RIGID BODY EQUATIONS ON SPACES OF PSEUDO-DIFFERENTIAL OPERATORS WITH RENORMALIZED TRACE

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JEAN-PIERRE MAGNOT[1] AND ENRIQUE G. REYES[2]

Abstract. We equip the regular Fréchet Lie group of invertible, odd-class, classical pseudodifferential operators $Cl^{0,*}_{odd}(M,E)$ —in which M is a compact smooth manifold and E a (complex) vector bundle over M— with pseudo-Riemannian metrics, and we use these metrics to introduce a class of rigid body equations. We prove the existence of a metric connection, we show that our rigid body equations determine geodesics on $Cl^{0,*}_{odd}(M,E)$, and we present rigorous formulas for the corresponding curvature and sectional curvature. Our main tool is the theory of renormalized traces of pseudodifferential operators on compact smooth manifolds.

Keywords: rigid body equations, pseudodifferential operators, renormalized traces.

MSC(2010):

1. Introduction

In this paper we continue our work on Mathematical Physics themes posed on spaces built out of pseudodifferential operators. In [22] we introduced a Kadomtsev-Petviashvili hierarchy with the help of odd-class non-formal pseudodifferential operators; here we consider analogs of the rigid body equation. Our motivation comes from the seminal work on "geometric hydrodynamics" by Arnold (see [2] and [14, 29] for recent reviews), and from the non-commutative version of the Korteweg-de Vries (KdV) equation considered by Berezin and Perelomov in [3]. In both cases, they begin their analysis by considering infinite-dimensional Lie groups and Lie algebras and, in Arnold's case, the existence of a metric on the space on which the equations are posed. Inspired by [22, 23], we observe herein that *there exist Fréchet Lie groups of non-formal pseudodifferential operators that can be equipped with pseudo-Riemannian metrics*, and that therefore we are in conditions of posing rigid body equations (following the Arnold and Berezin-Perelomov's nomenclature) on these highly interesting groups.

Our work is also an application of the theory of renormalized traces of pseudo-differential operators: our pseudo-Riemannian metrics exist because if we consider renormalized traces on *odd-class pseudodifferential operators* we obtain bona fide traces, and therefore we can mimic the construction of metrics familiar from mechanics on Lie groups (see [3, 14]) in this rigorous infinite-dimensional setting. More precisely, we observe herein that the set of invertible, odd-class, non-formal classical pseudodifferential operators on a compact manifold without boundary can be equipped with the structure of a regular Fréchet Lie group, and that properties of renormalized traces allow us to show that this group has the structure of an infinite-dimensional pseudo-Riemannian manifold. It is in this framework that we pose our rigid body equation.

Let us summarize the contents of this paper. We fix a compact smooth manifold without boundary M, and a (complex) vector bundle E over M. In Section 2 we present a quick survey of the properties of pseudodifferential operators that we use, including the Fréchet structure of the group of invertible and odd-class non-formal classical pseudodifferential operators acting on sections of E, a group we denote by $Cl^{0,*}_{odd}(M,E)$, see Theorem 2 below. We also introduce the Wodzicki residue and renormalized traces, and state the properties of renormalized traces that make them interesting objects for geometry if we restrict ourselves to odd-class pseudodifferential operators. In Section 3 we show how to construct non-degenerate pairings on the algebra of odd-class non-formal classical pseudodifferential operators using renormalized traces. This is the basic ingredient for our construction of pseudo-Riemannian metrics. In Section 4 we introduce our rigid body equations, and then in Section 5 we make some remarks on the pseudo-Riemannian geometry of the Fréchet Lie group $Cl^{0,*}_{odd}(M,E)$. In this section we show that, as in more classical contexts, our rigid body equations determine geodesics. We finish in Section 6 with some elementary examples of geodesic equations.

2. Preliminaries

2.1. Preliminaries on classical pseudodifferential operators.

We introduce groups and algebras of non-formal pseudodifferential operators needed to set up our equations. Basic definitions are valid for real or complex finite-dimensional vector bundles E over a compact manifold M without boundary whose typical fiber is a finite-dimensional real or complex vector space V. We begin with the following definition after [4, Section 2.1].

Definition 1. *The graded algebra of differential operators acting on the space of smooth sections $C^\infty(M,E)$ is the algebra $DO(E)$ generated by:*

- *Elements of $End(E)$, the group of smooth maps $E \to E$ leaving each fibre globally invariant and which restrict to linear maps on each fibre. This group acts on sections of E via (matrix) multiplication;*
- *The differentiation operators*

$$\nabla_X : g \in C^\infty(M,E) \mapsto \nabla_X g$$

where ∇ is a connection on E and X is a vector field on M.

Multiplication operators are operators of order 0; differentiation operators and vector fields are operators of order 1. In local coordinates, a differential operator of order k has the form $P(u)(x) = \sum p_{i_1\cdots i_r}(x)\nabla_{x_{i_1}}\cdots\nabla_{x_{i_r}}u(x)$, $r \leq k$, in which u is a (local) section and the coefficients $p_{i_1\cdots i_r}$ can be matrix-valued. The algebra $DO(M,E)$ is filtered by order: we note by $DO^k(M,E)$,$k \geq 0$, the differential operators of order less or equal than k.

Now we embed $DO(M,E)$ into the algebra of classical pseudodifferential operators $Cl(M,E)$. We need to assume that the reader is familiar with the basic facts on pseudodifferential operators defined on a vector bundle $E \to M$; these facts can be found for instance in [11], in the review [27, Section 3.3], and in the papers [5] and [31] in which the authors construct a global symbolic calculus for pseudodifferential operators showing, for instance, how the geometry of the base manifold M

furnishes an obstruction to generalizing local formulas of composition and inversion of symbols.

Notations. We note by $PDO(M,E)$ the space of pseudodifferential operators on smooth sections of E, see [27, p. 91]; by $PDO^o(M,E)$ the space of pseudodifferential operators of order o; and by $Cl(M,E)$ the space of classical pseudodifferential operators acting on smooth sections of E, see [27, pp. 89-91]. We also note by $Cl^o(M,E) = PDO^o(M,E) \cap Cl(M,E)$ the space of classical pseudodifferential operators of order o, and by $Cl^{o,*}(M,E)$ the group of units of $Cl^o(M,E)$.

A topology on spaces of classical pseudodifferential operators has been described by Kontsevich and Vishik in [17]: it is a Fréchet topology (and therefore it equips $Cl(M,E)$ with a smooth structure) such that each space $Cl^o(M,E)$ is closed in $Cl(M,E)$. This topology is discussed in [27, pp. 92-93], see also [6, 26, 28] for other descriptions. We use all along this work the Kontsevich-Vishik topology.

We set

$$PDO^{-\infty}(M,E) = \bigcap_{o\in\mathbb{Z}} PDO^o(M,E) \ .$$

It is well-known that $PDO^{-\infty}(M,E)$ is a two-sided ideal of $PDO(M,E)$, see e.g. [11, 28]. This fact allows us to define the quotients

$$\mathcal{F}PDO(M,E) = PDO(M,E)/PDO^{-\infty}(M,E) \ ,$$

$$\mathcal{F}Cl(M,E) = Cl(M,E)/PDO^{-\infty}(M,E) \ ,$$

and

$$\mathcal{F}Cl^o(M,E) = Cl^o(M,E)/PDO^{-\infty}(M,E) \ .$$

The script font $\mathcal{F}$ stands for *formal* pseudodifferential operators. The quotient $\mathcal{F}PDO(M,E)$ is an algebra isomorphic to the space of formal symbols, see [5], and the identification is a morphism of $\mathbb{C}$-algebras for the usual multiplication on formal symbols (appearing for instance in [11, Lemma 1.2.3] and [27, p. 89], and in [28, Section 1.5.2, Equation (1.5.2.3)] for the particular case of classical symbols).

