x_0}f_k$ is unbounded. Then the Cotton tensor C vanishes at x_0 .*

The Cotton tensor was introduced in Section 3. It is conformally invariant, what means that

$$C_{y_k}(D_{x_0}f_k(X_i), D_{x_0}f_k(X_j), D_{x_0}f_k(X_l)) = C_{x_0}(X_i, X_j, X_l)$$

for all triple (X_i, X_j, X_l) of $T_{x_0}M$. Now, let us put $X_i^{(k)} = (L_1^k)^{-1}.E_i(x_0)$ (resp. $Y_i^{(k)} = (L_2^k).F_i(x_k)$), for $i \in \{1, 2, 3\}$ (we keep the notations of the previous paragraph). Notice that the frame (X_1^k, X_2^k, X_3^k) (resp. (Y_1^k, Y_2^k, Y_3^k)) tends to a frame $(X_1^\infty, X_2^\infty, X_3^\infty)$ of $T_{x_0}M$ (resp. to a frame $(Y_1^\infty, Y_2^\infty, Y_3^\infty)$ of $T_{x_\infty}M$). The previous equality becomes $e^{\alpha_k}C_{y_k}(Y_i^k, Y_j^k, Y_l^k) = C_{x_0}(X_i^k, X_j^k, X_l^k)$.

When $(i, j, l) = (1, 2, 3)$ up to permutation, the sequence α_k is $\delta_k^+ + \delta_k^- + \sigma_k$. Hence, in this case, and under the hypothesis of the proposition, $\lim_{k \rightarrow +\infty} \alpha_k = -\infty$. At the limit, we get $C_{x_0}((X_1^\infty, X_2^\infty, X_3^\infty) = 0$.

But one can check that α_k tends to $-\infty$ for every triple (i, j, l) except $(i, j, l) = (1, 1, 1)$. This mean that when $(i, j, l) \neq (1, 1, 1)$, $C_{x_0}(X_i^\infty, X_j^\infty, X_l^\infty) = 0$. Since C is antisymmetric in its two first variables, we also have $C_{x_0}(X_1^\infty, X_1^\infty, X_1^\infty) = 0$. So, $C_{x_0} = 0$.

As a consequence of this proposition, we get that if the conjecture 7.3 is true, then it implies the generalized Lichnerowicz's conjecture in dimension 3.

When the dimension is greater than 3, we can do the same work on the Weyl tensor, instead of the Cotton tensor. It turns out that it is a little bit more tedious, but we get the:

Proposition 7.7. [17] *Let (f_k) be a sequence of conformal transformations of a Lorentz manifold (M, g) , whose dimension is greater or equal to four. We suppose that $\lim_{k \rightarrow +\infty} f_k(x_0) = x_\infty$ for some $x_\infty \in M$. We suppose also that (f_k) is stable at x_0 . Then, denoting by W the Weyl tensor of the conformal structure on M :*

- (i) *If $W_{x_\infty} = 0$, then $W_{x_0} = 0$.*
- (ii) *If $D_{x_0}f_k$ is unbounded, then $\text{Im}W_{x_0} \subset H_{x_0}^{<<}$ (here $\text{Im}W_{x_0}$ denotes the set of all possible values of the $(1, 3)$ tensor W at x_0).*
- (iii) *If (f_k) is strongly stable at x_0 , then $W_{x_0} = 0$.*

The conclusions of this proposition are not as strong as those in dimension three. Nevertheless, we see from point (i) that if one can prove the conjecture 7.3, and also prove that the Weyl tensor vanishes on the set $F_1^+ \cup \dots \cup F_s^+ \cup F_1^- \cup \dots \cup F_s^-$ (see the notations just before the conjecture), then it would imply the generalized Lichnerowicz's conjecture.

