Modelling and Prognostics of system degradation using Variance Gamma process

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Nowadays estimation of Remaining Useful Life (RUL) is essential to the prognostics and health management of high-priced systems. Initially, the properties of the VG process and estimation of the parameters are presented. Analytical approximation is also discussed and due to its complexity, a simulated method is proposed. Finally, the simulation method and the goodness of fit tests are used to predict the system life.

Keywords: Variance Gamma, Stochastic Models, Degradation Processes, Parameter Estimation, Remaining Useful Life, Prognostics.

1. Introduction

Reliability and safety of systems are the most important aspect confronting by the industries these days. The history of failure data of systems is one of the most significant informations required for degradation modelling and estimation of the Remaining Useful Life. Most aging failures could be attributed to some underlying degradation mechanism under which the damage accumulates over time. The failure of the product is determined depending on a predefined deterministic value of the failure provided by the product manufacturer, i.e. failure threshold. The recovery of real failure data from the industries or laboratories is challenging due to the strict confidentiality policies that lead to a significant demand for a high reliability degradation model. The degradation modelling techniques such as product design, testing, lifetime prediction, maintenance, cost planning, etc. were widely used in reliability engineering. Since degradation models play a significant role in reliability analysis, they are considered as the basis of product analysis and related decision-making.

Current researches indicate a growing interest in the application of degradation models in the estimation of reliability. It also indicates that substantial progress has been made in the application of degradation models in various industrial sectors (Gorjian et al. (2010), Wang et al. (2018)). Two main classes of degradation models are general path models and stochastic models Meeker et al. (2014). Stochastic processes such as Wiener process, Gamma process and Inverse

Gaussian process are good candidates for modelling degradation. As indicated by Ye and Xie (2015), the stochastic nature of these processes is capable of modelling the unexplained randomness of degradation over time due to non-observed environmental factors or unknown effects.

Gamma process is proven to be an effective tool in stochastic modelling of monotonic and gradual degradation. It is suitable for modelling gradual damage that accumulates over time in a sequence of tiny increments such as wear, fatigue, corrosion, crack growth, etc. (van Noortwijk (2009), Zhang et al. (2015), Pulcini (2016), Rodríguez-Picón et al. (2016), Yan et al. (2016)). In real applications, one can observe that various sources of variations affect the characteristic of a product's degradation process. Due to some non-observable effects, the degradation of a product can have a non-monotonous variation. A simple gamma process cannot capture such variations Rodríguez-Picón et al. (2016).

Similarly, the Wiener process has been widely used in non-monotonic degradation (Ye and Xie (2015), Karatzas and Shreve (1998), Øksendal (2003)) and in structural reliability, they are suitable to show alternately the increase and the decrease of the resistance of the structure. From a physical point of view, the degradation can be seen as an additive superposition of a large number of small external effects. Thus, the degradation increment can be normally distributed because of the central limit theorem. In this regard, this process is a good model for the degradation. Despite the fact that Wiener processes have been used to model many degradation phenomena, they are not

suitable for modelling processes of monotonous degradation, such as wear or cumulative damage.

The Black & Scholes (B & S) model is a non-linear and a non-monotonic process. One of its advantages comes from the analytical properties of the continuous time Brownian motion which allows for a simple computation of its formula. The B & S model has been increasingly criticized and the weaknesses emerge from an unrealistic set of assumptions such as constant volatility, i.e. implied data's volatility does not tend to vary which is not realistic. This will cause difficulties in precise estimation and prediction. The B & S model has its limitations when it comes to accurately predicting the model (Manaster and Koehler (1982), Karoui et al. (1998), Arriojas et al. (2007)). There are plenty of data that cannot be fitted using conventional Wiener, Gamma and B & S processes. Due to this, another stochastic process called Variance Gamma which can better replicate the non-monotonic degradation path is proposed here.

