Strong Incidence Colouring of Graphs

Brahim Benmedjdoub, Éric Sopena

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Abstract

An incidence of a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v, e) and (w, f) of G are adjacent whenever (i) $v = w$, or (ii) $e = f$, or (iii) $vw = e$ or f. An incidence p-colouring of G is a mapping from the set of incidences of G to the set of colours $\{1, \ldots, p\}$ such that every two adjacent incidences receive distinct colours. Incidence colouring has been introduced by Brualdi and Quinn Massey in 1993 and, since then, studied by several authors.

In this paper, we introduce and study the strong version of incidence colouring, where incidences adjacent to a same incidence must also get distinct colours. We determine the exact value of – or upper bounds on – the strong incidence chromatic number of several classes of graphs, namely cycles, wheel graphs, trees, ladder graphs and subclasses of Halin graphs.

Keywords: Strong incidence colouring; Incidence colouring; Tree; Halin graph; Ladder graph; Necklace; Double star; Wheel graph.

MSC 2010: 05C15.

1 Introduction

All graphs considered in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively, by $\Delta(G)$ the maximum degree of G, by $N(v)$ the set of vertices adjacent to the vertex v and by $\mathrm{dist}_G(u, v)$ the distance between vertices u and v in G.

An *incidence* of a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v, e) and (w, f) of G are *adjacent* whenever (i) $v = w$, or (ii) $e = f$, or (iii) $vw = e$ or f.

An *incidence p-colouring* of G is a mapping from the set of incidences of G to the set of colours $\{1, \ldots, p\}$ such that every two adjacent incidences receive distinct colours. The smallest p for which G admits an incidence p-colouring is the *incidence chromatic number* of G, denoted by $\chi_i(G)$. Incidence colourings were first introduced and studied by Brualdi and Quinn Massey [2]. Incidence colourings of various graph families have attracted much interest in recent years, see for instance [3, 4, 6, 7, 10, 11, 12].

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A *strong edge p-colouring* of G is a mapping from the set of edges of G to the set of colours $\{1, \ldots, p\}$ such that any two edges meeting at a common vertex, or being adjacent to a same edge of G, are assigned different colours. The smallest p for which G admits a strong edge p-colouring is the *strong chromatic index* of G, denoted by $\chi_s^{'}(G)$.

The strong version of incidence colouring is defined in a similar way. A *strong incidence p-colouring* of a graph G is a mapping from the set of incidences of G to a finite set of colours $\{1, \ldots, p\}$ such that any two incidences that are adjacent or adjacent to the same incidence receive distinct colours. The smallest p for which G admits a strong incidence p-colouring is the *strong incidence chromatic number*, denoted by $\chi_i^s(G)$.

Our paper is organised as follows. We first give some preliminary results in Section 2. We then study the strong incidence chromatic number of simple graph classes (stars, complete graphs, cycles, wheel graphs and trees) in Section 3, of ladder graphs in Section 4 and of subclasses of Halin graphs in Section 5. We finally propose some directions for future research in Section 6.

2 Preliminary results

We list in this section some basic results on the strong incidence chromatic number of various graph classes.

The *square* G^2 of a graph G is the graph defined by $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $\text{dist}_G(u, v) \leq 2$. A colouring of G^2 is called a *2-distance colouring* of G and the *2-distance chromatic number* of G is denoted by $\chi_2(G)$.

For any graph G, the *incidence graph* of G, denoted by I_G, introduced in [1], is the graph whose vertices are the incidences of G, two incidences being joined by an edge whenever they are adjacent. Clearly, every incidence colouring of G is nothing but a proper vertex colouring of I_G, and every strong incidence colouring of G is nothing but a 2-distance colouring of G, so that, $\chi_i(G) = \chi(I_G)$ and $\chi_i^s(G) = \chi_2(I_G)$. Moreover, since every strong incidence colouring is an incidence colouring, we have $\chi_i(G) \leq \chi_i^s(G)$ for every graph G.

For every vertex v in a graph G, we denote by $A^-(v)$ the set of incidences of the form (v, vu), and by $A^+(v)$ the set of incidences of the form (u, uv) (see Figure 1). We thus have $|A^-(v)| = |A^+(v)| = \deg(v)$ for every vertex v. Every edge uv of G has two incidences (u, uv) and (v, vu). We will say that two incidences are *strongly adjacent* if they are either adjacent or adjacent to a same incidence. The following observation will be useful.

Observation 1 *For every incidence (v, vu) in a graph G with maximum degree Δ, the set of incidences that are strongly adjacent to (v, vu) is*

$$\bigcup_{w \in N(v) \setminus u} A^+(w) \cup \bigcup_{w \in N(v)} A^-(w) \cup \bigcup_{w \in N(u) \setminus v} A^-(w),$$

whose cardinality is at most $3\Delta^2 - 2\Delta$.

Indeed, the cardinality of the set of incidences that are strongly adjacent to (v, vu) (see Figure 2) is

$$\sum_{w \in N(v) \setminus u} \deg(w) + \sum_{w \in N(v)} \deg(w) + \sum_{w \in N(u) \setminus v} \deg(w) \leq 2\Delta(\Delta - 1) + \Delta^2.$$

Therefore, if we colour G with $3\Delta^2 - 2\Delta + 1$ colours, then there exists at least one free colour that can be assigned to each incidence (v, vu) of G. We thus have the following upper bound on the strong incidence chromatic number of every graph.

Proposition 2 *For every graph G with maximum degree Δ, $\chi_i^s(G) \leq 3\Delta^2 - 2\Delta + 1$.*

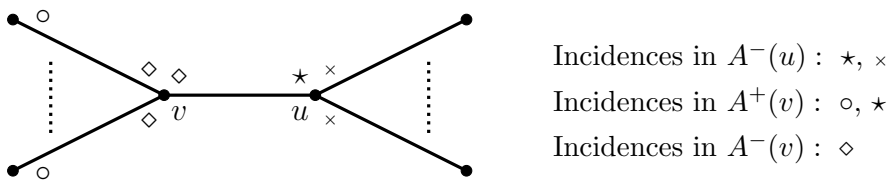


Figure 1: Adjacent incidences.

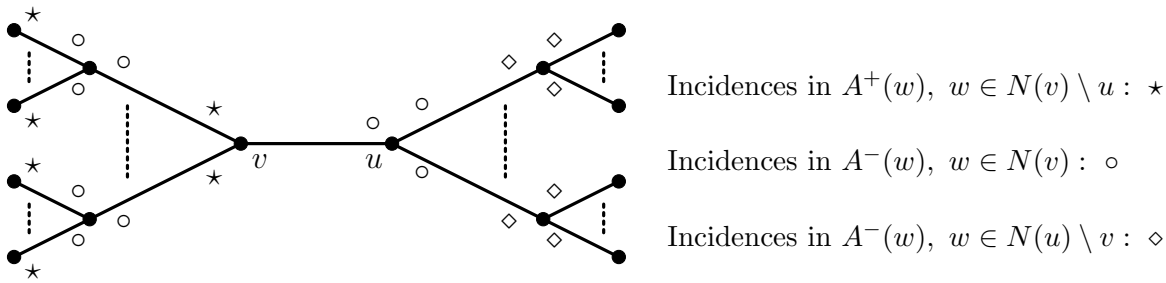


Figure 2: Strongly adjacent incidences.

For a given graph G with maximum degree Δ, we let

$$\sigma(G) = \max_{uv \in E(G)} \{2\deg_G(v) + \deg_G(u) - 1\}.$$

For every edge uv in $E(G)$, the incidences of the set $A^-(v) \cup A^+(v) \cup A^-(u)$, of cardinality $2\deg_G(v) + \deg_G(u) - 1$, are pairwise strongly adjacent, which means that they must be assigned distinct colours. Therefore, using Proposition 2, we have the following inequalities.

Proposition 3 *For every graph G with maximum degree Δ, $\sigma(G) \leq \chi_i^s(G) \leq 3\Delta^2 - 2\Delta + 1$.*

In the following proposition we give an upper bound on the strong incidence chromatic number of a graph G as a function of its strong chromatic index.

Proposition 4 *For every graph G, $\chi_i^s(G) \leq 2\chi_s^{'}(G)$.*

Proof. Let λ be a strong edge p-colouring of G. From this colouring, a strong incidence $2p$-colouring λ' is obtained using the set of $2p$ colours $\{1, 1' \ldots, p, p'\}$ as follows: for every edge $uv \in E(G)$, if $\lambda(uv) = k$, $k \in \{1, \ldots, p\}$, then $\lambda'(u, uv) = k$ and $\lambda'(v, vu) = k'$. Indeed, if $\lambda'(u, uv) = \lambda'(w, wx)$ for two incidences (u, uv) and (w, wx) of G, then $\lambda(uv) = \lambda(wx)$, which implies $\mathrm{dist}_G(uv, wx) \geq 3$, and thus $\mathrm{dist}_{I_G}((u, uv), (w, wx)) \geq 3$. $\square$

3 Simple graph classes

In this section, we determine the strong incidence chromatic number of stars, complete graphs, cycles, trees and wheel graphs.

We denote by $S_n,\ n \geq 1$, the star of order $n+1$, by $K_n,\ n \geq 1$, the complete graph of order n and by $K_{m,n},\ m \geq n \geq 2$, the complete bipartite graph with parts of size m and n. In [2], Brualdi and Massey showed that $\chi_i(S_n) = n+1,\ \chi_i(K_n) = n$ and $\chi_i(K_{m,n}) = m+2$, for all $m \geq n \geq 2$. Since all incidences of any graph in these classes of graphs are pairwise strongly adjacent, we have the following proposition.