7.2. Examples where stability imposes conformal flatness. In section 5.2, we studied several conformal transformations of Einstein's universe: translations, homotheties.... These transformations were proved to be strongly essential, and this for purely dynamical reasons, so that if any of these transformations preserves some conformal Lorentzian structure on Σ , it will be essential for that structure. This gives some hope to build non conformally flat structures on Σ which are essential. Indeed, let us take for example a translation acting on Σ . This translation preserves the canonical conformal structure, but maybe, it preserves other Lorentzian conformal structures on Σ . So, if among the preserved structures, we can pick a non conformally flat one, we would be done. The following proposition illustrates how to use the previous results on stable conformal transformations, to show that this method for getting counter-examples to generalized Lichnerowicz's conjecture is hopeless. We use the terminology of section 5.2.

Theorem 7.8. *Let T be a translation (resp. an homothety h_λ , resp. a transformation ψ^λ) acting on the manifold Σ . Then the only Lorentzian conformal structures on Σ which are preserved by T (resp. h_λ , resp. A^λ), are conformally flat.*

With the same methods, a similar statement could be obtained for the flows $\bar{\phi}^t$ and $\bar{\psi}^t$ of section 6.2.

Proof. We make the proof for a timelike translation T , and for a transformation ψ^λ , $\lambda < 0$. The other cases are left to the reader. Notice that since all these transformations are equicontinuous on a dense open subset of Σ , as follows from the conclusions of the dynamical lemmas of section 5.2, they are stable on a dense open subset. In dimension three, Theorem 7.8 is just a consequence of Proposition 7.6. We will now suppose that the dimension is at least four.

Let T be a timelike translation. We suppose that T acts as a conformal transformation of a Lorentzian metric g on Σ . We keep the notations of Lemma 5.3, so that T has a unique fixed point p_∞ . We consider that T is the time one of a flow T^t of timelike translations. This flow generates a vector field X_1 on Σ , with a unique singularity at p_∞ , and X_1 is preserved by T . Let us choose $T_2, \dots, T_n$, $(n-1)$ other translations, such that $T, T_2, \dots, T_n$ are lineary independent. These translations are time one of flows, which generate vector fields $X_2, \dots, X_n$ on Σ . These fields are preserved by T . Now, we saw in Lemma 5.3, that for any $x \in \Sigma \setminus C_\infty$, $\lim_{k \rightarrow +\infty} T^k \cdot x = p_\infty$. We deduce that $\lim_{k \rightarrow +\infty} D_x T^k(X_i(x)) = 0$, for $i = 1, \dots, n$. Now, $x \in \Sigma \setminus C_\infty$, so that $X_1(x), \dots, X_n(x)$ span the space $T_x \Sigma$. We infer that (T^k) is strongly stable at x . From Proposition 7.7, we conclude that $W_x = 0$, and since $\Sigma \setminus C_\infty$ is dense in Σ , W vanishes identically on Σ .

We now study the case of a transformation ψ^λ , $\lambda < 0$. We keep the notations of section 5.2.3, and from Lemma 5.5, we get that $(\psi^{k\lambda})_{k \in \mathbb{N}}$ is equicontinuous, hence stable on $\Omega_{\Delta-}$. Also, $(\psi^{-k\lambda})_{k \in \mathbb{N}}$ is equicontinuous, hence stable on $\Omega_{\Delta+}$. On $\Sigma \setminus C_\infty$, ψ^λ is conjugated via the stereographic projection to: $(x_1, \dots, x_n) \mapsto (e^{2\lambda}x_1, e^\lambda x_2, \dots, e^\lambda x_{n-1}, x_n)$. We see that the hyperplane $e_1^\perp$ is included in $H_x^{<<}$ for every $x \in \Sigma \setminus C_\infty$ (here $H_x^{<<}$ is the strongly stable space of the sequence

$(\psi^{k\lambda})_{k \in \mathbb{N}}$. In other words, and as a consequence of Proposition 7.7, we get that for every $p \in \Delta^-$, $p \neq p_\infty$, and every $x \in C(p) \setminus \Delta^-$, $Im_x W \subset T_x C(p)$. By continuity, this inclusion has to hold on the whole $C(p)$, and thus $W_p = 0$. We then get that for every $p \in \Delta^-$, $W_p = 0$. But now, we have that for every $x \in \Omega_{\Delta^+}$, $(\psi^{-k\lambda})_{k \in \mathbb{N}}$ is stable at x , $\lim_{k \rightarrow +\infty} \psi^{-k\lambda}.x = \pi^-(x)$, with $W_{\pi^-(x)} = 0$. We thus get from Proposition 7.7, point (i), that $W_x = 0$. Thus, $W = 0$ vanishes on Ω_{Δ^+} , and since this open subset is dense in Σ , we are done. $\square$

