On Modal Logics Characterized by Models with Relative Accessibility Relations: Part I

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Abstract

This work is divided in two papers (Part I and Part II). In Part I, we study a class of polymodal logics (called herein the class of "Rare-logics") for which the set of terms indexing the modal operators are hierarchized in two levels: the set of Boolean terms and the set of terms built upon the set of Boolean terms. By investigating different algebraic properties satisfied by the models of the Rare-logics, reductions for decidability are established by faithfully translating the Rare-logics into more standard modal logics. The main idea of the translation consists in eliminating the Boolean terms by taking advantage of the components construction and in using various properties of the classes of semilattices involved in the semantics. The novelty of our approach allows us to prove new decidability results (presented in Part II), in particular for information logics derived from rough set theory and we open new perspectives to define proof systems for such logics (presented also in Part II).

Key-words: polymodal logic, relative accessibility relation, translation.

1 Introduction

Background. During the last decade, the information logics derived from Pawlak's *information systems* [Paw81] have been the object of active research (see for example [Orło84, OP84, Vak91c]). An *information system* can be seen as a structure $\langle OB, AT \rangle$ such that

- OB is a non-empty set of *objects*;
- AT is a non-empty set of *attributes*;
- each attribute $at \in AT$, is a mapping $at : OB \rightarrow \mathcal{P}(Val_{at})$ where Val_{at} is a non-empty set of *values*.

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For each object o and for each attribute at, $at(o)$ is the set of *possible* values of o with respect to the attribute at. Usually, $at(o)$ is non-empty. Such structures are relevant to capture some aspects of incomplete information. In that setting, various relations between objects can be found in the literature. We recall some of them below. For any $o_1, o_2 \in OB$, $A \subseteq AT$,

- $o_1 ind(A) o_2 \stackrel{\text{def}}{\Leftrightarrow}$ for any $at \in A$, $at(o_1) = at(o_2)$ (see e.g. [Orło84, Rau84, Kon87, DO97]);
- $o_1 sim(A) o_2 \stackrel{\text{def}}{\Leftrightarrow}$ for any $at \in A$, $at(o_1) \cap at(o_2) \neq \emptyset$ (see e.g. [Vak91b, Vak91a, Kon97a, Vak97]);
- $o_1 fin(A) o_2 \stackrel{\text{def}}{\Leftrightarrow}$ for any $at \in A$, $at(o_1) \subseteq at(o_2)$ (see e.g. [Orło97]).

$o_1 ind(A) o_2$ can be read as follows: the objects o_1 and o_2 cannot be distinguished modulo the set of attributes A. Similarly, $o_1 sim(A) o_2$ iff o_1 and o_2 are similar modulo A. These relations are said to be *strong* because the quantification used in the definition is universal (see e.g. [Orło97]). The relations $wind(A), wsim(A), wfin(A)$ are defined by using the existential quantification (*weak* relations) instead of the universal quantification. Observe that for each $r \in \{ind, sim, fin\}$ and for $A, B \subseteq AT$,

(S) $r(A \cup B) = r(A) \cap r(B)$ and $r(\emptyset) = OB \times OB$;

(W) $wr(A \cup B) = wr(A) \cup wr(B)$ and $wr(\emptyset) = \emptyset$.

The polymodal logics obtained from the information systems are multimodal logics such that the relations in the Kripke-style semantical structures correspond to relations between objects in the underlying information systems. Hence, the relations are interdependent; for instance, if $B \subseteq A \subseteq AT$, then $ind(A) \subseteq ind(B)$. Moreover, $ind(A)$ [resp. $sim(A)$] is an equivalence relation [resp. a reflexive and symmetric relation].

The paper consists in studying the class of *Rare-logics* that contains different information logics from the literature. Some comments are in order to recall the pecularities of such logics. First, recall that numerous propositional modal logics can be defined from semantical structures of the form $\mathcal{M} = \langle W, (R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V \rangle$. $\mathcal{M}$ is a Kripke-style structure where W is a non-empty set, $(R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}$ is a family of binary relations over W, $\mathtt{M}_0$ is the set of modal constants (under which is built the set of modal expressions, say $\mathtt{M}$) and V is a *meaning function*. V is usually towfold: for each propositional variable $\mathtt{p}$, $V(\mathtt{p}) \subseteq W$ and for any $\mathtt{a} \in \mathtt{M}$, $V(\mathtt{a}) \in \mathcal{P}(W^2)$ with for any $\mathtt{c} \in \mathtt{M}_0$, $V(\mathtt{c}) = R_{\mathtt{c}}$. When $\mathtt{M}$ is a set of terms, there may be additional conditions to be satisfied, for instance

$$V(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n)) \stackrel{\text{def}}{=} f_\oplus(V(\mathtt{a}_1), \ldots, V(\mathtt{a}_n))$$

where $\oplus$ is a n-ary operator under which $\mathtt{M}$ is closed and $f_\oplus$ is a mapping $f_\oplus : \mathcal{P}(W^2)^n \to \mathcal{P}(W^2)$ interpreting the syntactic operator $\oplus$. The propositional dynamic logic PDL (see e.g. [Har84]) is one of the most known logics of this type. In the sequel, such logics are called *standard modal logics*.

By adding a Boolean dimension[1] to the standard modal logics we obtain a class of logics that includes numerous information logics described above. In

[1] This is different from the sense understood in [GV98].

order to define the logics of such a class, at the syntactical level, we replace $\mathtt{M}_0$ by a set of expressions of the form $r(\mathtt{A})$ with $\mathtt{A} \in \mathtt{P}$ where the set $\mathtt{P}$ is built upon a set $\mathtt{P}_0 = \{\mathtt{C}, \mathtt{D}, \ldots\}$ of *parameter constants* and is closed under the syntactic operators $\cap, \cup, -$. $\mathtt{A}$ is a Boolean term and is called a *parameter expression*. The symbol 'r' is an arbitrary symbol announcing a Boolean expression $\mathtt{A}$ in a modal expression. For instance, a modal expression $(\mathtt{c}_1 \circ \mathtt{c}_2) \cup \mathtt{c}_3$ can generate the modal expression $(r(\mathtt{C} \cap -\mathtt{D}) \circ r(\mathtt{D} \cup -\mathtt{D})) \cup r(\mathtt{C} \cap -\mathtt{D} \cap -\mathtt{C})$ for some Rare-logic. The semantical structures of such logics are of the form $\langle W, PAR, (R_P)_{P \subseteq PAR}, V\rangle$ where PAR is a non-empty set of *parameters*, V homomorphically maps every Boolean expression $\mathtt{A}$ to a subset of PAR and $V(r(\mathtt{A})) = R_{V(\mathtt{A})}$. By providing a richer structure to $\mathcal{P}(PAR)$ and $\mathcal{P}(W^2)$, the family $(R_P)_{P \subseteq PAR}$ can be seen as a mapping

$$R : \langle \mathcal{P}(PAR), \cup, \cap, -, \emptyset, PAR\rangle \rightarrow \langle \mathcal{P}(W^2), \cup, \cap, -, \emptyset, W^2\rangle$$

According to the algebraic properties of R, different *types* of logics are defined (see e.g. [Orło95]). For instance, we may require that R is a homomorphism of Boolean algebras but weaker conditions are also possible and relevant. For instance, R can be seen as a mapping from the lattice $\langle \mathcal{P}(PAR), \subseteq\rangle$ into the meet-semilattice $\langle \{R(P) : P \subseteq PAR\}, \subseteq\rangle$ with top element W^2 such that $R(P \cup P') = R(P) \cap R(P')$ and $R(\emptyset) = W^2$ (see the condition (S) above). As announced previously, logics of such a kind shall be called logics with Relative Accessibility RElations (Rare-logic for short) and as far as we know this term first appeared in [Orło88a]. Numerous logics from the literature can be seen as Rare-logics (see for example [Orło85, Kon87, Orło89, Kon97b]).

Our objectives. Our prime objective is to compare various classes of Rare-logics with their corresponding classes of standard modal logics. The criteria of comparison range from (un)decidability of satisfiability (finite model property, ...) to complete and sound axiomatization (presented in [DG]). By achieving such a comparison, we claim that we provide a framework for studying Rare-logics that allies generality and sheds a new light on the Rare-logics themselves. For instance by defining satisfiability-preserving transformation from Rare-logics into standard modal logics, we shall solve open problems about decidability issues (presented also in [DG]).

Our contribution. The main result of the paper states that there exist satisfiability-preserving transformations between Rare-logics from a large class (including most known Rare-logics from the literature) and corresponding standard modal logics. As a side-effect, the uniform proof of such a result provides the finite model property and the finite parameter set property. In [DG] we show that refinements are also proved to be correct in order to eliminate the universal modal operator for some cases. Furthermore, the extension of the proof technique is performed in order to define satisfiability-preserving transformations when nominals are added for atomic parameters and atomic propositions [DG]. By taking advantage of such translations, we define a general construction that allows to define an axiomatization of Rare-logics from calculi for the corresponding standard modal logics [DG]. Last, but not least, we provide a uniform proof of decidability for various Rare-logics from [Orło84, Orło93, Orło89, Orło88b, Bal96a, Bal97] for which the issue has been open up to now. Indeed, the decidability of the corresponding standard modal logics is used with possibly some adequate adjustments.

Related work. As far as we know, the notion of Rare-logic appeared in the literature in [Orło88a]. Different Rare-logics are described in [Orło88a] to model reasoning in presence of incomplete information. Proof systems for some Rare-logics can be found in [Bal96a, Bal96b, Bal97, Kon97a, Dem99, DG98]. Recently, in [BO99], a classification of logics with relative accessibility relations have been proposed. None of these works tackle the problem of relating Rare-logics with more standard modal logics in a systematic way. Moreover, in these works, decidability issues are not their main concern whereas in the present paper this is a crucial consequence of the faithful translations. In [GP90], the Boolean Modal Logic BML also admits in the language a family of operators $[\alpha]$ where α is a Boolean term interpreted as a *binary relation*. Like BML, a great deal of existing logics (PDL for instance) are determined by classes of multimodal frames such that the respective binary accessibility relations satisfy equations of the form $R_{\mathtt{a}_0} = f(R_{\mathtt{a}_1}, \ldots, R_{\mathtt{a}_n})$ where $\mathtt{a}_1, \ldots, \mathtt{a}_n$ are subexpressions of the modal expression $\mathtt{a}_0$ (possibly equal to $\mathtt{a}_0$) and f is a mapping of binary relations. Unlike these logics, the models of Rare-logics may satisfy the equation $R_{V(\mathtt{A} \cup \mathtt{B})} = R_{V(\mathtt{A})} \cap R_{V(\mathtt{B})}$ but nothing else might be said about $R_{V(-\mathtt{A})}$ or $R_{V(\mathtt{A} \cap \mathtt{B})}$. This is in sharp contrast with BML.

Plan of the paper. The rest of the paper is structured as follows. In Section 2, the class of standard modal logics into which the Rare-logics are translated is defined. In Section 3 the class of Rare-logics is introduced and we propose a rough classification. In Section 4, satisfiability-preserving transformations between Rare-logics and standard modal logics are defined. Section 5 contains concluding remarks.

2 Modal logics

We borrow (or adapt) various definitions from [Ven91]. Let $OP = \{\oplus_1, \ldots, \oplus_s\}$ be a (possibly empty) finite set of *operators*. An *OP-signature* $\mathtt{S}_{OP}$ is a triple $\mathtt{S}_{OP} = \langle OP, \rho^1_{OP}, \rho^2_{OP} \rangle$ such that for $i = 1, 2$, ρ^i_{OP} is a map $OP \rightarrow \omega \setminus \{0\}$. ρ^1_{OP} [resp. ρ^2_{OP}] assigns to each operator a *finite arity* [resp. a *finite modal arity*] -when $OP = \emptyset$, ρ^1_{OP} and ρ^2_{OP} take dummy values. A set $\mathtt{M} = \{\mathtt{a}, \mathtt{b}, \ldots\}$ of *modal expressions* is the smallest set that contains a non-empty countable set of *basic modal expressions* $\mathtt{M}_0$ and it is closed under OP with respect to the arity map ρ^1_{OP}.

A *modal similarity type* $\mathtt{S}$ is a pair $\mathtt{S} = \langle \{\langle \mathtt{a} \rangle : \mathtt{a} \in \mathtt{M}\}, \rho \rangle$ (depending on $\mathtt{S}_{OP}$) with $\rho : \mathtt{M} \rightarrow \omega \setminus \{0\}$ assigning a *finite modal arity* for each modal expression such that $\rho(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n)) = \rho^2_{OP}(\oplus)$ (here $\rho^1_{OP}(\oplus) = n$).

A *modal language* $\mathtt{L}$ is defined as a pair $\langle \mathtt{S}, \mathtt{For}_0 \rangle$ where $\mathtt{S}$ is modal similarity type and $\mathtt{For}_0 = \{\mathtt{p}, \mathtt{q}, \ldots\}$ is a countable set of propositional variables. The formulae $\mathtt{F}$ of $\mathtt{L}$ are inductively defined as follows:

$$\mathtt{F} ::= \mathtt{p} \mid \mathtt{F}_1 \wedge \mathtt{F}_2 \mid \neg \mathtt{F} \mid \langle \mathtt{a} \rangle (\mathtt{F}_1, \ldots, \mathtt{F}_n)$$

for $\mathtt{p} \in \mathtt{For}_0$, $\mathtt{a} \in \mathtt{M}$ with $\rho(\mathtt{a}) = n$. The set of $\mathtt{L}$-formulae is denoted $\mathtt{For}$. Standard abbreviations include $\perp$, $\top$, $[\mathtt{a}]$, $\vee$, $\Rightarrow$, $\Leftrightarrow$. For instance, when

$\rho(\texttt{a}) = n$, $[\texttt{a}](\texttt{F}_1, \ldots, \texttt{F}_n) \stackrel{\text{def}}{=} \neg\langle\texttt{a}\rangle(\neg\texttt{F}_1, \ldots, \neg\texttt{F}_n)$. In the proofs on induction on the structure of formulae, we might use $[\texttt{a}]$ instead of $\langle\texttt{a}\rangle$.

DEFINITION 2.1. As usual, we call $\mathcal{L} \subseteq \texttt{For}$ an L-*normal modal logic* $\stackrel{\text{def}}{\Leftrightarrow}$

- $\mathcal{L}$ is closed under *modus ponens*, *uniform substitution* and *universal generalization*, i.e. if $\texttt{F} \in \mathcal{L}$, then $[\texttt{a}](\texttt{F}_1, \ldots, \texttt{F}_{i-1}, \texttt{F}, \texttt{F}_{i+1}, \ldots, \texttt{F}_n) \in \mathcal{L}$ with $\rho(\texttt{a}) = n$ and $\texttt{F}_1, \texttt{F}_1, \ldots, \texttt{F}_{i-1}, \texttt{F}_{i+1}, \ldots, \texttt{F}_n \in \texttt{For}$;
- $\mathcal{L}$ contains all tautologies from propositional calculus and $[\texttt{a}](\texttt{p}_1, \ldots, \texttt{p}_{i-1}, \texttt{p} \Rightarrow \texttt{p}', \texttt{p}_{i+1}, \ldots, \texttt{p}_n) \Rightarrow ([\texttt{a}](\texttt{p}_1, \ldots, \texttt{p}_{i-1}, \texttt{p}, \texttt{p}_{i+1}, \ldots, \texttt{p}_n) \Rightarrow [\texttt{a}](\texttt{p}_1, \ldots, \texttt{p}_{i-1}, \texttt{p}', \texttt{p}_{i+1}, \ldots, \texttt{p}_n))$ with $\rho(\texttt{a}) = n$ and $\texttt{p}_1, \ldots, \texttt{p}_{i-1}, \texttt{p}_{i+1}, \ldots, \texttt{p}_n, \texttt{p}, \texttt{p}' \in \texttt{For}_0$.