Theorem 2. *The groups $Cl^{0,*}(M,E)$ and $\mathcal{F}Cl^{0,*}(M,E)$, in which $\mathcal{F}Cl^{0,*}(M,E)$ is the group of units of the algebra $\mathcal{F}Cl^0(M,E)$, are regular Fréchet Lie groups equipped with smooth exponential maps. Their Lie algebras are $Cl^0(M,E)$ and $\mathcal{F}Cl^0(M,E)$ respectively.*

Regularity is reviewed in [21, 22] and also in Paycha's lectures, see [27, p. 95]. The Lie group structure of $Cl^{0,*}(M,E)$ is discussed in [27, Proposition 4]. Theorem 2 is essentially proven in [19]: it is noted in this reference that the results of [12] imply that the group $Cl^{0,*}(M,E)$ (resp. $\mathcal{F}Cl^{0,*}(M,E)$) is open in $Cl^0(M,E)$ (resp. $\mathcal{F}Cl^0(M,E)$) and that therefore it is a regular Fréchet Lie group.

Now we will introduce our main classes of classical pseudodifferential operators. First of all we recall the following:

If $A \in Cl^o(M,E)$, the symbol $\sigma(A)(x,\xi)$ has an asymptotic expansion of the form

$$\sigma(A)(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{o-j}(A)(x,\xi) \ , \quad (x,\xi) \in T^*M \ , \tag{1}$$

in which each $\sigma_{o-j}(A)(x,\xi)$ satisfies the homogeneity condition

$$\sigma_{o-j}(A)(x,t\,\xi) = t^{o-j}\sigma_{o-j}(A)(x,\xi) \qquad \text{for every } t>0 \ .$$

The function $\sigma_o(A)(x,\xi)$ is the principal symbol of A. We define

Definition 3. *A classical pseudodifferential operator A on E is called*

- **odd class** *if and only if for all $n \in \mathbb{Z}$ and all $(x,\xi) \in T^*M$ we have:*
$$\sigma_n(A)(x,-\xi) = (-1)^n \sigma_n(A)(x,\xi) \ ,$$
and
- **even class** *if and only if for all $n \in \mathbb{Z}$ and all $(x,\xi) \in T^*M$ we have:*
$$\sigma_n(A)(x,-\xi) = (-1)^{n+1} \sigma_n(A)(x,\xi) \ .$$

Odd class pseudodifferential operators were introduced in [17, 18]; they are called "even-even pseudodifferential operators" in the treatise [28]. For instance, recalling Definition 1, we see that *all differential operators* are odd class.

Hereafter, the subscript $_{odd}$ (resp. $_{even}$) attached to a given space of (formal) pseudodifferential operators will refer to the set of all odd (resp. even) class (formal) pseudodifferential operators belonging to that space.

We need the following result, essentially present in [17, 28]:

Lemma 4. *$Cl_{odd}(M,E)$ and $Cl^0_{odd}(M,E)$ are associative algebras.*

Proof. We work locally. Let A, B be two odd class pseudodifferential operators of order m and m' respectively; the homogeneous pieces of the symbols of A, B, AB are related via (see [28, Section 1.5.2, Equation (1.5.2.3)])

$$\sigma_{m+m'-j}(AB)(x,\xi) = \sum_{|\mu|+k+l=j} \frac{1}{\mu!} \partial_\xi^\mu \sigma_{m-k}(A)(x,\xi) D_x^\mu \sigma_{m'-l}(B)(x,\xi) \ ,$$

in which $|\mu|$ is the length of the multi-index μ. We have, using the first equation appearing in Definition 3,

$$\partial_\xi^\mu \sigma_{m-k}(A)(x,-\xi) = (-1)^{m-k+|\mu|} \partial_\xi^\mu \sigma_{m-k}(A)(x,\xi)$$

and

$$D_x^\mu \sigma_{m'-l}(B)(x,-\xi) = (-1)^{m'-l} D_x^\mu \sigma_{m'-l}(B)(x,\xi) \ ,$$

so that

$$\sigma_{m+m'-j}(AB)(x,-\xi) = \sum_{|\mu|+k+l=j} \frac{1}{\mu!} (-1)^{m-k+|\mu|+m'-l} \partial_\xi^\mu \sigma_{m-k}(A)(x,\xi) D_x^\mu \sigma_{m'-l}(B)(x,\xi) \ .$$

Changing $+|\mu|$ for $-|\mu|$ in $(-1)^{m-k+|\mu|+m'-l}$ and using $|\mu|+k+l=j$ we obtain

$$\begin{aligned} \sigma_{m+m'-j}(AB)(x,-\xi) &= (-1)^{m+m'-j} \sum_{|\mu|+k+l=j} \frac{1}{\mu!} \partial_\xi^\mu \sigma_{m-k}(A)(x,\xi) D_x^\mu \sigma_{m'-l}(B)(x,\xi) \\ &= (-1)^{m+m'-j} \sigma_{m+m'-j}(AB)(x,\xi) \ , \end{aligned}$$

which proves the first claim. That $Cl^0_{odd}(M,E)$ is an associative algebra now follows from the standard fact that zero-order classical pseudodifferential operators form an algebra, see for instance the proof of Proposition 3 in [27]. □

The next proposition singles out an interesting Lie group included in $Cl_{odd}(M,E)$.

Proposition 5. *The algebra $Cl^0_{odd}(M,E)$ is a closed subalgebra of $Cl^0(M,E)$. Moreover, $Cl^{0,*}_{odd}(M,E)$ is*

- *an open subset of $Cl^0_{odd}(M,E)$ and,*

- *a regular Fréchet Lie group with Lie algebra $Cl^0_{odd}(M,E)$ and smooth Lie bracket $[A,B]=AB-BA$.*

Proof. We note by $\sigma(A)(x,\xi)$ the total formal symbol of $A \in Cl^0(M,E)$. We define the function

$$\phi : Cl^0(M,E) \to \mathcal{F}Cl^0(M,E)$$

as

$$\phi(A) = \sum_{n\in\mathbb{N}} \sigma_{-n}(x,\xi) - (-1)^n \sigma_{-n}(x,-\xi)\ .$$

This map is smooth, and

$$Cl^0_{odd}(M,E) = Ker(\phi),$$

which shows that $Cl^0_{odd}(M,E)$ is a closed subalgebra of $Cl^0(M,E)$. Moreover, if $H = L^2(M,E)$,

$$Cl^{0,*}_{odd}(M,E) = Cl^0_{odd}(M,E) \cap GL(H),$$

which proves that $Cl^{0,*}_{odd}(M,E)$ is open in the Fréchet algebra $Cl^0_{odd}(M,E)$, and it follows that it is a regular Fréchet Lie group by arguing along the lines of [12, 25]. □

We finish our preliminaries on pseudodifferential operators noting that at a formal level we have the splitting

$$\mathcal{F}Cl(M,E) = \mathcal{F}Cl_{odd}(M,E) \oplus \mathcal{F}Cl_{even}(M,E)\ ,$$

and the following composition rules for formal pseudodifferential operators $A \circ B$:

	A odd class	A even class
B odd class	$A \circ B$ odd class	$A \circ B$ even class
B even class	$A \circ B$ even class	$A \circ B$ odd class

2.2. **Renormalized traces.** Hereafter we assume that the typical fiber of the bundle E is a complex vector space, and that E is equipped with an Hermitian product $<\cdot,\cdot>$. An excellent review of this geometric set-up appears in [30, Chapters III, IV]. This product allows us to define the following L^2-inner product on sections of E:

$$\forall u,v \in C^\infty(M,E), \quad (u,v)_{L^2} = \int_M < u(x), v(x) > dx\ ,$$

where dx is a fixed Riemannian volume on M.

We need to use some further notions of the theory of pseudodifferential operators. First of all, we use the inner product just introduced to define *self-adjoint* and *positive* pseudodifferential operators. We also define *elliptic* pseudodifferential operators: a classical pseudodifferential operator P of order o is elliptic if its main symbol $\sigma_o(P)(x,\xi) : E_x \to E_x$ is invertible, see for instance [30, Chapter IV, Section 4] or [28, p. 92]; these pseudodifferential operators are also discussed quickly in [26, Definitions 6.17, 6.31]. We denote by $Ell(M,E)$ the space of all classical elliptic pseudodifferential operators.

