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Abstract

This paper presents a novel method for stability analysis of a wide class of linear, time-delay systems (TDS), including retarded non-neutral ones, as well as those incorporating incommensurate and distributed delays. The proposed method is based on frequency domain analysis and the application of Rouché's theorem. Given a parametrized TDS, and some parametric point for which the number of unstable poles is known, the proposed method is capable of identifying the maximum surrounding region in the parametric space for which the number of unstable poles remains invariant. First, a procedure for investigating stability along a line is developed. Then, the results are extended by the application of Hölder's inequality to investigating stability within a region. Contrary to existing approaches, the proposed method is uniformly applicable to parameters of different types (delays, distributed delay limits, time constants, etc.). Efficacy of the proposed method is demonstrated using illustrative examples.

Key words: Stability; Time-delay systems; Distributed-delay systems;

1 Introduction

Time delays are effectively used to model a wide range of physical, economic, social and biological phenomena. Examples include modeling industrial processes and their control, epidemic dynamics, operations research and computer network flows. TDS are linear but infinite-dimensional, rendering their behavioral analysis more challenging as compared to their finite-dimensional counterparts.

1.1 Literature Overview

Stability analysis of TDS has been a flourishing research field during the last several decades, with numerous approaches developed for different classes of TDS.

Historically, various analytical stability conditions have been proposed. Examples of such criteria can be found in Datko (1978); Su & Liu (1993); Busłowicz (2008). If applicable, the main advantage of analytical conditions is that they often strongly and unambiguously guarantee stability. However, they are usually quite complex, only sufficient, or they only apply to TDS of a specific form. Another important set of existing approaches is based on finding curves or surfaces in delay parametric space which delimit regions based on the number of unstable poles. This is proven to be equivalent to finding points for which the characteristic function contains at least one zero on the imaginary axis (Datko (1978); Cooke & Grossman (1982)). Such approaches have been successfully demonstrated for retarded systems with two and three independent delays (Hale & Huang (1993); Gu, Niculescu & Chen (2005); Sipahi & Olgac (2005); Gu & Naghnaeian (2011)), providing interesting and insightful graphical representations of the stability regions. A similar methodology is utilized in Morărescu, Niculescu & Gu (2006) to obtain the stability crossing set in a control problem which includes uncertainties in the delay parameters. Additionally, Sipahi & Delice (2009); Delice & Sipahi (2010) compute projections of stability-delimiting surfaces from higher dimensional parametric spaces onto two or three dimensional parametric spaces. A simpler problem regarding only one delay has been similarly treated in Olgac & Sipahi (2002).

A popular time-domain set of approaches uses Lyapunov-

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Krasovskii functionals and Razumikhin theorem to analyze both delay-dependent and delay-independent stability of TDS. One benefit of these methods is their applicability to broad classes of TDS including nonlinear ones. The main disadvantages of such approaches are conservatism and computational complexity. Different chosen functionals and method extensions/modifications have been studied, yielding different levels of pessimism and computational efficacy (see for example Fridman (2001); Kharitonov & Zhabko (2003); Parlakci (2007); Han (2009); Efimov & Fridman (2020)). Furthermore, interesting approaches targeting non-linear homogeneous TDS are discussed in Efimov, Polyakov, Perruquetti & Richard (2015), and a method that analyzes Lyapunov functions implicitly is proposed in Polyakov, Efimov, Perruquetti & Richard (2014).

A numerical "frequency-sweeping" framework is proposed in Chen & Latchman (1995), providing necessary and sufficient conditions for delay-independent stability of commensurate and non-commensurate retarded TDS. The same framework is extended in Niculescu & Che (1999), providing necessary and sufficient conditions for the delay-dependent cases as well. More recently, an improved frequency-sweeping approach has been presented in Li, Niculescu & Çela (2013, 2015), providing a framework for solving the complete stability of retarded systems with commensurate delays. Further improvements have been made in Li, Niculescu & Çela (2017), proposing an iterative approach which indirectly extends the framework to incommensurate TDS. The proposed methodology is effective for determining the number of unstable poles for an arbitrary parameter point.

Systems containing distributed delays pose different challenges due to their specific form. Interesting techniques for stability analysis of such systems can be found in Morărescu, Niculescu & Gu (2007); Gu, Kharitonov & Chen (2003); Zeng, He, Wu & She (2015).

Many stability analysis methods are built upon previously-obtained general behavioral analysis of TDS. Examples of such results can be found in Cooke & Grossman (1982); Bellman & Cooke (1963); Michiels & Niculescu (2007).

The high level of activity in the research field of TDS has also resulted in several monographs which provide good introductions to the subject and a plethora of stability analysis methods (Dugard & Verriest (1998); Gu et al. (2003); Niculescu & Gu (2004); Wu, He & She (2010); Fridman (2014); Michiels & Niculescu (2014)).

The method presented in this paper can be used to analyze stability with respect to both delay and non-delay parameters, provided that hypotheses defined in section 2 hold. The strength of the presented method, compared to various existing methods, can further be described by a combination of two facts. Firstly, the method presents necessary and sufficient conditions for stability equivalence, in the sense that it finds the maximum stability equivalence region around a given starting point. Secondly, it is applicable to a broad class of TDS, including retarded, incommensurate and distributed delays.

A simplified methodology, giving stability conditions along a parametric line in the case of a specific system involving two delays, was previously considered in Turkulov, Rapaić & Malti (2019).

1.2 Paper Outline

The paper is organized as follows: Section 2 defines problems considered in the remainder of the paper. The main results of the paper are presented in sections 3 and 4. Section 3 lays out the theory for extending the stability along a line, with additional special methods well-suited for retarded TDS. Section 4 extends the methodology to analyze stability within a region, again with an additional specific variant provided for retarded TDS. Both sections 3 and 4 include examples for retarded and non-retarded TDS. Finally, section 5 presents a short summary with several closing comments.

1.3 Notation

The paper utilizes standard mathematical notations. Symbol s denotes the Laplace variable. Angled brackets $\langle\cdot,\cdot\rangle$ represent the dot product. The p-norm of a vector $\mathbf{x}$ is denoted as $||\mathbf{x}||_p$. The set of non-negative real numbers is denoted as $\mathbb{R}_0^+$ and the set of non-negative integers by $\mathbb{N}_0$. Boundary of set $\mathcal{X}$ is denoted $\partial\mathcal{X}$ and the interior of set $\mathcal{X}$ is denoted $\mathrm{int}\left(\mathcal{X}\right)$. The Bromwich-Wagner contour enveloping the entire right half of the complex plane is denoted as $\mathcal{C}$ and defined as

$$\begin{aligned} &\mathcal{C} = \mathcal{C}_a \cup \mathcal{C}_c \\ &\mathcal{C}_a = \{s = j\omega \mid \omega \in \mathbb{R}\} \\ &\mathcal{C}_c = \left\{ s = \lim_{\rho\to\infty} \rho e^{j\varphi} \;\middle|\; \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\} \end{aligned} \tag{1}$$

The characteristic function of a linear TDS is denoted as

$$f : \mathbb{C} \times \mathcal{T} \to \mathbb{C}, \tag{2}$$

where $\mathcal{T} \subset (\mathbb{R}_0^+)^n$ denotes a parametric space. A parametric point is denoted as $\boldsymbol{\tau} = [\tau_1, \tau_2, \ldots, \tau_n] \in \mathcal{T}$. The gradient vector field of f over the parametric space is denoted as ∇f. $NU_f(\boldsymbol{\tau})$ denotes the number of zeroes of characteristic function $f(s, \boldsymbol{\tau})$ with non-negative real part, where each zero is counted as many times as its multiplicity.

