On Automatic Transitive Graphs

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Abstract. We study infinite automatic transitive graphs. In particular we investigate automaticity of certain Cayley graphs. We provide examples of infinite automatic transitive graphs that are not Cayley graphs. We prove that Cayley graphs of Baumslag–Solitar groups and the restricted wreath products of automatic transitive graphs with $\mathbb{Z}$ are automatic.

Keywords: finite automata, automatic graphs, transitive graphs

1 Introduction

This paper contributes to the field of automatic structures [11] with a particular emphasis on infinite automatic transitive graphs. Recall that a graph $G = (V, E)$ is **transitive** if for all vertices $u, v \in V$ there exists an automorphism of the graph that maps u into v. We postulate that the edges of our graphs are unordered pairs, and hence no loop exists from any vertex of the graph to itself.

Examples of transitive graphs are plentiful. The Rado graph, obtained as the Fraïssé limit of the class of all finite graphs, is transitive. So are Cayley graphs of finitely generated groups. Here we recall the definition of a Cayley graph. Let H be a finitely generated group with a finite set of generators X such that X does not contain the identity of the group. Define the Cayley graph of H with respect to X as follows: (a) the set of vertices of the Cayley graph is the set H, and (b) there is an edge between the vertices u and v if there exists an $x \in X$ such that $ux = v$ or $ux^{-1} = v$. Often Cayley graphs are directed and labelled graphs with labels from X, but for simplicity we omit the labels and directions on the edges.

We assume that the reader is familiar with the notion of finite automata and synchronous multi–head finite automata. Finite automata can be used to define the concept of an automatic graph. Call a graph $G = (V, E)$ **automatic** (FA–presentable) if it has an automatic presentation, i.e., if both the set of vertices V and the set of edges E are recognized by finite automata. The automaton that recognizes the set of edges is a synchronous 2–head finite automaton. We recall that for two strings $u_1 \dots u_k$ and $v_1 \dots v_m$ representing two vertices of V, the shorter one is padded with a padding symbol $\diamond$. Then such an automaton reads the string of paired symbols $(u_1, v_1)\dots(u_n, v_n)$ for $n = \max\{k, m\}$. Here we give two sets of general examples of automatic transitive graphs.

Example 1. Let H be a group generated by a finite set X. Consider its labelled and directed Cayley graph $\Gamma(H, X)$. Call the group H **automatic** if there exists a regular subset $L \subseteq (X \cup X^{-1})^*$ such that the natural mapping from L into G is bijective, and for any $x \in X$ the set of directed edges of $\Gamma(H, X)$ labelled by x is recognized by a synchronous 2–head finite automaton. For the theory of automatic groups the reader is referred to [5]. It is proved that FA–presentable finitely generated groups are virtually abelian [14]. This implies that the class of automatic groups properly contains FA–presentable finitely generated groups.

Example 2. As above, let H be a group generated by a finite set X. Consider its labelled and directed Cayley graph $\Gamma(H, X)$. We call the group H **Cayley automatic** [10] if there is a regular subset $L \subset \Sigma^*$ for some finite alphabet Σ that uniquely represents elements of H, and for every $x \in X$ the set of directed edges of $\Gamma(H, X)$ labelled by x is recognized by a synchronous 2–head finite automaton, i.e., $\Gamma(H, X)$ is FA–presentable. The class of Cayley automatic groups properly contains automatic groups and retains many of their properties. Clearly, a Cayley graph of a Cayley automatic group is automatic.

It is known that the Rado graph is not an automatic graph [12, 4], while the examples above show the abundance of transitive automatic graphs. Taking into account that the Rado graph is not automatic and the two examples above, we postulate the following conditions on transitive graphs G that will be assumed for the rest of the paper: (1) the graph G is infinite and connected, and (2) the degree of every vertex v in G is bounded. Since the graph is transitive, the second condition implies that all vertices of the graph have the same degree.

2 Contributions of The Paper

- We give an example of a sequence of automatic infinite transitive graphs that are not Cayley graphs. We also show that the limit of this sequence is an automatic transitive graph such that no Cayley graph is quasi–isometric to it. These examples show that the class of automatic transitive graphs properly contains the class of Cayley graphs of all Cayley automatic groups.
- In [10] it is proved that the Baumslag–Solitar groups $B(1, n)$ are Cayley automatic. We extend this result, and prove that all Baumslag–Solitar groups $B(m, n)$ are Cayley automatic. The proof is based on finding proper normal forms of group elements through HNN–extensions.
- In [10] it is proved that the wreath product of any finite group with the group $\mathbb{Z}$ of integers is Cayley automatic. These groups are examples of groups that are not automatic. We extend the result from [10] and prove that the wreath product of any Cayley automatic group with the group $\mathbb{Z}$ preserves Cayley automaticity. We also generalize this result to the class of locally finite automatic graphs. In addition, we make some relevant remarks on the Cayley automaticity of wreath products of finitely generated groups and infinite non–cyclic groups.