∇

In what follows, we consider L-normal modal logics having Kripke-style semantics. In order to interpret the operators from OP, some preliminary definitions are needed. A *relation operation* ϕ maps any set U to a mapping

$$\phi(U) : \mathcal{P}(U^{i_1}) \times \ldots \times \mathcal{P}(U^{i_n}) \to \mathcal{P}(U^{i_{n+1}})$$

with $i_1, \ldots, i_{n+1} \geq 1$ and $i_1, \ldots, i_{n+1}$ do not depend on U. $\langle i_1, \ldots, i_{n+1}\rangle$ is the *profile* of ϕ and n is its *arity*. Moreover, we require that if there is an 1-1 mapping $f : U \to U'$, then for any $\langle X_1, \ldots X_n\rangle \in \mathcal{P}(U^{i_1}) \times \ldots \times \mathcal{P}(U^{i_n})$, $f(\phi(U)(X_1, \ldots, X_n)) = \phi(U')(f(X_1), \ldots, f(X_n))$. A *set operation* ϕ is defined as a relation operation such that for any set U, $\phi(U)$ has the profile $\langle 1, \ldots, 1\rangle$. By abusing our notation, we often write $\phi(X_1, \ldots, X_n)$ instead of $\phi(U)(X_1, \ldots, X_n)$.

DEFINITION 2.2. A *dimension map* $\mathcal{D}$ for a modal language L is a partial function $\texttt{M} \cup OP \to (\omega \setminus \{0\}) \cup (\omega \setminus \{0\})^+$ such that

- for any $\texttt{c} \in \texttt{M}_0$, $\mathcal{D}(\texttt{c}) = \rho(\texttt{c}) + 1$ ($\texttt{c} \in \texttt{M}^{\mathcal{D}}$);
- for any $\oplus \in OP$, $\mathcal{D}(\oplus) = \langle i_1, \ldots, i_n, i_{n+1}\rangle \in (\omega\setminus\{0\})^+$ with $\rho^1_{OP}(\oplus) = n$, $i_{n+1} = \rho^2_{OP}(\oplus) + 1$;
- for any $\oplus \in OP$ and $\texttt{a}_1, \ldots, \texttt{a}_n \in \texttt{M}^{\mathcal{D}}$ suth that $\mathcal{D}(\oplus) = \langle \mathcal{D}(\texttt{a}_1), \ldots, \mathcal{D}(\texttt{a}_n), i_{n+1}\rangle$, we have $\mathcal{D}(\oplus(\texttt{a}_1, \ldots, \texttt{a}_n)) = i_{n+1}$ ($\oplus(\texttt{a}_1, \ldots, \texttt{a}_n) \in \texttt{M}^{\mathcal{D}}$).

∇

In the above Definition 2.2, $\texttt{M}^{\mathcal{D}}$ denotes the set of *well-formed* modal expressions with respect to L and $\mathcal{D}$. For the sake of economy, $\texttt{M}^{\mathcal{D}}$ is written M since we shall only deal with well-formed expressions in the sequel.

DEFINITION 2.3. Let $\mathcal{D}$ be a dimension map for L. An *operator interpretation* $\mathcal{I}$ maps the set OP into the set of relation operations such that for any $\oplus \in OP$, the profile of $\mathcal{I}(\oplus)$ is $\mathcal{D}(\oplus)$. ∇

Let L be a modal language and $\mathcal{D}$ be a dimension map for L. An L-*frame* $\mathcal{F}$ is a structure $\langle W, (R_{\texttt{c}})_{\texttt{c}\in\texttt{M}_0}\rangle$ such that W is a non-empty set and for any $\texttt{c} \in \texttt{M}_0$, $R_{\texttt{c}} \subseteq W^{\mathcal{D}(\texttt{c})}$. An L-*model* $\mathcal{M}$ *under some operator interpretation* $\mathcal{I}$ is a structure $\langle W, (R_{\texttt{c}})_{\texttt{c}\in\texttt{M}_0}, V\rangle$ such that $\mathcal{F} = \langle W, (R_{\texttt{c}})_{\texttt{c}\in\texttt{M}_0}\rangle$ is an L-frame ($\mathcal{M}$ is said to be *based on* $\mathcal{F}$) and V is a mapping $(\texttt{For}_0 \cup \texttt{M}) \to \bigcup_{i\geq 1} \mathcal{P}(W^i)$ such that

- $V(\mathtt{p}) \in \mathcal{P}(W)$ for any $\mathtt{p} \in \mathtt{For}_0$;
- $V(\mathtt{c}) = R_{\mathtt{c}}$ for any $\mathtt{c} \in \mathtt{M}_0$; $V(\mathtt{a}) \in \mathcal{P}(W^{\mathcal{D}(\mathtt{a})})$ for any $\mathtt{a} \in \mathtt{M}$;
- $V(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n)) = \mathcal{I}(\oplus)(W)(V(\mathtt{a}_1), \ldots, V(\mathtt{a}_n))$ for any $\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n) \in \mathtt{M}$.

The structure $\langle\{V(\mathtt{a}) : \mathtt{a} \in \mathtt{M}\}, \mathcal{I}(\oplus_1)(W), \ldots, \mathcal{I}(\oplus_s)(W)\rangle$ is therefore a sorted relation algebra of similarity type $\langle\rho^1_{OP}(\oplus_1), \ldots, \rho^1_{OP}(\oplus_s)\rangle$ where the sort of $V(\mathtt{a})$ for any $\mathtt{a} \in \mathtt{M}$ is its dimension $\mathcal{D}(\mathtt{a})$ (see e.g. [Orło88b]). Moreover, $\{V(\mathtt{a}) : \mathtt{a} \in \mathtt{M}\}$ is uniquely *generated* from $\{V(\mathtt{c}) : \mathtt{c} \in \mathtt{M}_0\}$ and $\mathcal{I}(\oplus_1)(W), \ldots, \mathcal{I}(\oplus_s)(W)$.

Let $\langle W, (R_{\mathtt{c}})_{\mathtt{c}\in\mathtt{M}_0}, V\rangle$ be an L-model under some $\mathcal{I}$. As usual, the formula F is *satisfied by the world* $u \in W$ *in* $\mathcal{M}$ $\stackrel{\text{def}}{\Leftrightarrow}$ $\mathcal{M}, u \models \mathtt{F}$ where the *satisfaction relation* $\models$ is inductively defined as follows:

- $\mathcal{M}, u \models \mathtt{p} \stackrel{\text{def}}{\Leftrightarrow} u \in V(\mathtt{p})$, for any $\mathtt{p} \in \mathtt{For}_0$;
- $\mathcal{M}, u \models \neg\mathtt{F} \stackrel{\text{def}}{\Leftrightarrow}$ not $\mathcal{M}, u \models \mathtt{F}$;
- $\mathcal{M}, u \models \mathtt{F} \wedge \mathtt{G} \stackrel{\text{def}}{\Leftrightarrow} \mathcal{M}, u \models \mathtt{F}$ and $\mathcal{M}, u \models \mathtt{G}$;
- $\mathcal{M}, u \models \langle\mathtt{a}\rangle(\mathtt{F}_1, \ldots, \mathtt{F}_n) \stackrel{\text{def}}{\Leftrightarrow} \exists u_1, \ldots, u_n \in W$ such that $(u, u_1, \ldots, u_n) \in V(\mathtt{a})$ and for $i \in \{1, \ldots, n\}$, $\mathcal{M}, u_i \models \mathtt{F}_i$ ($\rho(\mathtt{a}) = n$).

A formula F is *true* in an L-model $\mathcal{M}$ (denoted by $\mathcal{M} \models \mathtt{F}$) $\stackrel{\text{def}}{\Leftrightarrow}$ for any $u \in W$, $\mathcal{M}, u \models \mathtt{F}$. F is *valid* in an L-frame $\mathcal{F}$ (denoted by $\mathcal{F} \models \mathtt{F}$) $\stackrel{\text{def}}{\Leftrightarrow}$ $\mathcal{M} \models \mathtt{F}$ for any L-model based on $\mathcal{F}$ (under some $\mathcal{I}$).

Definition 2.4. Let $\mathcal{D}$ be a dimension map for the modal language L and $\mathcal{I}$ be an operator interpretation. Let $\mathcal{L}$ be an L-normal modal logic and $\mathcal{F}$ be an L-frame. $\mathcal{F}$ is an L-*frame for* $\mathcal{L}$ $\stackrel{\text{def}}{\Leftrightarrow}$ every formula F in $\mathcal{L}$ is valid in $\mathcal{F}$. ∇

A more standard notion of L-frame would be to consider the structures of the form $\langle W, (R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}}\rangle$. This is not done here since all the logics treated in the paper are determined: the structure $\langle W, (R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}}\rangle$ is uniquely generated from $\langle W, (R_{\mathtt{c}})_{\mathtt{c}\in\mathtt{M}_0}\rangle$ using the operator interpetation $\mathcal{I}$.

Definition 2.5. An L-normal modal logic $\mathcal{L}$ is said to be *semantically determined* $\stackrel{\text{def}}{\Leftrightarrow}$ there exists a structure $\langle\mathcal{D}, \mathcal{I}, \mathcal{C}\rangle$ such that $\mathcal{D}$ is a dimension map for L, $\mathcal{I}$ is an operator interpretation, $\mathcal{C}$ is a non-empty class of L-frames for L and $\mathcal{D}$, and for any L-formula F, $\mathtt{F} \in \mathcal{L}$ iff F is valid in all L-frames in $\mathcal{C}$. ∇

A semantically determined L-normal modal logic $\mathcal{L}$ is also represented by $\langle\mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}\rangle$. An $\mathcal{L}$-*model* is an L-model under $\mathcal{I}$ based on some $\mathcal{F} \in \mathcal{C}$. An L-formula F is said to be $\mathcal{L}$-*valid* $\stackrel{\text{def}}{\Leftrightarrow}$ $\mathtt{F} \in \mathcal{L}$ iff F is true in all $\mathcal{L}$-models. An L-formula F is said to be $\mathcal{L}$-*satisfiable* $\stackrel{\text{def}}{\Leftrightarrow}$ there exist an $\mathcal{L}$-model $\mathcal{M} = \langle W, (R_{\mathtt{c}})_{\mathtt{c}\in\mathtt{M}_0}, V\rangle$ and $u \in W$ such that $\mathcal{M}, u \models \mathtt{F}$. We say that G is a *logical* $\mathcal{L}$-*consequence* of the formula F (in symbols $\mathtt{F} \models_{\mathcal{L}} \mathtt{G}$) $\stackrel{\text{def}}{\Leftrightarrow}$ for any $\mathcal{L}$-model $\mathcal{M}$, $\mathcal{M} \models \mathtt{F}$ implies $\mathcal{M} \models \mathtt{G}$.

As usual, we say that the modal logic $\mathcal{L}$ has the *finite model property* (fmp) iff for each $\mathcal{L}$-satisfiable formula F has an $\mathcal{L}$-model such that W is finite. $\mathcal{L}$ is said to be *decidable* iff there is a decision procedure that for each L-formula F outputs "yes" if $\mathtt{F} \in \mathcal{L}$ and "no" otherwise. As usual, for semantically determined logics, satisfiability is decidable iff validity is decidable.

DEFINITION 2.6. A *standard modal logic* is a semantically determined $\mathtt{L}$-normal modal logic such that the set of basic modal expressions is a countable set $\mathtt{M}_0 = \{\mathtt{c}, \mathtt{d}, \ldots\}$ of constants. ∇

EXAMPLE 2.1. Let DAL $= \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C} \rangle$ be the standard modal logic (see e.g. [FdCO85]) such that $OP = \{\cap, \cup^*\}$ (respectively interpreted by $\mathcal{I}$ as the set intersection and the reflexive and transitive closure of the union). $\mathtt{M}_0$ is a countably infinite set $\{\mathtt{c}_0, \ldots, \mathtt{c}_n, \ldots\}$ of modal constants. Moreover, $\rho^2_{OP}(\cap) = \rho^2_{OP}(\cup^*) = 1$ and $\rho^1_{OP}(\cap) = \rho^1_{OP}(\cup^*) = 2$. For any frame $\mathcal{F} = \langle W, (R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0} \rangle$, $\mathcal{F} \in \mathcal{C} \stackrel{\text{def}}{\Leftrightarrow}$ each $R_{\mathtt{c}}$ is an equivalence relation on W. It is easy to see that for any DAL-model $\langle W, (R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V \rangle$, and for any modal expression $\mathtt{a}$, $V(\mathtt{a})$ is an equivalence relation on W.

A standard modal logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C} \rangle$ is said to be *closed under disjoint unions* $\stackrel{\text{def}}{\Leftrightarrow}$ for any $\mathcal{L}$-model $\mathcal{M}^1 = \langle W^1, (R^1_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V^1 \rangle$ and $\mathcal{M}^2 = \langle W^2, (R^2_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V^2 \rangle$, if $W^1 \cap W^2 = \emptyset$, then the structure $\mathcal{M} = \langle W, (R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V \rangle$ defined below is an $\mathcal{L}$-model:

- $W \stackrel{\text{def}}{=} W^1 \cup W^2$; for $\mathtt{p} \in \mathtt{For}_0$, $V(\mathtt{p}) = V^1(\mathtt{p}) \cup V^2(\mathtt{p})$;
- for $\mathtt{c} \in \mathtt{M}_0$, $R_{\mathtt{c}} \stackrel{\text{def}}{=} R^1_{\mathtt{c}} \cup R^2_{\mathtt{c}}$; for any $\mathtt{a} \in \mathtt{M}$, $V(\mathtt{a}) \stackrel{\text{def}}{=} V^1(\mathtt{a}) \cup V^2(\mathtt{a})$.

Similarly, a standard modal logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C} \rangle$ is said to be *closed under isomorphic copies* $\stackrel{\text{def}}{\Leftrightarrow}$ for any $\mathcal{L}$-model $\mathcal{M} = \langle W, (R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V \rangle$ and for any 1-1 mapping $f : W \to W'$ (naturally extended to $\mathcal{P}(W^n)$ for $n \geq 1$), the structure $\mathcal{M}' = \langle W', (R'_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V' \rangle$ defined below is an $\mathcal{L}$-model:

- for $\mathtt{p} \in \mathtt{For}_0$, $V'(\mathtt{p}) = f(V(\mathtt{p}))$;
- for $\mathtt{c} \in \mathtt{M}_0$, $R'_{\mathtt{c}} \stackrel{\text{def}}{=} f(R_{\mathtt{c}})$; for any $\mathtt{a} \in \mathtt{M}$, $V'(\mathtt{a}) \stackrel{\text{def}}{=} f(V(\mathtt{a}))$.

The condition for $\mathtt{a} \in \mathtt{M}$ immediately follows from the corresponding condition for $\mathtt{c} \in \mathtt{M}_0$ due to the corresponding condition in the definition of the relation operation. On the contrary, for disjoint unions the corresponding condition for $\mathtt{a} \in \mathtt{M}$ is not so immediate and provides some requirement to the operation interpretation. Observe also that for any $\mathtt{L}$-formula $\mathtt{F}$, for $u \in \{1, 2\}$ and for $w \in W^u$, $\mathcal{M}, w \models \mathtt{F}$ iff $\mathcal{M}^u, w \models \mathtt{F}$. Hence $\mathcal{M} \models \mathtt{F}$ iff $\mathcal{M}^1 \models \mathtt{F}$ and $\mathcal{M}^2 \models \mathtt{F}$. An analogous property holds true for isomorphic copies. Assuming that a standard modal logic is closed under disjoint unions and isomorphic copies is not a strong assumption (see e.g. [GT75]) and all the particular standard modal logics considered in the paper enter in this category. For instance, DAL is closed under disjoint unions and isomorphic copies. *Every* standard modal logic can be replaced by one closed under copies with the same set of valid (and satisfiable) formulas; and analogously for the closedness under disjoint unions but only under natural requirement on the operator interpretation.

A standard modal logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C} \rangle$ is said to be *closed under restrictions* $\stackrel{\text{def}}{\Leftrightarrow}$ for any $\mathcal{L}$-model $\mathcal{M} = \langle W, (R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V \rangle$ and for any $\emptyset \neq W' \subseteq W$, the structure $\mathcal{M}' = \langle W', (R'_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0}, V' \rangle$ defined below is an $\mathcal{L}$-model:

- for $\mathtt{p} \in \mathtt{For}_0$, $V'(\mathtt{p}) = V(\mathtt{p}) \cap W'$;
- for $\mathtt{c} \in \mathtt{M}_0$, $R'_{\mathtt{c}} \stackrel{\text{def}}{=} R_{\mathtt{c}} \cap (W')^{\mathcal{D}(\mathtt{c})}$;
- for any $\mathtt{a} \in \mathtt{M}$, $V'(\mathtt{a}) \stackrel{\text{def}}{=} V(\mathtt{a}) \cap (W')^{\mathcal{D}(\mathtt{a})}$.