The set of all parameter points of $f(s, \boldsymbol{\tau})$ sharing the same number of zeroes with non-negative real part as a starting point $\boldsymbol{\tau^0} \in \mathcal{T}$ is defined as

$$\mathcal{M}_f^{\#}(\boldsymbol{\tau^0}) = \{\boldsymbol{\tau} \in \mathcal{T} \quad | \quad NU_f(\boldsymbol{\tau}) = NU_f(\boldsymbol{\tau^0})\}. \qquad (3)$$

Define the maximum surrounding stability equivalence region of $\boldsymbol{\tau^0}$, $\mathcal{M}_f(\boldsymbol{\tau^0})$, as a set of points $\boldsymbol{\tau}$ satisfying the following conditions:

(1) $\boldsymbol{\tau} \in \mathcal{M}_f^{\#}(\boldsymbol{\tau^0}) \subset \mathcal{T}$
(2) There exists a path $\mathcal{P}$ which connects $\boldsymbol{\tau^0}$ with $\boldsymbol{\tau}$, such that $\mathcal{P} \subset \text{int}\left(\mathcal{M}_f^{\#}(\boldsymbol{\tau^0})\right)$.

2 Problem definition

Consider a linear TDS with a characteristic function

$$f = f(s, \boldsymbol{\tau}). \qquad (4)$$

Assuming that for some $\boldsymbol{\tau^0} \in \mathcal{T}$ the number of unstable poles $NU_f(\boldsymbol{\tau^0})$ is known, two versions of the problem are defined:

(P1) **Stability equivalence along a line**. Find the maximum segment $\mathcal{E} \subset \mathcal{T}$ along a predefined direction originating from $\boldsymbol{\tau^0}$ such that $NU_f(\boldsymbol{\tau^0}) = NU_f(\boldsymbol{\tau}), \forall \boldsymbol{\tau} \in \mathcal{E}$.
(P2) **Stability equivalence inside a region**. Find $\mathcal{M}_f(\boldsymbol{\tau^0})$, representing the maximum surrounding stability equivalence region surrounding $\boldsymbol{\tau^0}$.

Likewise, the paper presents two versions of the method (line-based and region-based) for solving both of the aforementioned problems.

For the method to be applicable, the following hypotheses must hold:

(H1) System characteristic function must be holomorphic in the closed right half complex-plane and continuous on the imaginary axis for all $\boldsymbol{\tau} \in \mathcal{T}$. These conditions hold for a majority of TDS, but they fail for most systems with spatially distributed and/or fractional dynamics.
(H2) The characteristic function must satisfy

$$\lim_{\rho \to \infty} \frac{\left|f(\rho e^{j\varphi}, \boldsymbol{\tau^A})\right|}{\left|\int_{\boldsymbol{\tau^A}}^{\boldsymbol{\tau^B}} \langle \nabla f(\rho e^{j\varphi}, \boldsymbol{\tau}), d\boldsymbol{\tau}\rangle\right|} = \infty, \qquad (5)$$

$\forall \boldsymbol{\tau^A}, \boldsymbol{\tau^B} \in \mathcal{T}$, $\forall \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where $\int_{\boldsymbol{\tau^A}}^{\boldsymbol{\tau^B}}$ denotes a line integral along a curve γ connecting points $\boldsymbol{\tau^A}$ and $\boldsymbol{\tau^B}$ such that $\gamma \subset \mathcal{T}$.

Lemma 1 *The hypotheses (H1) and (H2) hold true for all characteristic functions of the form*

$$f(s, \boldsymbol{\tau}) = s^m + \sum_{i=0}^{m-1} \alpha_i(\boldsymbol{\tau}) s^i e^{-s\beta_i(\boldsymbol{\tau})} \qquad (6)$$

where $p \geq 1$, $\alpha_i(\boldsymbol{\tau}), \beta_i(\boldsymbol{\tau}) : \mathcal{T} \to \mathbb{R}$ are differentiable functions $\forall i = 0, 1, \cdots, m-1$ and $\beta_i(\boldsymbol{\tau}) \geq 0$ for all $\boldsymbol{\tau} \in \mathcal{T}$.

PROOF. Observe that for all $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and all $\boldsymbol{\tau^A} \in \mathcal{T}$

$$\left|f(\rho e^{j\varphi}, \boldsymbol{\tau^A})\right| = \left|\rho^m e^{jm\varphi} + \sum_{i=0}^{m-1} \alpha_i(\boldsymbol{\tau^A}) e^{-\beta_i(\boldsymbol{\tau^A})\rho e^{j\varphi}} \rho^i e^{ij\varphi}\right| \underset{\rho \to \infty}{\sim} \rho^m. \qquad (7)$$

Further, observe that

$$\nabla f(s, \boldsymbol{\tau}) = D_1(s, \boldsymbol{\tau}) + D_2(s, \boldsymbol{\tau}) \qquad (8)$$

$$D_1(s, \boldsymbol{\tau}) = \sum_{i=0}^{m-1} s^i e^{-s\beta_i(\boldsymbol{\tau})} \nabla \alpha_i(\boldsymbol{\tau}) \qquad (9)$$

$$D_2(s, \boldsymbol{\tau}) = \sum_{i=0}^{m-1} -s^{i+1} e^{-s\beta_i(\boldsymbol{\tau})} \alpha_i(\boldsymbol{\tau}) \nabla \beta_i(\boldsymbol{\tau}) \qquad (10)$$

Denominator of (5) is bounded by

$$\left|\int_{\boldsymbol{\tau^A}}^{\boldsymbol{\tau^B}} \langle \nabla f(\rho e^{j\varphi}, \boldsymbol{\tau}), d\boldsymbol{\tau}\rangle\right| \leq I_1 + I_2, \qquad (11)$$

$$I_i = \int_{\boldsymbol{\tau^A}}^{\boldsymbol{\tau^B}} \left|\langle D_i(\rho e^{j\varphi}, \boldsymbol{\tau}), d\boldsymbol{\tau}\rangle\right|, \quad i = 1, 2 \qquad (12)$$

It is not difficult to see that $I_1 \underset{\rho \to \infty}{\sim} \rho^{m-1}$. For I_2, we have

$$I_2 \leq \sum_{i=0}^{m-1} \rho^{i+1} \int_{\boldsymbol{\tau^A}}^{\boldsymbol{\tau^B}} g_i(\boldsymbol{\tau})$$
$$g_i(\boldsymbol{\tau}) = \left|e^{-\rho \cos\varphi \beta_i(\boldsymbol{\tau})} \alpha_i(\boldsymbol{\tau}) \langle \nabla \beta_i(\boldsymbol{\tau}), d\boldsymbol{\tau}\rangle\right|. \qquad (13)$$

Applying Mean Value Theorem (MVT) to the integral of $g_i(\boldsymbol{\tau})$ yields

$$\int_{\boldsymbol{\tau^A}}^{\boldsymbol{\tau^B}} g_i(\boldsymbol{\tau}) = (\boldsymbol{\tau^A} - \boldsymbol{\tau^B}) g_i(\boldsymbol{\tau'}) \qquad (14)$$

for some $\boldsymbol{\tau'} \in (\boldsymbol{\tau^A}, \boldsymbol{\tau^B})$. If $g_i(\boldsymbol{\tau})$ is always zero, the proof is concluded. Otherwise, if $g_i(\boldsymbol{\tau}) > 0$ for some