3 Examples of Infinite Transitive Automatic Graphs

We start this section with a simple example of an automatic infinite transitive graph which is not a Cayley graph. This example showcases that the class of transitive infinite automatic graphs properly contains the class of Cayley graphs of all Cayley automatic groups.

The example is presented in [15]; but our description follows the beginning of § 2 in [3]. Consider the infinite 5–regular tree T_5 and the bipartite graph $K_{2,3}$. Define the graph $H_{2,3}$ as follows. First, replace vertices of T_5 by disjoint copies of $K_{2,3}$. Second, for each edge $\{u, v\}$ of T_5, identify a vertex of the $K_{2,3}$ corresponding to u with a vertex of the $K_{2,3}$ corresponding to v. No point in any $K_{2,3}$ is identified more than once, and a vertex in a class of size 2 is always identified with a vertex in a class of size 3 and vice versa. See Fig. 1.

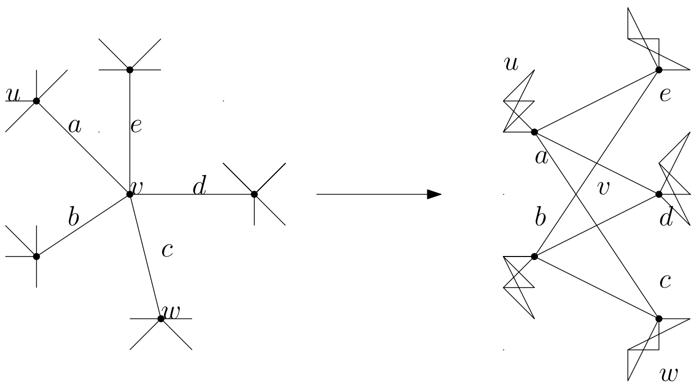

Fig. 1. Constructing the non–Cayley graph $H_{2,3}$ from T_5

Proposition 1. *The graph $H_{2,3}$ is a transitive infinite automatic graph that is not a Cayley graph for any finitely generated group.*

Proof. It is clear that $H_{2,3}$ is transitive. We prove that $H_{2,3}$ is not a Cayley graph for any finitely generated group. Recall that a graph H is a Cayley graph of some group if and only if there exists a subgroup S of automorphisms of H that acts freely and transitively on H. Here S acts transitively on H if for every pair u, v of vertices of H there exists an action of S that maps u to v; also, S acts freely on H if every nontrivial action of S moves every vertex of H.

Suppose that there exists a subgroup S of automorphisms of $H_{2,3}$ that acts freely and transitively on $H_{2,3}$. Let K be one of the $K_{2,3}$ making up $H_{2,3}$, and let $\{a, b\}$ and $\{c, d, e\}$ be the vertices in a class of size 2 and 3, respectively. It is not hard to see that any automorphism that sends an element of $\{c, d, e\}$ back into $\{c, d, e\}$ must fix K. Now let $\theta \in S$ be an automorphism such that $\theta(c) = d$;

therefore, it must swap a and b. Let $\theta' \in S$ be such that $\theta'(c) = e$; again, it must swap a and b. Then $\theta'\theta^{-1} \in S$ sends d to e and fixes a and b. This gives us a contradiction. Hence $H_{2,3}$ is not a Cayley graph for any finitely generated group.

To prove automaticity of $H_{2,3}$, we note that T_5 is an automatic graph. The definition of $H_{2,3}$ shows that we can identify the elements of $H_{2,3}$ and a quotient set $T_5 \times K_{2,3}/\equiv$, where $\equiv$ is recognizable by a finite automaton equivalence relation on $T_5 \times K_{2,3}$. An easy analysis shows that $H_{2,3}$ under the representation induced by $T_5 \times K_{2,3}/\equiv$ is automatic. □

The construction above can easily be generalized to build the automatic transitive graphs $H_{n,m}$ from the regular tree T_{n+m} and the bipartite graph $K_{n,m}$, where $n \neq m$ and $n \geqslant 2$ and $m \geqslant 3$:

Corollary 1. *The graphs $H_{n,m}$ are transitive infinite automatic graphs that are not Cayley graphs.* □

We now construct another transitive automatic graph $G^{\star}$. Our description follows [3]. The graph is obtained as a limit of automatic graphs $G_0, G_1, G_2, \ldots$ such that each G_i is quasi–isometric[1] to T_5. The conjecture that $G^{\star}$ is not quasi–isometric to any Cayley graph was stated in [3]. The conjecture was confirmed in [7].