Definition 6. *Q is a **weight** of order $q \in \mathbb{N}^*$ on E if and only if Q is a classical, elliptic, self-adjoint and positive pseudodifferential operator acting on smooth sections of E.*

Under these assumptions, the weight Q has a real discrete spectrum, and all its eigenspaces are finite dimensional. Moreover, for such a weight Q of order q, we can define complex powers of Q, see e.g. [6] or [26, Section 7.1] for a quick overview of technicalities and further references: the powers Q^{-s} of the weight Q are defined for $Re(s) > 0$ using a contour integral of the form

$$Q^{-s} = \int_\Gamma \lambda^s (Q - \lambda Id)^{-1} d\lambda \ ,$$

in which Γ is a contour around the real positive axis that appears precisely identified in [26, Section 7.1]. The pseudodifferential operator Q^{-s} is a classical pseudodifferential operator of order $-q\,s$.

Now we let A be a log-polyhomogeneous pseudodifferential operator, that is, A is a pseudodifferential operator such that its symbol is, locally, of the form

$$\sigma(x,\xi) \sim \sum_{j=0}^{o} \sum_{-\infty<k}^{o'} \sigma_{j,k}(A)(x,\xi)\ \log(|\xi|)^j \ ,$$

in which $\sigma_{j,k}(A)(x,\xi)$ are classical symbols, see [28, Section 2.6]. Within this general framework we introduce zeta functions and traces. The map

$$\zeta(A,Q,s) = s \in \mathbb{C} \mapsto \operatorname{tr}\left(AQ^{-s}\right) \in \mathbb{C} \ ,$$

in which tr is the classical trace of trace-class pseudo-differential operators, see [28, Section 1.3.5.1], is well-defined for $Re(s)$ large enough, and it extends to a meromorphic function on $\mathbb{C}$ with possibly a pole at $s = 0$ [26, 28]. When A is classical, this pole is a simple pole, and when A is classical, odd-class, and M is odd dimensional, $\zeta(A,Q,s)$ has no pole at $s = 0$.

Gilkey [11, Section 1.12.2] treats zeta functions and their relation to the heat kernel in detail; Scott [28, Section 1.5.7] deals with zeta functions in a very general setting: he extends the computations of Kontsevich and Vishik [17].

When A is a classical pseudodifferential operator, the Wodzicki residue, res_W, appearing in [32], see also [15], is directly linked with the simple pole of $\zeta(A,Q,.)$ at 0 by the residue formula

$$res_{s=0}\zeta(A,Q,s) = (1/q)\,\mathrm{res}_W A \ . \tag{2}$$

The Wodzicki residue is a higher dimensional analog of the Adler trace on formal symbols introduced in [1], but we remark that the former fails to be a direct extension of the latter. For example, $(1 + \frac{d}{dx})$ is invertible in $Cl(S^1,\mathbb{C})$, and since $(1 + \frac{d}{dx})^{-1}$ is odd class, we have on one hand that

$$res_W(1 + \frac{d}{dx})^{-1} = 0 \ ,$$

see *e.g.* [28, Section 1.5.8.2], but on the other hand, the formal symbol of $(1+\frac{d}{dx})^{-1}$ has a non-vanishing Adler trace.

Following [26, Chapter 7] and [28, Section 1.5.7], see also [6], we define renormalized traces of classical pseudodifferential operators as follows:

Definition 7. *Let A be a log-polyhomogeneous pseudo-differential operator and Q a fixed weight of order q. The finite part of $\zeta(A, Q, s)$ at $s = 0$ is called the renormalized trace of A. We denote it by $\mathrm{tr}^Q A$. If A is a classical pseudodifferential operator, then*

$$\mathrm{tr}^Q A = lim_{s\to 0}\left(tr(AQ^{-s}) - \frac{1}{q\,s}\mathrm{res}_W(A)\right).$$

If A is a trace-class pseudodifferential operator acting on $L^2(M, E)$? then $\mathrm{tr}^Q(A) = \mathrm{tr}(A)$, see *e.g.* [6]. However, generally speaking, the linear functional tr^Q *is not* a trace, this is, it does not vanish on commutators, although the linear map res_W determined by the Wodzicki residue does fulfil the trace property.

We state the main properties of res_W and of renormalized trace in Propositions 8 and 9.

Proposition 8.
(i) The Wodzicki residue res_W is a trace on the algebra of classical pseudodifferential operators $Cl(M, E)$, i.e. $\forall A, B \in Cl(M, E), \mathrm{res}_W[A, B] = 0$.
(ii) if $m = dimM$ and $A \in Cl(M, E)$,

$$\mathrm{res}_W A = \frac{1}{(2\pi)^n}\int_M \int_{|\xi|=1} tr\sigma_{-m}(x, \xi)d\xi dx$$

where σ_{-m} is the $(-m)$ positively homogeneous part of the symbol of A, see (1). In particular, res_W does not depend on the choice of Q, in spite of what (2) may suggest.

Proposition 9. *Let us fix a weight Q.*

- *Given two classical pseudo-differential operators A and B,*

$$\mathrm{tr}^Q[A, B] = -\frac{1}{q}\mathrm{res}(A[B, \log Q]). \tag{3}$$

- *Let us consider a family A_t of classical pseudo-differential operators of constant order, and a family Q_t of weights of constant order q, both of which are differentiable with respect to the Kontsevich and Vishik Fréchet structure on $Cl(M, E)$. Then,*

$$\frac{d}{dt}\left(tr^{Q_t} A_t\right) = tr^{Q_t}\left(\frac{d}{dt}A_t\right) - \frac{1}{q}\mathrm{res}\left(A_t(\frac{d}{dt}\log Q_t)\right). \tag{4}$$

- *If C is a classical elliptic injective operator or a diffeomorphism, and A is a classical pseudodifferential operator, $tr^{C^{-1}QC}\left(C^{-1}AC\right)$ is well-defined and equals $\mathrm{tr}^Q A$.*
- *Finally,*

$$\mathrm{tr}^Q A = \overline{\mathrm{tr}^Q A^*}.$$

In this proposition we have followed [6], and [20] for the third point.

We have stated that tr^Q is not a true trace; however, the renormalized trace of the bracket satisfies some interesting properties which we state following [19].

Definition 10. *Let E be a vector bundle over M let Q a weight and let $a \in \mathbb{Z}$. We define :*

$$\mathcal{A}_a^Q = \{B \in Cl(M,E) : [B, \log Q] \in Cl^a(M,E)\}.$$

Theorem 11.
(i) $\mathcal{A}_a^Q \cap Cl^0(M,E)$ is an subalgebra of $Cl(M,E)$ with unit.
(ii) Let $B \in Ell^*(M,E)$, $B^{-1}\mathcal{A}_a^Q B = \mathcal{A}_a^{B^{-1}QB}$.
(iii) Let $A \in Cl^b(M,E)$, and $B \in \mathcal{A}_{-b-2}^Q$, then $\mathrm{tr}^Q[A,B] = 0$. As a consequence,

$$\forall (A,B) \in Cl^{-\infty}(M,E) \times Cl(S^1,V), \quad \mathrm{tr}^Q[A,B] = 0 .$$

We are ready to state the properties of tr^Q that make odd-class pseudodifferential operators an interesting arena for infinite-dimensional mechanics.

Theorem 12. *Let $A, B \in Cl(M,E)$ and let Q be an odd-class weight of even order, e.g. $Q = \Delta$.*

- *If $(A,B) \in Cl_{odd}(M,E) \times Cl_{odd}(M,E)$, and if M is odd dimensional,*

$$\mathrm{tr}^Q([A,B]) = 0 .$$

- *If $(A,B) \in Cl_{even}(M,E) \times Cl_{odd}(M,E)$, and if M is even dimensional dimensional,*

$$\mathrm{tr}^Q([A,B]) = 0 .$$

Proof. The first item is due to Kontsevich and Vishik, see [17, 18]. We sketch a proof of this theorem following [28]:

If Q and B are odd class, with Q of even order, as in the statement of the theorem, $[B, \log Q] \in Cl_{odd}(M,E)$. Thus,

- If $A \in Cl_{odd}(M,E)$, then $A[B, \log Q] \in Cl_{odd}(M,E)$.
- If $A \in Cl_{even}(M,E)$, then $A[B, \log Q] \in Cl_{even}(M,E)$.