Proposition 5

1. *For every* $n \geq 1$, $\chi_i^s(S_n) = 2n$,
2. *for every* $n \geq 2$, $\chi_i^s(K_n) = 2|E(G)|$,
3. *for every* $m \geq n \geq 2$, $\chi_i^s(K_{m,n}) = 2nm$.

Let C_n, $n \geq 3$, denote the cycle of order n. Observe that $I_{C_n} = C_{2n}^2$. Therefore, a strong incidence colouring of the cycle C_n is a 2-distance colouring of C_{2n}^2, that is nothing but a proper colouring of $(C_{2n}^2)^2 = C_{2n}^4$. We thus have the following result.

Proposition 6 *For every integer* $n \geq 3$, $\chi_i^s(C_n) = \chi(C_{2n}^4)$.

By setting $a = 4$, the following theorem gives the value of $\chi(C_{2n}^4)$.

Theorem 7 (Prowse and Woodall [8]) *Let* n *and* a *be positive integers such that* $n \geq 2a$ *and* $n = q(a+1) + r$, *with* $q > 0$ *and* $0 \leq r \leq a$. *Then* $\chi(C_n^a) = a + 1 + \lceil r/q \rceil$.

Using Proposition 6 and Theorem 7, we can infer the value of the strong incidence chromatic number of any cycle.

Theorem 8 *Let* n *be a positive integer such that* $n \geq 4$ *and* $2n = 5q + r$, *with* $q > 0$ *and* $0 \leq r \leq 4$. *Then* $\chi_i^s(C_n) = 5 + \lceil r/q \rceil$.

We now determine the value of $\chi_i^s(W_n)$, where W_n, $n \geq 3$, is the wheel graph of order $n+1$, obtained from C_n by adding a universal vertex. It is easy to observe that $\chi_i(W_n) = n+1$ for every $n \geq 3$. Indeed, since the square of W_n is the complete graph K_{k+1}, we get

$$n + 1 = \Delta(W_n) + 1 \leq \chi_i(W_n) \leq \chi(W_n^2) = \chi(K_{n+1}) = n + 1.$$

Using Proposition 5 and Theorem 8, we can determine the strong incidence chromatic number of wheel graphs.

Theorem 9 *Let* n *be a positive integer such that* $n \geq 3$ *and* $2n = 5q + r$, *with* $q > 0$ *and* $0 \leq r \leq 4$. *Then* $\chi_i^s(W_n) = 5 + 2n + \lceil r/q \rceil$.

Proof. Since $W_3 = K_4$, the result holds by Proposition 5 for $n = 3$. Suppose now $n \geq 4$ and let T denote the spanning subgraph of W_n isomorphic to S_n. Since every incidence in T is strongly adjacent to every incidence not in T, we get $\chi_i^s(W_n) = \chi_i^s(C_{W_n}) + \chi_i^s(S_n)$ and the result follows from Proposition 5 and Theorem 8. □

We finally determine the strong incidence chromatic number of trees.

Theorem 10 *If* G *is a tree then* $\chi_i^s(G) = \max_{uv \in E(G)} \{2\deg_G(v) + \deg_G(u) - 1\} = \sigma(G)$.

Proof. By Proposition 3, we have $\chi_i^s(G) \geq \sigma(G)$. The other direction is proved by induction on $|V(G)|$. If G is a star, then $\sigma(G) = 2n$ and the result follows from Proposition 5. We can thus assume that G is not a star, so that $|V(G)| \geq 4$. Let u be a vertex of G of degree $p + 1 \geq 3$ having only one neighbour, denoted by u', which is not a leaf, and let $A = \{v_1, \ldots, v_p\}$ be the set of p leaves that are neighbours of u. Let λ be a strong incidence colouring of $G \setminus A$. Now, observe that each incidence of the form (u, uv_i), $1 \leq i \leq p$, has at most

$$2\deg_{G \setminus A}(u') = 2\deg_G(u') \leq \sigma(G) - \deg_G(u) + 1$$

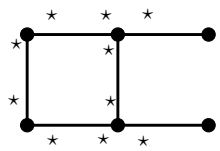


Figure 3: The graph L.

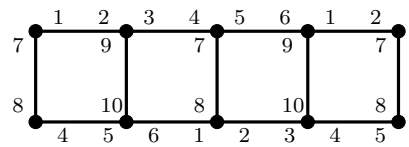


Figure 4: A strong incidence 10-colouring of L_5.

strongly adjacent incidences in $G \setminus A$, and each incidence of the form (v_i, v_iu), $1 \leq i \leq p$, has at most

$$\deg_{G\setminus A}(u') + 1 = \deg_G(u') + 1 \leq \sigma(G) - 2\deg_G(u) + 1$$

strongly adjacent incidences in $G \setminus A$, so that λ can be extended to a strong incidence colouring of G, starting by colouring the incidences of the form (u, uv_i), $1 \leq i \leq p$, and then, the incidences of the form (v_i, v_iu), $1 \leq i \leq p$. This concludes the proof. □

4 Ladder graphs

The *ladder graph*, denoted by L_h, is obtained from two paths of order h, $h \geq 1$, $P_h = v_1 \dots v_h$ and $P'_h = v'_1 \dots v'_h$ by adding the edges $v_iv'_i$, $1 \leq i \leq h$. In the following theorem, we give the value of $\chi_i^s(L_h)$.

Theorem 11 *For every integer $h \geq 3$, $\chi_i^s(L_h) = 10$.*

Proof. Note that when $h \geq 3$, L_h contains a subgraph isomorphic to the graph L (see Figure 3) which contains 10 pairwise strongly adjacent incidences (marked with a star), implying $\chi_i^s(L_h) \geq 10$ for every $h \geq 3$. To complete the proof, it suffices to give a strong incidence 10-colouring λ of L_h. Such a colouring can be obtained as follows (see Figure 4 for the case $h = 5$).

1. We sequentially colour the incidences of the path P_h using the pattern 123456.
2. We sequentially colour the incidences of the path P'_h using the pattern 456123.
3. For every i, $1 \leq i \leq h$, we set
   - $\lambda(v'_i, v'_iv_i) = 7$ and $\lambda(v_i, v_iv'_i) = 8$, if i is odd,
   - $\lambda(v'_i, v'_iv_i) = 9$ and $\lambda(v_i, v_iv'_i) = 10)$ if i is even.

The so-obtained colouring is clearly a strong incidence colouring of L_h. This concludes the proof. □

5 Subclasses of Halin graphs

Recall first that a *Halin graph* H is a planar graph obtained from a tree of order at least 4 with no vertex of degree 2, by adding a cycle connecting all its leaves [5]. We call this cycle the *outer cycle*

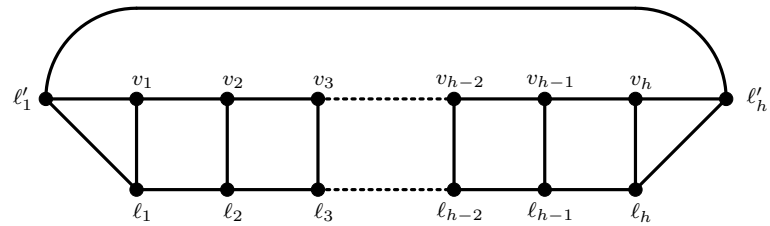


Figure 5: The graph N_h.

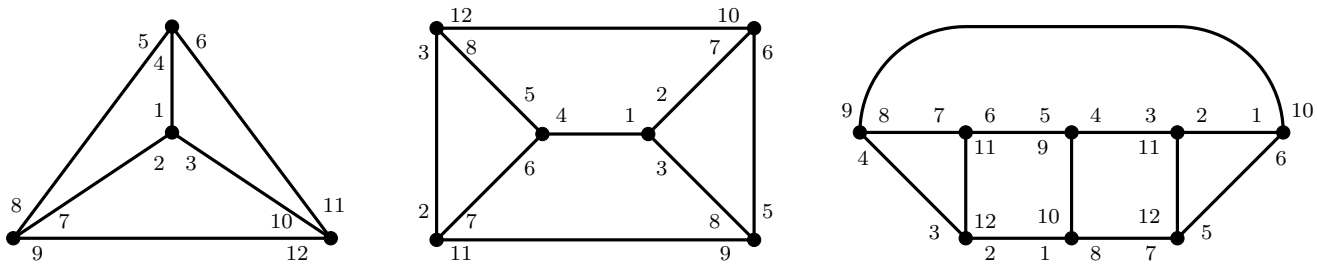


Figure 6: Strong incidence 12-colourings of N_1, N_2 and N_3.

of H. The subgraph T obtained by deleting all the edges of the outer cycle of H is thus a tree, called the *internal tree of* H.

In this section, we determine the exact value of – or upper bounds on – the strong incidence chromatic number of every Halin graph whose internal tree is either a comb or a double star.

5.1 Halin graphs whose internal tree is a comb

A tree is called a $(3,1)$-tree if the degree of each non-leaf vertex is 3. A caterpillar is a tree T such that, after deleting all its leaves, the remaining graph is a simple path called the *spine of* T. A *comb* is a caterpillar which is also a (3,1)-tree. It is easy to see that every Halin graph whose internal tree is a comb is a cubic Halin graph. In particular, if the spine has one vertex then this is the complete graph K_4.