First we recall the definition of the line graph of a digraph D. The **line graph** of a digraph $D = (V, E)$ is the digraph $D' = (V', E')$ with $V' = E$ and $E' = \{((u, v), (u', v')) \mid (u, v), (u', v') \in V' \;\wedge\; v = u'\}$. Now define the sequence $D_0, D_1, \ldots$ of graphs: D_0 is the 5–regular tree such that every node of the tree has exactly two ingoing edges and exactly three outgoing edges, and set D_{i+1} to be the line graph of D_i for $i \geq 0$.

Proposition 2. *[3, Proposition 3] The digraph D_n is isomorphic to the digraph whose vertices are the directed paths of length n in D_0, with an edge from $x_1x_2 \ldots x_{n+1}$ to $y_1y_2 \ldots y_{n+1}$ if $y_i = x_{i+1}$ for all $1 \leqslant i \leqslant n$.* □

Since D_0 is automatic, Proposition 2 implies that each digraph D_n is automatic.

Let G_n be the graph obtained from D_n by removing the directions from the edges of D_n. For instance, G_1 is isomorphic to $H_{2,3}$. Since each D_n is automatic, the graph G_n is also automatic. All the graphs $G_1, G_2, \ldots$ are transitive and they are not Cayley graphs. Moreover, each G_i is quasi–isometric to T_5. In [3] it is proved that the sequence $G_0, G_1, \ldots$ converges to G^{*}, and that the graph $G^{\star}$ can be described as follows.

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[1] We recall that two graphs G_1 and G_2 are quasi–isometric if there exist a map $\theta : V(G_1) \to V(G_2)$ and some $\lambda \geqslant 1$ such that $\frac{1}{\lambda} d_{G_1}(x, y) - \lambda \leqslant d_{G_2}(\theta(x), \theta(y)) \leqslant \lambda d_{G_1}(x, y) + \lambda$ for all $x, y \in V(G_1)$, and for any point $y \in V(G_2)$ there is some $x \in V(G_1)$ such that $d_{G_2}(\theta(x), y) \leqslant \lambda$; where $d_{G_1}(x, y)$ and $d_{G_2}(\theta(x), \theta(y))$ are the graph distances between x, y and $\theta(x), \theta(y)$ in G_1 and G_2, respectively. For more on quasi–isometries we refer the reader to [8].

Let X be a 3–regular tree where each node has in–degree 2 and out–degree 1. Let Y be a 4–regular tree in which each node has in–degree 1 and out–degree 3. Fix a vertex $O_1 \in V(X)$ and a vertex $O_2 \in V(Y)$. For each $x \in X$, set $r(x)$ to be the signed distance from O_1 to x, i.e., if the unique undirected path from O_1 to x in X has s forward edges and t backward edges then $r(x) = s - t$. Define $r(y)$ similarly for each $y \in Y$. The set of vertices of G^* is the set $\{(x, y) \in X \times Y : r(x) = r(y)\}$, and G^* has an arc from (x, y) to (x', y') if $(x, x') \in E(X)$ and $(y, y') \in E(Y)$. Thus, the results in [3] and [7] imply:

Theorem 1. *The structure $G^\star$ is a transitive automatic graph such that no Cayley graph is quasi–isometric to it.* □

4 Baumslag–Solitar Groups

The Baumslag–Solitar groups are defined as a particular class of two–generator one–relator groups [1]. They play an important role in combinatorial and geometric group theory.

Definition 1. *For given integers $m, n \geqslant 0$ the Baumslag–Solitar group $BS(m, n)$ is a two–generator one–relator group defined as follows:*

$$BS(m, n) = \langle a, t \,|\, t^{-1} a^m t = a^n \rangle.$$

We will further suppose that $m, n \neq 0$; if $m = 0$ or $n = 0$ then $BS(m, n)$ is isomorphic to a free product of a cyclic group and $\mathbb{Z}$. The groups $BS(m, n)$ are automatic if and only if $m = n$ [5]. It is also known that $BS(m, n)$ are all asynchronously automatic [5]. In this section we prove that the Baumslag–Solitar groups $BS(m, n)$ are Cayley automatic groups for all $m, n \in \mathbb{N}$.