The model $\mathcal{M}'$ is also denoted $\mathcal{M}_{|W'}$. DAL is also closed under restrictions.

3 Modal logics with relative accessibility relations

3.1 Definitions

A set of modal expressions is said to be *dedicated to Rare-logics* if each basic modal expression is of the form $r(\mathtt{A})$ where $\mathtt{A}$ is a *parameter expression* and 'r' is an arbitrary symbol fixed in the rest of the paper. The set $\mathtt{P} = \{\mathtt{A}, \mathtt{B}, \ldots\}$ of parameter expressions is the set $\bigcup\{\mathtt{P}_i : 1 \leq i \leq j\}$ such that each $\mathtt{P}_i$ is the smallest set containing a countable set $\mathtt{P}_0^i = \{\mathtt{C}, \mathtt{D}, \ldots\}$ of parameter constants and it is closed under the Boolean operators $\cap, \cup, -$. We assume that $\{\mathtt{P}_i : i \in \{1, \ldots, j\}\}$ is a partition of $\mathtt{P}$. We write[2] J to denote the set $\{1, \ldots, j\}$. A modal similarity type $\mathtt{S} = \langle\{\langle\mathtt{a}\rangle : \mathtt{a} \in \mathtt{M}\}, \rho\rangle$ *dedicated to Rare-logics* is defined as in Section 2 except that for any $i \in J$ and for any $\mathtt{A}, \mathtt{B} \in \mathtt{P}_i$, $\rho(r(\mathtt{A})) = \rho(r(\mathtt{B}))$. A modal language dedicated to Rare-logics is defined in the natural way.

Notational convention For any syntactic category $\mathtt{X}$ and any syntactic object $\mathtt{O}$, we write $\mathtt{X}(\mathtt{O})$ to denote the set composed of elements of $\mathtt{X}$ occurring in $\mathtt{O}$. For instance $\mathtt{P}_0^i(\mathtt{A})$ denotes the set of parameter constants from $\mathtt{P}_0^i$ that occur in the parameter expression $\mathtt{A} \in \mathtt{P}_i$.

Example 3.1. Let $\mathtt{L}$ be a modal language dedicated to Rare-logics such that $\{\cap, \cup^*\} \subseteq OP$, $\rho^1_{OP}(\cap) = \rho^1_{OP}(\cup^*) = 2$, $\rho^2_{OP}(\cap) = \rho^2_{OP}(\cup^*) = 1$ and for $\mathtt{A} \in \mathtt{P}_1$, $\rho(r(\mathtt{A})) = 1$. Here is an $\mathtt{L}$-formula: $([r(\mathtt{C}) \cup^* r(\mathtt{C})]\mathtt{p} \Rightarrow [r(\mathtt{C} \cap \mathtt{C}) \cap r(\mathtt{C})]\mathtt{p}) \wedge ([r(-\mathtt{C})]\mathtt{p} \Leftrightarrow [r(-(\mathtt{C} \cup \mathtt{C})) \cap r(-(\mathtt{C} \cap \mathtt{C}))]\mathtt{p})$.

The notions of dimension map $\mathcal{D}$, operator interpretation $\mathcal{I}$, $\mathtt{L}$-normal modal logic $\mathcal{L}$ are defined for the languages dedicated to Rare-logics as in Section 2, with possibly some obvious adaptations. For any $i \in J$, we write $\mathcal{D}(i)$ to denote the natural number such that for any $\mathtt{A} \in \mathtt{P}_i$, $\mathcal{D}(r(\mathtt{A})) = \mathcal{D}(i)$.

Definition 3.1. Let $\mathtt{L}$ be a modal language dedicated to Rare-logics and $\mathcal{D}$ be a dimension map for $\mathtt{L}$. An $\mathtt{L}$-*frame* (for Rare-logics) is a structure $\langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}\rangle$ such that

- W is a non-empty set;
- for any $i \in J$, PAR_i is a non-empty set of *parameters*;
- for any $P \subseteq PAR_i$, $R^i_P \subseteq W^{\mathcal{D}(i)}$.

∇

Definition 3.2. A $\mathtt{P}$-*valuation* V is a map $V : \mathtt{P} \rightarrow \mathcal{P}(PAR_1) \cup \ldots \cup \mathcal{P}(PAR_j)$ such that for any $i \in J$ and for any $\mathtt{A}_1, \mathtt{A}_2 \in \mathtt{P}_i$, $V(\mathtt{A}_1) \in \mathcal{P}(PAR_i)$; $V(-\mathtt{A}_1) = PAR_i \setminus V(\mathtt{A}_1)$; $V(\mathtt{A}_1 \cap \mathtt{A}_2) = V(\mathtt{A}_1) \cap V(\mathtt{A}_2)$; $V(\mathtt{A}_1 \cup \mathtt{A}_2) = V(\mathtt{A}_1) \cup V(\mathtt{A}_2)$. ∇

By an $\mathtt{L}$-*model* $\mathcal{M}$ *under the operator interpretation* $\mathcal{I}$ dedicated to Rare-logics, we understand a structure $\langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}, V\rangle$ such that $\mathcal{F} = \langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}\rangle$ is an $\mathtt{L}$-frame dedicated to Rare-logics and V is a mapping $(\mathtt{For}_0 \cup \mathtt{P} \cup \mathtt{M}) \rightarrow (\bigcup_{i\in J} \mathcal{P}(PAR_i) \cup \bigcup_{i\geq 1} \mathcal{P}(W^i))$ such that

[2] Numerous results in the paper can be straightforwardly extended when J is countable. For the sake of clarity, herein we omit these results.

- $V(\mathtt{p}) \in \mathcal{P}(W)$ for any $\mathtt{p} \in \mathtt{For}_0$;
- $V(\mathtt{A}) \in \bigcup_{i \in J} \mathcal{P}(PAR_i)$ for any $\mathtt{A} \in \mathtt{P}$ and V restricted to $\mathtt{P}$ is a $\mathtt{P}$-valuation (see Definition 3.2);
- $V(\mathtt{a}) \in \mathcal{P}(W^{\mathcal{D}(\mathtt{a})})$ for any $\mathtt{a} \in \mathtt{M}$; $V(r(\mathtt{A})) = R^i_{V(\mathtt{A})}$ for any $\mathtt{A} \in \mathtt{P}_i$, $i \in J$;
- $V(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n)) = \mathcal{I}(\oplus)(W)(V(\mathtt{a}_1), \ldots, V(\mathtt{a}_n))$.

For the sake of comparison the models dedicated to Rare-logics are richer than those for standard modal logics since the sets of parameters are structured: the Boolean algebra $\langle \mathcal{P}(PAR_i), \cup, \cap, -, \emptyset, PAR_i \rangle$ is explicitly used in the conditions involved to define the different Rare-logics (see Definition 3.3 below). This could be related to the various parameter signatures that can be found in the literature for polymodal logics (see e.g. [HO91, dG94]).

REMARK. Each basic modal expression is of the form $r(\mathtt{A})$ where $\mathtt{A} \in \mathtt{P}$ and 'r' is an arbitrary symbol. Other symbols replacing 'r' can be found in the literature: '*ind*' [resp. '*sim*'] where $V(r(\mathtt{A}))$ is an *indiscernibility relation* [resp. a *similarity relation*] -see e.g. [Orło90]. It is possible to partially get rid of this symbol announcing a Boolean expression by *decorating* the operators $\cap, \cup, -$ used to build the elements of $\mathtt{P}$ (by considering for instance $\cap^b, \cup^b, -^b$). Alternatively, we could have written $\{\mathtt{A}\}$ instead of $r(\mathtt{A})$. Indeed, the role of the syntactic construction $r(.)$ is to express a parameter expression that represents an index of a relation in the models. So, the syntactic construction $r(.)$ plays the rôle of parantheses since it makes the reading of modal expressions non ambiguous. In the modal expression $r(\mathtt{C}_1 \cap \mathtt{C}_2) \cap r(\mathtt{C}_3)$, the first occurrence of $\cap$ represents intersection of sets of parameters represented by $\mathtt{C}_1$ and $\mathtt{C}_2$, but the second occurrence of $\cap$ represents intersection of relations indexed with $\mathtt{C}_1 \cap \mathtt{C}_2$ and $\mathtt{C}_3$ respectively.

In the sequel we shall omit to specify the operator interpretation $\mathcal{I}$ for the models when it is clear from the context or when it is irrelevant. For any $i \geq 1$, we write Fr^i to denote the class of structures $\langle W, R \rangle$ where W is a non-empty set and $R \subseteq W^i$. The relation $\models$ with $\mathtt{L}$-models dedicated to Rare-logics is defined as in Section 2 as well as the other related notions.

DEFINITION 3.3. Let $\mathtt{L}$ be a modal language dedicated to Rare-logics. An $\mathtt{L}$-normal modal logic $\mathcal{L}$ is said to be a *Rare-logic of type* $\mathtt{T}$ $\stackrel{\text{def}}{\Leftrightarrow}$ there is a structure $\langle \mathcal{D}, \mathcal{I}, \mathcal{C}, (X_i)_{i \in J} \rangle$ such that

- $\mathcal{D}$ is a dimension map for $\mathtt{L}$;
- $\mathcal{I}$ is an operator interpretation;
- $\mathcal{C}$ is a non-empty class of $\mathtt{L}$-frames for $\mathtt{L}$ and $\mathcal{D}$ such that for any $\mathtt{F} \in \mathtt{For}$, $\mathtt{F} \in \mathcal{L}$ iff for any $\mathcal{F} \in \mathcal{C}$, $\mathcal{F} \models \mathtt{F}$;
- for any $i \in J$, $\emptyset \neq X_i \subseteq Fr^{\mathcal{D}(i)}$;
- for any $\mathtt{L}$-frame $\mathcal{F}$, $\mathcal{F} \in \mathcal{C}$ iff for any $i \in J$ and for any $P \subseteq PAR_i$, $\langle W, R^i_P \rangle \in X_i$ and $\mathcal{F}$ satisfies the condition $C_{\mathtt{T}}$.

∇

The condition $C_{\mathtt{T}}$ is simply a technical means to capture various kinds of possible requirements on $\mathcal{F}$. Most of the time, it encodes that for $i \in J$,

$(R^i_P)_{P \subseteq PAR_i}$ has a certain algebraic structure. So, T is just a label to name a condition $C_{\mathtt{T}}$.

A Rare-logic of type T is also noted $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, (X_i)_{i \in J}, \mathtt{T} \rangle$. There is some redundancy in the definition of such a structure since the class $\mathcal{C}$ is uniquely determined by $(X_i)_{i \in J}$ and $C_{\mathtt{T}}$. This is done for the sake of clarity.

The other definitions for the Rare-logics are natural adaptations of those for standard modal logics except for the notion of finite model property. A Rare-logic $\mathcal{L}$ has the *finite model property* $\stackrel{\text{def}}{\Leftrightarrow}$ every $\mathcal{L}$-satisfiable formula is satisfied in some $\mathcal{L}$-model $\langle W, (PAR_i)_{i \in J}, (R^1_P)_{P \subseteq PAR_1}, \ldots, (R^j_P)_{P \subseteq PAR_j}, V \rangle$ such that W is finite and for any $i \in J$, there is a *finite* subset $Z_i \subseteq PAR_i$ such that for any $P \subseteq PAR_i$, $R^i_P = R^i_{P \cap Z_i}$. This does not necessarily implies that each PAR_i is finite.

The conditions $C_{\mathtt{T}}$ can be of different kinds as defined below
($\mathcal{F} = \langle W, (PAR_i)_{i \in J}, (R^1_P)_{P \subseteq PAR_1}, \ldots, (R^j_P)_{P \subseteq PAR_j} \rangle$ is an L-frame):

- $\mathcal{F}$ always satisfies the condition C_0 (equivalent to `true`);
- Let ϕ_1 be a set operation of arity $n \geq 1$ and for any $i \in J$, ϕ^i_2 be a relation operation of profile $\langle \mathcal{D}(i), \ldots, \mathcal{D}(i) \rangle$ and of arity n. $\mathcal{F}$ satisfies the condition $C_{[\phi_1, \phi^1_2, \ldots, \phi^j_2]} \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$ and for any $P_1, \ldots, P_n \subseteq PAR_i$, $R^i_{\phi_1(PAR_i)(P_1, \ldots, P_n)} = \phi^i_2(W)(R^i_{P_1}, \ldots, R^i_{P_n})$.
- Let $k \in \{0, 1\}$. $\mathcal{F}$ satisfies the condition $C_{\emptyset^k} \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$, if $k = 0$, then $R^i_\emptyset = \emptyset$ otherwise $R^i_\emptyset = W^{\mathcal{D}(i)}$. $\mathcal{F}$ satisfies the condition $C_{-\emptyset^k} \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$, if $k = 0$, then $R^i_{PAR_i} = \emptyset$ otherwise $R^i_{PAR_i} = W^{\mathcal{D}(i)}$.
- Let $\{\mathtt{T}_1, \ldots, \mathtt{T}_k\}$ be a finite set of types. $\mathcal{F}$ satisfies the condition $C_{\langle \mathtt{T}_1, \ldots, \mathtt{T}_k \rangle} \stackrel{\text{def}}{\Leftrightarrow}$ for any $l \in \{1, \ldots, k\}$ $\mathcal{F}$ satisfies the condition $C_{\mathtt{T}_l}$. Each $\mathtt{T}_i$, $1 \leq i \leq k$, is called a *component* of $\langle \mathtt{T}_1, \ldots, \mathtt{T}_k \rangle$.
- As particular cases of the types defined above can be found the following types that shall be central in our investigations:
  - $\mathcal{F}$ satisfies the condition $C_1 \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$ and any $P, P' \subseteq PAR_i$, $R^i_{P \cup P'} = R^i_P \cap R^i_{P'}$ and $R^i_\emptyset = W^{\mathcal{D}(i)}$. $\langle \{R^i_P : P \subseteq PAR_i\}, \supseteq \rangle$ is a meet-semilattice with a top element (see e.g. [DP90])
  - $\mathcal{F}$ satisfies the condition $C_2 \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$ and any $P, P' \subseteq PAR_i$, $R^i_{P \cup P'} = R^i_P \cup R^i_{P'}$ and $R^i_\emptyset = \emptyset$. $\langle \{R^i_P : P \subseteq PAR_i\}, \subseteq \rangle$ is a join-semilattice with a bottom element.
  - $\mathcal{F}$ satisfies the condition $C_3 \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$ and for any $P, P' \subseteq PAR_i$, $R^i_{P \cap P'} = R^i_P \cap R^i_{P'}$ and $R^i_{PAR_i} = W^{\mathcal{D}(i)}$. $\langle \{R^i_P : P \subseteq PAR_i\}, \subseteq \rangle$ is a meet-semilattice with a top element.
  - $\mathcal{F}$ satisfies the condition $C_4 \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$ and for any $P, P' \subseteq PAR_i$, $R^i_{P \cap P'} = R^i_P \cup R^i_{P'}$ and $R^i_{PAR_i} = \emptyset$. $\langle \{R^i_P : P \subseteq PAR_i\}, \supseteq \rangle$ is a join-semilattice with a bottom element.
  - $\mathcal{F}$ satisfies the condition $C_5 \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$ and for any $P \subseteq PAR_i$, $R^i_P = R^i_{PAR_i \setminus P}$;
  - $\mathcal{F}$ satisfies the condition $C_6 \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$ and for any $P \subseteq PAR_i$, $R^i_P = W^{\mathcal{D}(i)} \setminus R^i_{PAR_i \setminus P}$;

– $\mathcal{F}$ satisfies the condition $C_7 \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$ and for any $P, P' \subseteq PAR_i$, $R^i_{P \cup P'} = R^i_P \cup^* R^i_{P'}$ ($\cup^*$ is the transitive and reflexive closure of the union) and $R^i_\emptyset = \emptyset$.

So, for instance Definition 3.3 and the conditions above imply that for any Rare-logic $\mathcal{L}$ of type 1, $\langle W, W^{\mathcal{D}(i)} \rangle \in X_i$ for any $\mathcal{L}$-model

$$\mathcal{M} = \langle W, (PAR_i)_{i \in J}, (R^1_P)_{P \subseteq PAR_1}, \ldots, (R^j_P)_{P \subseteq PAR_j}, V \rangle,$$

and $i \in J$ since $\langle W, W^{\mathcal{D}(i)} \rangle = \langle W, R^i_\emptyset \rangle$. So, there is no logic of type 1 such that for some $i \in J$, $\langle W, W^{\mathcal{D}(i)} \rangle \not\in X_i$. Similarly, for any Rare-logic $\mathcal{L}$ of type 7, the relation R^i_P of the $\mathcal{L}$-models are reflexive and transitive since for any $i \in J$ and any $P \subseteq PAR_i$, $R^i_P = R^i_P \cup^* R^i_P$.