$\boldsymbol{\tau} \in (\boldsymbol{\tau^A}, \boldsymbol{\tau^B})$, it holds that $g_i(\boldsymbol{\tau'}) > 0$ and $\nabla\beta_i(\boldsymbol{\tau'}) \neq 0$. Since $\beta_i(\boldsymbol{\tau})$ is differentiable and non-negative, $\beta_i(\boldsymbol{\tau'}) > 0$, leading to

$$\lim_{\rho\to\infty} I_2 = 0. \tag{15}$$

Therefore,

$$\left| \int_{\boldsymbol{\tau^A}}^{\boldsymbol{\tau^B}} \langle \nabla f(\rho e^{j\varphi}, \boldsymbol{\tau}), d\boldsymbol{\tau} \rangle \right| \underset{\rho\to\infty}{\sim} \rho^{m-1}, \tag{16}$$

leading to (5) and concluding the proof. □

Throughout the paper, special attention will be given to TDS of retarded type, modeled by

$$\dot{\boldsymbol{x}}(t) = A_0 \boldsymbol{x}(t) + \sum_{i=1}^{n'} A_i \boldsymbol{x}(t - \tau_i') \tag{17}$$

where $\tau_i' \geq 0, i = 1, \cdots, n$ are the corresponding delays. Although more specific than (4), such systems still model a broad class of problems, while their specific form eases the application of the proposed method and increases its computational feasibility. The characteristic function of such systems is given by

$$f'(s, \boldsymbol{\tau'}) = \det\left(sI - A_0 - \sum_{i=1}^{n'} A_i e^{-s\tau_i'} \right). \tag{18}$$

By substituting the original delay parameters $\boldsymbol{\tau'}$ with a derived set of parameters $\boldsymbol{\tau}$, it can straightforwardly be shown that (18) can be expressed as

$$f(s, \boldsymbol{\tau}) = f'(s, \boldsymbol{\tau'}) = \sum_{i=1}^{n} P_i(s) e^{-s\tau_i}, \tag{19}$$

where P_i are polynomials and $n \geq n'$. Expression (19) is a special case of (6).

Finally, it is important to stress that the stability addressed in this paper is the exponential stability. A linear TDS is considered to be exponentially stable if and only if all zeroes of its characteristic function have strictly negative real parts. A similar method investigating BIBO stability of fractional non-commensurate systems subject to perturbations in differentiation orders is proposed in Malti & Rapaić (2017); Rapaić & Malti (2019).

3 Stability equivalence along a line

In this section, a solution to (P1) is given. Let us characterize variations of $\boldsymbol{\tau}$ along a line starting from $\boldsymbol{\tau^0}$ by a single scalar non-negative parameter θ as

$$\boldsymbol{\tau}(\theta) = \boldsymbol{\tau^0} + \theta \boldsymbol{\tau^d}, \tag{20}$$

where $\boldsymbol{\tau^d}$ is an arbitrarily chosen unit direction vector. Define the starting value of θ as $\theta_0 = 0$, corresponding to $\boldsymbol{\tau}(0) = \boldsymbol{\tau^0}$. For simplicity, the characteristic function in this section is expressed as

$$f(s, \boldsymbol{\tau}(\theta)) \equiv f(s, \theta). \tag{21}$$

The problem (P1) reduces to finding the maximum value of θ for which the number of non-negative zeroes of f is preserved. Such stability-limiting value of θ is defined as

$$\theta_{lim} = \sup\left\{ \theta' \,\middle|\, NU_f(\boldsymbol{\tau^0}) = NU_f(\boldsymbol{\tau}(\theta)), \forall\theta \in [\theta_0, \theta') \right\}. \tag{22}$$

3.1 *Sufficient condition*

As a first step towards finding θ_{lim}, sufficient stability equivalence condition along a line is provided.

Theorem 2 *Let f be defined as in (21) and satisfying (H1) and (H2). Let $\theta_0 \geq 0$ be an initial point for which $NU_f(\boldsymbol{\tau}(\theta_0))$ is known, such that $f(j\omega, \boldsymbol{\tau}(\theta_0)) \neq 0, \forall\omega \in \mathbb{R}$. Let $\boldsymbol{\tau}(\theta)$ be defined as in (20), and $\Delta\theta = \theta - \theta_0$. Then,*

$$NU_f(\boldsymbol{\tau}(\theta_0)) = NU_f(\boldsymbol{\tau}(\beta)), \forall\beta \in [\theta_0, \theta] \tag{23}$$

holds if

$$\Delta\theta < \min_{\omega\in\mathbb{R}_0^+} \frac{|f(j\omega, \theta_0)|}{\max\limits_{\theta_0\leq\beta\leq\theta} \left| \frac{\partial f}{\partial\theta}(j\omega, \theta = \beta) \right|}. \tag{24}$$

PROOF. Due to (H1), Rouché's theorem can be applied to f as

$$|f(s,\theta) - f(s,\theta_0)| < |f(s,\theta_0)|, \forall s \in \mathcal{C} \Rightarrow NU_f(\boldsymbol{\tau}(\theta_0)) = NU_f(\boldsymbol{\tau}(\theta)). \tag{25}$$

Furthermore, the fundamental theorem of calculus can be applied to (25), resulting in

$$\left| \int_{\theta_0}^{\theta} \frac{\partial f}{\partial\theta}(s, \theta = \beta) d\beta \right| < |f(s,\theta_0)|, \forall s \in \mathcal{C} \Rightarrow NU_f(\boldsymbol{\tau}(\theta_0)) = NU_f(\boldsymbol{\tau}(\theta)). \tag{26}$$

Due to (H2), inequality (26) holds $\forall s \in \mathcal{C}_c$. Taking the symmetry of $f(s,\theta)$ into account, further analysis needs to be performed only for $s = j\omega, \omega \in \mathbb{R}_0^+$. Notice that

$$\left| \int_{\theta_0}^{\theta} \frac{\partial f}{\partial\theta}(j\omega, \theta = \beta) d\beta \right| \leq \int_{\theta_0}^{\theta} \left| \frac{\partial f}{\partial\theta}(j\omega, \theta = \beta) \right| d\beta \leq (\theta - \theta_0) \max_{\theta_0\leq\beta\leq\theta} \left| \frac{\partial f}{\partial\theta}(j\omega, \theta = \beta) \right|. \tag{27}$$

Introducing the conservative bound (27) to (26) results in

$$(\theta-\theta_0)\max_{\theta_0\le\beta\le\theta}\left|\frac{\partial f}{\partial\theta}(j\omega,\theta=\beta)\right| < |f(j\omega,\theta_0)|, \forall\omega\in\mathbb{R}_0^+ \Rightarrow NU_f(\boldsymbol{\tau}(\theta_0)) = NU_f(\boldsymbol{\tau}(\theta)). \quad (28)$$

It can trivially be observed that

$$m(\theta) = (\theta-\theta_0)\max_{\theta_0\le\beta\le\theta}\left|\frac{\partial f}{\partial\theta}(j\omega,\theta=\beta)\right| \quad (29)$$

is a non-decreasing function with respect to θ. Consequently, if inequality (28) holds for some value of θ, it also holds for all values of $\beta\in[\theta_0,\theta]$. Based on that fact and by introducing $\Delta\theta = \theta-\theta_0$, we obtain

$$\Delta\theta < \frac{|f(j\omega,\theta_0)|}{\max\limits_{\theta_0\le\beta\le\theta}\left|\frac{\partial f}{\partial\theta}(j\omega,\theta=\beta)\right|}, \forall\omega\in\mathbb{R}_0^+ \Rightarrow NU_f(\boldsymbol{\tau}(\theta_0)) = NU_f(\boldsymbol{\tau}(\beta)), \forall\beta\in[\theta_0,\theta]. \quad (30)$$

Steps smaller than $\Delta\theta$ retain stability if (30) holds $\forall\omega\in\mathbb{R}_0^+$. Thus, a valid step limit can be obtained by finding the minimum of (30) with respect to ω (the worst-case scenario), resulting in (24) and concluding the proof. □

3.2 Necessary and sufficient conditions

Under assumptions of Theorem 2, by applying (24), one can obtain a stability equivalence interval defined by the endpoint

$$\theta_1 = \theta_0 + \Delta\theta. \quad (31)$$

The method can now be applied again, taking previously obtained value θ_1 as the new starting point. Formally, the method can be iterated a certain number of times

$$\theta_{k+1} = \theta_k + \Delta\theta_k, \quad \forall k\in\mathbb{N}_0. \quad (32)$$

It will be proven that the sequence θ_k converges to θ_{lim} when θ_{lim} exists, and diverges when θ_{lim} does not exist. For the purpose of the following discussion, it is important to note that the denominator of (24) is always finite and positive $\forall\omega\in\mathbb{R}$.