In [10] it is proved that the Baumslag–Solitar groups $BS(1, n), n \in \mathbb{N}$ are Cayley automatic. The proof is based on representing group elements as linear functions acting on the real line. We extend this theorem to the class of all Baumslag–Solitar groups. Our proof is based on a normal form for elements of $BS(m, n)$ obtained from representing $BS(m, n)$ as a Higman–Neumann–Neumann (HNN) extension.

We now recall a general construction for HNN extension and the normal form theorem for HNN extension; we will follow [13, Chapter IV, § 2]. The HNN extension of G relative to subgroups $A, B \leqslant G$ and an isomorphism $\phi : A \to B$ is the group

$$G^* = \langle G, t; t^{-1} a t = \phi(a), a \in A \rangle,$$

where t is usually called a stable letter. The normal form theorem for HNN extension, as presented in [13], is as follows. For the proof see the statement (II) of Theorem 2.1 in [13]:

Theorem 2. *Suppose we have fixed all the representatives for right cosets of subgroups A and B in G, where the identity $1 \in G$ represents the subgroups A and B. Then every element w of G^* has a unique representation as $w = g_0 t^{\epsilon_1} \cdots t^{\epsilon_\ell} g_\ell$ where*

– *g_0 is an arbitrary element of G,*
– *if $\epsilon_i = -1$, then g_i is a representative of a right coset of A in G,*
– *if $\epsilon_i = +1$, then g_i is a representative of a right coset of B in G, and*
– *there is no consecutive subsequence $t^{\epsilon}, 1, t^{-\epsilon}$.* □

Replacing representatives of right cosets by representatives of left cosets, with 1 representing both A and B, one obtains the following corollary of Theorem 2.

Corollary 2. *Suppose that we have fixed all the representatives for left cosets of A and B in G with the identity $1 \in G$ representing A and B. Then every element w of G^* has a unique representation as $w = g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} g_0$, where*

– *g_0 is an arbitrary element of G,*
– *if $\epsilon_i = -1$, then g_i is a representative of a left coset of B in G,*
– *if $\epsilon_i = +1$, then g_i is a representative of a left coset of A in G,*
– *there is no consecutive subsequence $t^{\epsilon}, 1, t^{-\epsilon}$.* □

Note that the Baumslag–Solitar group $BS(m,n)$ is the HNN extension of $\mathbb{Z} = \langle a \rangle$ relative to subgroups $m\mathbb{Z}$ and $n\mathbb{Z}$ and the isomorphism $\phi : m\mathbb{Z} \to n\mathbb{Z}$ that maps a^m to a^n. Using Corollary 2 we will show that $BS(m,n)$ is a Cayley automatic group.

Theorem 3. *The group $BS(m,n)$ is a Cayley automatic group.*

Proof: We put $1, \dots, a^{m-1}$ and $1, \dots, a^{n-1}$ to be the representatives for the left cosets of subgroups $m\mathbb{Z}$ and $n\mathbb{Z}$ in $\mathbb{Z}$, respectively. By Corollary 2, every element $w \in BS(m,n)$ has a unique representation as

$$w = g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} g_0, \tag{1}$$

where $g_0 = a^k$ for some $k \in \mathbb{Z}$, and if $\epsilon_i = -1$ then $g_i \in \{1, a, \dots, a^{n-1}\}$, if $\epsilon_i = +1$ then $g_i \in \{1, a, \dots, a^{m-1}\}$, and there is no consecutive subsequence $t^{\epsilon}, 1, t^{-\epsilon}$.