From the above list of conditions, one can easily imagine other algebraic structures (see e.g. [Pag97, Dün97]), for instance

- $\mathcal{F}$ satisfies the condition $C_{cl} \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$, $\langle \{R^i_P : P \subseteq PAR_i\}, \subseteq \rangle$ is a complete lattice;
- $\mathcal{F}$ satisfies the condition $C_b \stackrel{\text{def}}{\Leftrightarrow}$ for any $i \in J$,

$$\langle \{R^i_P : P \subseteq PAR_i\}, \cup, \cap, -, \emptyset, \bigcup \{R^i_P : P \subseteq PAR_i\} \rangle$$

  is a Boolean algebra.

In [Bal97], particular cases of the type $[\phi_1, \phi^1_2, \ldots, \phi^j_2]$ are considered. Conditions with one-way inclusion are also considered in [Bal97] (for instance, $R^i_{P \cup P'} \subseteq R^i_P \cap R^i_{P'}$). These conditions are out of the scope of the present paper. Various obvious generalizations of the above conditions are possible. This is not considered here in order to avoid the boredom of repetitive definitions.

The relations in the frames dedicated to Rare-logics have to satisfy local conditions (for instance, each R^i_P is an equivalence relation) and global conditions (for instance, $\langle \{R^i_P : P \subseteq PAR_i\}, \subseteq \rangle$ is a complete lattice). The Rare-logics of type 1-7 can be found in [Orło93, Kon97b, Bal96b, Bal97]. By way of example, in [Orło95] the condition C_1 corresponds to the families of *strong relations* (see also [Orło88a]).

Definition 3.4. Let $\mathcal{L}_k$ be the Rare-logic $\langle \mathtt{L}_k, \mathcal{D}_k, \mathcal{I}_k, \mathcal{C}_k, (X^k_i)_{i \in J}, \mathtt{T}_k \rangle$ for $k = 1, 2$. $\mathcal{L}_1$ and $\mathcal{L}_2$ are *similar* $\stackrel{\text{def}}{\Leftrightarrow}$ all their components are equal except possibly their types $\mathtt{T}_k$ and the $\mathcal{C}_k$'s. ∇

So, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are similar, then $(X^1_i)_{i \in J} = (X^2_i)_{i \in J}$. However, $\mathcal{C}_1$ and $\mathcal{C}_2$ may be different since $\mathcal{C}_k$ is uniquely determined by $(X^k_i)_{i \in J}$ *and* by $\mathtt{T}_k$.

Example 3.2. Consider the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, (X_i)_{i \in J}, \mathtt{T} \rangle$ of type 1 such that $OP = \{\cup, \circ, ^*\}$, $j = 1$, $\mathcal{D}(1) = 2$, $\rho^1_{OP}(\cup) = \rho^1_{OP}(\circ) = 2$, $\rho^1_{OP}(^*) = 1$, $\rho^2_{OP}(\cup) = \rho^2_{OP}(\circ) = \rho^2_{OP}(^*) = 1$ and $X_1 = Fr^2$. Its language is that of Propositional Dynamic Logic PDL (see e.g. [Pra80]) where the *program constants* have been substituted by basic modal expressions of the form $r(\mathtt{A})$. $\mathcal{I}$ is defined such that $\cup$ [resp. $\circ$, *] is interpreted as union [resp. composition, Kleene star].

A Rare-logic $\mathcal{L}$ has the *[resp. strong] finite parameter sets property* (fpsp) [resp. (sfpsp)] $\stackrel{\text{def}}{\Leftrightarrow}$ for any $\mathcal{L}$-satisfiable formula $\mathtt{F}$, there exist an $\mathcal{L}$-model

$$\mathcal{M} = \langle W, (PAR_i)_{i \in J}, (R_P^1)_{P \subseteq PAR_1}, \ldots, (R_P^j)_{P \subseteq PAR_j}, V \rangle$$

and $w \in W$ such that $\mathcal{M}, w \models \mathtt{F}$ and for any $i \in J$, PAR_i is finite [resp. and $PAR_1 = \ldots = PAR_j$].

Proposition 3.1 below can be easily proved and shall be often used in the rest of the paper.

PROPOSITION 3.1. Let $\mathcal{L}_1, \mathcal{L}_2$ be two logics semantically determined (either standard modal logics or Rare-logics) and $\mathtt{F}_i$ be an $\mathtt{L}_i$-formula for $i = 1, 2$. Assume there exists a map Φ between the set $\{\mathtt{a}_1, \ldots, \mathtt{a}_n\}$ of modal expressions in $\mathtt{F}_1$ and the set of modal expressions in $\mathtt{F}_2$ such that $\mathtt{F}_2$ is obtained from $\mathtt{F}_1$ by substituting simultaneously every occurrence of $\mathtt{a}_i$ by $\Phi(\mathtt{a}_i)$ for $i \in \{1, \ldots, n\}$. Let $\mathcal{M}_i$ be an $\mathcal{L}_i$-model for $i = 1, 2$ such that $W_1 = W_2$; for any propositional variable $\mathtt{p}$ occurring in $\mathtt{F}_1$, $V_1(\mathtt{p}) = V_2(\mathtt{p})$ and for any $i \in \{1, \ldots, n\}$, $V_1(\mathtt{a}_i) = V_2(\Phi(\mathtt{a}_i))$. Then, for any $u \in W_1$, $\mathcal{M}_1, u \models \mathtt{F}_1$ iff $\mathcal{M}_2, u \models \mathtt{F}_2$.

Similarly, Proposition 3.2 below shall be used in the sequel and it can be easily proved.

PROPOSITION 3.2. Let $\mathcal{L}$ be a logic semantically determined (either a standard modal logic or a Rare-logic) and let $\mathtt{a}, \mathtt{b} \in \mathtt{M}$ such that for all $\mathcal{L}$-models $\langle \ldots, V \rangle$, $V(\mathtt{a}) = V(\mathtt{b})$ (in the sequel noted $\mathtt{a} \equiv_{\mathcal{L}} \mathtt{b}$). Let $\mathtt{F}$ be an $\mathtt{L}$-formula and $\mathtt{G}$ obtained from $\mathtt{F}$ by substituting some occurrences of $\mathtt{a}$ by $\mathtt{b}$. Then, $\mathtt{F} \Leftrightarrow \mathtt{G}$ is $\mathcal{L}$-valid.

3.2 Dual types of Rare-logics: a simplification

The classes of Rare-logics of type $\mathtt{T} \in \{1, \ldots, 4\}$ are central in this paper since they contain various logics from the literature: the so-called *information logics* (see e.g. [Orło88b]) that have been defined to capture some aspects of reasoning with incomplete information. In the sequel we shall show that various properties relevant for the mechanization of such logics are identical between Rare-logics of type 1 [resp. 2] and Rare-logics of type 3 [resp. 4].

PROPOSITION 3.3. (**Duality of types**) Let $\mathcal{L}_1$ be the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}_1, (X_i)_{i \in J}, \mathtt{T}_1 \rangle$ of type 1 [resp. 2] and $\mathcal{L}_2$ be the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}_2, (X_i)_{i \in J}, \mathtt{T}_2 \rangle$ of type 3 [resp. 4] such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are similar in the sense of Definition 3.4. There exists a linear-time mapping f_{dual} from the set of $\mathtt{L}$-formulae into the set of $\mathtt{L}$-formulae such that,

(I) any $\mathtt{L}$-formula $\mathtt{F}$ is $\mathcal{L}_1$-satisfiable [resp. $\mathcal{L}_2$-satisfiable] iff $f_{dual}(\mathtt{F})$ is $\mathcal{L}_2$-satisfiable [resp. $\mathcal{L}_1$-satisfiable];

(II) $\mathcal{L}_1$ has the fmp [resp. fpsp, sfpsp] iff $\mathcal{L}_2$ has the fmp [resp. fpsp, sfpsp].

The map f_{dual} is actually simple. $f_{dual}(\mathtt{F})$ is obtained from $\mathtt{F}$ by substituting every occurrence of $r(\mathtt{A})$ by $r(-\mathtt{A})$. Actually, from a complete and sound proof system for $\mathcal{L}_1$ [resp. $\mathcal{L}_2$] one can build a complete and sound proof system for $\mathcal{L}_2$ [resp. $\mathcal{L}_1$] by using the mapping f_{dual} under reasonable hypothesis. In the sequel the Rare-logics of type 3 and 4 are not treated in the technical developments.

3.3 Rare-logics versus standard modal logics

This section is devoted to the definition of standard modal logics from Rare-logics.

DEFINITION 3.5. Let $\mathcal{L}$ be the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, (X_i)_{i \in J}, \mathtt{T} \rangle$. The *standard modal logic* $\mathcal{L}_d$ *from* $\mathcal{L}$ is the structure $\langle \mathtt{L}_d, \mathcal{D}_d, \mathcal{I}_d, \mathcal{C}_d \rangle$ defined below:

- $\mathtt{L}_d$ (whose set of modal constants is $\mathtt{M}_0$) is obtained from $\mathtt{L}$ by replacing the set $\{r(\mathtt{A}) : \mathtt{A} \in \mathtt{P}_0^i\}$ (partial set of basic modal expressions) by a countably infinite set of modal constants $\mathtt{M}_0^i = \{\mathtt{c}_0^i, \ldots, \mathtt{c}_n^i, \ldots\}$ such that for $k \in \omega$, $\rho(\mathtt{c}_k^i) = \mathcal{D}(i) - 1$ and $\{\mathtt{M}_0^i : i \in J\}$ is a partition of $\mathtt{M}_0$;
- $\mathcal{I}_d \stackrel{\text{def}}{=} \mathcal{I}$;
- $\mathcal{D}_d$ is the unique dimension map for $\mathtt{L}_d$ that agrees with $\mathcal{D}$ for OP;
- Let $\mathcal{F} = \langle W, (R_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0} \rangle$ be an $\mathtt{L}_d$-frame and $[\phi_1^1, \phi_2^{1,1}, \ldots, \phi_2^{1,j}], \ldots [\phi_1^l, \phi_2^{l,1}, \ldots, \phi_2^{l,j}]$ be components of $\mathtt{T}$. $\mathcal{F} \in \mathcal{C}_d \stackrel{\text{def}}{\Leftrightarrow}$ for $i \in J$, the relations generated from $\{R_{\mathtt{c}} : \mathtt{c} \in \mathtt{M}_0^i\}$ with $\phi_2^{1,i}(W), \ldots, \phi_2^{l,i}(W)$ belong to X_i.

∇

The standard modal logic from the Rare-logic defined in Example 3.2 is the logic PDL.

Two standard modal logics are said to be *similar* $\stackrel{\text{def}}{\Leftrightarrow}$ there exist two similar Rare-logics for which they are the respective standard modal logics. Equivalently, they share the same $\mathtt{L}$, $\mathcal{D}$ and $\mathcal{I}$.

EXAMPLE 3.3. Let $\mathcal{L}_1$ be the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}_1, X_1, 1 \rangle$ [resp. $\mathcal{L}_2$ be the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}_2, X_1, 2 \rangle$] with $OP = \{\cap, \cup^*\}$ (respectively interpreted as the intersection and the reflexive and transitive closure of the union on binary relations), X_1 be the set of S5-frames (equivalence relations), $j = 1$, $\mathcal{D}(1) = 2$. The corresponding standard modal logic $\mathcal{L}_{1d}$ is the logic DAL (see Example 2.1) whereas the corresponding standard modal logic $\mathcal{L}_{2d}$ is the logic DALLA defined in [Gar86]. Hence DAL and DALLA are *similar standard modal logics* in the sense above.

We have seen that various conditions $C_{\mathtt{T}}$ use universal and empty relations $W^{\mathcal{D}(i)}$ and $\emptyset$. In the rest of this section, we shall introduce definitions about universal operators -see e.g. [GP92]- that shall simplify some forthcoming developments.

DEFINITION 3.6. Let $\mathcal{L}_d$ be a standard modal logic. For any finite subset Y of $\{\mathtt{0}_i : i \geq 2\} \cup \{\mathtt{U}_i : i \geq 2\}$, we write $\mathcal{L}_d^Y$ to denote the standard modal logic obtained from $\mathcal{L}_d$ where the nullary modal operators in Y have been added to the language and by extending the $\mathtt{L}_d$-frames with the conditions $V(\mathtt{0}_i) = \emptyset$ and $V(\mathtt{U}_i) = W^i$ ($\rho(\mathtt{U}_i) = \rho(\mathtt{0}_i) = i - 1$). Moreover, we denote by $\mathcal{L}_d^{Y^-}$ the logic obtained from $\mathcal{L}_d^Y$ by only allowing in the language the occurrences of the $\mathtt{U}_i$'s of the form $[\mathtt{U}_i]$ or $\langle \mathtt{U}_i \rangle$ and the occurrences of the $\mathtt{0}_i$'s of the form $[\mathtt{0}_i]$ or $\langle \mathtt{0}_i \rangle$. ∇

The elements of Y are therefore considered in $\mathcal{L}_d^Y$ as nullary modal operators and not as basic modal expressions. However, we need to extend the definition of set of operators from Section 2 to allow nullary operators. The notion of operator interpretation has also to be extended appropriately.

Just one more definition about the universal operators.

DEFINITION 3.7. Let $\mathcal{L}_d^Y$ be a standard modal logic and $\mathtt{c}$ be in Y. $\mathcal{L}_d^Y$ is said to be $\mathtt{c}$-*simplifiable* $\stackrel{\text{def}}{\Leftrightarrow}$ there exists an effective procedure $f_{\mathtt{C}} : \mathtt{For} \rightarrow \mathtt{For}$ such that for any $\mathtt{F} \in \mathtt{For}$:

- the only occurrences of $\mathtt{c}$ in $f_{\mathtt{C}}(\mathtt{F})$ are in the context $[\mathtt{c}]$ or $\langle \mathtt{c} \rangle$;
- $f_{\mathtt{C}}(\mathtt{F}) \Leftrightarrow \mathtt{F}$ is $\mathcal{L}_d^Y$-valid (which entails $f_{\mathtt{C}}(\neg \mathtt{F}) \Leftrightarrow \neg f_{\mathtt{C}}(\mathtt{F})$ is $\mathcal{L}_d^Y$-valid).

∇

Definition 3.7 does not state that if $\mathcal{L}_d^Y$ is $\mathtt{c}$-simplifiable, then for any $\mathtt{a} \in \mathtt{M}$ such that $\mathtt{a} \equiv_{\mathcal{L}_d^Y} \mathtt{c}$, for any formula $\mathtt{F}$, $\mathtt{a}$ occurs in $f_{\mathtt{C}}(\mathtt{F})$ in the context of $[\mathtt{a}]$ or $\langle \mathtt{a} \rangle$. The standard universal operator $[\mathtt{U}]$ that can be found in the literature (see e.g. [GP92]) corresponds to $[\mathtt{U}_2]$ in our terminology. We usually omit the indices in $\mathtt{U}_i$ or in $\mathtt{O}_i$ when the context is clear. For instance, for the standard modal logic DAL -see Example 3.3 and Example 2.1 - $\mathrm{DAL}^{\{\mathtt{U}\}}$ is $\mathtt{U}$-simplifiable. This is simply due to the fact that for all $R \subseteq W \times W$, $R \cup^* W \times W = W \times W \cup^* R = W \times W$ and $R \cap W \times W = W \times W \cap R = R$. In this case, $f_{\mathtt{U}}(\mathtt{F})$ can be computed in quadratic time in the size of the $\mathrm{DAL}^{\mathtt{U}}$-formula $\mathtt{F}$.