Symmetry properties show that in both cases

$$\int_{|\xi|=1} \sigma_{-m}\left(A[B, \log Q]\right) = 0 ,$$

and the result follows by applying the local formula for the Wodzicki residue. □

Corollary 13. *Let $Q = f(\Delta)$ in which f is any analytic function such that Q is a weight, and assume that A, B and C are classical pseudodifferential operators either in the odd-class or in the even-class. If the product ABC is odd class and M is odd dimensional, or if the product ABC is even class and M is even dimensional, then*

$$tr^Q(ABC) = tr^Q(CAB) = tr^Q(BCA) .$$

3. Renormalized traces determine non-degenerate pairings

In this short but crucial section we give an extension of a result from [23, Section 3.2] which connects the foregoing discussion with Hermitian geometry.

Theorem 14. *We consider a weight Q and a fixed classical pseudodifferential operator $Q_0 \in Cl(M,E)$.*

(1) *The sesquilinear map*

$$(.,.)_{Q,Q_0} : (A,B) \in Cl(M,E) \times Cl(M,E) \mapsto \mathrm{tr}^Q\left(AQ_0B^*\right)$$

is non-degenerate if and only if Q_0 is injective.

(2) *Moreover, if* Q_0 *is self-adjoint, then* $(.,.)_{Q,Q_0}$ *is Hermitian, this is,*

$$(B,A)_{Q,Q_0} = \overline{(A,B)}_{Q,Q_0} .$$

(3) *As a consequence, the Hilbert-Schmidt positive definite Hermitian product*

$$(A,B)_{HS} = \mathrm{tr}\,(AB^*)$$

which determines a positive definite metric on $Cl^{-1-dimM}(M,E)$*, extends to a Hermitian form*

$$(\cdot\,,\cdot)_\Delta = (\cdot\,,\cdot)_{\Delta,Id}$$

- *which is a non-degenerate form on* $Cl(M,E)$
- *whose real part defines a* $(\mathbb{R}-)$ *bilinear, symmetric non-degenerate form on* $Cl(M,E)$.

The same properties hold true if we replace $Cl(M,E)$ *by* $Cl^0(M,E)$ *in statements* (1), (2), (3).

Proof. (1) First, let us assume that Q_0 is not injective. Let $y \neq 0 \in KerQ_0$ and let $A = p_y$ be L^2 orthogonal projection on the 1-dimensional vector space spanned by y. Then $AQ_0 = 0$, and $\forall B \in Cl(M,V)$ we have $(A,B)_{Q,Q_0} = 0$, so that $(\cdot\,,\cdot)_{Q,Q_0}$ is degenerate.

Let us now assume that Q_0 is injective. Then, $\forall A \neq 0 \in Cl(M,E)$, $AQ_0 \neq 0$. The formula $(A,B)_{Q,Q_0} = \mathrm{tr}^Q\,(AQ_0B^*)$ certainly defines a sesquilinear form, let us prove that it is non-degenerate. Let $A \in Cl(M,E)$, and let $u \in C^\infty(M,E) \cap (ImAQ_0 - \{0\})$. We assume that u is the image of a function x such that $||x||_{L^2} = 1$, and we let p_x be the L^2- orthogonal projection on the $\mathbb{C}-$vector space spanned by x. Finally, we also let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal base with $e_0 = x$. For $\mathfrak{Re}(s) \geq \frac{2ord(A)+2ordQ_0+1+dimM}{q}$, we observe that the operators $AQ_0\,(AQ_0p_x)^*\,Q^{-s}$, $AQ_0\,(AQ_0p_x)^*\,Q^{-s/2}$, $Q^{-s/2}$, $(AQ_0p_x)^*\,Q^{-s/2}$ and $Q^{-s/2}AQ_0$ are Hilbert-Schmidt class. Thus, applying commutation relations of the usual trace of trace-class operators, we obtain the following:

$$\begin{aligned}
\phi(s) = \mathrm{tr}\left(AQ_0\,(AQ_0p_x)^*\,Q^{-s}\right) &= \mathrm{tr}\left(Q^{-s/2}AQ_0\,(AQ_0p_x)^*\,Q^{-s/2}\right)\\
&= \mathrm{tr}\left((AQ_0p_x)^*\,Q^{-s/2}.Q^{-s/2}AQ_0\right)\\
&= \mathrm{tr}\left((AQ_0p_x)^*\,Q^{-s}AQ_0\right) .
\end{aligned}$$

We will simplify this expression in order to show that the meromorphic continuation of $\phi(s)$ has no poles and a non-zero value at $s = 0$. The meromorphic continuation to $\mathbb{C}$ of $s \mapsto \mathrm{tr}\left((AQ_0p_x)^*\,Q^{-s}AQ_0\right)$ exists and it coincides with the meromorphic continuation of $s \mapsto \phi(s)$; in particular they coincide at $s = 0$. Moreover,

$$\begin{aligned}
\mathrm{tr}\left((AQ_0p_x)^*\,Q^{-s}AQ_0\right) &= \sum_{k\in\mathbb{N}}((AQ_0p_x)^*\,Q^{-s}AQ_0e_k, e_k)_{L^2}\\
&= \sum_{k\in\mathbb{N}}(Q^{-s}AQ_0e_k, AQ_0p_xe_k)_{L^2}\\
&= (Q^{-s}AQ_0x, AQ_0x)_{L^2}\\
&= (Q^{-s/2}u, Q^{-s/2}u)_{L^2} .
\end{aligned}$$

Since $\lim_{s\to 0} Q^{-s/2} = Id$ for weak convergence, the limit of the last term is $||u||^2_{L^2} \neq 0$. The operator AQ_0p_x is a smoothing (rank 1) operator and hence it belongs to $Cl(M,E)$, which ends the proof.

(2) Let $(A,B) \in Cl(M,E)$. We calculate directly using Proposition 9:

$$(B,A)_{Q,Q_0} = \mathrm{tr}^Q(BQ_0A^*) = \mathrm{tr}^Q\left((AQ_0B^*)^*\right) = \overline{\mathrm{tr}^Q\left(AQ_0B^*\right)} = \overline{(A,B)}_{Q,Q_0} \; .$$

(3) It follows from the two previous items.

We finish the proof by remarking that our foregoing arguments hold true when considering only bounded classical pseudodfferential operators. □

Remark 15. *We remark that if Q_0 is self-adjoint and injective, the polarization identity*

$$\mathfrak{Re}(A,B)_{Q,Q_0} = \frac{1}{4}\left[(A+B,A+B)_{Q,Q_0} - (A-B,A-B)_{Q,Q_0}\right]$$

implies that $\mathfrak{Re}(.,.)_{Q,Q_0}$ is a symmetric and non-degenerate real-valued bilinear form. This bilinear form is not positive- definite, see a direct calculation for $M = S^1$ in [23, Section 3].

4. Rigid body equations

In this and the next section we work with the regular Fréchet Lie group $Cl^{0,*}_{odd}(M,E)$ and its Lie algebra $Cl^0_{odd}(M,E)$. Our main claim is that this Lie group is a non-trivial differential geometric framework on which we can pose equations of mechanics in the spirit of Arnold, see [2]. Our main references for this section are [13, 14, 21] and [29].

We remark that in Subsection 4.2 (and also in Section 5) we consider pseudo-Riemannian metrics on $Cl^{0,*}_{odd}(M,E)$ induced by twisting the non-degenerate bilinear forms constructed in Section 3. Our metrics are defined using *right* translations, see e.g. [14]. This convention forces us to re-define the ad_X morphism on $Cl^0_{odd}(M,E)$ as $ad_X Y = -[X,Y]$.