The right–multiplication by the generator a transforms the normal form of w as follows:

$$g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^k \xrightarrow{\times a} g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^{k+1}. \tag{2}$$

Let $k = mp + r$ where $p \in \mathbb{Z}$ and $r \in \{0, \dots, m-1\}$. The right–multiplication by the generator t transforms a normal form of w as follows:

– if $r \neq 0$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^r t a^{np}, \tag{3}$$

– if $r = 0$, and $\ell \geqslant 1, \epsilon_1 = -1$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_2 t^{\epsilon_2} g_1 t^{-1} a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_2 t^{\epsilon_2} (g_1 a^{np}), \tag{4}$$

– if $r = 0$ and $\ell \geqslant 1, \epsilon_1 = 1$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_1 t a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_1 t \, 1 \, t a^{np}, \tag{5}$$

– if $r = 0$ and $\ell = 0$ then

$$a^k \xrightarrow{\times t} a^r t a^{np}. \tag{6}$$

Combining (1) with (2)–(6), we see that $BS(m, n)$ is Cayley automatic. We note that in order to construct corresponding finite automata, one can use a m–ary representation of $k \in \mathbb{Z}$ for which the map $k = mp + r \to np$ is recognizable by a finite automaton. $\square$

Remark 1. We note that the map $k = mp + r \to mp$ is recognizable by a finite automaton for a unary representation of k. It follows from the proof of Theorem 3 that, if $m = n$, then $BS(m, n)$ is automatic.

Remark 2. We recall that $BS(m, n)$ is not automatic if $m \neq n$. Indeed, a group that is automatic satisfies a quadratic isoperimetric inequality (for the proof see, e.g., [5, Theorem 2.3.12]). However, by choosing a proper family of cycles in the Cayley graph, it can be shown that $BS(m, n)$ does not satisfy a polynomial isoperimetric inequality (see the explanation in [5, § 7.4]).

Remark 3. It is known that $BS(m, n)$ is an asynchronously automatic group (see the explanation in [5, § 7.4]). This fact can be alternatively derived from Theorem 3. To show it, one can observe that the map $k = mp + r \to np$ is recognizable by an asynchronous automaton for the unary representation of k.

5 Wreath Products

We start by recalling the definition of the restricted wreath products of two groups A and B; we will follow [9, § 6.2].

Let A and B be groups. We denote by $A^{(B)}$ the group of all functions $f : B \to A$ having finite support and the usual multiplication rule; recall that a function $f : B \to A$ has finite support if $f(x) \neq e$ for only finite number of $x \in B$, where e is the identity of A. Let us define a homomorphism $\tau : B \to \mathrm{Aut}(A^{(B)})$ as follows: for a given $f \in A^{(B)}$, an automorphism $\tau(b)$ maps f to $f^b \in A^{(B)}$, where $f^b(x) = f(bx)$ for all $x \in B$. For given groups A and B, the restricted wreath product $A \wr B$ is defined to be the semidirect product $A^{(B)} \rtimes_\tau B$. Thus, $A \wr B$ is the Cartesian product $B \times A^{(B)}$ with multiplication given by

$$(b, f) \cdot (b', f') = (bb', f^{b'} f'), \tag{7}$$

where $f^{b'}(x) = f(b'x)$.

For our purposes we will use the converse order for representing elements of a wreath product. Namely, we will represent an element of $A \wr B$ as a pair (f, b), where $f \in A^{(B)}$ and $b \in B$. For this representation the group multiplication is given by

$$(f, b) \cdot (f', b') = (f f'^{b^{-1}}, bb'), \tag{8}$$

where $f'^{b^{-1}}(x) = f'(b^{-1}x)$. There exist natural embeddings $B \to A \wr B$ and $A^{(B)} \to A \wr B$ mapping b to $(\mathbf{e}, b)$ and f to (f, e) respectively, where $\mathbf{e}$ is the identity of $A^{(B)}$ and e is the identity of B. For the sake of simplicity, we will identify B and $A^{(B)}$ with the corresponding subgroups of $A \wr B$.

5.1 Wreath Products of Finitely Generated Groups and $\mathbb{Z}$

For a given finitely generated group G, let us consider the wreath product $G \wr \mathbb{Z}$. We represent an element of $G \wr \mathbb{Z}$ as a pair (f, z), where $f \in G^{(\mathbb{Z})}$ and $z \in \mathbb{Z}$. By (8), the multiplication in $G \wr \mathbb{Z}$ on pairs (f, z) and (f', z') is given by:

$$(f, z)(f', z') = (f f'^{-z}, z + z'),$$

where $f'^{-z}(x) = f'(x - z)$.