8. Essential actions of simple groups on compact manifolds

Let us finish by quoting some positive results toward generalized Lichnerowicz's conjecture, when one looks at essential conformal actions of simple Lie groups. When a simple Lie group acts on a manifold, preserving a rigid geometric structure, some conditions are imposed on its rank. This is due to the following result of R.J. Zimmer:

Theorem 8.1. [37] *Let G be a simple Lie group, acting non trivially on a compact manifold M , and preserving an H structure, where H is a real algebraic group. Then $\text{rank}_{\mathbb{R}} G \leq \text{rank}_{\mathbb{R}} H$.*

A pseudo-Riemannian metric of signature (p, q) , $p \leq q$, is an H -structure with $H = SO(p, q)$. Thus Zimmer's theorem ensures that a simple Lie group acting on a compact manifold, by isometries of a metric of signature (p, q) , must have real rank at most p .

A pseudo-Riemannian conformal structure of signature (p, q) , $p \leq q$, is an H -structure for $H = \mathbb{R}_+^* \times SO(p, q)$, whose rank is $p + 1$. Thus a simple Lie group acting on a compact manifold M , by conformal transformations of a metric of signature (p, q) , $p \leq q$, must have rank at most $p + 1$.

From this, one deduces that a conformal action of a simple group G on a compact pseudo-Riemannian manifold (M, g) of signature (p, q) , $p \leq q$, is automatically essential as soon as $\text{rank}_{\mathbb{R}} G = p + 1$. What does occur in this case? The first related result was obtained by U.Bader and A.Nevo (to avoid trivialities, we suppose implicitly that $\dim(M) \geq 3$ in all what follows):

Theorem 8.2. [5] *Let G be a connected simple Lie group, acting conformally on a compact pseudo-Riemannian manifold (M, g) of signature (p, q) , $1 \leq p \leq q$. If the rank of G is $p + 1$, then:*

- G is locally isomorphic to $SO^o(p + 1, k + 1)$, for some k such that $p \leq k \leq q$.
- There exists a closed G -orbit, which is conformally equivalent to a finite cover of $\mathbf{Ein}^{p, k}$.

Using the conclusions of [5], their result is refined in [20] to get:

Theorem 8.3. [20] *Let G be a connected simple Lie group acting smoothly and conformally on a smooth compact pseudo-Riemannian manifold M of type (p, q) with $p \geq 2$. If the rank of G equals $p + 1$, then:*

- The group G is locally isomorphic to $SO^o(p+1, k+1)$ for some k such that $p \leq k \leq q$.
- Up to finite cover, M is conformally equivalent to the space $\mathbf{Ein}^{p,k}$.

The situation is a little bit more subtle in Lorentzian signature, but fully understood.

Theorem 8.4. [20] *Let G be a connected simple Lie group of rank 2, acting smoothly and conformally on a smooth compact Lorentz manifold M of dimension n . Then:*

- The group G is locally isomorphic to $SO^o(2, k)$ for some k such that $3 \leq k \leq n$.
- M is, up to finite cover, a complete conformally flat structure on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, i.e M is a quotient of $\widetilde{\mathbf{Ein}}_n$ (the universal cover of $\mathbf{Ein}_n$) by an infinite cyclic group Γ .
- The possible groups Γ are those generated by any element in a product $\mathbb{Z}^* \times O(n-k) \subset \widetilde{O^o(2, n)}$ (the universal cover of $O^o(2, n)$), where the $\mathbb{Z}$ factor is the center of $\widetilde{O(2, n)}$.

Despite these results, the essential conformal actions of simple Lie groups on compact manifolds is still not fully understood, even in the Lorentzian case.

9. References

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