4 Satisfiability-preserving maps

4.1 Preliminary constructions on semilattices

We consider algebras $\langle D, \sqcup \rangle$ of similarity type $\langle 2 \rangle$ such that $\sqcup$ is commutative, associative, idempotent with a zero element e. In particular, any join-semilattice [resp. meet-semilattice] $\langle D, \leq \rangle$ with a bottom element $\perp$ [resp. with a top element $\top$], can be seen as an algebra of that kind by defining for any $a, b \in D$, $a \sqcup b \stackrel{\text{def}}{=} a \vee b$ [resp. $a \sqcup b \stackrel{\text{def}}{=} a \wedge b$]. Similarly, any algebra $\langle D, \sqcup \rangle$ of the above kind can be seen for instance as a join-semilattice by defining for any $a, b \in D$, $a \leq b \stackrel{\text{def}}{\Leftrightarrow} a \sqcup b = b$. By abusing our notation, we write $\langle D, \sqcup, e \rangle$ to denote the algebras of the above class and we call them *semilattices with zero element e*. We establish results about semilattices that are useful to show the correctness of transformations from satisfiability for Rare-logics into satisfiability for the corresponding standard modal logics (see Section 4.3). Indeed, we shall mainly investigate Rare-logics for which the sets of relations of the models are semilattices with zero element. In [Nov97, Jär97], similar algebraic structures are considered for studying dependence spaces and information systems.

Let $\mathtt{L}$ be a modal language dedicated to Rare-logics (we mainly use the notations from Section 3) and let $\mathtt{M}_{0d} = \{\mathtt{c}_0, \ldots, \mathtt{c}_n, \ldots\}$ be a countable set of modal constants. Let $\mathtt{C}_1, \ldots, \mathtt{C}_n$ be elements of $\mathtt{P}_0^i$ for some $i \in J$. For any integer $k \in \{0, \ldots, 2^n - 1\}$, we write $\mathtt{A}_k^{i*}$ to denote the Boolean expression (also called a *component*)

$$\mathtt{A}_k^{i*} \stackrel{\text{def}}{=} \mathtt{A}_1 \cap \ldots \cap \mathtt{A}_n$$

where for any $s \in \{1, \ldots, n\}$, $\mathtt{A}_s \stackrel{\text{def}}{=} \mathtt{C}_s$ if $bit_s(k) = 0$ ($bit_s(k)$ denoting the sth bit in the binary representation of k) otherwise $\mathtt{A}_s \stackrel{\text{def}}{=} -\mathtt{C}_s$. For instance, $\mathtt{A}^{i*}_{2^n-2} \stackrel{\text{def}}{=} -\mathtt{C}_1 \cap \ldots \cap -\mathtt{C}_{n-1} \cap \mathtt{C}_n$. For any $\mathtt{P}$-valuation V, the family

$$\{V(\mathtt{A}^{i*}_k) : k \in \{0, \ldots, 2^n - 1\}\}$$

is a partition of PAR_i. We write $\mathtt{A} \equiv \mathtt{B}$ [resp. $\mathtt{A} \equiv \perp$] for $\mathtt{A}, \mathtt{B} \in \mathtt{P}_i \stackrel{\text{def}}{\Leftrightarrow}$ for any $\mathtt{P}$-valuation V, $V(\mathtt{A}) = V(\mathtt{B})$ [resp. $V(\mathtt{A}) = \emptyset$]. For any parameter expressions $\mathtt{A} \in \mathtt{P}_i$ such that $\mathtt{P}^i_0(\mathtt{A}) \subseteq \{\mathtt{C}_1, \ldots, \mathtt{C}_n\}$ either $\mathtt{A} \equiv \perp$ or there is a unique non-empty set $Y = \{\mathtt{A}^{i*}_{i_1}, \ldots, \mathtt{A}^{i*}_{i_l}\}$ such that $\mathtt{A} \equiv \mathtt{A}^{i*}_{i_1} \cup \ldots \cup \mathtt{A}^{i*}_{i_l}$. There exists an effective procedure that computes Y in deterministic exponential-time in the size of $\mathtt{A}$ (for some reasonably succinct encoding). Proposition 4.1 below states how to transform a family $(X_P)_{P \subseteq PAR_i}$ into a family $(Y_{\mathtt{C}})_{\mathtt{C} \in \mathtt{M}_{0d}}$ when both families can be seen as the carrier sets of some semilattices.

PROPOSITION 4.1. (**From semilattices to families**) Let $\mathtt{C}_1, \ldots, \mathtt{C}_n$ be distinct elements of $\mathtt{P}^i_0$ for some $i \in J$ ($n > 0$), V be the restriction to $\mathtt{P}_i$ of a $\mathtt{P}$-valuation and $\langle \{X_P : P \subseteq PAR_i\}, \sqcup, e\rangle$ be a semilattice with zero element e such that
(H1) $X_\emptyset = e$ and (H2) for any $P, P' \subseteq PAR_i$, $X_{P \cup P'} = X_P \sqcup X_{P'}$.
Then, there is a family $(Y_{\mathtt{C}})_{\mathtt{C} \in \mathtt{M}_{0d}}$ such that

(I) $\{Y_{\mathtt{C}} : \mathtt{c} \in \mathtt{M}_{0d}\}$ is a finite subset of $\{X_P : P \subseteq PAR_i\}$;

(II) If $\mathtt{A}$ is a parameter expression built upon the parameter constants $\mathtt{C}_1, \ldots, \mathtt{C}_n$ such that $\mathtt{A} \equiv \mathtt{A}^{i*}_{i_1} \cup \ldots \cup \mathtt{A}^{i*}_{i_l}$, then $X_{V(\mathtt{A})} = Y_{\mathtt{C}_{i_1}} \sqcup \ldots \sqcup Y_{\mathtt{C}_{i_l}}$.

PROOF:The family $(Y_{\mathtt{C}})_{\mathtt{C} \in \mathtt{M}_{0d}}$is easy to define.

- for any $k \in \{0, \ldots, 2^n - 1\}$, $Y_{\mathtt{C}_k} \stackrel{\text{def}}{=} X_{V(\mathtt{A}^{i*}_k)}$;
- for any $\mathtt{c} \in \mathtt{M}_{0d} \setminus \{\mathtt{c}_0, \ldots, \mathtt{c}_{2^n-1}\}$, $Y_{\mathtt{C}} \stackrel{\text{def}}{=} Y_{\mathtt{C}_1}$ (arbitrary value).

By way of example we check that (II) holds ((I) is immediate). Take $\mathtt{A} \equiv \mathtt{A}^{i*}_{i_1} \cup \ldots \cup \mathtt{A}^{i*}_{i_l}$.

$X_{V(\mathtt{A})} = X_{V(\mathtt{A}^{i*}_{i_1}) \cup \ldots \cup V(\mathtt{A}^{i*}_{i_l})}$ (V is the restriction of a $\mathtt{P}$-valuation)
$= X_{V(\mathtt{A}^{i*}_{i_1})} \sqcup \ldots \sqcup X_{V(\mathtt{A}^{i*}_{i_l})}$ (by hypothesis (H2) and by the properties of $\sqcup$)
$= Y_{\mathtt{C}_{i_1}} \sqcup \ldots \sqcup Y_{\mathtt{C}_{i_l}}$ (by definition)

Q.E.D.

Proposition 4.1 and Proposition 4.2 below are used for defining faithful translations between Rare-logics of type 1,2,3,4 and 7 and their corresponding standard modal logics.

Proposition 4.2 below states how to transform a family $(Y_{\mathtt{C}})_{\mathtt{C} \in \mathtt{M}_{0d}}$ into a family $(X_P)_{P \subseteq PAR_i}$ when both families can be seen as the carrier sets of semilattices with zero element.

PROPOSITION 4.2. (**From families to semilattices**) Let $\mathtt{C}_1, \ldots, \mathtt{C}_n$ be distinct elements of $\mathtt{P}^i_0$ for some $i \in J$ ($n > 0$), $\langle Y, \sqcup, e\rangle$ be a semilattice with zero element and $(Y_{\mathtt{C}})_{\mathtt{C} \in \mathtt{M}_{0d}}$ be a family such that $\{Y_{\mathtt{C}} : \mathtt{c} \in \mathtt{M}_{0d}\} \subseteq Y$. Then, there is a subalgebra $\langle \{X_P : P \subseteq PAR_i\}, \sqcup, e\rangle$ of $\langle Y, \sqcup, e\rangle$ satisfying (H1)-(H2) from Proposition 4.1 and there is a restriction V of a $\mathtt{P}$-valuation to $\mathtt{P}_i$ such that

(I) PAR_i is finite and $card(PAR_i) = 2^n$;

(II) The statement (II) from Proposition 4.1 holds;

(III) for any $P \subseteq PAR_i$, there exists a (possibly empty) finite subset $\{x_1, \ldots, x_k\}$ of PAR_i such that $X_P = X_{\{x_1\}} \sqcup \ldots \sqcup X_{\{x_k\}} \sqcup X_\emptyset$.

PROOF: Let $\langle \{X_P : P \subseteq PAR_i\}, e, \sqcup \rangle$ and V be defined as follows:

- $PAR_i \stackrel{\text{def}}{=} \{0, \ldots, 2^n - 1\}$;
- $X_\emptyset \stackrel{\text{def}}{=} e$ and for any $\emptyset \neq P \subseteq PAR_i$, $X_P \stackrel{\text{def}}{=} \sqcup_{k \in P} Y_{c_k}$ ($\sqcup$ is commutative, associative and each P is finite);
- for any $s \in \{1, \ldots, n\}$, $V(\mathtt{C}_s) \stackrel{\text{def}}{=} \{k \in PAR_i : bit_s(k) = 0\}$ (for the other parameter constants V is not constrained provided it is the restriction of a $\mathtt{P}$-valuation which is always possible).

$\langle \{X_P : P \subseteq PAR_i\}, \sqcup, e \rangle$ and V satisfy the required conditions. (I) and (III) are by an easy verification.
(II) First, observe that for any $k \in \{0, \ldots, 2^n - 1\}$, $V(\mathtt{A}_k^{i*}) = \{k\}$.

$$\begin{aligned} X_{V(\mathtt{A})} &= X_{V(\mathtt{A}_{i_1}^{i*} \cup \ldots \cup \mathtt{A}_{i_l}^{i*})} \text{ (normal form of } \mathtt{A}) \\ &= X_{V(\mathtt{A}_{i_1}^{i*}) \cup \ldots \cup V(\mathtt{A}_{i_l}^{i*})} \text{ (} V \text{ is the restriction of a } \mathtt{P}\text{-valuation)} \\ &= X_{V(\mathtt{A}_{i_1}^{i*})} \sqcup \ldots \sqcup X_{V(\mathtt{A}_{i_l}^{i*})} = X_{\{i_1\}} \sqcup \ldots \sqcup X_{\{i_l\}} = Y_{\mathtt{C}_{i_1}} \sqcup \ldots \sqcup Y_{\mathtt{C}_{i_l}} \end{aligned}$$

Q.E.D.

In Proposition 4.1 and in Proposition 4.2 the X_P's and $Y_\mathtt{C}$'s are not necessarily relations. We are going to take advantage of these propositions when dealing with possible-world semantics for *polymodal modal* logics.

4.2 The normalization process N

In the rest of Section 4, unless otherwise stated, $\mathcal{L}$ is a Rare-logic of type $\mathtt{T} = \langle [\cup, \phi_2^1, \ldots, \phi_2^j], \emptyset^{i_\mathcal{L}} \rangle$ such that

- $i_\mathcal{L} \in \{0, 1\}$ and $\cup$ is the binary set operation for union;
- for $i \in J$, ϕ_2^i is a binary relation operation of profile $\langle \mathcal{D}(i), \mathcal{D}(i), \mathcal{D}(i) \rangle$ such that for any set W, $\langle \mathcal{P}(W^{\mathcal{D}(i)}), \phi_2^i(W), e_\mathcal{L}^i \rangle$ is a semilattice with zero element; $e_\mathcal{L}^i \stackrel{\text{def}}{=} \emptyset$ if $i_\mathcal{L} = 0$ otherwise $e_\mathcal{L}^i \stackrel{\text{def}}{=} W^{\mathcal{D}(i)}$;
- for $i \in J$, for any non-empty set W, $\langle W, e_\mathcal{L}^i \rangle \in X_i$ (we omit to write that $e_\mathcal{L}^i$ depends on W);
- for $i \in J$, there is $\oplus^i \in OP$ such that $\mathcal{I}(\oplus^i) = \phi_2^i$, in particular $\rho_{OP}^1(\oplus^i) = 2$ and $\rho_{OP}^2(\oplus^i) = \mathcal{D}(i) - 1$.

The assumptions capture Rare-logics of type 1,2 and 7. Moreover, by Proposition 3.3 results for Rare-logics of type 3 and 4 can be obtained easily.

Let $\mathtt{F}$ be an $\mathtt{L}$-formula such that for any $i \in J$, $\mathtt{P}_0^i(\mathtt{F}) = \{\mathtt{C}_1^i, \ldots, \mathtt{C}_{n_i}^i\}$ The degenerate case when $\mathtt{P}_0^i(\mathtt{F}) = \emptyset$ is omitted herein but it poses no extra difficulties. Indeed, assume that $\mathtt{P}_0^i(\mathtt{F}) = \emptyset$ for some $i \in J$. Then $\mathtt{F} \wedge [r(\mathtt{C}_1^i)](\overbrace{\top, \ldots, \top}^{\mathcal{D}(i)-1 \text{ times}}) \Leftrightarrow \mathtt{F}$ is $\mathcal{L}$-valid. So, for the forthcoming developments we can assume that $n_i \neq 0$ for $i \in J$.

As done in Section 4.1, for any $i \in J$ and any $k \in \{0, \ldots, 2^{n_i} - 1\}$ we write $\mathtt{A}_k^{i*}$ to denote the Boolean expression $\mathtt{A}_1 \cap \ldots \cap \mathtt{A}_{n_i}$ where for any $s \in \{1, \ldots, n_i\}$, $\mathtt{A}_s \stackrel{\text{def}}{=} \mathtt{C}_s^i$ if $bits(k) = 0$ otherwise $\mathtt{A}_s \stackrel{\text{def}}{=} -\mathtt{C}_s^i$. Let $r(\mathtt{A})$ be a basic modal expression occurring in $\mathtt{F}$ such that $\mathtt{A} \in \mathtt{P}_i$ for some $i \in J$ and $\mathtt{A} \equiv \mathtt{A}_{i_1}^{i*} \cup \ldots \cup \mathtt{A}_{i_l}^{i*}$ for some $\{i_1, \ldots, i_l\} \subseteq \{0, \ldots, 2^{n_i} - 1\}$. The first normal form of $r(\mathtt{A})$, written $\mathtt{N}_1(r(\mathtt{A}))$, is the basic modal expression $r(\mathtt{A}_{i_1}^{i*} \cup \ldots \cup \mathtt{A}_{i_l}^{i*})$. It is similar to the canonical disjunctive normal form for the propositional calculus (see e.g. [Lem65]). In the case when $\mathtt{A} \equiv \perp$, $\mathtt{N}_1(r(\mathtt{A})) \stackrel{\text{def}}{=} r(\mathtt{C}_1^i \cap -\mathtt{C}_1^i)$. We write $\mathtt{N}_1(\mathtt{F})$ to denote the formula obtained from $\mathtt{F}$ by substituting $r(\mathtt{A})$ by $\mathtt{N}_1(r(\mathtt{A}))$. $\mathtt{N}_1(\mathtt{F})$ is unique modulo associativity and commutativity of $\cup$ and $\cap$ (which is harmless in the sequel). Observe that $\mathtt{F} \Leftrightarrow \mathtt{N}_1(\mathtt{F})$ is $\mathcal{L}$-valid.

Until now, we did not use any assumption about $\mathcal{L}$. The second normal form of $\mathtt{F}$, written $\mathtt{N}_2(\mathtt{F})$, is the formula obtained from $\mathtt{N}_1(\mathtt{F})$ where each occurrence of $r(\mathtt{A}_{i_1}^{i*} \cup \ldots \cup \mathtt{A}_{i_l}^{i*})$ has been substituted by $r(\mathtt{A}_{i_1}^{i*}) \oplus^i \ldots \oplus^i r(\mathtt{A}_{i_l}^{i*})$. One can easily show that $\mathtt{F} \Leftrightarrow \mathtt{N}_2(\mathtt{F})$ is $\mathcal{L}$-valid. The second normalization process $\mathtt{N}_2$ depends on the type of the Rare-logic $\mathcal{L}$ (because of the specificity of the operator $\oplus^i$). We are now in position to define the mapping $\mathtt{N}$ from the set of $\mathcal{L}$-formulae into the set of $\mathcal{L}_d^{\{\mathtt{U}_{\mathcal{D}(1)}, \ldots, \mathtt{U}_{\mathcal{D}(j)}\}}$-formulae if $i_{\mathcal{L}} = 1$ and into the set of $\mathcal{L}_d^{\{\mathtt{O}_{\mathcal{D}(1)}, \ldots, \mathtt{O}_{\mathcal{D}(j)}\}}$-formulae in case $i_{\mathcal{L}} = 0$ where $\mathcal{L}_d$ is the standard modal logic from $\mathcal{L}$ -see Definition 3.5. For the sake of simplicity, the logic $\mathcal{L}_d^{\{\mathtt{U}_{\mathcal{D}(1)}, \ldots, \mathtt{U}_{\mathcal{D}(j)}\}}$ [resp. $\mathcal{L}_d^{\{\mathtt{O}_{\mathcal{D}(1)}, \ldots, \mathtt{O}_{\mathcal{D}(j)}\}}$] is denoted $\mathcal{L}_d^{\star}$.