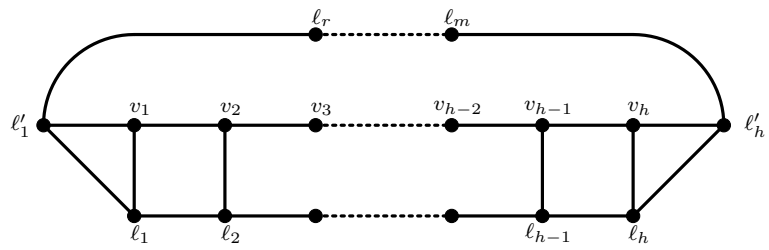
For every integer $h \geq 1$, we construct a Halin graph H_h of order $2h+2$ whose internal tree T_h is a comb, using the construction given in [9]. Let $P_h = v_1 v_2 \ldots v_h$ be the spine of H_h. We denote by ℓ_1 and ℓ'_1 (resp. ℓ_h and ℓ'_h) the two leaves of v_1 (resp. v_h), by ℓ_i the unique leaf of v_i, $2 \leq i \leq h-1$, and by C_h the outer cycle of H_h.

Let $\mathcal{H}_h^c$ be the set of all Halin graphs whose internal tree is a comb of order $2h+2$. A Halin graph H_h such that $C_h = \ell_1 \ell_2 \ldots \ell_h \ell_1$ is called a *necklace*. We denote by N_h the (unique) necklace of order $2h+2$. Observe that $\mathcal{H}_h^c = \{N_h\}$ for every h, $1 \leq h \leq 3$.

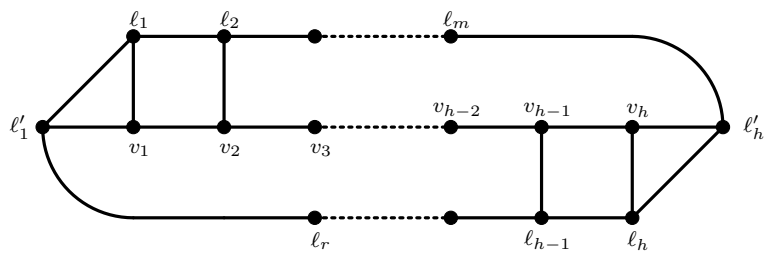
It is easy to see that all incidences of N_1 are pairwise strongly adjacent. Therefore, $\chi_i^s(N_1) = 12$. If $G = N_2$, then the incidences of the set

$$A^-(v_1) \cup A^+(v_1) \cup A^-(v_2) \cup \{(\ell'_1, \ell'_1\ell'_2),\ (\ell'_2, \ell'_2\ell'_1),\ (\ell_1, \ell_1\ell_2), (\ell_2, \ell_2\ell_1)\},$$

of cardinality 12, are pairwise strongly adjacent. Hence, we have $\chi_i^s(N_2) \geq 12$. Also, the cardinality



(a)



(b)

Figure 7: Each graph $H \in \mathcal{H}_h^c \setminus \{N_h\}$ has the form (a) or the form (b)

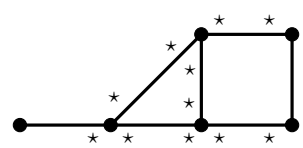


Figure 8: The graph F.

of the set of the incidences of N_3 is 24. Therefore, if we colour the graph with 11 colours, then at least two colours must be repeated at least three times or at least one colour must be repeated at least four times. It is tedious but not difficult to check that this is not possible. Hence, $\chi_i^s(N_3) \geq 12$. Strong incidence 12-colourings of $N_h,\ 1 \leq h \leq 3$, are given in Figure 6.

Suppose now $h \geq 4$. In [9], Shiu, Lam and Tam proved the following theorem.

Theorem 12 (Shiu, Lam and Tam [9]) *If $H \in \mathcal{H}_h^c,\ h \geq 4$, then $6 \leq \chi_s(H) \leq 8$.*

By Proposition 4 and Theorem 12, we get $\chi_i^s(H) \leq 16$, for every graph $H \in \mathcal{H}_h^c,\ h \geq 4$. We prove that if H is not a necklace then this bound can be decreased to 14.

Theorem 13 *If $H \in \mathcal{H}_h^c \setminus \{N_h\},\ h \geq 4$, then $11 \leq \chi_i^s(H) \leq 14$.*

Proof. Let $H \in \mathcal{H}_h^c \setminus \{N_h\}$. By exchanging if necessary the leaves ℓ_1 and ℓ'_1, or ℓ_h and ℓ'_h, we can assume that H has either the form depicted in Figure 7(a) or the form depicted in Figure 7(b), where the edges $v_i\ell_i, 3 \leq i \leq h-2$, may be either upward or downward. In both cases, H contains a subgraph isomorphic to the graph F (see Figure 8) which contains 11 pairwise strongly adjacent incidences (marked with a star), implying $\chi_i^s(H) \geq 11$.

We now construct a strong incidence 14-colouring λ of H assuming that H has either the form depicted in Figure 7(a) or the form depicted in Figure 7a(b), which means that the incidence $(v_2, v_2\ell_2)$ (resp. $(v_{h-1}, v_{h-1}v_h)$) is not strongly adjacent with the incidence $(\ell'_1, \ell'_1\ell_1)$ (resp. $(\ell'_h, \ell'_h\ell_h)$). Such a colouring can be obtained as follows (see Figure 9).

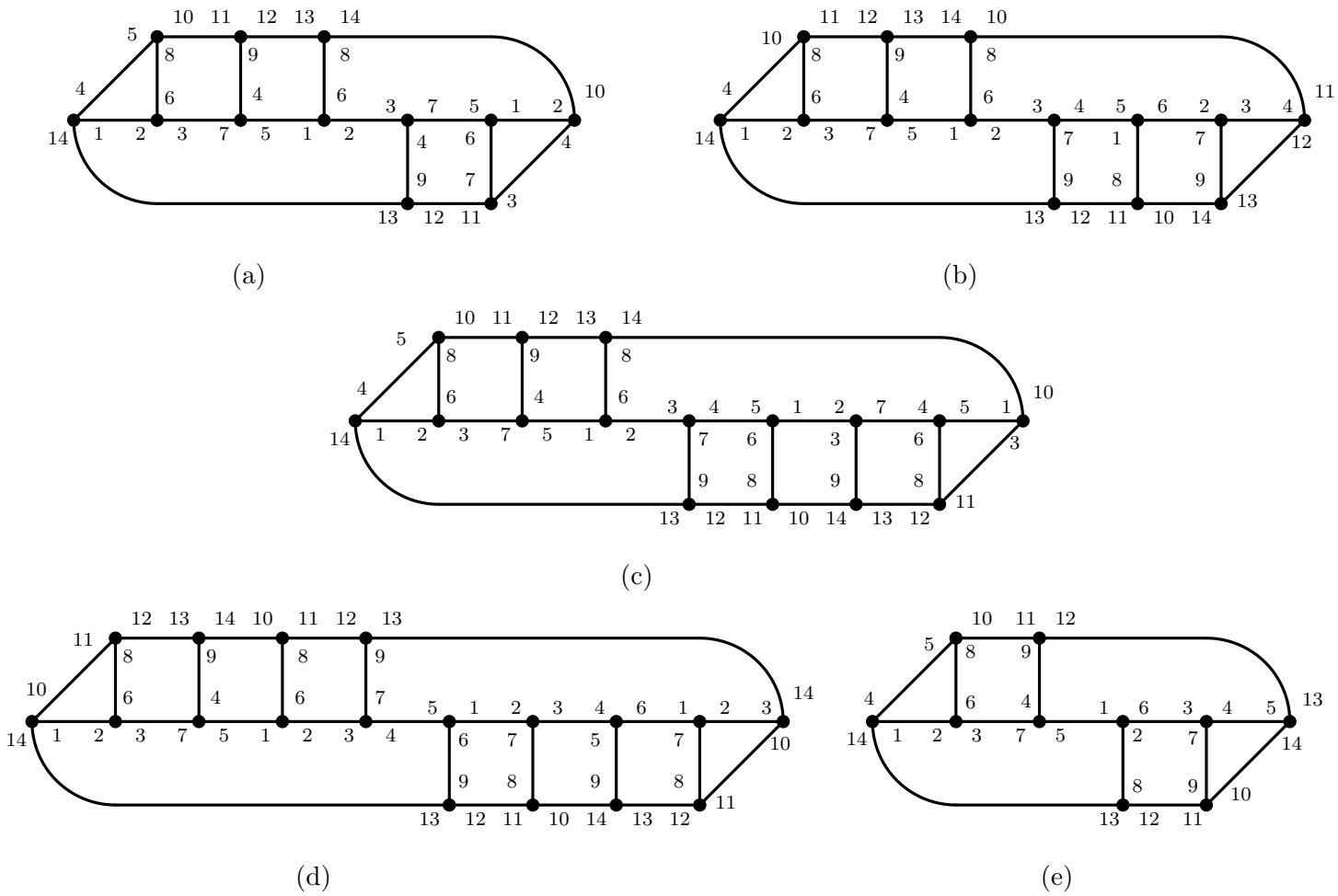


Figure 9: Strong incidence 14-colourings of graphs in $\mathcal{H}_h^c$, $4 \leq h \leq 8$.