Lemma 3 *Let θ_{lim} be defined in* (22) *and let a sequence θ_k be obtained by* (32)*, with increments $\Delta\theta_k$ fulfilling* (24)*. If θ_{lim} exists, then $\theta_k\le\theta_{lim}, \forall k\in\mathbb{N}_0$.*

PROOF. The lemma is directly satisfied by Theorem 2, as (24) presents sufficient stability equivalence condition for each iteration k. □

Lemma 4 *Let θ_{lim} be defined in* (22) *and let a sequence θ_k be obtained by* (32)*, with increments $\Delta\theta_k$ fulfilling* (24)*. If $\lim\limits_{k\to\infty}\theta_k$ exists, then $\lim\limits_{k\to\infty}\theta_k = \theta_{lim}$.*

PROOF. Proof by contradiction. Assume that $\lim_{k\to\infty}\theta_k$ converges to some $\theta_\# < \theta_{lim}$. As a consequence of Lemma 3, such $\theta_\#$ must be smaller than θ_{lim}. The bare existence of a convergence limit implies that values $\Delta\theta_k$ get arbitrary small as $k\to\infty$. This, combined with (24) implies that the value of

$$\min_\omega |f(j\omega,\theta_k)| \quad (33)$$

becomes arbitrary small as $k\to\infty$ and $\theta\to\theta_\#$. However, it is not possible that (33) becomes arbitrarily small near $\theta_\#$ because:

(1) Function $|f(j\omega,\theta)|$ is continuous with regards to θ.
(2) By definition (22), θ_{lim} is the smallest value of $\theta\in[\theta_0,\infty)$ for which $\exists\omega\in\mathbb{R}$ such that

$$|f(j\omega,\theta)| = 0. \quad (34)$$

Thus, $\exists\alpha\in\mathbb{R}^+$ and $\exists\varepsilon\in\mathbb{R}^+$ such that

$$\min_\omega |f(j\omega,\theta)| > \alpha, \forall\theta\in(\theta_\#-\varepsilon,\theta_\#+\varepsilon), \quad (35)$$

contradicting the assumption that $\theta_k\to\theta_\#$. In other words, values of $\Delta\theta_k$ cannot be arbitrarily small in the neighborhood of any $\theta_\# < \theta_{lim}$. □

Lemma 5 *Let θ_{lim} be defined in* (22) *and let a sequence θ_k be obtained by* (32)*, with increments $\Delta\theta_k$ fulfilling* (24)*. If θ_k is not strictly increasing, then θ_{lim} exists and $\exists k_0$ such that $\theta_k = \theta_{lim}, \forall k\ge k_0$.*

PROOF. From (24), $\Delta\theta_k\ge 0$, hence the sequence θ_k is non-decreasing. Assume that for some k_0, $\theta_{k_0} = \theta_{k_0+1}$. From (32) it can be seen that $\theta_{k_0} = \theta_{k_0+1}$ if and only if $\Delta\theta_{k_0} = 0$. This further implies from (24) that $|f(j\omega,\theta_{k_0})| = 0$, proving the existence of θ_{lim} as defined in (22). From (22) and Lemma 3, the only such possible value of θ_{k_0} is θ_{lim}. Furthermore, it can trivially be observed that $\Delta\theta_k = 0, \forall k\ge k_0$, implying $\theta_k = \theta_{lim}, \forall k\ge k_0$, thus concluding the proof. □

Combining Lemmas 3-5, the following theorem concerning convergence of (32) can be proven.

Theorem 6 *Let θ_{lim} be defined in* (22) *and let a sequence θ_k be obtained by* (32)*, with increments $\Delta\theta_k$ fulfilling* (24)*. If θ_{lim} exists, then θ_k converges to θ_{lim}. Otherwise, if θ_{lim} does not exist, then θ_k diverges.*

PROOF. Assume that θ_{lim} exists. From (24), the sequence θ_k is non-decreasing. If θ_k is not strictly increasing, then θ_k trivially converges to θ_{lim} as a consequence of Lemma 5. Otherwise, θ_k is strictly increasing. From Lemma 3, the sequence θ_k will never overshoot θ_{lim}. Hence, the sequence θ_k must converge to a value in the interval $[\theta_0, \theta_{lim}]$. From Lemma 4, the only such a possible value of convergence is θ_{lim}.

On the other hand, assume that θ_{lim} does not exist. Similarly as in Lemma 4, the convergence of non-decreasing sequence θ_k would imply that the values of $\min_\omega |f(j\omega, \theta_k)|$ get arbitrary small as $k \to \infty$. This is not possible because the non-existence of θ_{lim} implies that $\exists \alpha > 0$ such that

$$\min_{\omega} |f(j\omega, \theta_k)| > \alpha, \forall \theta > \theta_0. \tag{36}$$

Thus, the steps $\Delta\theta_k$ cannot become arbitrarily small, concluding the proof. □

Theorem 6 shows that the sequence θ_k, defined by (32) and (24), presents necessary and sufficient conditions for finding the stability limit $\theta_{lim} \leq \infty$.

Implementation issues While it is natural to implement iterative algorithms on a digital computer, such implementations introduce issues related to finite precision and representation of large numbers. If θ_k is convergent, it is expected, as θ_k iteratively increases, that the steps $\Delta\theta_k$ start converging towards zero. Since the algorithm cannot be run indefinitely and the computer precision is finite, a termination criterion needs to be introduced. The simplest one is to terminate the algorithm when the step $\Delta\theta_k$ becomes smaller than a prescribed value δ, although more sophisticated termination criteria can be established as well. A similar problem occurs if θ_k is divergent, since it might lead to $\Delta\theta_k > \delta, \forall k \in \mathbb{N}_0$. Therefore, the algorithm is also terminated if θ_k becomes larger than a prescribed value Θ. Finally, (24) depends on finding the global minimum of a function. A wrongly evaluated minimum might lead to an incorrect jump $\Delta\theta_k$, invalidating the results. Thus, care must be taken when performing the necessary global optimizations in order to avoid invalid jumps. To that end, a scaling factor $\eta \in (0, 1)$ could be introduced in in order to scale down the obtained jumps $\Delta\theta_k$, reducing the likelihood of invalid results. Taking that into account, the final algorithm is presented as Algorithm 1.