Let $g_1, \dots, g_n$ be generators of G and $\mathbb{Z} = \langle t \rangle$; let us identify t with $1 \in \mathbb{Z}$. We denote by $f_1, \dots, f_n$ the functions from $G^{(\mathbb{Z})}$ such that $f_i(0) = g_i$ and $f_i(x) = e$ for all $x \neq 0 \in \mathbb{Z}$, $i = 1 \dots n$, where e is the identity of G. The pairs $(f_1, 0), \dots, (f_n, 0)$ and $(\mathbf{e}, t)$ generate $G \wr \mathbb{Z}$. The right–multiplications of a pair (f, z) by these generators are given by:

$$(f, z)(\mathbf{e}, t) = (f, z + 1), \tag{9}$$

$$(f, z)(f_i, 0) = (f f_i^{-z}, z),\ i = 1, \dots, n, \tag{10}$$

where $f_i^{-z}(z) = g_i$, $f_i^{-z}(x) = e$ for all $x \neq z$ and $\mathbf{e}$ is the identity of $G^{(\mathbb{Z})}$; one can obtain $f f_i^{-z}$ from f by changing the value of the function f at the point z to the one equal to $f(z) g_i$.

It is useful to imagine an element $(f, z) \in G \wr \mathbb{Z}$ as a bi–infinite string of elements of G, such that only a finite number of elements are not equal to $e \in G$, with the tape head in position z pointing at the corresponding element of this string. Then the identity $(\mathbf{e}, 0)$ of $G \wr \mathbb{Z}$ corresponds to the string where all elements are equal to $e \in G$ with the tape head in the origin $0 \in \mathbb{Z}$.

For the sake of convenience, let t and f_i denote the corresponding elements $(\mathbf{e}, t)$ and (f_i, e) in $G \wr \mathbb{Z}$. Then, in the wreath product $G \wr \mathbb{Z}$, the right–multiplication by t shifts the tape head to the right by one whilst that of by t^{-1} shifts the tape head to the left by one; the right–multiplication by f_i changes the value of the element that the tape head is pointed at by multiplying it by g_i. Therefore, a word $t^k f_{i_1} \cdots f_{i_\ell} t^{-k}$ can be interpreted as follows: the tape head makes k steps from the origin either to the right or left, for $k > 0$ or $k < 0$, respectively, then the k^{th} element is changed from e to $g_{i_1} \cdots g_{i_\ell}$, and then the tape head returns back to the origin.

For each $g \in G$ let us fix a geodesic word $g_{i_1}^{\pm 1} \cdots g_{i_\ell}^{\pm 1}$ representing g with respect to the generators $g_1, \dots, g_n$. There exist two natural ways to represent uniquely elements of $G \wr \mathbb{Z}$ [2]. For a given $w \in G \wr \mathbb{Z}$ let the right–first normal form $rf(w)$ be the following:

$$rf(w) = (t^{i_1} v_{i_1} t^{-i_1}) \cdots (t^{i_k} v_{i_k} t^{-i_k})(t^{-j_1} v_{j_1} t^{j_1}) \cdots (t^{-j_\ell} v_{j_\ell} t^{j_\ell}) t^m, \tag{11}$$

where $i_k > \ldots i_1 \geqslant 0$, $j_l > \ldots j_1 > 0$, $m \in \mathbb{Z}$, and $v_{i_1}, \ldots, v_{i_k}, v_{j_1}, \ldots, v_{j_\ell}$ are the corresponding geodesic words. The left–first normal form $lf(w)$ is given by:

$$lf(w) = (t^{-j_1} v_{j_1} t^{j_1}) \cdots (t^{-j_\ell} v_{j_\ell} t^{j_\ell})(t^{i_1} v_{i_1} t^{-i_1}) \cdots (t^{i_k} v_{i_k} t^{-i_k}) t^m. \tag{12}$$

In the right–first normal form the tape head first moves to the right from the origin whilst in the left–first form the tape head first moves to left from the origin. By [2], either $rf(w)$ or $lf(w)$, taken in the reduced form, provide the minimal length representative of w.

The normal forms (11) and (12) give us a clue how to show that the wreath product $G \wr \mathbb{Z}$ is Cayley automatic if G itself is Cayley automatic.

Theorem 4. *For a given Cayley automatic finitely generated group G the wreath product $G \wr \mathbb{Z}$ is Cayley automatic.*

Proof. Let G be a Cayley automatic group with a finite set of generators $S = \{g_1, \ldots, g_n\}$. Let P be a regular language of normal forms that provide an automatic representation of the labelled and directed Cayley graph $\Gamma(G, S)$. Without loss of generality we will suppose that each element $w \in G$ has a unique word $v \in P$ as a representative. In order to prove that $G \wr \mathbb{Z}$ is Cayley automatic, we need to construct a regular language Q representing elements of $G \wr \mathbb{Z}$ and describe automata recognizing right–multiplications by the generators $(\mathbf{e}, t)$ and $(f_i, 0)$, see (9) and (10) respectively.