- We colour the incidences of the path ℓ'_1, ℓ'_h sequentially, from $(\ell'_1, \ell'_1 v_1)$ to $(\ell'_h, \ell'_h v_h)$ using the pattern 12345.
- For every integer i, $1 \leq i \leq h$, we set $\lambda(v_i, v_i\ell_i) = 6$ if i is odd and $\lambda(v_i, v_i\ell_i) = 7$ otherwise.
- We colour circularly the incidences of the form $(\ell_i, \ell_i v_i)$, $1 \leq i \leq h$ according to their order in the outer cycle C_{2h+2} by alternating the colours 8 and 9.
- We exchange the colours of the two incidences $(v_2, v_2\ell_2)$ and (v_2, v_2v_1), and the colours of the two incidences $(v_{h-1}, v_{h-1}v_h)$ and $(v_{h-1}, v_{h-1}\ell_{h-1})$.
- We now colour the incidences of the outer cycle C_{2h+2} according to the value of $h \mod 5$:
  - $h = 5k$, $k \geq 1$ (see Figure 9(a) for the case $h = 5$).
    We first set $\lambda(\ell'_1, \ell'_1\ell_1) = \lambda(v_2, v_2\ell_2)$, $\lambda(\ell_1, \ell_1\ell'_1) = \lambda(v_2, v_2v_3)$, $\lambda(\ell'_h, \ell'_h\ell_h) = \lambda(v_{h-1}, v_{h-1}\ell_{h-1})$ and $\lambda(\ell_h, \ell_h\ell'_h) = \lambda(v_{h-1}, v_{h-1}v_{h-2})$. We then sequentially colour the uncoloured incidences of C_{2h+2}, starting from $(\ell_1, \ell_1\ell_2)$, using the pattern 10.11.12.13.14.
  - $h = 5k + 1$, $k \geq 1$ (see Figure 9(b) for the case $h = 6$).
    We first set $\lambda(\ell'_1, \ell'_1\ell_1) = \lambda(v_2, v_2\ell_2)$. We then sequentially colour the uncoloured incidences of C_{2h+2}, starting from $(\ell_1, \ell_1\ell'_1)$, using the pattern 10.11.12.13.14.
  - $h = 5k + 2$, $k \geq 1$ (see Figure 9(c) for the case $h = 7$).
    We first set $\lambda(\ell'_1, \ell'_1\ell_1) = \lambda(v_2, v_2\ell_2)$, $\lambda(\ell_1, \ell_1\ell'_1) = \lambda(v_2, v_2v_3)$ and $\lambda(\ell'_h, \ell'_h\ell_h) = \lambda(v_{h-1}, v_{h-1}\ell_{h-1})$. We then sequentially colour the uncoloured incidences of C_{2h+2}, starting from $(\ell_1, \ell_1\ell_2)$, using the pattern 10.11.12.13.14.
  - $h = 5k + 3$, $k \geq 1$ (see Figure 9(d) for the case $h = 8$).
    We sequentially colour the incidences of C_{2h+2}, starting from $(\ell'_1, \ell'_1\ell_1)$, using the pattern 10.11.12.13.14.
  - $h = 5k + 4$, $k \geq 0$ (see Figure 9(e) for the case $h = 4$).
    We first set $\lambda(\ell'_1, \ell'_1\ell_1) = \lambda(v_2, v_2\ell_2)$ and $\lambda(\ell_1, \ell_1\ell'_1) = \lambda(v_2, v_2v_3)$. We then sequentially colour the uncoloured incidences of C_{2h+2}, starting from $(\ell_1, \ell_1\ell_2)$, using the pattern 10.11.12.13.14.

In each case, the so-obtained colouring is clearly a strong incidence colouring of $G \in \mathcal{H}^c_h$. This completes the proof. □

We Now determine the value of the strong incidence chromatic number of necklaces.

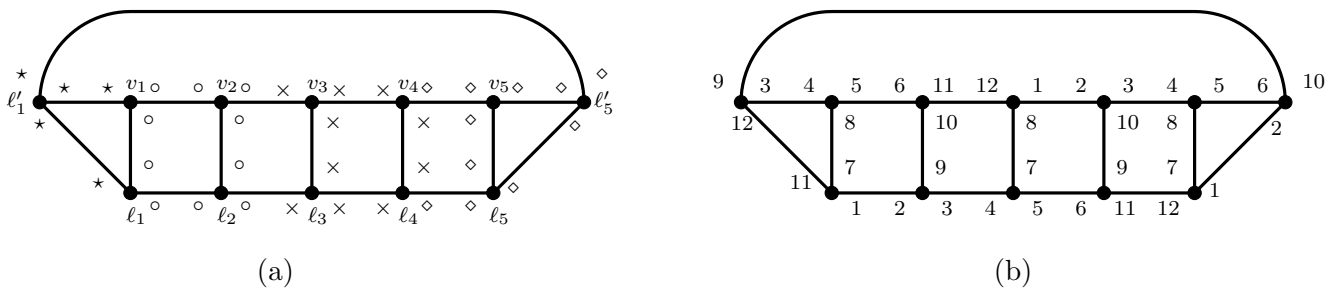
Theorem 14 *For every necklaces N_h, $h \geq 1$, we have*

$$\chi^s_i(N_h) = \begin{cases} 12 & \text{if } h = 1, 2, 3, 5, \\ 11 & \text{otherwise.} \end{cases}$$

Proof. As we showed in the proof of Theorem 13, N_h contains a subgraph isomorphic to the graph F (see Figure 8), which implies $\chi^s_i(N_h) \geq 11$ for every $h \geq 1$. The values of $\chi^s_i(N_h)$, $1 \leq h \leq 3$, were given in the beginning of the subsection.

If $h = 5$, the cardinality of the set of the incidences of N_5 is 36. Therefore, if we colour the graph with 11 colours, then at least three colours must be repeated at least four times, or one colour must be repeated at least five times and one colour must be repeated at least four times, or at least one colour must be repeated at least six times. We will prove that at most two colours can be repeated four times, which will imply $\chi^s_i(N_5) \geq 12$.

The set of incidences of N_5 can be partitioned into four sets, the set of incidences marked with a star, the set of incidences marked with a diamond, the set of incidences marked with a circle



(a) (b)

Figure 10: The necklace N_5 (for the proof of Theorem 14).

and the set of incidences marked with a cross (See Figure 10(a)). Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of any incidence of N_5 can be repeated at most four times. Moreover, any colour repeated four times must be used exactly once in each of these sets. We now prove that among the colours of the incidences marked with a star in Figure 10(a), only two colours can be repeated four times.

- The incidence $(\ell'_1, \ell'_1\ell'_5)$.
  The set of the incidences that are not strongly adjacent with this incidence can be partitioned into the two sets $\{(v_2, v_2\ell_2),\ (v_2, v_2v_3),\ (\ell_2, \ell_2v_2),\ (\ell_2, \ell_2\ell_3),\ (v_3, v_3v_2),\ (\ell_2, \ell_2\ell_3)\}$ and $\{(v_3, v_3\ell_3),\ (v_3, v_3v_4),\ (\ell_3, \ell_3v_3),\ (\ell_3, \ell_3\ell_4),\ (v_4, v_4v_3),\ (v_4, v_4\ell_4),\ (v_4, v_4v_5),\ (\ell_4, \ell_4\ell_3),\ (\ell_4, \ell_4v_4), (\ell_4, \ell_4\ell_5)\}$. Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of the incidence $(\ell'_1, \ell'_1\ell'_5)$ can be repeated only two more times.
- The incidence (ℓ'_1, ℓ'_1v_1).
  The set of the incidences that are not strongly adjacent with this incidence can be partitioned into the two sets $\{(\ell_2, \ell_2v_2),\ (\ell_2, \ell_2\ell_3),\ (v_3, v_3v_2),\ (v_3, v_3\ell_3),\ (v_3, v_3v_4),\ (\ell_3, \ell_3\ell_2),\ (\ell_3, \ell_3v_3),\ (\ell_3, \ell_3\ell_4)\}$ and $\{(v_4, v_4v_3),\ (v_4, v_4\ell_4),\ (v_4, v_4v_5),\ (\ell_4, \ell_4\ell_3), (\ell_4, \ell_4v_4), (\ell_4, \ell_4\ell_5), (v_5, v_5v_4), (v_5, v_5\ell_5),\ (\ell_5, \ell_5\ell_4), (\ell_5, \ell_5v_5), \}$. Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of the incidence (ℓ'_1, ℓ'_1v_1) can be repeated only two more times.
- By symmetry, the case of the incidence $(\ell'_1, \ell'_1\ell_1)$ is similar to the case of the incidence (ℓ'_1, ℓ'_1v_1).

Based on the above, only the colours of the two incidences $(\ell_1, \ell_1\ell'_1)$ and $(v_1, v_1\ell'_1)$ can be repeated four times. Hence, $\chi_i^s(N_5) \geq 12$. A strong incidence 12-colouring of N_5 is given in Figure 10(b), so that $\chi_i^s(N_5) = 12$.

Finally, if $h = 4$ or $h \geq 6$ then it suffices to construct a strong incidence 11-colouring of N_h. Such a colouring can be obtained as follows (see Figure 11). We consider two cases, depending on the parity of h.