4.1. **The Hamiltonian construction.**

We consider the trace tr^Δ on the regular Lie algebra $Cl^0_{odd}(M,E)$ and the pairing

$$\langle A,B\rangle = (A,B)_{\Delta,Q_0} = tr^\Delta(A\,Q_0\,B^*)\;,$$

in which Q_0 is injective (and hence the pairing is non-degenerate) and self-adjoint (and hence the pairing is Hermitian), and we also consider its real part $\mathfrak{Re}\,\langle A,B\rangle$. The next lemma is a direct consequence of Theorem 14:

Lemma 16. *Let us assume that Q_0 is an injective and self-adjoint classical pseudodifferential operator.*

1. *The $\mathbb{C}$-valued pairing $\langle A,B\rangle = tr^\Delta(A\,Q_0\,B^*)$ on $Cl^0_{odd}(M,E)$ is Hermitian and non-degenerate.*
2. *The real-valued pairing $\mathfrak{Re}\,\langle A,B\rangle$ is bilinear, symmetric and non-degenerate for any choice of self-adjoint and injective operator $Q_0 \in Cl(M,E)$.*

This lemma allows us to consider the *regular dual space* of $Cl^0_{odd}(M,E)$, namely,

$$Cl^0_{odd}(M,E)' = \left\{\mu \in L(Cl^0_{odd}(M,E),\mathbb{C}) : \mu = \langle A,\cdot\rangle \text{ for some } A \in Cl^0_{odd}(M,E)\right\}\;.$$

We can equip $Cl^0_{odd}(M,E)'$ with a Fréchet structure simply by transferring the structure of $Cl^0_{odd}(M,E)$, since there is a bijection between $Cl^0_{odd}(M,E)'$ and $Cl^0_{odd}(M,E)$.

We consider smooth polynomial functions on $Cl^0_{odd}(M,E)'$ of the form

$$f(\mu) = \sum_{k=0}^{n} a_k \, tr^{\Delta}(P^k) \ , \tag{5}$$

in which $a_k \in \mathbb{C}$ and P is determined by the equation $\mu = \langle P, \cdot \rangle$.

If f is such a smooth function on $Cl^0_{odd}(M,E)'$, we define the functional derivative of f, $\delta f/\delta\mu \in Cl^0_{odd}(M,E)$, via the equation

$$\left\langle \nu \, , \frac{\delta f}{\delta \mu} \right\rangle = (d\,f)_\mu(\nu) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} f(\mu + \epsilon\nu) \ ,$$

and, motivated by [1, 3], we equip $Cl^0_{odd}(M,E)'$ with a Poisson bracket (which acts on polynomial functions) as follows, see [9, 21]:

For smooth functions $f, g : Cl^0_{odd}(M,E)' \to \mathbb{C}$ and $\mu \in Cl^0_{odd}(M,E)'$, we set

$$\{f \, , \, g\}(\mu) = + \left\langle \mu \ , \ \left[\frac{\delta\, f}{\delta\, \mu} \, , \, \frac{\delta\, g}{\delta\, \mu} \right] \right\rangle \ . \tag{6}$$

The plus sign is due to our right translation convention, see [14, Remark 9.12]. Next, let us fix a smooth function $H : Cl^0_{odd}(M,E)' \to \mathbb{C}$. The bracket (6) determines a derivation X_H on functions $f : Cl^0_{odd}(M,E)' \to \mathbb{C}$ via

$$X_H(\mu) \cdot f = \{f, H\}(\mu) \tag{7}$$

for all $\mu \in Cl^0_{odd}(M,E)'$, and we can pose Hamilton's equations

$$\frac{d}{dt}(f \circ \mu) = X_H(\mu) \cdot f \tag{8}$$

on $Cl^0_{odd}(M,E)'$. For $\mu(t) = \langle P(t), \cdot \rangle \in Cl^0_{odd}(M,E)'$ they become

$$\left\langle \frac{d\mu}{dt}, \frac{\delta\, f}{\delta\, \mu} \right\rangle = + \left\langle \mu \ , \ \left[\frac{\delta\, f}{\delta\, \mu} \, , \, \frac{\delta\, H}{\delta\, \mu} \right] \right\rangle \ ,$$

this is,

$$\left\langle \frac{dP}{dt}, Q \right\rangle = + \left\langle P \ , \ \left[Q \, , \, \frac{\delta\, H}{\delta\, \mu} \right] \right\rangle \tag{9}$$

for $Q = \frac{\delta\, f}{\delta\, \mu} \in Cl^0_{odd}(M,E)$.

This is a "weak version" of the Euler equation appearing in Berezin and Perelomov [3]. In our Hermitian context we do not have infinitesimal Ad-invariance, and so we obtain (9) instead of a standard equation such as

$$\frac{d\,P}{dt} = \left[P \, , \frac{\delta\, H}{\delta\, \mu} \right] \ ,$$

see [3, Equation (8)]. For example, if we take $\mu = \langle P, \cdot \rangle$ and $H_k(\mu) = tr^{\Delta}\left(P^k\right)$, $k = 1, 2, 3, \cdots$, we can easily check that (assuming existence of Q_0^{-1})

$$\frac{\delta H_k}{\delta \mu} = k Q_0^{-1} (P^*)^{k-1} \ ,$$

and Equation (9) on $Cl^0_{odd}(M,E)$ become

$$\left\langle \frac{dP}{dt}, Q \right\rangle = k \left\langle P \ , \ \left[Q \, , \, Q_0^{-1} (P^*)^{k-1} \right] \right\rangle \ . \tag{10}$$

Remark 17. *The presence of the operator Q_0^{-1} requires us to be careful. An injective self-adjoint pseudodiferential operator Q_0 has:*

(1) *an inverse Q_0^{-1} which is itself an injective self adjoint pseudodifferential operator if and only if Q_0 is not smoothing*
(2) *an inverse $Q_0^{-1} \notin Cl(M,E)$ if and only if Q_0 is smoothing.*

The second case is the one which needs more attention. Indeed, if Q_0 is smoothing, e.g. $Q_0 = e^{-\Delta}$, then its formal symbol vanishes. This explains why its inverse cannot be a pseudodifferential operator. However, Q_0 is a self-adjoint injective compact operator. Hence, via spectral analysis, it is easy to define Q_0^{-1} which is an unbounded operator with L^2-dense domain in $C^\infty(M,E)$. Therefore, Equation (10) is always well-stated.

The foregoing equations are equations on the (regular) dual of the Lie algebra $Cl^0_{odd}(M,E)$. We can work directly on the Lie algebra $Cl^0_{odd}(M,E)$ and we can use more general pairings if we proceed as follows.

We assume that there exists an operator $\mathbb{A} : Cl^0_{odd}(M,E) \to Cl^0_{odd}(M,E)$ such that the new pairing

$$\langle X, Y \rangle_{\mathbb{A}} = \langle X, \mathbb{A}(Y) \rangle$$

is Hermitian and non-degenerate. We think of $\mathbb{A}$ as a twist of our previous pairing or, motivated by [2, 14, 16], see also [13], as an "inertia operator". We also consider the real part of $\langle \cdot \, , \, \cdot \rangle_{\mathbb{A}}$,

$$\mathfrak{Re} \, \langle X, Y \rangle_{\mathbb{A}} = \mathfrak{Re} \, \langle X, \mathbb{A}(Y) \rangle$$

for $X, Y \in Cl^0_{odd}(M,E)$. Since $\langle \cdot \, , \, \cdot \rangle_{\mathbb{A}}$ is Hermitian and non-degenerate, this new pairing is a symmetric and non-degenerate real-valued bilinear form which allows us to consider (in view of Remark 15) pseudo-Riemannian geometry. In order to do so, we define a new adjoint map motivated by Arnold, see [2], as

$$\mathfrak{Re} \, \langle [X,Y], Z \rangle_{\mathbb{A}} = -\mathfrak{Re} \, \langle Y, ad_{\mathbb{A}}(X) \, Z \rangle_{\mathbb{A}} = -\mathfrak{Re} \, \langle ad_{\mathbb{A}}(X) \, Z, Y \rangle_{\mathbb{A}} \ , \tag{11}$$

so that $ad_{\mathbb{A}}(X)$ is the adjoint of ad_X in accordance with our sign convention, see also [13, Section 2]. We compute $ad_{\mathbb{A}}$ explicitly as follows:

$$\begin{aligned}
-\mathfrak{Re} \langle Y, ad_{\mathbb{A}}(X) \, Z \rangle_{\mathbb{A}} &= -\mathfrak{Re} \, \langle ad_X Y, Z \rangle_{\mathbb{A}} = \mathfrak{Re} \, \langle [X,Y], \mathbb{A}(Z) \rangle \\
&= \mathfrak{Re} \, tr^\Delta ([X,Y] \, Q_0 \, \mathbb{A}(Z)^*) \\
&= \mathfrak{Re} \, tr^\Delta \left(XYQ_0 \mathbb{A}(Z)^* - YXQ_0 \, \mathbb{A}(Z)^* \right) \\
&= \mathfrak{Re} \, tr^\Delta \left(YQ_0 \mathbb{A}(Z)^* X - YXQ_0 \, \mathbb{A}(Z)^* \right) \\
&= \mathfrak{Re} \, tr^\Delta \left(Y (Q_0 \, \mathbb{A}(Z)^* X - XQ_0 \, \mathbb{A}(Z)^*) \right) \\
&= \mathfrak{Re} \, tr^\Delta \left(Y \, [Q_0 \, \mathbb{A}(Z)^*, X] \right) \\
&= \mathfrak{Re} \, tr^\Delta \left(YQ_0 Q_0^{-1} \, [Q_0 \, \mathbb{A}(Z)^*, X] \right) \ .
\end{aligned}$$