Without any loss of generality (see Definition 3.5), we can assume that for any $i \in J$ we have in the language of $\mathcal{L}_d^{\star}$ the following stock $\{\mathtt{c}_0^i, \ldots, \mathtt{c}_{2^{n_i}-1}^i\}$ of distinct modal constants such that for $k \in \{0, \ldots, 2^{n_i} - 1\}$, $\mathcal{D}(\mathtt{c}_k^i) = \mathcal{D}(i)$ (here we keep the notations of the previous sections). The normal form of $\mathtt{F}$, written $\mathtt{N}(\mathtt{F})$, is the $\mathcal{L}_d^{\star}$-formula obtained from $\mathtt{N}_2(\mathtt{F})$ by substituting every occurrence of $r(\mathtt{C}_1^i \cap -\mathtt{C}_1^i)$ by $\mathtt{U}_{\mathcal{D}(i)}$ if $i_{\mathcal{L}} = 1$ [resp. by $\mathtt{O}_{\mathcal{D}(i)}$ if $i_{\mathcal{L}} = 0$] and every occurrence of $r(\mathtt{A}_k^{i*})$ by $\mathtt{c}_k^i$ for any $k \in \{0, \ldots, 2^{n_i} - 1\}$. $\mathtt{N}(\mathtt{F})$ is defined modulo the renaming of modal constants (which is harmless in the sequel). Observe that $\mathtt{N}(\mathtt{F})$ can be computed by an effective procedure.

EXAMPLE 4.1. Let $\mathcal{L}$ be a Rare-logic of type 1 such that $OP = \{\cap, \cup\}$ with $j = 1$, $\mathcal{D}(1) = 3$, $\rho_{OP}^1(\cap) = \rho_{OP}^1(\cup) = 2$ and $\rho_{OP}^2(\cap) = \rho_{OP}^2(\cup) = 2$. Let $\mathtt{F}$ be the following $\mathcal{L}$-formula:

$$\mathtt{F} = [r(\mathtt{C})](\mathtt{p}, \mathtt{r}) \wedge [r(\mathtt{C} \cap \mathtt{D})](\mathtt{q}, \mathtt{p}) \wedge [r(\mathtt{C} \cap \mathtt{D} \cap -\mathtt{C}) \cup r(\mathtt{D})](\mathtt{p}, \mathtt{r})$$

Here are the successive normal forms of $\mathtt{F}$:

- $\mathtt{N}_1(\mathtt{F}) = [r((\mathtt{C} \cap \mathtt{D}) \cup (\mathtt{C} \cap -\mathtt{D}))](\mathtt{p}, \mathtt{r}) \wedge [r(\mathtt{C} \cap \mathtt{D})](\mathtt{q}, \mathtt{p}) \wedge [r(\mathtt{C} \cap -\mathtt{C}) \cup r((\mathtt{C} \cap \mathtt{D}) \cup (-\mathtt{C} \cap \mathtt{D}))](\mathtt{p}, \mathtt{r})$;
- $\mathtt{N}_2(\mathtt{F}) = [r(\mathtt{C} \cap \mathtt{D}) \cap r(\mathtt{C} \cap -\mathtt{D})](\mathtt{p}, \mathtt{r}) \wedge [r(\mathtt{C} \cap \mathtt{D})](\mathtt{q}, \mathtt{p}) \wedge [r(\mathtt{C} \cap -\mathtt{C}) \cup (r(\mathtt{C} \cap \mathtt{D}) \cap r(-\mathtt{C} \cap \mathtt{D}))](\mathtt{p}, \mathtt{r})$;
- $\mathtt{N}(\mathtt{F}) = [\mathtt{c}_0 \cap \mathtt{c}_1](\mathtt{p}, \mathtt{r}) \wedge [\mathtt{c}_0](\mathtt{q}, \mathtt{p}) \wedge [\mathtt{U}_3 \cup (\mathtt{c}_0 \cap \mathtt{c}_2)](\mathtt{p}, \mathtt{r})$.

Example 4.1 should not mislead the reader about the complexity of the normalization process. Actually the normal form mapping may increase exponentially the size of the formulae although the number of subformulae is constant. The technique of components has been firstly used for information logics by Konikowska (see e.g. [Kon97a]) in order to define Rasiowa-Sikorski-style proof systems (dual tableaux) for relative similarity logics.

4.3 The satisfiability-preserving map $\mathtt{N}$

Proposition 4.3 is one of the main results of the paper. Indeed it states that $\mathtt{N}$ is a satisfiability-preserving transformation from $\mathcal{L}$-satisfiability into $\mathcal{L}_d^\star$-satisfiability. It shall be extended to a larger language and it has various consequences related to the finite model property, to complexity upper bounds and to the design of proof systems for Rare-logics.

PROPOSITION 4.3. (**Faithfulness of** $\mathtt{N}$) Let $\mathtt{F}$ be an $\mathcal{L}$-formula. The statements below are equivalent:

(I) $\mathtt{F}$ is $\mathcal{L}$-satisfiable;

(II) $\mathtt{N(F)}$ is $\mathcal{L}_d^\star$-satisfiable.

The proof of Proposition 4.3 consists in taking advantage of the semilattice structure of the families of relations in the $\mathcal{L}$-models.

PROOF: (I) $\rightarrow$ (II) Assume that $\mathtt{F}$ is $\mathcal{L}$-satisfiable. So there exist an $\mathcal{L}$-model $\mathcal{M} = \langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \models \mathtt{F}$.

For any $i \in J$, $\langle\{R^i_P : P \subseteq PAR_i\}, \phi^i_2(W), e^i_{\mathcal{L}}\rangle$ and the restriction of V to $\mathtt{P}_i$ satisfy the hypotheses of Proposition 4.1 with the set $\mathtt{P}^i_0(\mathtt{F})$ of parameter constants -if $\mathtt{P}^i_0(\mathtt{F}) = \emptyset$, then take $\{\mathtt{C}^i_1\}$. Hence, by Proposition 4.1, for any $i \in J$ there is a family $(Y^i_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}^i_0}$ satisfying the conditions (I) and (II) from Proposition 4.1. Let $\mathcal{M}'$ be the $\mathcal{L}_d^\star$-model $\langle W, (R_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}_0}, V'\rangle$ such that,

- for any $\mathtt{p} \in \mathtt{For}_0$, $V'(\mathtt{p}) \stackrel{\text{def}}{=} V(\mathtt{p})$ and for any $i \in J$ and for any $\mathtt{c} \in \mathtt{M}^i_0$ $R_{\mathtt{C}} \stackrel{\text{def}}{=} Y^i_{\mathtt{C}}$
- for any well-formed modal expression $\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_s)$, $V'(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_n)) \stackrel{\text{def}}{=} \mathcal{I}(\oplus)(W)(V'(\mathtt{a}_1), \ldots, V'(\mathtt{a}_n))$.

For $i \in J$, any relation generated from $\{R_{\mathtt{C}} : \mathtt{c} \in \mathtt{M}^i_0\}$ with $\phi^i_2(W)$ is equal to R^i_P for some $P \subseteq PAR_i$. Therefore $\mathcal{M}'$ is really an $\mathcal{L}_d^\star$-model. Moreover, for any $\mathtt{A} \in \mathtt{P}_i(\mathtt{F})$ such that $\mathtt{A} \not\equiv \perp$, $V(r(\mathtt{A})) = V'(\mathtt{c}^i_{i_1} \oplus^i \ldots \oplus^i \mathtt{c}^i_{i_l})$ where $\mathtt{A}^{i*}_{i_1} \cup \ldots \cup \mathtt{A}^{i*}_{i_l}$ is the normal form of $\mathtt{A}$. Consider the surjective map $\Phi : \mathtt{M(F)} \rightarrow \mathtt{M}_d(\mathtt{N(F)})$ ($\mathtt{M}_d$ denotes the set of modal expressions of $\mathcal{L}_d^\star$) such that for $\mathtt{a} \in \mathtt{M(F)}$, $\Phi(\mathtt{a})$ is obtained from $\mathtt{a}$ by substituting simultaneously,

- each $r(\mathtt{A})$ such that $\perp \equiv \mathtt{A} \in \mathtt{P}_i(\mathtt{F})$ by $\mathtt{U}_{\mathcal{D}(i)}$ if $i_{\mathcal{L}} = 1$ [resp. by $\mathtt{0}_{\mathcal{D}(i)}$ if $i_{\mathcal{L}} = 0$];
- each $r(\mathtt{A})$ such that $\mathtt{A}^{i*}_{i_1} \cup \ldots \cup \mathtt{A}^{i*}_{i_l} \equiv \mathtt{A} \in \mathtt{P}_i(\mathtt{F})$ by $\mathtt{c}^i_{i_1} \oplus^i \ldots \oplus^i \mathtt{c}^i_{i_l}$.

By Proposition 3.1, $\mathcal{M}, w \models \mathtt{F}$ iff $\mathcal{M}', w \models \mathtt{N(F)}$. Hence $\mathcal{M}', w \models \mathtt{N(F)}$.

(II) $\rightarrow$ (I) Now assume that $\mathtt{N(F)}$ is $\mathcal{L}_d^\star$-satisfiable. So there exist an $\mathcal{L}_d^\star$-model $\mathcal{M} = \langle W, (R_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}_0}, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \models \mathtt{N(F)}$. For any $i \in J$, $(R_{\mathtt{C}})_{\mathtt{C}\in\mathtt{M}^i_0}$ satisfies the hypothesis of Proposition 4.2 with the semilattice $\langle\mathcal{P}(W^{\mathcal{D}(i)}), \phi^i_2(W), e^i_{\mathcal{L}}\rangle$. By Proposition 4.2, for any $i \in J$, there exist a structure $\langle\{R^i_P : P \subseteq PAR_i\}, \phi^i_2(W), e^i_{\mathcal{L}}\rangle$ and the restriction V_i of some $\mathtt{P}$-valuation such that

- PAR_i is finite and $card(PAR_i) = 2^{n_i}$;
- the range of V_i is $\mathcal{P}(PAR_i)$;

- $\langle \{R^i_P : P \subseteq PAR_i\}, \phi^i_2(W), e^i_{\mathcal{L}}\rangle$ is a semilattice with zero element $e^i_{\mathcal{L}} = W^{\mathcal{D}(i)}$ [resp. $= \emptyset$] when $i_{\mathcal{L}} = 1$ [resp. when $i_{\mathcal{L}} = 0$];
- $R^i_\emptyset = e^i_{\mathcal{L}}$ and for any $P, P' \subseteq PAR_i$, $R^i_{P\cup P'} = \phi^i_2(W)(R^i_P, R^i_{P'})$.

Let $\mathcal{M}' = \langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}, V'\rangle$ be the $\mathcal{L}$-model such that

- V' restricted to $\mathtt{P}_i$ is V_i; for any $\mathtt{p} \in \mathtt{For}_0$, $V'(\mathtt{p}) \stackrel{\text{def}}{=} V(\mathtt{p})$;
- for any well-formed modal expression $\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_s)$,

$$V'(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_s)) \stackrel{\text{def}}{=} \mathcal{I}(\oplus)(W)(V'(\mathtt{a}_1), \ldots, V'(\mathtt{a}_s))$$

It is easy to check that $\mathcal{M}'$ is an $\mathcal{L}$-model. It is a routine task to check that for any $\mathtt{A} \in \mathtt{P}_i(\mathtt{F})$ such that $\mathtt{A} \not\equiv \perp$, $V'(r(\mathtt{A})) = V(\mathtt{c}^i_{i_1} \oplus^i \ldots \oplus^i \mathtt{c}^i_{i_l})$ -with the usual notation for the normal form of $\mathtt{A}$. Hence by applying Proposition 3.1 (with the map Φ described in the proof (I) $\rightarrow$ (II)), we conclude that $\mathcal{M}', w \models \mathtt{F}$ iff $\mathcal{M}, w \models \mathtt{N}(\mathtt{F})$. Q.E.D.

Proposition 4.3 entails that $\mathcal{L}^\star_d$ is decidable only if $\mathcal{L}$ is decidable. Proposition 4.4 will help stating the converse.

PROPOSITION 4.4. (**Reducing $\mathcal{L}^\star_d$ to $\mathcal{L}$**) There exists a polynomial-time transformation ("many-one reduction" [Pap94]) from $\mathcal{L}^\star_d$-satisfiability into $\mathcal{L}$-satisfiability.

PROOF: We define a map $\mathtt{N}^{-1}$ from the set of $\mathcal{L}^\star_d$-formulae into the set of $\mathcal{L}$-formulae such that $\mathtt{F}$ is $\mathcal{L}^\star_d$-satisfiable iff $\mathtt{N}^{-1}(\mathtt{F})$ is $\mathcal{L}$-satisfiable. So, let $\mathtt{F}$ be an $\mathcal{L}^\star_d$-formula such that for any $i \in J$, $\mathtt{M}^i_0(\mathtt{F}) = \{\mathtt{c}^i_0, \ldots, \mathtt{c}^i_{N_i}\}$. Let n_i be the smallest natural number such that $2^{n_i} - 1 \geq N_i$. Take n_i parameter constants from the language of $\mathcal{L}$, say $\mathtt{C}^i_1, \ldots, \mathtt{C}^i_{n_i}$. If $\mathtt{M}^i_0(\mathtt{F}) = \emptyset$, then we just consider $\mathtt{C}^i_1$. For any $k \in \{0, \ldots, N_i\}$ we define the parameter expression $\mathtt{A}^{i*}_k$ as follows: $\mathtt{A}^{i*}_k \stackrel{\text{def}}{=} \mathtt{A}_1 \cap \ldots \cap \mathtt{A}_n$ where for any $s \in \{1, \ldots, n\}$, $\mathtt{A}_s \stackrel{\text{def}}{=} \mathtt{C}^i_s$ if $bit_s(k) = 0$ otherwise $\mathtt{A}_s \stackrel{\text{def}}{=} -\mathtt{C}^i_s$. $\mathtt{N}^{-1}(\mathtt{F})$ is obtained from $\mathtt{F}$ by substituting each occurrence of $\mathtt{c}^i_k$ by $r(\mathtt{A}^{i*}_k)$ and each occurrence of $\mathtt{U}_l$ [resp.$\mathtt{O}_l$] by $r(\mathtt{C}^{l'}_1 \cap -\mathtt{C}^{l'}_1)$ where $\mathcal{D}(l') = l$ and $i_{\mathcal{L}} = 0$ [resp. $i_{\mathcal{L}} = 1$]. It is easy to see that $\mathtt{N}_2(\mathtt{N}^{-1}(\mathtt{F})) = \mathtt{N}^{-1}(\mathtt{F})$ and $\mathtt{N}(\mathtt{N}^{-1}(\mathtt{F}))$ is $\mathcal{L}^\star_d$-satisfiable iff $\mathtt{F}$ is $\mathcal{L}^\star_d$-satisfiable since $\mathtt{F}$ and $\mathtt{N}(\mathtt{N}^{-1}(\mathtt{F}))$ are equal modulo the renaming of modal constants. Hence $\mathtt{N}(\mathtt{N}^{-1}(\mathtt{F}))$ is $\mathcal{L}^\star_d$-satisfiable iff $\mathtt{N}^{-1}(\mathtt{F})$ is $\mathcal{L}$-satisfiable (by Proposition 4.3) iff $\mathtt{F}$ is $\mathcal{L}^\star_d$-satisfiable. Q.E.D.

Using the construction of the proof of Proposition 4.3, one can prove the proposition below.