- h is even.
  - We first colour the subgraph S_h induced by the set of vertices $\{v_1, \ldots,\ v_h,\ \ell_1, \ldots,\ \ell_h\}$ (which is isomorphic to the ladder graph L_h), as in the proof of Theorem 11.
  - We then modify the colouring of the subgraph S_h and we complete the colouring of N_h according to the value of $h \mod 6$:
    * $h = 6k$, $k \geq 1$ (see Figure 11(a) for the case $h = 6$).
      We set $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 11$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 3$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 2$, $\lambda(\ell'_h, \ell'_hv_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 11$, $\lambda(\ell_h, \ell_h\ell'_h) = 5$.
    * $h = 6k + 2$, $k \geq 1$ (see Figure 11(b) for the case $h = 8$).
      We set $\lambda(\ell_1, \ell_1\ell_2) = 11$, $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1v_1) = 9$, $\lambda(v_1, v_1\ell'_1) =$

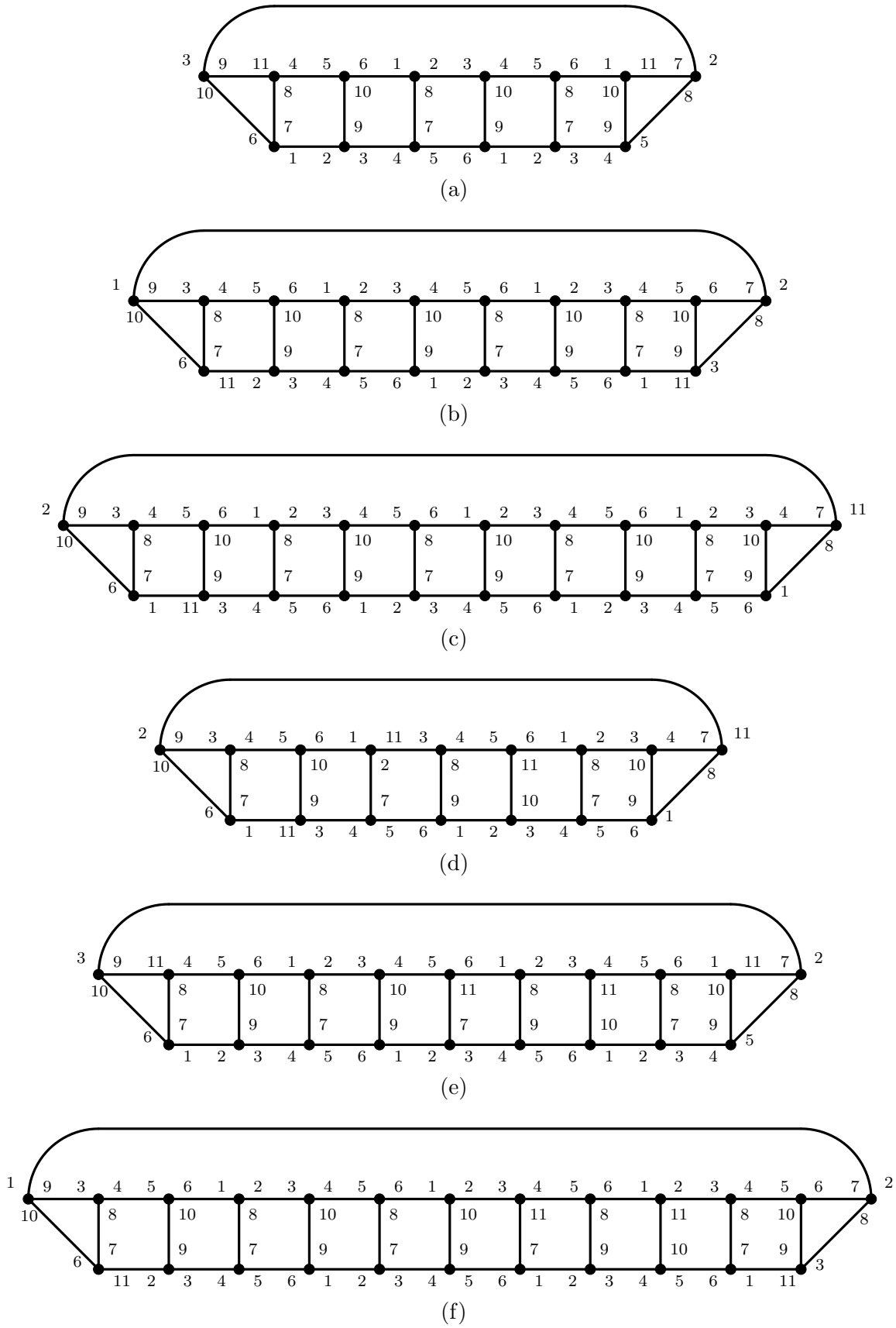


Figure 11: Strong incidence 11-colourings of N_h, $6 \leq h \leq 11$.

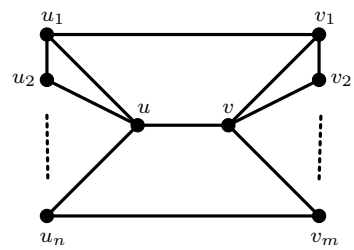


Figure 12: The Halin graph $HD_{m,n}$.

3, $\lambda(\ell'_1, \ell'_1\ell'_h) = 1$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 2$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 6$, $\lambda(\ell_h, \ell_h\ell'_h) = 3$.

  * $h = 6k + 4$, $k \geq 0$ (see Figure 11(c) for the case $h = 10$).
  We set $\lambda(\ell_2, \ell_2\ell_1) = 11$, $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1 v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 3$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 2$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 11$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 4$, $\lambda(\ell_h, \ell_h\ell'_h) = 1$.

- h is odd.
  - We first colour the subgraph S_h induced by the vertices $\{v_1, \ldots, v_h, \ell_1, \ldots, \ell_h\}$ (which is isomorphic to the ladder graph L_h), as in the proof of Theorem 11, and we make the following modifications.
  We set $\lambda(v_{h-4}, v_{h-4}\ell_{h-4}) = 11$, $\lambda(v_{h-3}, v_{h-3}\ell_{h-3}) = 8$, $\lambda(v_{h-2}, v_{h-2}\ell_{h-2}) = 11$, $\lambda(\ell_{h-2}, \ell_{h-2}v_{h-2}) = 10$, $\lambda(v_{h-1}, v_{h-1}\ell_{h-1}) = 8$, $\lambda(\ell_{h-1}, \ell_{h-1}v_{h-1}) = 7$, $\lambda(v_h, v_h\ell_h) = 10$, $\lambda(\ell_h, \ell_h v_h) = 9$.
  - We modify the colouring of the subgraph S_h and we complete the colouring of N_h according to the value of $h \mod 6$:
    * $h = 6k + 1$, $k \geq 1$ (see Figure 11(d) for the case $h = 7$).
    We set $\lambda(\ell_2, \ell_2\ell_1) = 11$, $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1 v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 3$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 2$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 11$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 4$, $\lambda(\ell_h, \ell_h\ell'_h) = 1$. if $h = 7$ the we exchange the colours of the incidences (v_3, v_3v_4) and $(v_3, v_3\ell_3)$.
    * $h = 6k + 3$, $k \geq 1$ (see Figure 11(e) for the case $h = 9$).
    We set $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1 v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 11$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 3$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 2$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 11$, $\lambda(\ell_h, \ell_h\ell'_h) = 5$.
    * $h = 6k + 5$, $k \geq 1$ (see Figure 11(f) for the case $h = 11$).
    We set $\lambda(\ell_1, \ell_1\ell_2) = 11$, $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1 v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 3$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 1$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 2$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 6$, $\lambda(\ell_h, \ell_h\ell'_h) = 3$.

In each case, the so-obtained colouring is clearly a strong incidence colouring of N_h. This completes the proof. □

5.2 Halin graphs whose internal tree is a double star

The *double star*, denoted by $S_{m,n}$, $m \geq n \geq 2$, is the graph obtained from the stars S_m and S_n by adding an edge joining the central vertex of S_m to the central vertex of S_n. The Halin graph $HD_{m,n}$ (see Figure 12) is the Halin graph whose internal tree is the double star $S_{m,n}$ and whose outer cycle is $u_1 \ldots u_n v_m \ldots v_1 u_1$. We denote by P the path $v_1 \ldots v_m$ and by P' the path $u_1 \ldots u_n$.

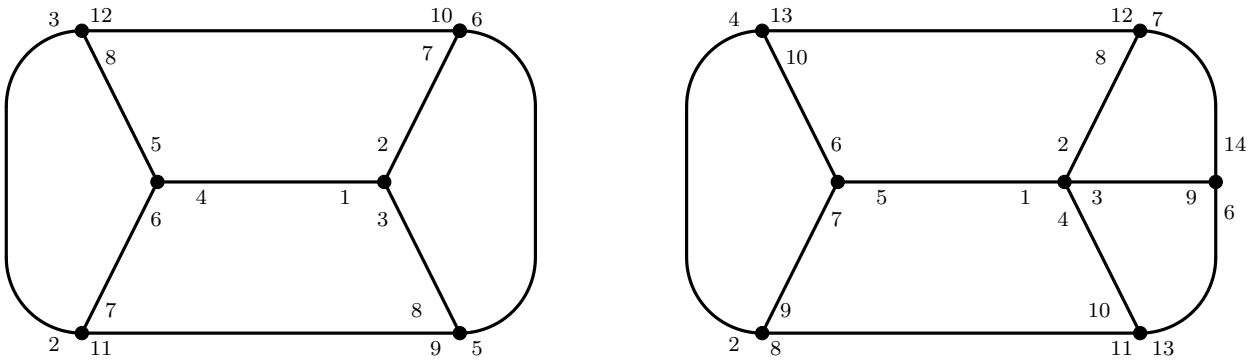


Figure 13: Strong incidence colourings of $HD_{2,2}$ and $HD_{3,2}$.

It is easy to see that for every graph $HD_{m,n}$, $m \geq n \geq 2$, the incidences of the set

$$A^-(v) \cup A^+(v) \cup A^-(u) \cup \{(v_1, v_1u_1)\},$$

of cardinality $2\deg(v) + \deg(u) = \sigma(HD_{m,n}) + 1$, are pairwise strongly adjacent. Therefore, we have the following inequality.