We set

$$-\mathbb{A}(R)^* = Q_0^{-1} \, [Q_0 \, \mathbb{A}(Z)^*, X] \ . \tag{12}$$

Then, $-\mathfrak{Re}\langle Y, ad_{\mathbb{A}}(X)\, Z\rangle_{\mathbb{A}} = -\mathfrak{Re}\, tr^{\Delta}\,(Y\, Q_0\, \mathbb{A}(R)^*) = -\mathfrak{Re}\, \langle Y, R\rangle_{\mathbb{A}}$, and therefore $ad_{\mathbb{A}}(X)Z = R$. We compute R quite easily. Equation (12) implies

$$\mathbb{A}(R) = [X, Q_0\, \mathbb{A}(Z)^*]^*\, Q_0^{-1} = [\mathbb{A}(Z)\, Q_0, X^*]\, Q_0^{-1} \ ,$$

and so we conclude that

$$ad_{\mathbb{A}}(X)\, Z = \mathbb{A}^{-1}\left([\mathbb{A}(Z)\, Q_0, X^*]\, Q_0^{-1}\right) = -\mathbb{A}^{-1}\left([ad_{\mathbb{A}(Z)Q_0} X^*] Q_0^{-1}\right) \ . \tag{13}$$

4.2. **Euler-Lagrange equations.** Now we use $ad_{\mathbb{A}}$ and the bilinear form $\mathfrak{Re}\,\langle\cdot,\cdot\rangle_{\mathbb{A}}$ to write down equations of motion on the Lie group $Cl^{0,*}_{odd}(M,E)$. Our equations are Euler-Lagrange equations arising from a natural action functional. We follow, roughly, Taylor's lecture notes [29].

We set $\langle\cdot|\cdot\rangle = \mathfrak{Re}\,\langle\cdot,\cdot\rangle_{\mathbb{A}}$ just to simplify our notation. First of all, we extend the symmetric and non-degenerate bilinear form $\langle\cdot|\cdot\rangle$ to a pseudo-Riemannian metric on $G = Cl^{0,*}_{odd}(M,E)$ via right translation:

$$g(P)(V,W) = \langle T_P R_{P^{-1}} V | T_P R_{P^{-1}} W\rangle \ , \tag{14}$$

in which $P \in G$, $W, V \in T_P G$, and $R_{P^{-1}}$ is right translation. We simplify this expression using the identification $V = (P + \epsilon Q_1)'(0)$ and $W = (P + \epsilon Q_2)'(0)$ for $Q_1, Q_2 \in Lie(G)$; we obtain

$$g(P)(V,W) = \left\langle Q_1 P^{-1} | Q_2 P^{-1}\right\rangle \ .$$

Now we set up the kinetic energy Lagrangian functional on curves in G,

$$I[P(t)] = \int_a^b g(P(t))(\dot{P}(t), \dot{P}(t))dt = \int_a^b \left\langle \dot{P}(t)P(t)^{-1} | \dot{P}(t)P(t)^{-1}\right\rangle dt \ ,$$

in which $\dot{P}(t)$ is now considered an an element of $Lie(G)$ for each t, and we find the corresponding equation for critical points of I. We assume that $t \mapsto P(t)$ is a critical, and we deform this curve slightly via $P(t) \mapsto P(t) + \epsilon\eta(t)Q$, with $\eta(a) = \eta(b) = 0$ and $Q \in Lie(G)$ in such a way that that this deformed curve lies in G equipped with its Fréchet topology (recall that $Cl^{0,*}(M,E)$ is open in $Cl^{0,*}(M,E)$). Because $t \mapsto P(t)$ is critical, we have

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I[P(t) + \epsilon\eta(t)Q] = 0 \ .$$

Hereafter we omit t dependence for clarity. We have:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I[P(t)+\epsilon\eta(t)Q] = \int_a^b \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left\langle (\dot{P} + \epsilon\dot{\eta}Q)(P + \epsilon\eta Q)^{-1} | (\dot{P} + \epsilon\dot{\eta}Q)(P + \epsilon\eta Q)^{-1}\right\rangle dt = 0 \ ,$$

this is,

$$\int_a^b \left\langle \dot{\eta}QP^{-1} - \dot{P}P^{-1}\eta QP^{-1} | \dot{P}P^{-1}\right\rangle dt = \int_a^b \dot{\eta}\left\langle QP^{-1}|\dot{P}P^{-1}\right\rangle dt - \int_a^b \eta\left\langle \dot{P}P^{-1}QP^{-1}|\dot{P}P^{-1}\right\rangle dt = 0 \ .$$

We integrate by parts and use the boundary conditions for η; we obtain

$$-\int_a^b \eta\left\langle QP^{-1}|\dot{P}P^{-1}\right\rangle^{\cdot} dt - \int_a^b \eta\left\langle \dot{P}P^{-1}QP^{-1}|\dot{P}P^{-1}\right\rangle dt = 0 \ ,$$

this is,

$$\int_a^b \eta\left\{\left\langle QP^{-1}\dot{P}P^{-1}|\dot{P}P^{-1}\right\rangle - \left\langle QP^{-1}|(\dot{P}P^{-1})^{\cdot}\right\rangle\right\} dt - \int_a^b \eta\left\langle \dot{P}P^{-1}QP^{-1}|\dot{P}P^{-1}\right\rangle dt = 0 \ .$$

Since $\eta(t)$ is arbitrary, we find the equation of motion

$$\left\langle QP^{-1}\dot{P}P^{-1}|\dot{P}P^{-1}\right\rangle - \left\langle QP^{-1}|(\dot{P}P^{-1})^{\cdot}\right\rangle - \left\langle \dot{P}P^{-1}QP^{-1}|\dot{P}P^{-1}\right\rangle = 0 \tag{15}$$

in which Q is an arbitrary element of $Lie(G)$.

Since $\dot{P}P^{-1}$ and QP^{-1} belong to $Lie(G)$, we can write $\dot{P}P^{-1} = X$ and $QP^{-1} = W$ for $X, W \in Lie(G)$. Equation (15) becomes

$$\langle WX|X\rangle - \left\langle W|\dot{X}\right\rangle - \langle XW|X\rangle = 0$$

for all $W \in Lie(G)$, this is,

$$\langle [W, X]|X\rangle = \left\langle W|\dot{X}\right\rangle \tag{16}$$

for all $W \in Lie(G)$. As pointed out in [29], if we solve for X in (16), the curve $P(t)$ is recovered via $\dot{P}(t) = X(t)P(t)$. Thus, Equation (16) —an equation posed on $Lie(G)$— determines a family of curves on G. It remains to find a "strong" formulation of (16). We go back to the notation used in Subsection 4.1. Equation (16) becomes

$$\mathfrak{Re}\,\langle ad_X W, X\rangle_{\mathbb{A}} = \mathfrak{Re}\left\langle W, \dot{X}\right\rangle_{\mathbb{A}} \ ,$$

and therefore, using the operator $ad_{\mathbb{A}}$ we obtain

$$\mathfrak{Re}\,\langle W, ad_{\mathbb{A}}(X)X\rangle_{\mathbb{A}} = \mathfrak{Re}\left\langle W, \dot{X}\right\rangle_{\mathbb{A}} \ .$$