COROLLARY 4.5. (**Decidability and finite model property correspondences**)

(I) $\mathcal{L}$ is decidable [resp. has the fmp] iff $\mathcal{L}^\star_d$ is decidable [resp. has the fmp].

(II) If $i_{\mathcal{L}} = 1$ [resp. $i_{\mathcal{L}} = 0$] and for any $i \in J$, $\mathcal{L}^\star_d$ is $\mathtt{U}_{\mathcal{D}(i)}$-simplifiable [resp. $\mathtt{O}_{\mathcal{D}(i)}$-simplifiable] then $\mathcal{L}$ is decidable iff $\mathcal{L}^{\star -}_d$ is decidable and $\mathcal{L}$ has the fmp iff $\mathcal{L}^{\star -}_d$ has the fmp.

EXAMPLE 4.2. Let $\mathcal{L}$ be the Rare-logic $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, X_1, 1\rangle$ such that $OP = \{\cap, \circ, ^*, \cup, ^{-1}\}$, $X_1 = Fr^2$ and $\cap$ [resp. $\circ$, *, $\cup$, $^{-1}$] is interpreted as the intersection [resp. composition, Kleene star, union, converse]. The Propositional Dynamic Logic PDL with the operators $\circ, ^*, \cup, ^{-1}, \cap$ is a fragment of $\mathcal{L}_d^{\star}$. Since such an extension of PDL does not have the finite model property -see e.g. [Vak92]-, by Corollary 4.5, $\mathcal{L}$ does not have the finite model property.

By way of example, we present some sets of operators that allow the simplifications assumed in Corollary 4.5(II).

PROPOSITION 4.6. (**Particular cases for simplification**) Let $\mathcal{L}_d = \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}\rangle$ be a standard modal logic such that $OP \subseteq \{\cap, \cup, \cup^*, \circ, -, ^{-1}\}$ (interpreted respectively as intersection, union, reflexive and transitive closure of union, composition, complement and converse) and any relation of the $\mathcal{L}_d$-models is binary.

(I) If $\circ \not\in OP$, then $\mathcal{L}_d^{\{\mathtt{U}_2\}}$ is $\mathtt{U}_2$-simplifiable;

(II) If for any $\mathcal{L}_d$-model $\langle W, (R_{\mathtt{c}})_{\mathtt{c}\in \mathtt{M}_0}, V\rangle$ and any $\mathtt{a} \in \mathtt{M}$, $V(\mathtt{a})$ and $V(\mathtt{a})^{-1}$ are serial, then $\mathcal{L}_d^{\{\mathtt{U}_2\}}$ is $\mathtt{U}_2$-simplifiable;

(III) If $- \not\in OP$, then $\mathcal{L}_d^{\{\mathtt{O}_2\}}$ is $\mathtt{O}_2$-simplifiable.

Observe that since $[\mathtt{O}_2]\mathtt{G} \Leftrightarrow \top$ is $\mathcal{L}_2^{\{\mathtt{O}_2\}}$-valid, in Proposition 4.6(III), one can effectively get rid of any occurrence of $\mathtt{O}_2$.

PROOF: (I) $\mathcal{L}_d^{\{\mathtt{U}_2\}}$ is $\mathtt{U}_2$-simplifiable is a consequence of the $\mathcal{L}_d^{\{\mathtt{U}_2\}}$-validity of the following formulas:

(i) $[\mathtt{a}]\mathtt{F} \Leftrightarrow [\mathtt{a}']\mathtt{F}$ where $\mathtt{a}'$ is obtained from $\mathtt{a}$ by replacing some occurrences of $\mathtt{U}_2 \cap \mathtt{b}$ by $\mathtt{b}$ [resp. $\mathtt{U}_2 \cup \mathtt{b}$ by $\mathtt{U}_2$, $\mathtt{U}_2 \cup^* \mathtt{b}$ by $\mathtt{U}_2$, $(\mathtt{U}_2)^{-1}$ by $\mathtt{U}_2$];

(ii) $[\mathtt{a}]\mathtt{F} \Leftrightarrow [\mathtt{a}']\mathtt{F}$ where $\mathtt{a}'$ is obtained from $\mathtt{a}$ by replacing some occurrences of $-\mathtt{U}_2 \cap \mathtt{b}$ by $-\mathtt{U}_2$ [resp. $-\mathtt{U}_2 \cup \mathtt{b}$ by $\mathtt{b}$, $-\mathtt{U}_2 \cup^* \mathtt{b}$ by $\mathtt{b} \cup^* \mathtt{b}$, $(-\mathtt{U}_2)^{-1}$ by $-\mathtt{U}_2$, $-(-\mathtt{U}_2)$ by $\mathtt{U}_2$];

Observe that the above replacements in (i) and (ii) do not introduce new modal operators. Now, using (i) and (ii), one can show that for any modal expression $\mathtt{a}$, $[\mathtt{a}]\mathtt{F}$ is equivalent to a formula $[\mathtt{a}']\mathtt{F}$ such that either $\mathtt{a}' = \mathtt{U}_2$ or $\mathtt{a}' = -\mathtt{U}_2$ or $\mathtt{a}'$ does not contain any occurrence of $\mathtt{U}_2$. Now, since $[-\mathtt{U}_2]\mathtt{H} \Leftrightarrow \top$ is $\mathcal{L}_d$-valid, we can easily prove that $\mathcal{L}_d^{\{\mathtt{U}_2\}}$ is $\mathtt{U}_2$-simplifiable.
(II) In the present case, $\circ \in OP$. We shall complete the points (i) and (ii) from (I).

(iii) $[\mathtt{a}]\mathtt{F} \Leftrightarrow [\mathtt{a}']\mathtt{F}$ where $\mathtt{a}'$ is obtained from $\mathtt{a}$ by replacing some occurrences of $-\mathtt{U}_2 \circ \mathtt{b}$ by $-\mathtt{U}_2$ [resp. $\mathtt{b} \circ -\mathtt{U}_2$ by $-\mathtt{U}_2$, $\mathtt{U}_2 \circ \mathtt{b}$ by $\mathtt{U}_2$, $\mathtt{b} \circ \mathtt{U}_2$ by $\mathtt{U}_2$].

The last two replacements are correct since by assumption, $V(\mathtt{b})$ and $V(\mathtt{b})^{-1}$ are serial binary relations.
(III) The fact that $\mathcal{L}_d^{\{\mathtt{O}_2\}}$ is $\mathtt{O}_2$-simplifiable follows from the $\mathcal{L}_d^{\{\mathtt{O}_2\}}$-validity of the following formulas:

(iv) $[\mathtt{a}]\mathtt{F} \Leftrightarrow [\mathtt{a}']\mathtt{F}$ where $\mathtt{a}'$ is obtained from $\mathtt{a}$ by replacing some occurrences of $\mathtt{O}_2 \cap \mathtt{b}$ by $\mathtt{O}_2$ [resp. $\mathtt{O}_2 \cup \mathtt{b}$ by $\mathtt{b}$, $\mathtt{O}_2 \cup^* \mathtt{b}$ by $\mathtt{b} \cup^* \mathtt{b}$, $(\mathtt{O}_2)^{-1}$ by $\mathtt{O}_2$].

Q.E.D.

EXAMPLE 4.3. The Rare-logic $\mathcal{L}$ defined by $\langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, X_1, 2\rangle$ with $j = 1$, $\mathcal{D}(1) = 2$, $OP = \{\cup, \circ, ^*, -\}$ (interpreted in the standard way) and $X_1 = Fr^2$ is undecidable (consequence of the undecidability of PDL with complement [Har84]).

Strong finite parameter set property Proposition 4.7 below states that the families $(PAR_i)_{i \in J}$ of parameters can be strongly constrained.

PROPOSITION 4.7. (**Strong finite parameter set property**) Let $\mathtt{F}$ be an $\mathcal{L}$-formula and for $i \in J$, $n_i = max(1, card(\mathtt{P}_0^i(\mathtt{F})))$. The following statements are equivalent:

(I) $\mathtt{F}$ is $\mathcal{L}$-satisfiable;

(II) for any equivalence relation S on J there is an $\mathcal{L}$-model

$$\langle W, (PAR_i)_{i \in J}, (R_P^1)_{P \subseteq PAR_1}, \ldots, (R_P^j)_{P \subseteq PAR_j}, V\rangle$$

such that:

- for some $w \in W$, $\mathcal{M}, w \models \mathtt{F}$;
- for any $i, i' \in J$, $\langle i, i'\rangle \in S$ implies $PAR_i = PAR_{i'}$;
- for any $i \in J$, $card(PAR_i) = max\{2^{n_{i'}} : i' \in S(i)\}$ (and therefore PAR_i is finite).

PROOF: (II) $\rightarrow$ (I) Obvious.
(I) $\rightarrow$ (II) Suppose $\mathtt{F}$ is $\mathcal{L}$-satisfiable. So, there exist an $\mathcal{L}$-model $\mathcal{M} = \langle W, (PAR_i)_{i \in J}, (R_P^1)_{P \subseteq PAR_1}, \ldots, (R_P^j)_{P \subseteq PAR_j}, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \models \mathtt{F}$. Let $\mathcal{M}'$ be the $\mathcal{L}_d^{\star}$-model built in the proof of Proposition 4.3 (for the part (I) $\rightarrow$ (II)). By construction, $\mathcal{M}', w \models \mathtt{N}(\mathtt{F})$. Now, let

$$\mathcal{M}'' = \langle W'', (PAR_i'')_{i \in J}, (R_P^{''1})_{P \subseteq PAR_1''}, \ldots, (R_P^{''j})_{P \subseteq PAR_j''}, V''\rangle$$

be the $\mathcal{L}$-model built in the proof of Proposition 4.3 (for the part (II) $\rightarrow$ (I)) from $\mathcal{M}'$. $\mathcal{M}$ [resp. $\mathcal{M}'$] in the proof of Proposition 4.3[(II) $\rightarrow$ (I)] is replaced here by $\mathcal{M}'$ [resp. $\mathcal{M}''$]. By construction, $\mathcal{M}'', w \models \mathtt{F}$ and for $i \in J$, $card(PAR_i'') = 2^{n_i}$. Now let us build the $\mathcal{L}$-model

$$\mathcal{M}''' = \langle W''', (PAR_i''')_{i \in J}, (R_P^{'''1})_{P \subseteq PAR_1'''}, \ldots, (R_P^{'''j})_{P \subseteq PAR_j'''}, V'''\rangle$$

as follows:

- $W''' \stackrel{\text{def}}{=} W''$;
- for $i \in J$, we write $rep(i)$ to denote some element of J such that $2^{n_{rep(i)}} = max\{2^{n_{i'}} : i' \in S(i)\}$. Since $card(PAR_i'') \leq card(PAR_{rep(i)}'')$ and $PAR_{rep(i)}''$ is finite, there is a surjective map $f_i : PAR_{rep(i)}'' \rightarrow PAR_i''$. $rep(i)$ is defined as a canonical representant of the class $S(i)$ and therefore for $i', i'' \in S(i)$, $rep(i') = rep(i'')$. As is usual, for $x \in PAR_i''$, $f_i^{-1}(x) \stackrel{\text{def}}{=} \{y \in PAR_{rep(i)}'' : f_i(y) = x\}$ and for $X \subseteq PAR_i''$, $f_i^{-1}(X) \stackrel{\text{def}}{=} \{y \in PAR_{rep(i)}'' : f_i(y) \in X\}$.

- for $i \in J$, $PAR'''_i \stackrel{\text{def}}{=} PAR''_{rep(i)}$;
- for $i \in J$, for $P \subseteq PAR_i$, $R'''^i_P \stackrel{\text{def}}{=} R''^i_{f_i(P)}$;
- the restriction of V''' to $\mathtt{For}_0$ is defined as the restriction of V'' to $\mathtt{For}_0$.

Consequently one can check that for any $P \subseteq PAR''_i$, $R'''^i_{f_i^{-1}(P)} = R''^i_P$. Let us check that $R'''^i_{P_1 \cup P_2} = R'''^i_{P_1} \phi^i_2(W''') R'''^i_{P_2}$:

$$R'''^i_{P_1 \cup P_2} = R''^i_{f_i(P_1 \cup P_2)} = R''^i_{f_i(P_1) \cup f_i(P_2)} = R''^i_{f_i(P_1)} \phi^i_2(W'') R''^i_{f_i(P_2)}$$

Now since, $R''^i_{f_i(P_1)} = R'''^i_{P_1}$, $R''^i_{f_i(P_2)} = R'''^i_{P_2}$ and $W'' = W'''$, we are done.

The valuation V''' is defined as follows: for $\mathtt{C} \in \mathtt{P}^i_0(\mathtt{F})$ $(i \in J)$ $V'''(\mathtt{C}) \stackrel{\text{def}}{=} f_i^{-1}(V''(\mathtt{C}))$. Observe that for any $P, P' \subseteq PAR_i$,

- $f_i^{-1}(PAR''_i \setminus P) = PAR'''_{rep(i)} \setminus f_i^{-1}(P)$ (f_i is surjective);
- $f_i^{-1}(P \cup P') = f_i^{-1}(P) \cup f_i^{-1}(P')$;
- $f_i^{-1}(P \cap P') = f_i^{-1}(P) \cap f_i^{-1}(P')$;

So, for $\mathtt{A} \in \mathtt{P}_i(\mathtt{F})$, $V'''(\mathtt{A}) = f_i^{-1}(V''(\mathtt{A}))$. So,

$$V'''(r(\mathtt{A})) = R'''^i_{V'''(\mathtt{A})} = R'''^i_{f_i^{-1}(V''(\mathtt{A}))} = R''^i_{V''(\mathtt{A})} = V''(r(\mathtt{A}))$$

From Proposition 3.1, it follows that $\mathcal{M}''', w \models \mathtt{F}$.

Q.E.D.

By fixing $S = J \times J$ in Proposition 4.7 we obtain that,

Corollary 4.8. $\mathcal{L}$ has the strong finite parameter set property.

Rare-logics of type 0 Until now, we have not dealt with the Rare-logics of type 0, the reason being that we can easily translate them into standard modal logics. We can show using the normal form $\mathtt{N}_1$ that for any Rare-logic $\mathcal{L}$ of type 0, $\mathcal{L}$ is decidable [resp. has the fmp] iff $\mathcal{L}_d$ is decidable [resp. has the fmp]. As a corollary of the proof, $\mathcal{L}$ have the sfpsp.

Rare-logics and Boolean homomorphisms In this paragraph, consider a Rare-logic $\mathcal{L}$ of type[3] $\langle 1,4,6 \rangle$ [resp. $\langle 2,3,6 \rangle$] such that for any $i \in J$, there is an intersection operator $\cap_i \in OP$ such that the profile of $\mathcal{I}(\cap_i)$ is $\langle \mathcal{D}(i), \mathcal{D}(i), \mathcal{D}(i) \rangle$, there is a union operator $\cup_i \in OP$ such that the profile of $\mathcal{I}(\cup_i)$ is $\langle \mathcal{D}(i), \mathcal{D}(i), \mathcal{D}(i) \rangle$ and there is a complement operator $-_i$ such that the profile of $\mathcal{I}(-_i)$ is $\langle \mathcal{D}(i), \mathcal{D}(i) \rangle$. Moreover, we assume that for $i \in J$, $X_i = Fr^{\mathcal{D}(i)}$. $\cap_i$ [resp. $\cup_i$, $-_i$] are indeed interpreted as set intersection [resp. set union, set complement]. Analogous types of logics have been introduced in [Bal97]. It is worth observing that either $C_{\langle 1,4 \rangle}$ or $C_{\langle 2,3 \rangle}$ entails C_6.

Let $\mathtt{F}$ be an $\mathcal{L}$-formula such that for any $i \in J$, $\mathtt{P}^i_0(\mathtt{F}) = \{\mathtt{C}^i_1, \ldots, \mathtt{C}^i_{n_i}\}$. The first normal form of $r(\mathtt{A})$ occurring in $\mathtt{F}$ is $\mathtt{N}_1(r(\mathtt{A}))$ inductively defined as follows:

- $\mathtt{N}_1(r(\mathtt{C})) = r(\mathtt{C})$ for $\mathtt{C} \in \mathtt{P}^i_0$; $\mathtt{N}_1(r(-\mathtt{A})) = -\mathtt{N}_1(r(\mathtt{A}))$;

[3] The Rare-logics of type $\langle 2,4 \rangle$ and $\langle 1,3 \rangle$ collapse to degenerate cases.