Proposition 15 *For every two integers m and n, $m \geq n \geq 2$, $\chi_i^s(HD_{m,n}) \geq 2m + n + 3 = \sigma(HD_{m,n}) + 1$.*

We first define a partial colouring λ of $HD_{m,n}$, for every $m \geq n \geq 3$, as follows.

- The incidences (v, vu), (v, vv_1), $(v, vv_2), \ldots,$ (v, vv_m) are coloured with the colours 1, 2, $3, \ldots,$ $m + 1$, respectively.
- The incidences (u, uv), (u, uu_1), $(u, uu_2), \ldots,$ (u, uu_n) are coloured with the colours $m + 2$, $m + 3$, $m + 4, \ldots,$ $m + n + 2$, respectively.
- The incidences (v_1, v_1v), $(v_2, v_2v), \ldots,$ (v_m, v_mv) are coloured with the colours $m+n+3$, $m+n+4, \ldots,$ $2m + n + 2$, respectively.
- The incidence (v_1, v_1u_1) is coloured with the colour $2m + n + 3$.

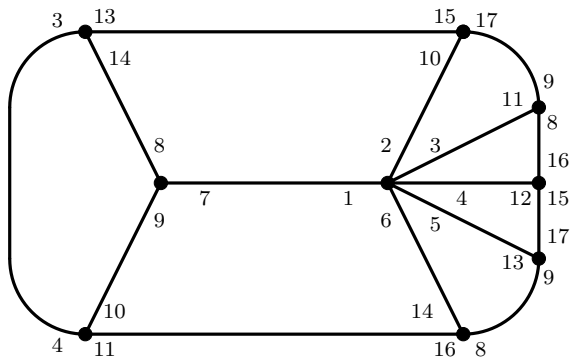
In the next lemmas, we will extend λ to a colouring of $HD_{m,n}$, according to the values of m and n.

Lemma 16 *For every integer $m \geq 2$,*

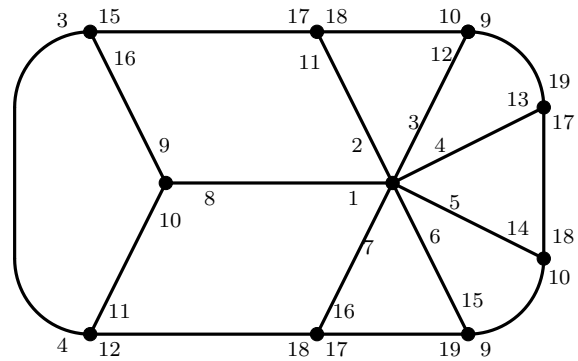
$$\chi_i^s(HD_{m,2}) = \begin{cases} \sigma(HD_{m,2}) + 4 & \text{if } m = 2, \\ \sigma(HD_{m,2}) + 3 & \text{otherwise.} \end{cases}$$

Proof. Strong incidence colourings of $HD_{2,2}$ and $HD_{3,2}$ are given in Figure 13. Suppose now $m \geq 4$. Observe that each colour of the set $\{1, \ldots, m+1, m+2, m+5, \ldots, 2m+4\}$ of cardinality $2m+2$ is forbidden on the incidences of P, due to the partial colouring λ. Therefore, $\chi_i^s(HD_{m,2}) \geq 2m+7 = \sigma(HD_{m,2})+3$. To colour the incidences of the path P, we use the three colours $m+3$, $m+4$ and $2m + 5$, and two additional colours $2m + 6$ and $2m + 7$.

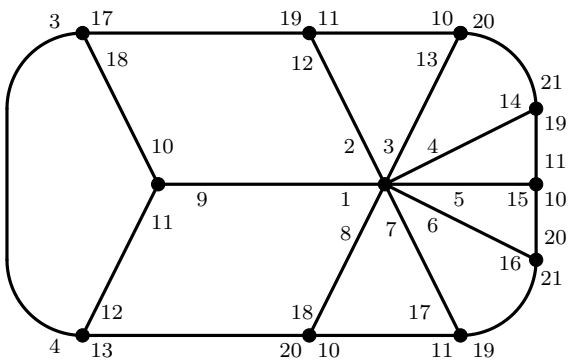
- We will sequentially colour the path P, according to the value of $m \mod 5$, as follows.



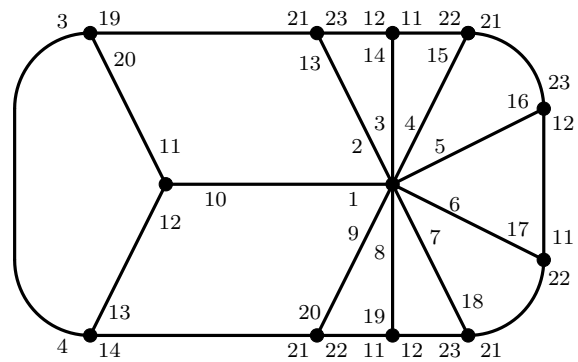
(a)



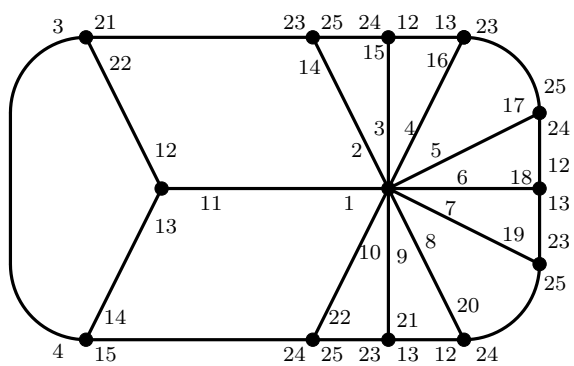
(b)



(c)



(d)



(e)

Figure 14: Strong incidence colourings of $HD_{m,2},\ 5 \leq m \leq 9$.

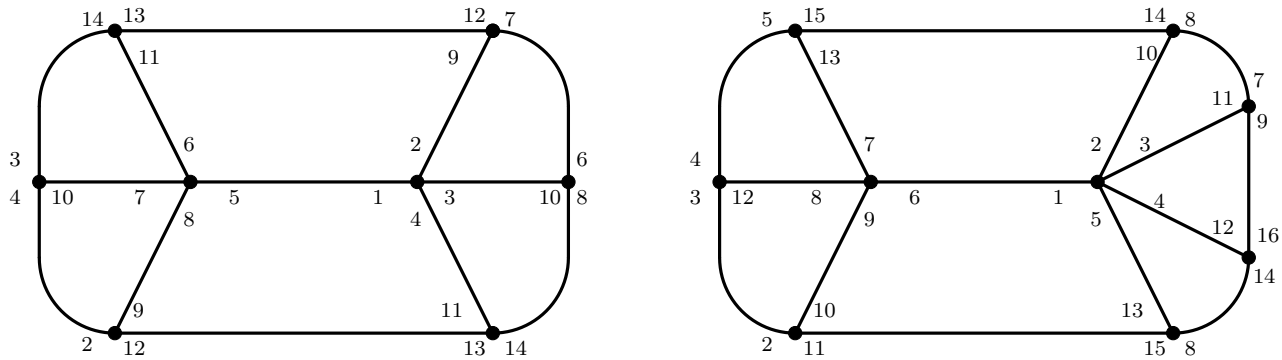


Figure 15: Strong incidence colourings of $HD_{3,3}$ and $HD_{4,3}$.

- $m = 5k$, $k \geq 1$ (see Figure 14(a) for the case $m = 5$).
  We use the pattern $(2m+7)(m+4)(m+3)(2m+6)(2m+5)$.
- $m = 5k+1$, $k \geq 1$ (see Figure 14(b) for the case $m = 6$).
  We use the pattern $(2m+6)(m+4)(m+3)(2m+7)(2m+5)$.
- $m = 5k+2$, $k \geq 1$ (see Figure 14(c) for the case $m = 7$).
  We use the pattern $(m+4)(m+3)(2m+6)(2m+7)(2m+5)$.
- $m = 5k+3$, $k \geq 1$ (see Figure 14(d) for the case $m = 8$).
  We use the pattern $(2m+7)(m+4)(m+3)(2m+6)(2m+5)$.
- $m = 5k+4$, $k \geq 1$ (see Figure 14(e) for the case $m = 9$).
  We use the pattern $(2m+7)(2m+6)(m+3)(m+4)(2m+5)$.

- We then set $\lambda(v_m, v_m u_n) = 2m+5$ if $m = 5k+3$, and $\lambda(v_m, v_m u_n) = 2m+6$ otherwise.
- We finally complete the colouring of $HD_{m,2}$ by assigning the colours 3, 4, $2m+4$, $m+5$, $2m+3$ and $m+6$ to the incidences (u_1, u_1u_2), (u_2, u_2u_1), (u_1, u_1u), (u_2, u_2u), (u_1, u_1v_1) and (u_2, u_2v_m), respectively.