Non-degeneracy of the inner product $\mathfrak{Re}\,\langle\cdot,\cdot\rangle_{\mathbb{A}}$ implies that $X(t) \in Cl^0_{odd}(M, E)$ satisfies the non-linear equation

$$\frac{d}{dt}X = ad_{\mathbb{A}}(X(t))X(t) \ . \tag{17}$$

We note the formal similarity between (16) and the Hamiltonian equation (9). Due to this fact, we naturally call (16), or (17), the Euler equation on $Cl^0_{odd}(M, E)$. We have proven the following theorem:

Theorem 18. *The Euler equation*

$$\frac{d}{dt}X = \mathbb{A}^{-1}\left([\mathbb{A}(X)\,Q_0, X^*]\,Q_0^{-1}\right) \tag{18}$$

on $Cl^0_{odd}(M, E)$, is the Euler-Lagrange equation of the kinetic energy action functional on the Fréchet Lie group $Cl^{0,*}_{odd}(M, E)$ equipped with the pseudo-Riemannian metric (14)*.*

Equation (17) is formally analogue to the Euler equation posed on a Lie group G equipped with a *Riemannian* metric. In this Riemannian case, our foregoing computations translate *mutatis mutandis* into the well-known fact that Euler equations determine geodesics on G, see for instance [14, 16, 29] and references therein.

Remark 19. *If we take $Q_0 = Id$ and we pose Equation* (18) *on the subgroup of self-adjoint operators, we obtain*

$$\mathbb{A}\left(\frac{d}{dt}X\right) = [\mathbb{A}(X), X] = -ad_{\mathbb{A}(X)}X \ ,$$

an equation that looks exactly as the classical Euler equation in $so(3)$, see [14, Theorem 7.2].

5. Pseudo-Riemannian Geometry on $Cl^{0,*}_{odd}(M,E)$

In this section we review some basics facts of the pseudo-Riemannian geometry of the regular Fréchet group $Cl^{0,*}_{odd}(M,E)$, motivated by Arnold's classical paper [2]. We fix an inertia operator $\mathbb{A}$ and we consider the pseudo-Riemannian metric on $Cl^{0,*}_{odd}(M,E)$ induced by right translation of the non-degenerate and symmetric bilinear form $\mathfrak{Re}\langle \cdot\,,\,\cdot, \rangle_{\mathbb{A}}$, see Equation (14).

We note that there exist some difficulties in describing the whole space of connection 1-forms

$$\Omega^1(Cl^{0,*}_{odd}(M,V), Cl^0_{odd}(M,V))\ .$$

Indeed, to our knowledge, the space of smooth linear maps acting on $Cl^0_{odd}(M,V)$ is actually not well-understood. In the classical setting of a *Riemannian* (e.g. finite dimensional, or Hilbert) Lie group G with Lie algebra $\mathfrak{g}$, the Levi-Civita connection 1-form (i.e metric-compatible and torsion-free) reads as

$$\theta_X Y = \frac{1}{2}\left\{ad_X Y - ad^*_X Y - ad^*_Y X\right\}\ ,$$

in which ad^* is the adjoint of ad with respect to the metric of G and X, Y are *left invariant* vector fields, see [10, Proposition 1.7]. It is possible to go beyond this well-known result, and extend it to (pseudo-)Riemannian right-invariant metrics, if an adjoint for ad is known. Formal calculations have been already carried out, see for example the classical Arnold's paper [2] or [13, Section 2] and references therein, but in the context of pseudodifferential operators, finding a rigorous (*i.e.* truly smooth) adjoint of the adjoint map, as described in [23], remains a difficult task. We can bypass this difficulty here, since we already have $ad_{\mathbb{A}}$ at our disposal.

Theorem 20. *Let $(X,Y) \in Cl^0_{odd}(M,E)^2$. We define, using right invariance, the connection 1-form*

$$\theta_X Y = \frac{1}{2}\left\{-ad_X Y + ad_{\mathbb{A}}(X)Y + ad_{\mathbb{A}}(Y)X\right\}\ .$$

Then we have that:

(a) $\forall (X,Y) \in Cl^0_{odd}(M,E)^2$, $\theta_X Y - \theta_Y X = -ad_X Y$ *(Torsion-free)*
(b) $\forall (X,Y,Z) \in Cl^0_{odd}(M,E)^3$, $\mathfrak{Re}\langle \theta_X Y, Z\rangle_{\mathbb{A}} = -\mathfrak{Re}\langle Y, \theta_X Z\rangle_{\mathbb{A}}$ *(Pseudo-Riemannian metric compatibility)*

Moreover, $\theta : (X,Y) \mapsto \theta_X Y$ is the only bilinear map which satisfies these two properties.

Proof. As in the previous section, in this proof we set $\langle \cdot|\cdot\rangle = \mathfrak{Re}\langle \cdot,\cdot\rangle_{\mathbb{A}}$ for ease of notation.

We first check that θ satisfies (a) and (b). By direct computation, we have:

$$\begin{aligned}
\theta_X Y - \theta_Y X &= \frac{1}{2}\left\{-ad_X Y + ad_{\mathbb{A}}(X)Y + ad_{\mathbb{A}}(Y)X\right\} \\
&\quad -\frac{1}{2}\left\{-ad_Y X + ad_{\mathbb{A}}(Y)X + ad_{\mathbb{A}}(X)Y\right\} \\
&= -ad_X Y,
\end{aligned}$$

which proves (a). We now compute using Equation (11):

$$\begin{aligned}
2\langle \theta_X Y|Z\rangle &= \langle -ad_X Y + ad_{\mathbb{A}}(X)Y + ad_{\mathbb{A}}(Y)X|Z\rangle \\
&= \langle -ad_X Y|Z\rangle + \langle ad_{\mathbb{A}}(X)Y|Z\rangle + \langle ad_{\mathbb{A}}(Y)X|Z\rangle \\
&= -\langle ad_{\mathbb{A}}(X)Z|Y\rangle + \langle ad_X Z|Y\rangle + \langle ad_Y Z|X\rangle \\
&= -\langle ad_{\mathbb{A}}(X)Z|Y\rangle + \langle ad_X Z|Y\rangle - \langle ad_Z Y|X\rangle \\
&= -\langle ad_{\mathbb{A}}(X)Z|Y\rangle + \langle ad_X Z|Y\rangle - \langle ad_{\mathbb{A}}(Z)X|Y\rangle \\
&= -2\langle Y|\theta_X Z\rangle ,
\end{aligned}$$

which proves (b).

Now, let

$$\Theta : (X,Y) \in Cl^0_{odd}(M,E)^2 \mapsto \Theta_X Y \in Cl^0_{odd}(M,E)$$

be a bilinear form satisfying (a) and (b). Then

$$\langle \Theta_X Y|Z\rangle + \langle Y|\Theta_X Z\rangle = 0 \ ,$$

$$\langle \Theta_Z X|Y\rangle + \langle X|\Theta_Z Y\rangle = 0 \ ,$$

$$\langle \Theta_Y Z|X\rangle + \langle Z|\Theta_Y X\rangle = 0 \ .$$

From the third line and (a) we get that

$$\langle Z|\Theta_X Y\rangle = -\langle \Theta_Y Z|X\rangle + \langle Z|[X,Y]\rangle$$

and from the first line we get that

$$\langle \Theta_X Y|Z\rangle = -\langle Y|\Theta_X Z\rangle \ .$$

Combining these two equalities, and exploiting properties (a) and (b), we have:

$$\begin{aligned}
2\langle \Theta_X Y|Z\rangle &= -\langle \Theta_Y Z|X\rangle + \langle Z|[X,Y]\rangle - \langle Y|\Theta_X Z\rangle \\
&= -\langle [Y,Z] + \Theta_Z Y|X\rangle + \langle Z|[X,Y]\rangle - \langle Y|[X,Z] + \Theta_Z X\rangle \\
&= -\langle [Y,Z]|X\rangle + \langle Z|[X,Y]\rangle - \langle Y|[X,Z]\rangle \\
&= 2\langle \theta_X Y|Z\rangle .
\end{aligned}$$

Since $\langle \cdot|\cdot\rangle$ is non-degenerate, this equality ends the proof. □

It is important for us to highlight the fact that the proof of Theorem 20 goes through because we can use the smooth adjoint $ad_{\mathbb{A}}$. Now, using $\theta_X Y$ we can define the curvature operator and sectional curvature of the Lie group $Cl^{0,*}_{odd}(M,E)$ as follows:

The curvature operator for the connection θ is given, at the identity of $Cl^{0,*}_{odd}(M,E)$, by

$$\mathrm{R}_{\mathbb{A}}(X,Y) = [\ \theta_X\ ,\ \theta_Y\ ] - \theta_{[X,Y]}$$

for every X and Y in $Cl^0_{odd}(M,E)$, see also [10, Equation (1.10)]. Hence, the sectional curvature associated to the biplane generated by X and Y is

$$K_{\mathbb{A}}(X,Y) = -\frac{\langle \mathrm{R}_{\mathbb{A}}(X,Y)X \mid Y\rangle}{|X \wedge Y|^2_{\mathbb{A}}} \tag{19}$$

whenever the area of the parallelogram spanned by X, Y, $|X\wedge Y|_{\mathbb{A}}$, is different from zero.