To obtain the alphabet of Q, let us add to the alphabet of P four additional symbols: A, B, C and $\#$. For a given element of $G \wr \mathbb{Z}$ we use the symbol A to specify the position of the origin $0 \in \mathbb{Z}$, the symbol C to specify the position of the tape head $m \in \mathbb{Z}$, and the symbol B will be used instead of A and C in case the positions of the origin and the tape head coincide: $m = 0$. The symbol $\#$ is used to separate subwords representing elements of G; at the positions corresponding to the origin and the tape head, the symbols A, C or B should be used instead of $\#$. Given an element of $G \wr \mathbb{Z}$, we represent it as a finite string of the following form:

$$v_{-j} \# \ldots \# v_{-1} A v_0 \# v_1 \# \ldots \# v_{m-1} C v_m \# v_{m+1} \ldots \# v_i, \tag{13}$$

where $v_{-j}, \ldots, v_i$ are the corresponding words of the language P and v_{-j} and v_i represent the leftmost and rightmost nontrivial elements of G. The language of such finite strings Q is regular.

The right–multiplication by $(\mathbf{e}, t)$ corresponds to the set of 2–tuples of the form:

$$\begin{pmatrix} \ldots \ \# \ v_{m-1} \ C \ v_m \ \# \ v_{m+1} \ \# \ \ldots \\ \ldots \ \# \ v_{m-1} \ \# \ v_m \ C \ v_{m+1} \ \# \ \ldots \end{pmatrix},$$

which is, clearly, recognizable by a 2–tape synchronous finite automaton.

The right–multiplication by $(f_i, 0)$ corresponds to the set of 2–tuples of the form:

$$\begin{pmatrix} \dots \# v_{m-1}\, C\, v_m \# v_{m+1} \# \dots \\ \dots \# v_{m-1}\, C\, u_m \# v_{m+1} \# \dots \end{pmatrix}. \tag{14}$$

The relation given by the set of 2–tuples of the form $\begin{pmatrix} v_m \\ u_m \end{pmatrix}$ is recognizable by a 2–tape synchronous finite automaton by the initial assumption that G is Cayley automatic. The difference between the lengths of the words v_m and u_m is bounded by some constant C for all generators $g_i \in S$. Therefore, the relation (14) is recognizable by a finite automaton. □

The following remark concerns asynchronously automatic groups.

Remark 4. The wreath product $G \wr \mathbb{Z}$ is not an asynchronously automatic group for any nontrivial group G. Indeed, if a group is asynchronous automatic then it is finitely presented (see [5, Theorem 7.3.4]). But, the wreath product $G \wr \mathbb{Z}$ is finitely presented only if G is trivial.

5.2 Generalization

The notion of wreath products for groups can naturally be extended for graphs. Indeed, let B be a graph and A be a graph with a distinguished vertex $a_0 \in V(A)$. For a function $f : V(B) \to V(A)$, the support of f is the set $\{b \in V(B) : f(b) \neq a_0\}$.

Definition 2. *([6, Definition 2.1]) The wreath product $A \wr B$ of graphs A and B is the graph the vertices of which are pairs (f, b), where $b \in V(B)$ and $f : V(B) \to V(A)$ is a function with a finite support. Two vertices (f_1, b_1) and (f_2, b_2) are joined by an edge if either*

- *$b_1 = b_2$ and $f_1(x) = f_2(x)$ for all $x \neq b_1$ and there is an edge in A between $f_1(b_1)$ and $f_2(b_2)$, or*
- *$f_1(x) = f_2(x)$ for all $x \in V(B)$ and there is an edge in B between b_1 and b_2.*

Note that in this definition we do not require that the graphs A and B are transitive. However, it is not hard to see that if A and B are transitive then $A \wr B$ is transitive.

Let G_1 and G_2 be Cayley graphs of two finitely generated groups H_1 and H_2, respectively, and the distinguished vertex of G_1 be the identity of H_1. Then it is easy to see that the wreath product of graphs $G_1 \wr G_2$ is a Cayley graph of the wreath product of groups $H_1 \wr H_2$.

We denote by $\Gamma(\mathbb{Z})$ the Cayley graph of $\mathbb{Z}$ with respect to the standard generator $t = 1 \in \mathbb{Z}$. Recall that a graph is locally finite if the degree of every vertex of the graph is finite. Theorem 4 can be straightforwardly generalized as follows.