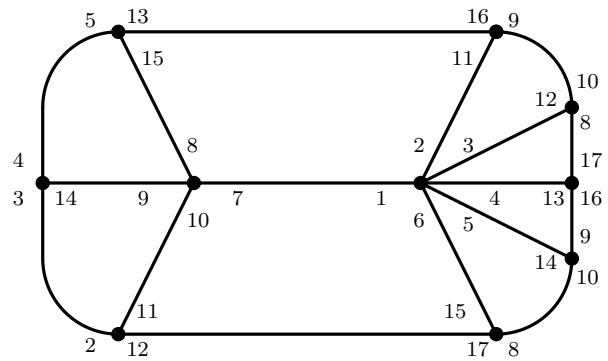
This concludes the proof. □

Lemma 17 *For every integer $m \geq 3$,*

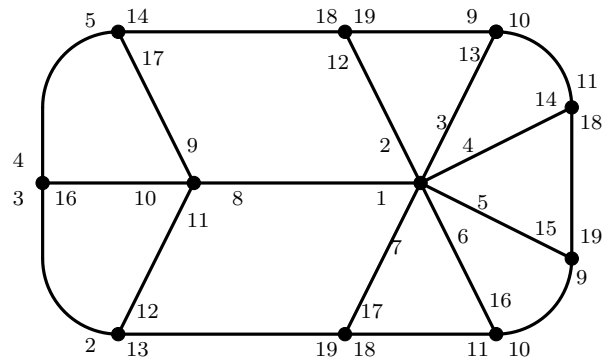
$$\chi_i^s(HD_{m,3}) = \begin{cases} \sigma(HD_{m,3}) + 3 & \text{if } m = 3 \text{ or } m = 4, \\ \sigma(HD_{m,3}) + 2 & \text{otherwise.} \end{cases}$$

Proof. Strong incidence colourings of $HD_{3,3}$ and $HD_{4,3}$ are given in Figure 15. Suppose now $m \geq 5$. Observe that each colour of the set $\{1, \ldots, m+1, m+2, m+6, \ldots, 2m+5\}$ of cardinality $2m+2$ is forbidden on the incidences of P, due to the partial colouring λ. Therefore, $\chi_i^s(HD_{m,3}) \geq 2m+7 = \sigma(HD_{m,3}) + 2$. To colour the incidences of the path P, we use the four colours $m+3$, $m+4$, $m+5$ and $2m+6$, and an additional colour $2m+7$.

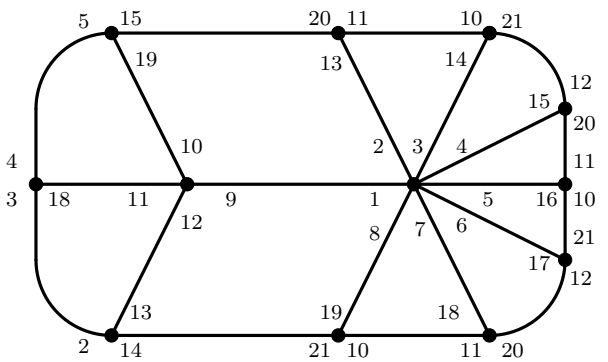
- We will sequentially colour the incidences of path P, starting from the incidence (v_1, v_1v_2), according to the value of $m \mod 5$, as follows.
  - $m = 5k$, $k \geq 1$ (see Figure 16(a) for the case $m = 5$).
    We use the pattern $(m+4)(m+5)(m+3)(2m+7)(2m+6)$.



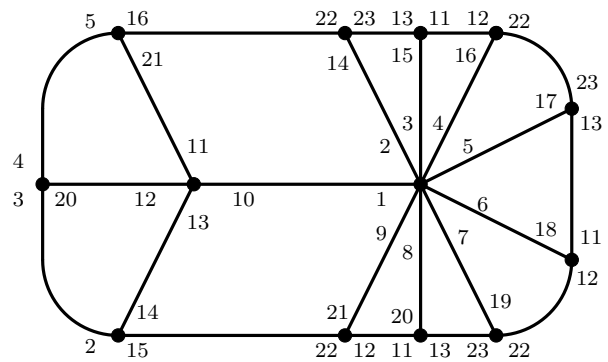
(a)



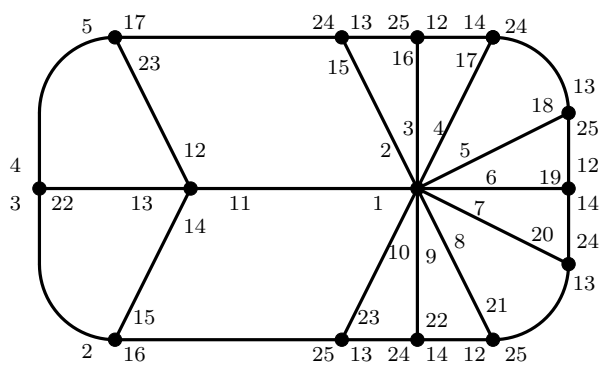
(b)



(c)



(d)



(e)

Figure 16: Strong incidence colourings of $HD_{m,3}$, $5 \leq m \leq 9$.

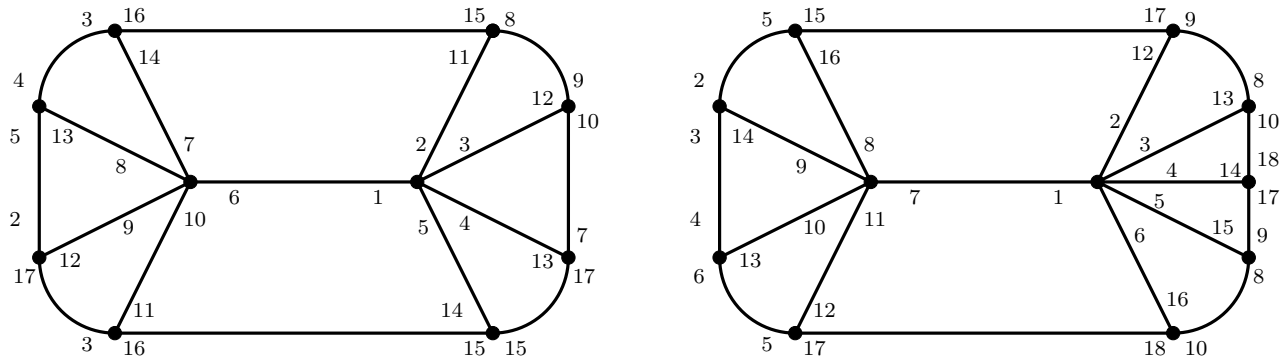


Figure 17: Strong incidence colourings of $HD_{4,4}$ and $HD_{5,4}$.

- $m = 5k + 1$, $k \geq 1$ (see Figure 16(b) for the case $m = 6$).
  We use the pattern $(2m + 7)(m + 3)(m + 4)(m + 5)(2m + 6)$.
- $m = 5k + 2$, $k \geq 1$ (see Figure 16(c) for the case $m = 7$).
  We use the pattern $(m + 4)(m + 3)(2m + 7)(m + 5)(2m + 6)$.
- $m = 5k + 3$, $k \geq 1$ (see Figure 16(d) for the case $m = 8$).
  We use the pattern $(2m + 7)(m + 5)(m + 3)(m + 4)(2m + 6)$.
- $m = 5k + 4$, $k \geq 1$ (see Figure 16(e) for the case $m = 9$).
  We use the pattern $(m + 4)(2m + 7)(m + 3)(m + 5)(2m + 6)$.

- We then set $\lambda(v_m, v_m u_n) = 2m + 6$ if $m = 5k + 3$, and $\lambda(v_m, v_m u_n) = 2m + 7$ otherwise.
- We finally complete the colouring of $HD_{m,3}$ by assigning the colours 5, 4, 3, 2, $2m + 5$, $2m + 4$, $m + 6$, $m + 8$ and $m + 7$ to the incidences (u_1, u_1u_2), (u_2, u_2u_1), (u_2, u_2u_3), (u_3, u_3u_2), (u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_1, u_1v_1) and (u_3, u_3v_m), respectively.

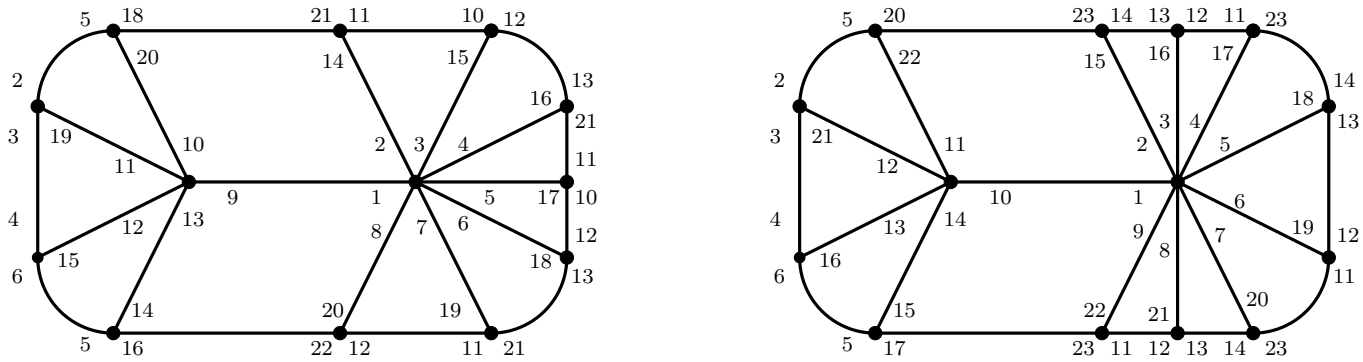
This concludes the proof. □

Lemma 18 *For every integer $m \geq 4$,*

$$\chi_i^s(HD_{m,4}) = \begin{cases} \sigma(HD_{m,4}) + 3 & \textit{if } m = 4, \\ \sigma(HD_{m,4}) + 1 & \textit{if } m \equiv 3 \pmod 5, \\ \sigma(HD_{m,4}) + 2 & \textit{otherwise.} \end{cases}$$