- $\mathtt{N}_1(r(\mathtt{A}_1 \cup \mathtt{A}_2)) = \mathtt{N}_1(r(\mathtt{A}_1)) \cap \mathtt{N}_1(r(\mathtt{A}_2))$ [resp. $= \mathtt{N}_1(r(\mathtt{A}_1)) \cup \mathtt{N}_1(r(\mathtt{A}_2))$];
- $\mathtt{N}_1(r(\mathtt{A}_1 \cap \mathtt{A}_2)) = \mathtt{N}_1(r(\mathtt{A}_1)) \cup \mathtt{N}_1(r(\mathtt{A}_2))$ [resp. $= \mathtt{N}_1(r(\mathtt{A}_1)) \cap \mathtt{N}_1(r(\mathtt{A}_2))$].

We write $\mathtt{N}_1(\mathtt{F})$ to denote the formula obtained from $\mathtt{F}$ by substituting every occurrence of $r(\mathtt{A})$ by $\mathtt{N}_1(r(\mathtt{A}))$. For any $i \in J$, we can assume that in the language of $\mathcal{L}_d$, the following stock $\{\mathtt{c}^i_1, \ldots, \mathtt{c}^i_{n_i}\}$ of distinct modal constants exists. $\mathtt{N}(\mathtt{F})$ is obtained from $\mathtt{N}_1(\mathtt{F})$ by substituting every occurrence of $r(\mathtt{C}^i_k)$ by $\mathtt{c}^i_k$.

Proposition 4.9. (**Faithfulness of** $\mathtt{N}$) Let $\mathtt{F}$ be an $\mathcal{L}$-formula. The statements below are equivalent:

(I) $\mathtt{F}$ is $\mathcal{L}$-satisfiable;

(II) $\mathtt{N}(\mathtt{F})$ is $\mathcal{L}_d$-satisfiable.

Proof: (I) $\rightarrow$ (II) Assume that $\mathtt{F}$ is $\mathcal{L}$-satisfiable. So there exist an $\mathcal{L}$-model

$$\langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}, V\rangle$$

and $w \in W$ such that $\mathcal{M}, w \models \mathtt{N}_1(\mathtt{F})$ (by Proposition 3.2). Let $\mathcal{M}'$ be the $\mathcal{L}_d$-model $\langle W, (R_\mathtt{C})_{\mathtt{C}\in\mathtt{M}_0}, V'\rangle$ such that,

- for any $i \in J$, for any $k \in \{1, \ldots, n_i\}$, $R_{\mathtt{C}^i_k} \stackrel{\text{def}}{=} R^i_{V(\mathtt{C}^i_k)}$;
- for any $\mathtt{c} \in \mathtt{M}^i_0 \setminus \{\mathtt{c}^i_1, \ldots, \mathtt{c}^i_{n_i}\}$, $R_\mathtt{C} \stackrel{\text{def}}{=} R^i_{V(\mathtt{C}^i_1)}$ (if $\mathtt{P}^i_0(\mathtt{F}) = \emptyset$, then take some arbitrary $\mathtt{C}^i_1 \in \mathtt{P}^i_0$);
- for any well-formed modal expression $\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_s)$,

$$V'(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_s)) \stackrel{\text{def}}{=} \mathcal{I}(\oplus)(W)(V'(\mathtt{a}_1), \ldots, V'(\mathtt{a}_s))$$

- for any $\mathtt{p} \in \mathtt{For}_0$, $V'(\mathtt{p}) \stackrel{\text{def}}{=} V(\mathtt{p})$.

One can check that $\mathcal{M}'$ is an $\mathcal{L}_d$-model (see the conditions of closure under $\cap$, $\cup$, $-$). It is a routine task to check that $\mathcal{M}', w \models \mathtt{N}(\mathtt{F})$.
(II) $\rightarrow$ (I) Assume that $\mathtt{N}(\mathtt{F})$ is $\mathcal{L}_d$-satisfiable. So there exist an $\mathcal{L}_d$-model $\mathcal{M} = \langle W, (R_\mathtt{C})_{\mathtt{C}\in\mathtt{M}_0}, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \models \mathtt{N}(\mathtt{F})$. Let $\mathcal{M}' = \langle W, (PAR_i)_{i\in J}, (R^1_P)_{P\subseteq PAR_1}, \ldots, (R^j_P)_{P\subseteq PAR_j}, V'\rangle$ be the $\mathcal{L}$-model such that

- for any $i \in J$, $PAR_i \stackrel{\text{def}}{=} W^{\mathcal{D}(i)}$;
- for any $\mathtt{p} \in \mathtt{For}_0$, $V'(\mathtt{p}) \stackrel{\text{def}}{=} V(\mathtt{p})$;
- for any $i \in J$ and $P \subseteq PAR_i$, $R^i_P \stackrel{\text{def}}{=} \begin{cases} W^{\mathcal{D}(i)} \setminus P \text{ if } \mathcal{L} \text{ is of type } \langle 1,4,6\rangle \\ P \text{ if } \mathcal{L} \text{ is of type } \langle 2,3,6\rangle \end{cases}$
- for any $i \in J$, $k \in \{1, \ldots, n_i\}$, $V'(\mathtt{C}^i_k) \stackrel{\text{def}}{=} \begin{cases} W^{\mathcal{D}(i)} \setminus R_{\mathtt{C}^i_k} \text{ if } \mathcal{L} \text{ is of type } \langle 1,4,6\rangle \\ R_{\mathtt{C}^i_k} \text{ if } \mathcal{L} \text{ is of type } \langle 2,3,6\rangle \end{cases}$
- for any well-formed expression $\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_s)$,

$$V'(\oplus(\mathtt{a}_1, \ldots, \mathtt{a}_s)) \stackrel{\text{def}}{=} \mathcal{I}(\oplus)(W)(V'(\mathtt{a}_1), \ldots, V'(\mathtt{a}_s))$$

The originality of the construction of $\mathcal{M}'$ rests on the definition of the PAR_i's. Observe that $R^i_{V'(\mathtt{C}^i_k)} = R_{\mathtt{C}^i_k}$. Indeed, for instance when $\mathcal{L}$ is of type $\langle 2,3,6\rangle$, each family $(R^i_P)_{P\subseteq PAR_i}$ can be seen as the identity isomorphism on the Boolean algebra $\langle \mathcal{P}(W^{\mathcal{D}(i)}), \cap, \cup, -, \emptyset, W^{\mathcal{D}(i)}\rangle$. It is easy to check that $\mathcal{M}'$ is an $\mathcal{L}$-model and $\mathcal{M}', w \models \mathtt{N}_1(\mathtt{F})$. Q.E.D.

Moreover, using arguments similar to those in Proposition 4.4, one can show that there is a polynomial-time transformation from $\mathcal{L}_d$-satisfiability into $\mathcal{L}$-satisfiability.

PROPOSITION 4.10. (**Decidability and finite model property correspondences**) Let $\mathcal{L}$ be a Rare-logic satisfying the conditions at the beginning of the present paragraph. Then, $\mathcal{L}$ is decidable [resp. has the fmp and the fpsp] iff $\mathcal{L}_d$ is decidable [resp. has the fmp].

EXAMPLE 4.4. Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{D}, \mathcal{I}, \mathcal{C}, X_1, \mathtt{T}\rangle$ be the Rare-logic of type $\langle 1,4,6\rangle$ [resp. $\langle 2,3,6\rangle$] such that $j = 1$, $\mathcal{D}(1) = 2$ and $OP = \{\cap, \cup, -\}$ respectively interpreted as intersection, union and complement operators. The logic $\mathcal{L}_d$ is the Boolean Modal Logic BML defined in [GP90] that is known to be decidable and satisfies the fmp [GP90]. By Proposition 4.10, $\mathcal{L}$ has the fpsp and the $\mathcal{L}$-satisfiability problem is decidable. If we add to OP the composition operator $\circ$ and the Kleene star $\star$, then the $\mathcal{L}$-satisfiability problem becomes undecidable since $\mathcal{L}_d$ is an extension of PDL with complement that is known to be undecidable [Har84].

5 Concluding remarks

In the paper, we have studied a class of polymodal logics with relative accessibility relations, the Rare-logics. Particular instances are the information logics from [Orło84, Orło88a, Orło95, Kon97a, Bal97]. We have shown how to translate Rare-logics into more standard modal logics (see extensions into combinatory dynamic logics in [DG]) and the other way around. Various kinds of algebraic properties for the families of relations in the models have been taken into account. The translations are interesting for their own sake, for instance they help understanding what is brought by adding a Boolean dimension to a logic.

In [DG] we are able to prove new decidability results about some Rare-logics in a unifying framework. The flexibility of the translations allows an extension when nominals are included in the language for atomic propositions and above all for atomic parameters (it is technically more involved). Some refinements to eliminate the universal operator are also presented in [DG]. Most of our results have a semantical flavour except that we shall also define translations of calculi between Rare-logics and corresponding standard modal logics.

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References

[Bal96a] Ph. Balbiani. A modal logic for data analysis. In W. Penczek and A. Szalas, editors, *MFCS'96, Krakow*, pages 167–179. LNCS 1113, Springer-Verlag, 1996.

[Bal96b] Ph. Balbiani. Modal logics with relative accessibility relations. In D. Gabbay and H.J. Ohlbach, editors, *FAPR'96, Bonn*, pages 29–41. LNAI, Springer-Verlag, 1996.

[Bal97] Ph. Balbiani. Axiomatization of logics based on Kripke models with relative accessibility relations. In *[Oe97]*, pages 553–578, 1997.

[BO99] Ph. Balbiani and E. Orlowska. A hierarchy of modal logics with relative accessibility relations. *Journal of Applied Non-Classical Logics, special issue in the Memory of George Gargov*, 1999. To appear.

[Dem99] S. Demri. A logic with relative knowledge operators. *Journal of Logic, Language and Information*, 8(2):167–185, 1999.

[DG] S. Demri and D. Gabbay. On modal logics characterized by models with relative accessibility relations: Part II. *this journal*. To appear.

[dG94] M. d'Agostino and D. Gabbay. A generalization of analytic deduction via labelled deductive systems. part I: Basic substructural logics. *Journal of Automated Reasoning*, 13:243–281, 1994.

[DG98] S. Demri and R. Goré. Display calculi for logics with relative accessibility relations. Technical Report TR-ARP-04-98, Automated Reasoning Project, Australian National University, May 1998.

[DO97] S. Demri and E. Orłowska. Logical analysis of indiscernibility. In *[Oe97]*, pages 347–380, 1997.

[DP90] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge Mathematical Textbooks. Cambridge University Press, 1990.

[Dün97] I. Düntsch. Rough sets and algebras of relations. In *[Oe97]*, pages 95–108, 1997.

[FdCO85] L. Fariñas del Cerro and E. Orłowska. DAL - A logic for data analysis. *Theoretical Computer Science*, 36:251–264, 1985.

[Gar86] G. Gargov. Two completeness theorems in the logic for data analysis. Technical Report 581, ICS, Polish Academy of Sciences, Warsaw, 1986.

[GP90] G. Gargov and S. Passy. A note on boolean modal logic. In P. Petkov, editor, *Summer School and Conference on Mathematical Logic '88*, pages 299–309. Plenum Press, 1990.

[GP92] V. Goranko and S. Passy. Using the universal modality: gains and questions. *Journal of Logic and Computation*, 2(1):5–30, 1992.

[GT75] R. Goldblatt and S. Thomason. Axiomatic classes in propositional modal logic. In J. Crossley, editor, *Algebra and Logic*, pages 163–173. Springer-Verlag, Lecture Notes in Mathematics 450, 1975.

[GV98] V. Goranko and D. Vakarelov. Hyperboolean algebras and hyperboolean modal logic. Technical Report 1/98, ??, January 1998.

[Har84] D. Harel. Dynamic logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, Volume II*, pages 497–604. Reidel, Dordrecht, 1984.

[HO91] A. Herzig and H.J. Ohlbach. Parameter structures for parametrized modal operators. In *IJCAI '91*, 1991.

[Jär97] J. Järvinen. Representation of information systems and dependences spaces, and some basic algorithms. Licentiate's thesis, 1997.

[Kon87] B. Konikowska. A formal language for reasoning about indiscernibility. *Bulletin of the Polish Academy of Sciences*, 35:239–249, 1987.

[Kon97a] B. Konikowska. A logic for reasoning about relative similarity. *Studia Logica*, 58(1):185–226, 1997.

[Kon97b] B. Konikowska. A logic for reasoning about similarity. In *[Oe97]*, pages 462–491, 1997.

[Lem65] E. Lemmon. *Beginning Logic.* Chapman and Hall, 1965.

[Nov97] M. Novotny. Applications of dependence spaces. In *[Oe97]*, pages 247–289, 1997.

[Oe97] E. Orłowska (ed.). *Incomplete Information: Rough Set Analysis.* Studies in Fuzziness and Soft Computing. Physica-Verlag, Heidelberg, 1997.

[OP84] E. Orłowska and Z. Pawlak. Expressive power of knowledge representation systems. *Int. Journal Man-Machines Studies*, 20:485–500, 1984.

[Orło84] E. Orłowska. Logic of indiscernibility relations. In A. Skowron, editor, *5th Symposium on Computation Theory, Zaborów, Poland*, pages 177–186. LNCS 208, Springer-Verlag, 1984.

[Orło85] E. Orłowska. Logic of nondeterministic information. *Studia Logica*, 44:93–102, 1985.

[Orło88a] E. Orłowska. Kripke models with relative accessibility and their applications to inferences from incomplete information. In G. Mirkowska and H. Rasiowa, editors, *Mathematical Problems in Computation Theory*, pages 329–339. Banach Center Publications, Volume 21 PWN - Polish Scientific Publishers, Warsaw, 1988.

[Orło88b] E. Orłowska. Logical aspects of learning concepts. *Journal of Approximate Reasoning*, 2:349–364, 1988.

[Orło89] E. Orłowska. Logic for reasoning about knowledge. *Zeitschr. f. Math. Logik und Grundlagen d. Math.*, 35:559–568, 1989.

[Orło90] E. Orłowska. Kripke semantics for knowledge representation logics. *Studia Logica*, 49(2):255–272, 1990.

[Orło93] E. Orłowska. Reasoning with incomplete information: rough set based information logics. In V. Alagar, S. Bergler, and F.Q. Dong, editors, *Incompleteness and Uncertainty in Information Systems Workshop*, pages 16–33. Springer-Verlag, October 1993.

[Orło95] E. Orłowska. Information algebras. In *AMAST'95, Montreal*, pages 50–65. LNCS 639, Springer-Verlag, 1995.

[Orło97] E. Orłowska. Introduction: What you always wanted to know about rough sets. In *[Oe97]*, pages 1–20, 1997.

[Pag97] P. Pagliani. Rough set theory and logic-algebraic structures. In *[Oe97]*, pages 109–192, 1997.

[Pap94] Ch. Papadimitriou. *Computational Complexity.* Addison-Wesley Publishing Compoany, 1994.

[Paw81] Z. Pawlak. Information systems theoretical foundations. *Information Systems*, 6(3):205–218, 1981.

[Pra80] V. Pratt. Applications of modal logic to programming. *Studia Logica*, 39:257–274, 1980.

[Rau84] C. Rauszer. An equivalence between indiscernibility relations in information systems and a fragment of intuitionistic logic. In A. Skowron, editor, *5th Symposium on Computation Theory, Zaborów, Poland*, pages 298–317. LNCS 208, Springer-Verlag, 1984.

[Vak91a] D. Vakarelov. Logical analysis of positive and negative similarity relations in property systems. In M. de Glas and D. Gabbay, editors, *First World Conference on the Fundamentals of Artificial Intelligence*, 1991.

[Vak91b] D. Vakarelov. A modal logic for similarity relations in Pawlak knowledge representation systems. *Fundamenta Informaticae*, 15:61–79, 1991.

[Vak91c] D. Vakarelov. Modal logics for knowledge representation systems. *Theoretical Computer Science*, 90:433–456, 1991.

[Vak92] D. Vakarelov. A modal logic for cyclic repeating. *Information and Computation*, 101:103–122, 1992.

[Vak97] D. Vakarelov. Information systems, similarity and modal logics. In *[Oe97]*, pages 492–550, 1997.

[Ven91] Y. Venema. *Many-dimensional modal logic*. PhD thesis, FWI, Amsterdam University, September 1991.