Remark 7 *In this section, the stability equivalence has been extended along a straight line defined by* (20)*. However, the line can be replaced by any curve parametrized by a scalar* θ*, provided that* $\boldsymbol{\tau}(\theta_0 = 0) = \boldsymbol{\tau^0}$*. For example, one might analyze stability along an arc of an n-sphere.*

Require: $\delta > 0, \Theta > 0, \theta_0 \in [0, \Theta), \eta \in (0, 1)$
  $\theta_k := \theta_0$
  $\Delta\theta_k := \infty$
  while $\Delta\theta_k > \delta$ and $\theta_k < \Theta$ **do**
    $\Delta\theta_k := \eta \min_{\omega} \dfrac{|f(j\omega, \theta_k)|}{\max_{\theta_k \leq \beta \leq \theta} \left| \frac{\partial f}{\partial \theta}(j\omega, \theta = \beta) \right|}$
    $\theta_k := \theta_k + \Delta\theta_k$
    $k := k + 1$
  end while
  $\theta_{lim} = \theta_k$

Algorithm 1. Computation of θ_{lim}

Example 8 *Consider a distributed delay system modeled by*

$$\dot{x}(t) = -\int_{-\tau}^{0} e^{k\alpha} x(t + \alpha) d\alpha. \tag{37}$$

Its stability is investigated with respect to τ *and* k*.*

System characteristic function is given by

$$f(s, \tau, k) = s^2 + sk + 1 - e^{-\tau(s+k)}, \tag{38}$$

fulfilling (H1) and (H2). Algorithm 1 is applied to manually chosen starting points $(0.05, 1)$, $(4.9, 0.1)$, $(8, 0.04)$, $(11.3, 0.08)$, $(14.5, 0.04)$, $(17.5, 0.08)$, for which the number of unstable poles, respectively $0, 2, 4, 2, 4, 2$, has been determined using Cauchy's argument principle. The obtained lines for which the number of unstable poles is preserved are plotted in figure 1. The algorithm diverges in the positive (north and east) directions of both axes, indicating that the stability limit is ∞ (Theorem 6). Although applying the algorithm to a plethora of rays gives a good sketch of stability/instability regions, the result does not guarantee stability equivalence in a dense set of (τ, k). This shortcoming is overcome in section 4 by analyzing stability inside a region. □

3.3 Application to retarded TDS

Although applicable to a wide class of linear systems, the proposed algorithm is particularly simple in case of non-neutral retarded TDS, which characteristic equation is given by (19). Based on (20), each element of $\boldsymbol{\tau}$ can directly be expressed as

$$\tau_i = \tau_i^0 + \theta\tau_i^d, \tag{39}$$

which can be plugged into (19), resulting in a characteristic function of the form

$$f(s, \theta) = \sum_{i=0}^{m} f_i(s) e^{-s\theta a_i}, \tag{40}$$

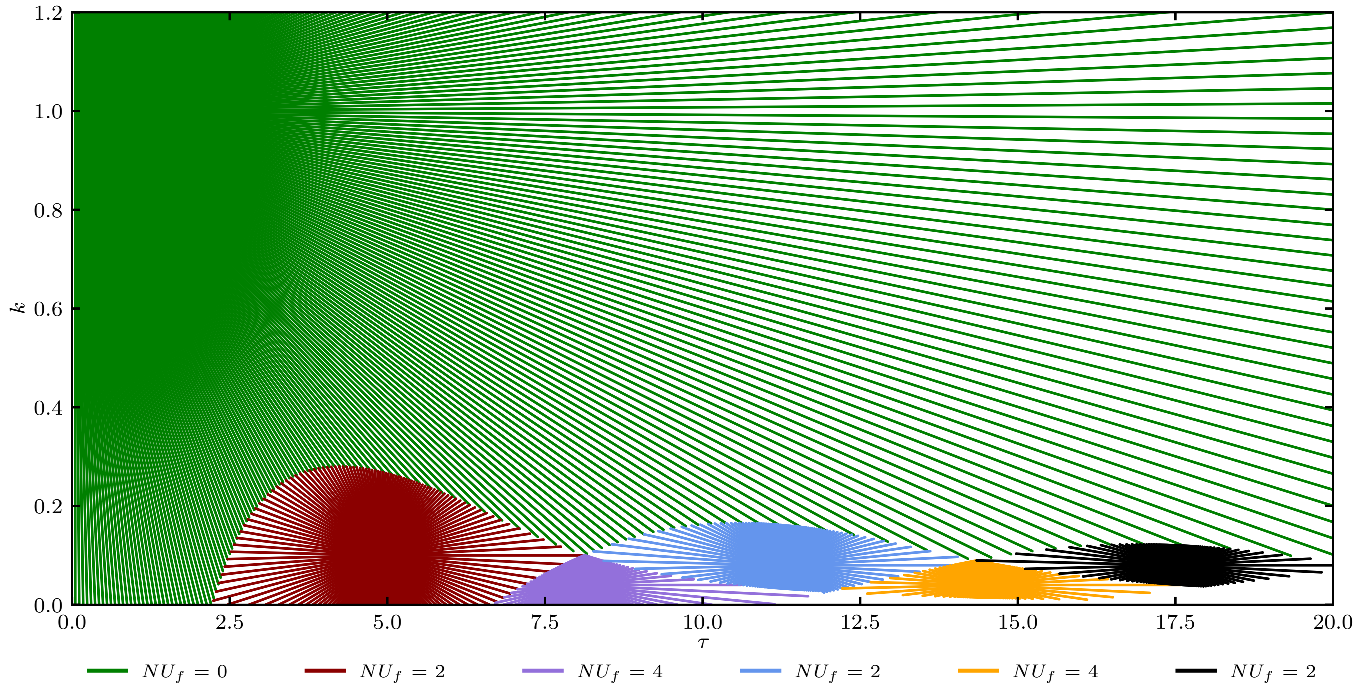

Fig. 1. Stability analysis of example 8

where m is the system order, a_i are real scalars and $f_i(s)$ are complex functions independent of θ. This result is important because of the convenient form of (40). Namely, in order to implement the general method (24), evaluation of $|f(j\omega, \theta_0)|$ and $\max_{\theta_0 \leq \beta \leq \theta} |\frac{\partial f}{\partial \theta}(j\omega, \beta)|$ is required. The former expression is directly evaluated from (40). For the latter one, observe that

$$\max_{\theta_0 \leq \beta \leq \theta} \left| \frac{\partial f}{\partial \theta}(j\omega, \theta = \beta) \right| \leq \max_{\theta_0 \leq \beta \leq \theta} \sum_{i=0}^{n} \left| a_i j\omega f_i(j\omega) e^{-j\omega a_i \theta} \right| = \sum_{i=0}^{n} a_i \omega |f_i(j\omega)|, \tag{41}$$

which yields an elegant expression, albeit conservative. Similarly as before, the conservatism is not problematic; it only decreases the size of algorithm steps $\Delta\theta$, but does not change its end result. Hence, the following corollary to Theorem 2 can be formulated.

Corollary 9 *Let f be defined as in* (19)*. Let $\theta_0 \geq 0$ be an initial point for which $NU_f(\boldsymbol{\tau}(\theta_0))$ is known, such that $f(j\omega, \boldsymbol{\tau}(\theta_0)) \neq 0, \forall \omega \in \mathbb{R}$. Let $\boldsymbol{\tau}(\theta)$ be defined as in* (20)*, and $\Delta\theta = \theta - \theta_0$. Then,*

$$NU_f(\boldsymbol{\tau}(\theta_0)) = NU_f(\boldsymbol{\tau}(\beta)), \forall \beta \in [\theta_0, \theta] \tag{42}$$

holds if

$$\Delta\theta \leq \min_{\omega \in \mathbb{R}_0^+} \frac{|f(j\omega, \theta)|}{\sum_{i=0}^{n} a_i \omega |f_i(j\omega)|}, \tag{43}$$

where $f_i(s)$ and a_i are parameters resulting from parametrization of f as in (40)*.*

This corollary allows an easier implementation of Algorithm 1, because the denominator of (43) is already maximized, as compared to (24). Moreover, the expression on the right-hand side of (43) is independent of $\Delta\theta$, allowing direct evaluation of a permissible jump. Other aspects of Algorithm 1, such as the termination criterion, remain unchanged.

As a final note, this specialized version of the method is not limited to retarded TDS. In fact, it can be applied to any system that has a characteristic function as in (40).

Example 10 *Consider a system with a characteristic function given by*

$$f(s, \boldsymbol{\tau}) = s^2 + 2se^{-s\tau_1} + e^{-s\tau_2}. \tag{44}$$

Its stability is investigated with resepct to $\boldsymbol{\tau} = [\tau_1, \tau_2]$.