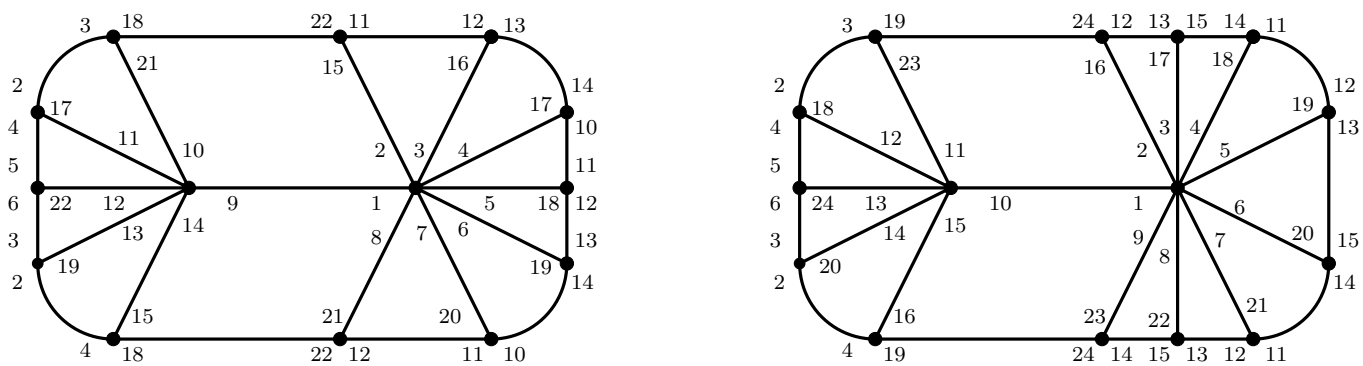
Proof. Strong incidence colourings of $HD_{4,4}$ and $HD_{5,4}$ are given in Figure 17. Suppose now $m \geq 5$. Observe that each colour of the set $\{1, \ldots, m+1, m+2, m+7, \ldots, 2m+6\}$ of cardinality $2m+2$ is forbidden on the incidences of P and the incidence $(v_m, v_m u_n)$, due to the partial colouring λ. Therefore, $\chi_i^s(HD_{m,4}) \geq 2m+7 = \sigma(HD_{m,4})+1$. It should be noted that if $\lambda(v_m, v_m u_n) = 2m+7$, then we can colour the path P with the colours $m + 3$, $m + 4$, $m + 5$, $m + 6$ and $2m + 7$ if and only if $m \equiv 3 \pmod 5$. We will sequentially colour the incidences of the path P, starting from the incidence (v_1, v_1v_2), as well as the incidence $(v_m, v_m u_n)$, according to the value of $m \mod 5$, as follows.

- $m \neq 5k + 3$ (see Figure 18(a) for the case $m = 7$).
  We use the pattern $(m+4)(m+3)(m+5)(m+6)(2m+7)$ for P and we set $\lambda(v_m, v_m u_n) = 2m+8$.



(a) (b)

Figure 18: Strong incidence colourings of $HD_{7,4}$ and $HD_{8,4}$.



(a) (b)

Figure 19: Strong incidence colourings of $HD_{7,5}$ and $HD_{8,5}$.

- $m = 5k + 3$, $k \geq 1$ (see Figure 18(b) for the case $m = 8$).
  We use the pattern $(m+6)(m+5)(m+4)(m+3)(2m+7)$ for P and we set $\lambda(v_m, v_m u_n) = 2m+7$.

We finally colour the remaining incidences of $HD_{m,4}$ by assigning the colours 5, 2, 3, 4, 6, 5, $2m+6$, $2m+5$, $m+8$, $m+7$, $2m+4$ and $m+9$ to the incidences (u_1, u_1u_2), (u_2, u_2u_1), (u_2, u_2u_3), (u_3, u_3u_2), (u_3, u_3u_4), (u_4, u_4u_3), (u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_4, u_4u), (u_1, u_1v_1) and (u_4, u_4v_m), respectively. This concludes the proof. □

Lemma 19 *For every integer $m \geq 5$, $\chi_i^s(HD_{m,5}) = \sigma(HD_{m,5}) + 1$.*

Proof. By Proposition 15, $\chi_i^s(HD_{m,5}) \geq 2m + 8 = \sigma(HD_{m,5}) + 1$. To complete the proof, we give a strong incidence $(2m + 8)$-colouring by extending λ to a colouring of $HD_{m,5}$ as follows.

- We first sequentially colour the incidences of the path P, starting from the incidence (v_1, v_1v_2), according to the value of $m \mod 5$ as follows.
  - $m \neq 5k + 3$, $k \geq 1$ (see Figure 19(a) for the case $m = 7$).
    We use the pattern $(m + 4)(m + 5)(m + 6)(m + 7)(m + 3)$.

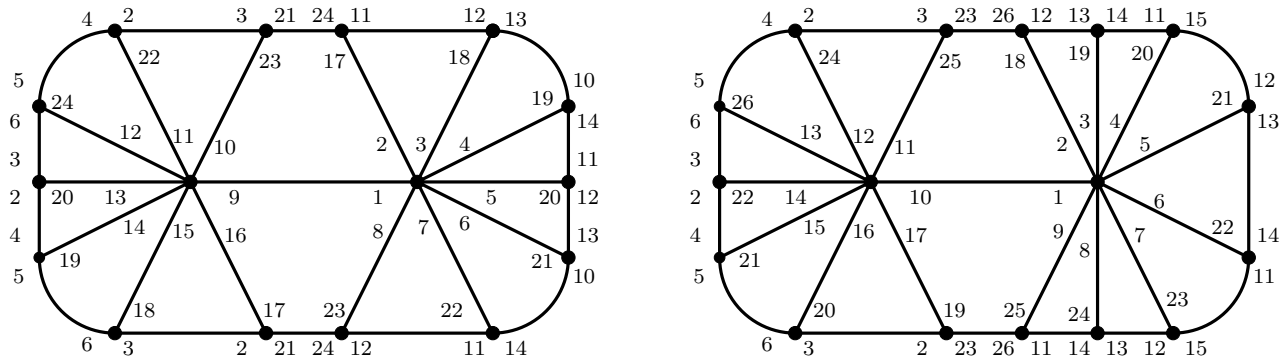


Figure 20: Strong incidence colourings of $HD_{7,7}$ and $HD_{8,7}$.

- $m = 5k + 3$, $k \geq 1$ (see Figure 19(b) for the case $m = 8$).
  We use the pattern $(m + 4)(m + 5)(m + 7)(m + 6)(m + 3)$.

- We then sequentially colour the incidences of the path P', starting from the incidence (u_1, u_1u_2), using the pattern 32456.
- We finally colour the remaining incidences of $HD_{m,5}$ by assigning the colours $2m + 7$, $m + 10$, $2m+8$, $m+12$, $m+8$, $m+11$, $m+11$ and $2m+8$ to the incidences (u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_4, u_4u), (u_5, u_5u) (u_1, u_1v_1), (u_5, u_5v_m) and (v_m, v_mu_n), respectively.

This concludes the proof. □

Lemma 20 *For every two integers m and n, $m \geq n \geq 6$, $\chi_i^s(HD_{m,n}) = \sigma(HD_{m,n}) + 1$.*

Proof. By Proposition 15, $\chi_i^s(HD_{m,n}) \geq 2m + n + 3 = \sigma(HD_{m,n}) + 1$. To complete the proof, we give a strong incidence $(2m + n + 3)$-colouring by extending λ to a colouring of $HD_{m,n}$ as follows (see Figure 20).

- We first sequentially colour the incidences of the path P, starting from the incidence (v_1, v_1v_2), using the pattern $(m + 4)(m + 5)(m + 6)(m + 3)(m + 7)$.
- We then sequentially colour the incidences of the path P', starting from the incidence (u_1, u_1u_2), using the pattern 32456.
- We finally set $\lambda(u_i, u_iu) = 2m + n + 3 - i$, for every i, $i \in \{1, 2, 4, \ldots, n\}$, $\lambda(u_3, u_3u) = 2m + n + 3$, $\lambda(u_n, u_nv_m) = \lambda(u_1, u_1v_1) = 2m + n$ and $\lambda(v_m, v_mu_n) = 2m + n + 3$.

This concludes the proof. □

Putting together Lemmas 16, 17, 18, 19 and 20, we finally get the following theorem.

Theorem 21 *For every two integers m and n, $m \geq n \geq 3$,*

$$\chi_i^s(HD_{m,n}) = \begin{cases} \sigma(HD_{m,n}) + 4 & \text{if } n = 2 \text{ and } m = 2, \\ \sigma(HD_{m,n}) + 3 & \text{if } n = 2 \text{ and } m \neq 2, \\ & \text{or } (n, m) \in \{(3,\ 3),\ (3,\ 4),\ (4,\ 4)\}, \\ \sigma(HD_{m,n}) + 2 & \text{if } n = 3 \text{ and } m \notin \{3,\ 4\}, \\ & \text{or } n = 4 \text{ and } m \neq 4 \text{ and } m \not\equiv 3 \pmod 5, \\ \sigma(HD_{m,n}) + 1 & \text{otherwise.} \end{cases}$$

6 Discussion

In this paper, we have introduced and studied the strong version of incidence colouring. We have determined the exact value of – or upper bounds on – the strong incidence chromatic number of several classes of graphs, namely cycles, wheel graphs, trees, ladder graphs and some subclasses of Halin graphs. We leave as open problems the following questions.

1. What is the best possible upper bound on the strong incidence chromatic number of graphs with bounded maximum degree? In particular, what about graphs with maximum degree 3?
2. What is the best possible upper bound on the strong incidence chromatic number of Halin graphs?
3. What is the best possible upper bound on the strong incidence chromatic number of d-degenerated graphs?

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