These constructions yield Theorem 5 of Arnold's [2]. Using our foregoing notation this theorem reads as follows, see [13, Proposition 2.1]:

Theorem 21. *Let $\mathbb{A}$ be an inertia operator and set $\mathfrak{N}(X,Y) = \frac{1}{2}\,(ad_{\mathbb{A}}(X)Y + ad_{\mathbb{A}}(Y)X)$. Given X and Y in $Cl^0_{odd}(M,E)$ we have the identity*

$$\mid X \wedge Y \mid^2_A K_{\mathbb{A}}(X,Y) =$$
$$-\frac{3}{4}\langle\, [X,Y] \,|\, [X,Y]\, \rangle + \frac{1}{2}\langle\, [X,Y] \mid ad_{\mathbb{A}}(X)Y - ad_{\mathbb{A}}(Y)X\, \rangle$$
$$+ \langle\, \mathfrak{N}(X,Y) \,|\, \mathfrak{N}(X,Y)\, \rangle - \langle\, \mathfrak{N}(X,X) \mid \mathfrak{N}(Y,Y)\, \rangle \ .$$

We remark once again that this theorem is a rigorous statement on the sectional curvature of the Fréchet Lie group $Cl^{0,*}_{odd}(M,E)$, not a formal result as [13, Proposition 2.1]. We finish this section computing geodesics:

Let us set $G = Cl^{0,*}_{odd}(M,E)$ and $Lie(G) = Cl^0_{odd}(M,E)$. We recall that a *spray* over G is a vector field $S : TG \to TTG$ satisfying $T\pi_G \circ S = Id_{TG}$, in which $\pi_G : TG \to G$ is the canonical projection, and the homogeneity condition

$$(T\mu_t)\cdot S(v) = \frac{1}{t}S(tv)$$

for $t \neq 0$, in which $\mu_t : TG \to TG$ is the smooth function $\mu_t(v) = t\,v$. In the present case we use $TG = G \times Lie(G)$ and $TTG = G \times Lie(G) \times Lie(G) \times Lie(G)$. Then

$$S : TG \to TTG\,, \qquad S(g,X) = (g,X,X,ad_{\mathbb{A}}(X)X) = (g,X,X,\theta_X X)$$

for $g \in G$ and $X \in Lie(G)$, is a spray on G, as it can be easily checked (see [24, Section 1.21]). In actual fact, it can be proven that the spray S is precisely the metric spray corresponding to our right-invariant metric (14), see [7, Section 6.2]. The integral curves of S are the *geodesics* corresponding to the spray S. We obtain that $(g(t),X(t))$ is a geodesic for the spray S if and only if

$$\begin{aligned} \frac{dg}{dt} &= X \\ \frac{dX}{dt} &= ad_{\mathbb{A}}(X)X\ . \end{aligned}$$

The second equation is exactly the Euler-Lagrange equation (17).

6. Examples on the n-dimensional torus

In this section, we specialize to $M = \mathbb{T}_n = (S^1)^n$ equipped with its product metric, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We recall that the Laplace operator is

$$\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\ .$$

6.1. **When Q_0 is a heat operator.** We set $\mathbb{A} = Id$. Let $s \in \mathbb{R}^*_+$. We define $Q_0 = e^{-s\Delta}$. This is an injective smoothing operator, hence it is both odd and even class. We apply our previous computations to $\mathbb{T}_n$ for any $n \in \mathbb{N}^*$ and we obtain

$$\langle A,B\rangle = tr^{\Delta}\left(Ae^{-s\Delta}B\right) = tr\left(Ae^{-s\Delta}B\right)\ .$$

Now we need to define formally $Q_0^{-1} = e^{s\Delta}$ which is not a pseudodifferential operator but it can be rigorously defined, as we discussed in Remark 17.

We obtain the formulas:

$$ad_{\mathbb{A}}(X)Z = [Ze^{-s\Delta}, X^*]e^{s\Delta}$$

and

$$\theta_X Y = \frac{1}{2}\left\{[X,Y]+[Ye^{-s\Delta},X^*]e^{s\Delta}+[Xe^{-s\Delta},Y^*]e^{s\Delta}\right\} ,$$

and the geodesic equation (17) reads

$$\frac{dX}{dt} = [Xe^{-s\Delta},X^*]e^{s\Delta} .$$

Let us now test these three equations taking $X \in C^\infty(\mathbb{T}_n,\mathbb{C})$ and expanding it with respect to the Fourier basis. First, for $n=1$, i.e. for $\mathbb{T}_n = S^1$. Let $(l,m,p)\in\mathbb{Z}^3$. We obtain

$$\begin{aligned}\left(ad_A(z^l)z^m\right) z^p &= z^m e^{-s\Delta} z^{-l} e^{s\Delta} z^p - z^{-l+m+p}\\ &= \left(e^{s\left(p^2-(p-l)^2\right)}-1\right) z^{-l+m+p}\\ &= \left(e^{s\left(2pl-l^2\right)}-1\right) z^{-l+m+p} ,\end{aligned}$$

and

$$(\theta_{z^l} z^m)\, z^p = \frac{-1}{2}\left\{\left(-e^{s\left(2pl-l^2\right)}+1\right) z^{-l+m+p} + \left(-e^{s\left(2pm-m^2\right)}+1\right) z^{l-m+p}\right\} .$$

The same kind of relations can be implemented for $n>1$, by considering tensor products.

6.2. **When Q_0 is a power of the Laplacian.** We set $\mathbb{A} = Id$. We now investigate $Q_0 = (\Delta+\pi)^{(n+1)/2}$, where π is the L^2- orthogonal projection on the kernel of the Laplacian. We remark that Q_0 is injective and self-adjoint of order $n+1$. Moreover, if n is odd, then Q_0 is odd class, and if n is even, then Q_0 is even class. In this class of examples, we get an operator Q_0^{-1} which is a pseudo-differential operator of order $-n-1$, in the same class as Q_0. Hence the following formulas are fully valid in $Cl^0_{odd}(\mathbb{T}_n,\mathbb{C})$:

$$ad_A(X)Z = [ZQ_0,X^*]Q_0^{-1} \in Cl^{-1}_{odd}(\mathbb{T}_n,\mathbb{C})$$

$$\theta_X Y = \frac{1}{2}\left\{[X,Y]-[YQ_0,X^*]Q_0^{-1}-[XQ_0,Y^*]Q_0^{-1}\right\} .$$

and the geodesic equation reads

$$\frac{dX}{dt} = [XQ_0,X^*]Q_0^{-1} = XQ_0X^*Q_0^{-1} - XX^* ,$$

where the right-hand side is an operator of order -1 (and hence compact). Let $M = S^1$ and let us restrict ourselves to $X \in C^\infty(S^1,\mathbb{C})$. Then, for $p\in\mathbb{Z}$ we obtain

$$\frac{dX}{dt}(z^p) = X(\Delta+\pi)X^*(\Delta+\pi)^{-1}(z^p) - XX^*z^p.$$

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