Since the system is retarded, the simplified version of the algorithm (using Corollary 9) is applied. The algorithm is initialized at five different points, namely $(0, 0)$, $(1.75, 1.2)$, $(0.88, 2.73)$, $(0.2, 3.1)$, $(0.71, 3.22)$, for which the number of unstable poles, respectively 0, 2, 2, 2, 4, is determined using the methodology from Li et al. (2017). The results are displayed on figure 2 and compared to the stability crossing set (SCS) obtained by Gu et al. (2005) for verification purposes. □

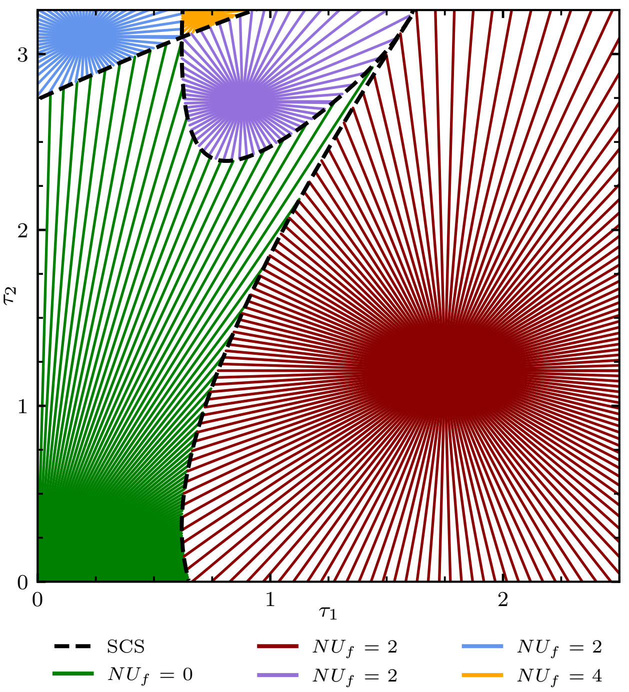

Fig. 2. Stability analysis of example 10

4 Stability equivalence within a region

In this section, a solution to (P2) is given.

4.1 Sufficient condition

Theorem 11 *Let f be defined as in (4) and satisfying (H1) and (H2). Let $\boldsymbol{\tau}^{\mathbf{0}} \in \mathcal{T}$ be any parameter point satisfying $f(j\omega, \boldsymbol{\tau}^{\mathbf{0}}) \neq 0, \forall \omega \in \mathbb{R}$. Let p and q be real scalars satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 \leq p, q \leq \infty. \tag{45}$$

Then, $\forall \varepsilon_q$ such that

$$0 < \varepsilon_q < \min_{\omega \in \mathbb{R}_0^+} \frac{|f(j\omega, \boldsymbol{\tau}^{\mathbf{0}})|}{\max\limits_{||\mathbf{v}||_q < \varepsilon_q} ||\nabla f(j\omega, \boldsymbol{\tau}^{\mathbf{0}} + \mathbf{v})||_p}, \tag{46}$$

the following holds

$$NU_f(\boldsymbol{\tau}^{\mathbf{0}}) = NU_f(\boldsymbol{\tau}), \forall \boldsymbol{\tau} \in \left\{ \boldsymbol{\tau}^{\mathbf{0}} + \mathbf{v} \,\middle|\, ||\mathbf{v}||_q \leq \varepsilon_q \right\}. \tag{47}$$

PROOF. To build towards the proof, it is beneficial to start by analyzing stability equivalence of two arbitrary parameter points. To that end, define a parameter point $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau}(\mathbf{v}) = \boldsymbol{\tau}^{\mathbf{0}} + \mathbf{v}, \tag{48}$$

where $\boldsymbol{\tau}^{\mathbf{0}} \in \mathcal{T}$ represents a chosen starting point and $\mathbf{v}$ represents a change vector. The objective is to discuss the stability equivalence of parameter points $\boldsymbol{\tau}^{\mathbf{0}} = \boldsymbol{\tau}(\mathbf{0})$ and $\boldsymbol{\tau}(\mathbf{v})$. From Rouché's theorem, it is known that stability equivalence of these points is guaranteed if

$$|f(s, \boldsymbol{\tau}(\mathbf{v})) - f(s, \boldsymbol{\tau}(\mathbf{0}))| < |f(s, \boldsymbol{\tau}(\mathbf{0}))|, \forall s \in \mathcal{C}. \tag{49}$$

For simplicity, $f(s, \boldsymbol{\tau}(\mathbf{v}))$ is denoted $f(\mathbf{v})$ in the remainder of this section. The LHS of (49) can further be elaborated to obtain

$$|f(\mathbf{v}) - f(\mathbf{0})| = \left| \int_\gamma \left\langle \nabla f(\mathbf{r}), d\mathbf{r} \right\rangle \right| = \left| \int_0^1 \left\langle \nabla f(\mathbf{r}(\beta)), \mathbf{r}' \right\rangle d\beta \right| \leq \int_0^1 \left| \left\langle \nabla f(\mathbf{r}(\beta)), \mathbf{r}' \right\rangle \right| d\beta, \tag{50}$$

where $\mathbf{r}(\beta)$ represents parametrization of curve γ which connects the $\mathbf{0}$ vector with $\mathbf{v}$ for $\beta \in [0, 1]$ and $\mathbf{r}'$ represents the derivative of $\mathbf{r}(\beta)$ with respect to β. Introducing the conservative bound (50) in (49) and utilizing (H2) and the symmetry of f yields

$$\int_0^1 \left| \left\langle \nabla f(\mathbf{r}(\beta)), \mathbf{r}' \right\rangle \right| d\beta < |f(\mathbf{0})|, s = j\omega, \forall \omega \in \mathbb{R}_0^+. \tag{51}$$

By defining the curve γ as $\mathbf{r}(\beta) = \beta \mathbf{v}$,

$$\int_0^1 \left| \left\langle \nabla f(\mathbf{r}(\beta)), \mathbf{r}' \right\rangle \right| d\beta = \int_0^1 \left| \left\langle \nabla f(\beta \mathbf{v}), \mathbf{v} \right\rangle \right| d\beta. \tag{52}$$

The application of Hölder's inequality further yields

$$\int_0^1 \left| \left\langle \nabla f(\beta \mathbf{v}), \mathbf{v} \right\rangle \right| d\beta \leq \int_0^1 ||\nabla f(\beta \mathbf{v})||_p \, ||\mathbf{v}||_q \, d\beta. \tag{53}$$

The results presented so far guarantee stability equivalence for a specific change vector $\mathbf{v}$. To guarantee stability equivalence within a region, choose some $\varepsilon_q > 0$ and define $\mathcal{W}$ as

$$\mathcal{W}(\boldsymbol{\tau}) = \left\{ (\boldsymbol{\tau} + \mathbf{v}) \in \mathcal{T} \,\middle|\, ||\mathbf{v}||_q \leq \varepsilon_q \right\}. \tag{54}$$

A region of parameter points around $\boldsymbol{\tau}^{\mathbf{0}}$ can thus be defined as $\mathcal{W}(\boldsymbol{\tau}^{\mathbf{0}})$. The shape of the region is defined by q, and the size of the region is defined by the value of ε_q. Notice that for any $\mathbf{v}$ which satisfies $||\mathbf{v}||_q < \varepsilon_q$, it is possible to substitute (53) with a more conservative