Theorem 5. *For any given locally finite automatic graph Γ the wreath product $\Gamma \wr \Gamma(\mathbb{Z})$ is automatic.* □

5.3 Is The Wreath Product $\mathbb{Z}_2 \wr \mathbb{Z}^2$ Cayley Automatic?

The analysis of Cayley automaticity of a wreath product $G \wr H$ becomes more complex if H is an not a cyclic group. For example, we do not know whether the groups $\mathbb{Z}_2 \wr \mathbb{F}_2$ and $\mathbb{Z}_2 \wr \mathbb{Z}^2$ are Cayley automatic or not. In this subsection we show that if $\mathbb{Z}_2 \wr \mathbb{Z}^2$ is Cayley automatic then its Cayley automatic presentation is not natural in the sense described below.

Consider a wreath product $G \wr H$, where G is a Cayley automatic group and H is an infinite Cayley automatic non–cyclic group, say $H = \mathbb{Z}^2$. It seems reasonable to attempt to construct a Cayley automatic representation of $G \wr H$ following the approach used for $G \wr \mathbb{Z}$. For any bijective map $\tau : \mathbb{Z} \to H$, a regular language representing the elements of $G \wr H$ can be constructed composing the representation of $G \wr \mathbb{Z}$ given in the proof of Theorem 4 and the bijection τ. Assume that $\tau(0)$ is the identity of H. Then a regular language representing $G^{(H)} \trianglelefteq G \wr H$ consists of all words of Q having the symbol B as a subword that correspond to the configurations when the tape head is pointing at the identity of H. Moreover, if G is a finite group, it gives a regular representation of the subgroup $G^{(H)}$ for which the group operation is recognizable by a finite automaton, as long as B symbols are aligned. However, as it follows from Proposition 3 below, such representations of $G \wr H$ fail to be Cayley automatic.

Assume that P is a regular language that gives a Cayley automatic representation of $\mathbb{Z}_2 \wr \mathbb{Z}^2$. We suppose that each element of $\mathbb{Z}_2 \wr \mathbb{Z}^2$ has a unique representative in P. Let $f_0 \in \mathbb{Z}_2^{(\mathbb{Z}^2)}$ be the function such that $f_0(0,0) = 1$ and $f_0(z_1, z_2) = 0$ if $(z_1, z_2) \neq (0,0)$; let $r = (1,0) \in \mathbb{Z}^2$ and $u = (0,1) \in \mathbb{Z}^2$. These are generators of $\mathbb{Z}_2 \wr \mathbb{Z}^2$. Cayley automaticity of $\mathbb{Z}_2 \wr \mathbb{Z}^2$ implies that the right–multiplications by f_0, r and u are recognizable by finite automata.

Proposition 3. *Assume that $\mathbb{Z}_2 \wr \mathbb{Z}^2$ is Cayley automatic with respect to P such that the subset $P' \subset P$ of representatives of the subgroup $\mathbb{Z}_2^{(\mathbb{Z}^2)}$ is a regular language. Then the group operation in $\mathbb{Z}_2^{(\mathbb{Z}^2)}$ is not recognizable by a finite automaton.*

Proof. We will prove the proposition by contradiction. Suppose that the group operation in $\mathbb{Z}_2^{(\mathbb{Z}^2)}$ is recognizable by a finite automaton.

For a given $n \in \mathbb{N}$, we denote by $H_n \leqslant \mathbb{Z}_2^{(\mathbb{Z}^2)}$ the subgroup of functions $f \in \mathbb{Z}_2^{(\mathbb{Z}^2)}$ having $\operatorname{supp} f \subset \{(i,j) \mid -n \leqslant i,j \leqslant n\}$. Since the group operation in $\mathbb{Z}_2^{(\mathbb{Z}^2)}$ and right–multiplications by u, r and f_0 are recognizable by finite automata, it directly follows from the constant growth lemma (see, e.g., [10, Lemma 14.1]) that there exists a constant C such that $|f|_{P'} \leqslant Cn$ for all $f \in H_n$, where $|f|_{P'}$ denotes the length of the representative of f in P'. On the other hand, the number of elements in H_n is equal to $2^{(2n+1)^2}$. Thus, we get a contradiction and, therefore, the group operation in $\mathbb{Z}_2^{(\mathbb{Z}^2)}$ is not recognizable by a finite automaton. □

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