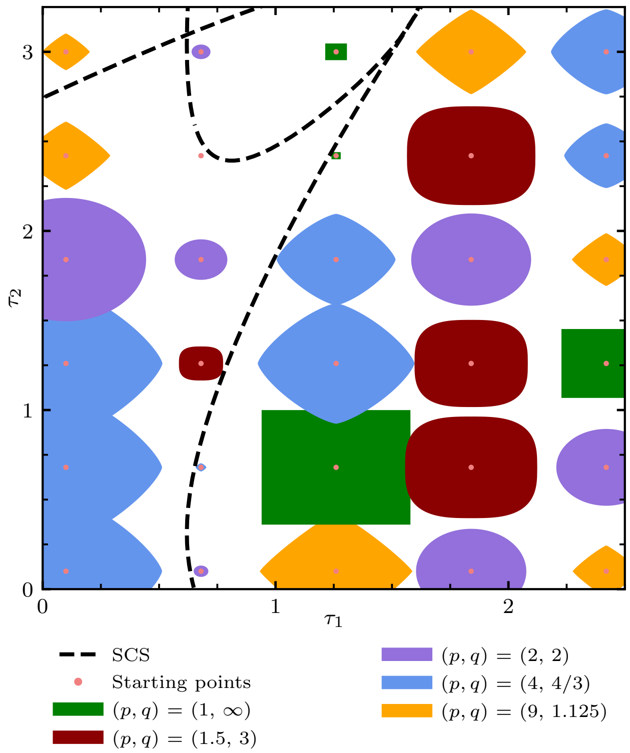

Fig. 3. Results of applying (56) to example 10

expression

$$\int_0^1 ||\nabla f(\beta \mathbf{v})||_p \, ||\mathbf{v}||_q \, d\beta \;\leq\; \max_{||\mathbf{v}||_q < \varepsilon_q} ||\nabla f(\mathbf{v})||_p \, \varepsilon_q. \tag{55}$$

Finally, (51) and (55) imply that a permissible stability equivalence region defined by ε_q can be obtained by

$$\varepsilon_q < \min_{\omega \in \mathbb{R}_0^+} \frac{|f(j\omega, \mathbf{0})|}{\max_{||\mathbf{v}||_q < \varepsilon_q} ||\nabla f(j\omega, \mathbf{v})||_p}, \tag{56}$$

concluding the proof. $\square$

The application of (56) is analogous to performing a single step of the line version of the algorithm. Figure (3) shows the results of applying (56) to example 10, with different shapes corresponding to different combinations of (p, q) and different starting points. The number of unstable poles is equivalent for all the points inside each individual region.

It remains now to set up a framework for connecting all the shapes within the maximum surrounding stability equivalence region around any $\boldsymbol{\tau^0} \in \mathcal{T}$, namely $\mathcal{M}_f(\boldsymbol{\tau^0})$.

4.2 Necessary and sufficient conditions

Analogously to the line-based version of the method, necessary and sufficient conditions are established for characteristic functions to have the same number of non-negative zeroes when their parameters vary in a region. First, choose p and q satisfying (45) and assume that applying (56) on a parameter point $\boldsymbol{\tau}$ yields a stability equivalence region $\mathcal{W}(\boldsymbol{\tau})$. Choose a starting point $\boldsymbol{\tau}^0$ and define a set $\mathcal{S}_0$ as

$$\mathcal{S}_0 = \{\boldsymbol{\tau^0}\}. \tag{57}$$

Construct a monotonously growing sequence of sets

$$\mathcal{S}_{k+1} = \mathcal{S}_k \cup \bigcup_{\boldsymbol{\tau} \in \partial \mathcal{S}_k} \mathcal{W}(\boldsymbol{\tau}), \quad \forall k \in \mathbb{N}_0. \tag{58}$$

It is now established that S_k converges to $\mathcal{M}_f(\boldsymbol{\tau^0})$.

Theorem 12 *Let f be defined as in* (4) *and satisfying (H1) and (H2). Let p and q be real scalars satisfying* (45)*. Let $\boldsymbol{\tau^0} \in \mathcal{T}$ be any parameter point satisfying $f(j\omega, \boldsymbol{\tau^0}) \neq 0, \forall \omega \in \mathbb{R}$. Define $\mathcal{S}_0$ as in* (57)*, $\mathcal{S}_k$ as in* (58)*, and $\mathcal{W}(\boldsymbol{\tau})$ as in* (54)*, where steps ε_q satisfy* (46)*. Then,*

$$\limsup_{k \to \infty} \mathcal{S}_k = \mathcal{M}_f(\boldsymbol{\tau^0}). \tag{59}$$

PROOF. Choose any point $\boldsymbol{\tau^*} \in \mathcal{M}_f(\boldsymbol{\tau^0})$. By definition of $\mathcal{M}_f(\boldsymbol{\tau^0})$, there exists a path $\mathcal{P}$ defined by a continuous function $g : [0, 1] \to \mathcal{P} \subset \operatorname{int}\left(\mathcal{M}_f(\boldsymbol{\tau^0})\right)$ such that $g(0) = \boldsymbol{\tau^0}$ and $g(1) = \boldsymbol{\tau^*}$. Define the sequence

$$m_k = \max\left\{ x \in [0, 1] \;\middle|\; g(x) \in \mathcal{S}_k \right\}. \tag{60}$$

For any fixed k, the set S_k is closed and bounded, and therefore compact. Consequently, the maximum in (60) is well-defined. Define the sequence $\boldsymbol{\tau^k} = g(m_k)$, which represents the farthest point along the path $\mathcal{P}$ (referenced from $\boldsymbol{\tau^0}$) such that $\boldsymbol{\tau^k} \in \mathcal{S}_k$ at iteration k. There are two possible scenarios:

(1) $\boldsymbol{\tau^{k-1}} < \boldsymbol{\tau^*}$, implying $\boldsymbol{\tau^k} \in \partial \mathcal{S}_k$. In this scenario, $\boldsymbol{\tau^k}$ is one of the points on which (58) is evaluated at iteration k.
(2) $\boldsymbol{\tau^{k-1}} = \boldsymbol{\tau^*}$, implying that the endpoint $\boldsymbol{\tau^*}$ has already been reached.

Let us further analyze scenario (1). Since $\boldsymbol{\tau^k} \in \operatorname{int}\left(\mathcal{M}_f(\boldsymbol{\tau^0})\right)$, it holds that $|f(j\omega, \boldsymbol{\tau^k})| > 0, \forall k \in \mathbb{N}_0, \forall \omega \geq 0$, further implying that the resulting ε_q from (56) is strictly positive $\forall k \in \mathbb{N}_0$. Consequently, either $m_k = 1$, or $m_k < m_{k+1}$, meaning that $\boldsymbol{\tau^k}$ gets strictly closer to $\boldsymbol{\tau}^*$ along $\mathcal{P}$ at each successive iteration unless $\boldsymbol{\tau^k} = \boldsymbol{\tau^*}$. Thus, $\exists k_0$ such that $\boldsymbol{\tau^*} \in \mathcal{S}_k, \forall k \geq k_0$.

Since the same reasoning can be applied to any chosen point $\boldsymbol{\tau}^* \in \mathcal{M}_f(\boldsymbol{\tau}^0)$, the proof is concluded. $\square$

Remark 13 *Of course, in any practical implementation it is impossible to construct the sequence S_k in the described manner. Instead, the construction of S_{k+1} is based on a finite set of samples belonging to ∂S_k, which can be achieved in several ways.*

Figure 4 shows the results of applying (58) to example 8. The obtained stability regions are similar regardless which starting point is chosen in the interior of the represented regions. [1]

4.3 Application to retarded TDS

Analogously to the line version of the method, the convenient form of non-neutral retarded TDS characteristic function given by (19) can be utilized to further simplify (56). To evaluate $||\nabla f(j\omega, \boldsymbol{\tau})||_p$ in (56), it is beneficial to first find the upper bounds for partial derivatives of f with regards to each parameter component τ_i. Assuming $\omega \geq 0$, observe that

$$\left|\frac{\partial f}{\partial \tau_i}(j\omega, \boldsymbol{\tau})\right| = \left| -j\omega P_i(j\omega)e^{-j\omega\tau_i}\right| = \omega\Big|P_i(j\omega)\Big| \quad (61)$$

which allows to set a conservative upper bound

$$||\nabla f(j\omega, \boldsymbol{\tau})||_p = \left(\sum_{i=1}^{n}\Big(\omega\big|P_i(j\omega)\big|\Big)^p\right)^{\frac{1}{p}} \quad (62)$$

which does not depend on $\boldsymbol{\tau}$!

Corollary 14 *Let f be defined as in* (19) *and let $\boldsymbol{\tau}^0 \in \mathcal{T}$ be any parameter point satisfying $f(j\omega, \boldsymbol{\tau}^0) \neq 0, \forall \omega \in \mathbb{R}$. Let p and q be real scalars satisfying* (45)*. Then, $\forall \varepsilon_q$ such that*

$$0 < \varepsilon_q \leq \min_{\omega \geq 0} \frac{|f(j\omega, \boldsymbol{\tau}^0)|}{\left(\sum_{i=1}^{n}\big(\omega|P_i(j\omega)|\big)^p\right)^{\frac{1}{p}}}, \quad (63)$$

the following holds

$$NU_f(\boldsymbol{\tau}^0) = NU_f(\boldsymbol{\tau}), \forall \boldsymbol{\tau} \in \left\{\boldsymbol{\tau}^0 + \mathbf{v} \;\middle|\; ||\mathbf{v}||_q \leq \varepsilon_q\right\}. \quad (64)$$

Figure 5 shows the result of iterative applications of (63) to example 10.

[1] If a starting point is on the boundary of two regions, then condition $f(j\omega, \boldsymbol{\tau}^0) \neq 0$ is not satisfied and theorems 2, 6, 11 and 12 cannot be applied.

5 Conclusions and discussions

This paper presents a new methodology for analyzing stability of linear TDS. Necessary and sufficient conditions for expanding the stability equivalence along a line and inside a region are provided. Special attention is given to the class of retarded TDS. The effectiveness of the method is displayed on two examples. Example 8 presents a distributed delay system, the stability of which is being analyzed with regards to one delay and one non-delay parameter. To the best of authors' knowledge, there are no effective methods for analyzing interactions of delay and non-delay parameters, particularly in the case of distributed TDS. Example 10 presents a simple retarded TDS with two delays, the results of which are compared to existing state-of-the-art methods.

The resulting methods can be used to build a complete framework for iteratively exploring stability regions. There are three general options for building such a framework:

(1) Utilize only the line-based algorithm. While this option may provide a good sketch of the stability region such as the ones shown in figures 1 and 2, it only guarantees stability on the lines themselves. This can especially be problematic due to the delay-interference phenomenon described in Michiels & Niculescu (2007), although running the algorithm from a plethora of starting points in various directions alleviates this issue to a certain extent.
(2) Utilize only the region-based method. This approach overcomes issues that option (1) suffers from and is the one used by the authors for obtaining figures 4 and 5.
(3) Combine both line-based and region-based methods. Benefit of a combined approach is the generally better performance of the line-based algorithm for extending stability as much as possible in a certain direction, while obtaining the generally more favorable stability equivalence regions from the region-based method.

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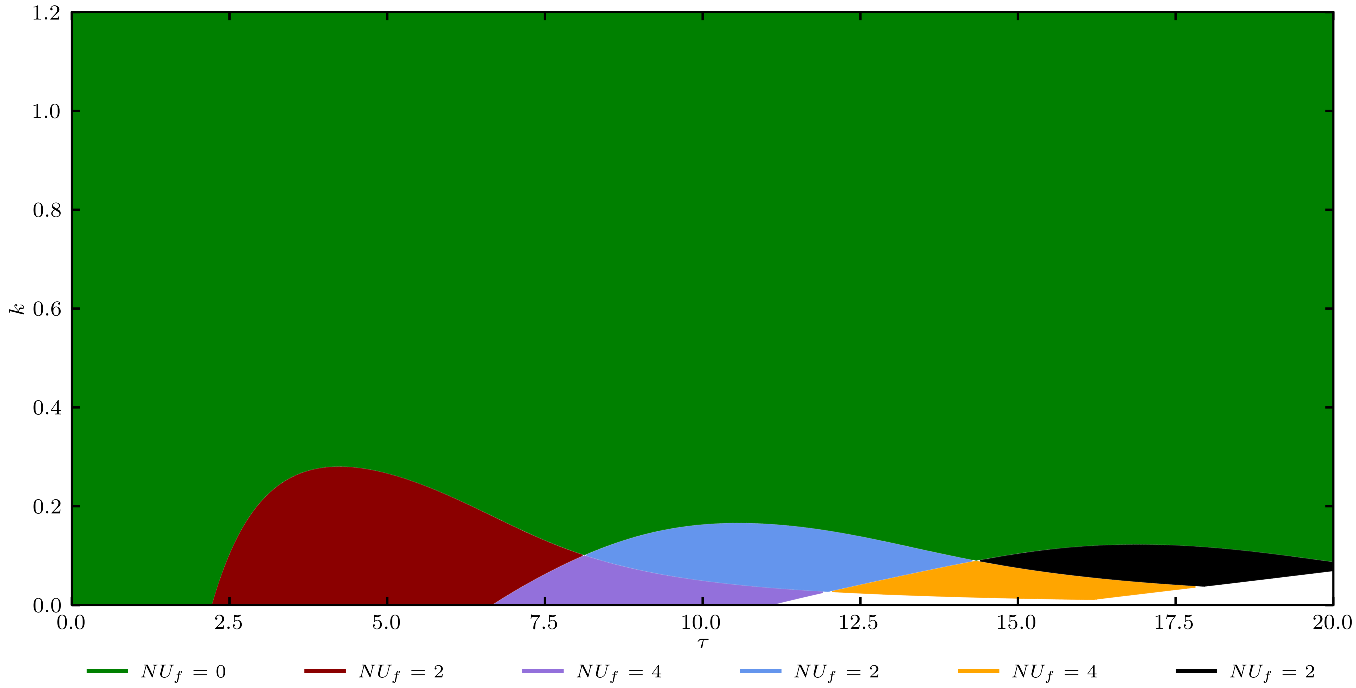

Fig. 4. Results of iteratively applying (56) to example 8

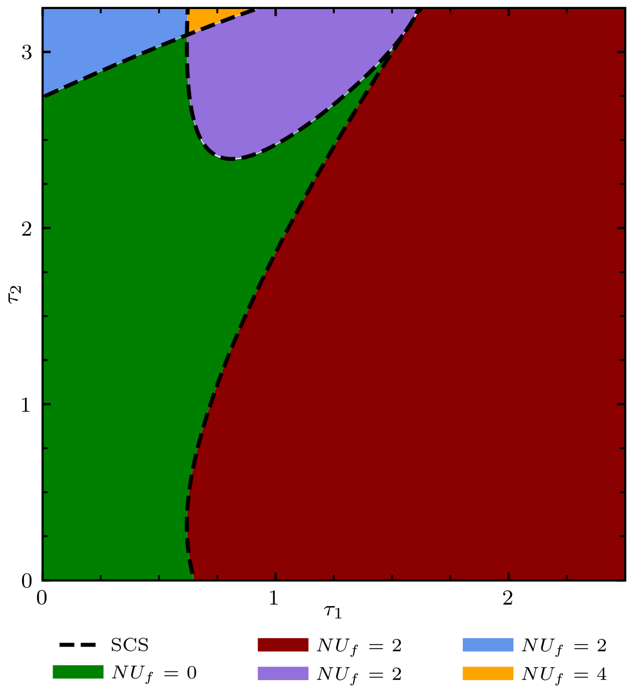

Fig. 5. Results of iteratively applying (63) to example 10

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