Variance Gamma process is initially introduced in the financial analysis and it was brought in as an extension of the Brownian motion. The first complete presentation of the model in its simplified symmetric form is explained in Madan and Seneta (1990) and has been considered in Madan and Milne (1991) and Madan et al. (1998). Some of its properties were discussed in Madan and Seneta (1987) and empirical comparisons with other models were presented. This stochastic process contains two more parameters compared to the geometric Brownian motion, which help in the control of the kurtosis and the skewness. This Lévy process can be written as a Brownian motion evaluated at random times. The Variance Gamma process can be obtained by replacing the time in the Brownian motion with a Gamma process. The selection of the degradation model is crucial and it is determined by the shape of the degradation paths. Due to its two presentations, VG can be used to model any kind of non-monotonic degradation phenomenon as far as the increments follow the VG distribution.

In this study, the Variance Gamma process will be defined in Section 2 and the estimation of the Variance Gamma parameters will be carried out in Section 3. A comparative study will be performed between Variance Gamma and other processes such as Wiener, Black & Scholes in fitting simulated data. An analytical approximation of the distribution of the failure time (FT) will also be presented and discussed. A simulation-based method will be proposed to obtain the failure time distribution. It will be fitted to different distributions and goodness of fit measures will be applied in Section 4. Finally, the simulated method and the goodness of fit tests will be used to establish the system life forecasting in Section 5.

2. Variance Gamma Process

As mentioned before, the shape of the degradation paths is crucial to identify the degradation model. In this study, the hydraulic pump is mentioned as an example and motivation. In such case, the state of the system (presented by Variance Gamma) represent the difference between the incoming sources (presented by a positive gamma) and the leakage (presented by a negative gamma). The result path demonstrates a non-monotonic and a non-linear behaviour which can be replicated using Variance Gamma process. In this section, the Variance Gamma process and its properties are presented.

2.1. Variance Gamma as time changed Brownian motion

Here, the Variance Gamma (VG) process is introduced as an extension of Brownian motion which can be obtained by evaluating wiener process at random times defined by a gamma process. In other terms, the time in the Brownian motion will be replaced with a gamma process (Geman et al. (2001)). Let B be a brownian motion with positive parameters, the drift θ and volatility σ and a standard Brownian motion $W(t)$. It is defined as:

$$B(t; \theta, \sigma) = \theta t + \sigma W(t) \quad (1)$$

The gamma process of independent gamma distributed increments on a time interval $(t, t+h)$, $\gamma(t; \mu, \nu)$ with μ as the mean rate and ν as the variance rate was considered. The VG process is defined as:

$$X(t; \sigma, \nu, \theta) = B(\gamma(t; \mu, \nu), \theta, \sigma) \quad (2)$$

The density function of VG process at a time t can be expressed as a normal density function conditional on the realisation of the time change given by the gamma distribution (Madan and Seneta (1990)).

The VG process has four parameters: σ the volatility of the Brownian motion, ν the variance rate of the gamma time changes and θ the drift in the Brownian motion with drift and μ the mean rate. The process offers two additional dimensions of control on the distribution over and above that of the volatility. The parameters θ and ν control the skewness and kurtosis respectively. The density function for the VG process at time t can be given by this expression (Scott et al. (2011)):

$$f(x; \mu, \sigma^2, \theta, \nu) = \frac{2\exp(\theta(x - \mu)/\sigma^2)}{\sigma\sqrt{2\pi}(\frac{1}{\nu})^\nu \theta(\nu)} \left(\frac{|x - \mu|}{\sqrt{2\theta^2\nu + \sigma^2}} \right)^{\nu - \frac{1}{2}} K_{\nu - \frac{1}{2}} \left(\frac{|x - \mu|\sqrt{2\sigma^2\nu + \theta^2}}{\sigma^2} \right) \quad (3)$$

where $x \in \mathbb{R} \setminus \mu$, $K_\nu(\cdot)$ is the modified Bessel function of the second kind with index ν , μ is the location parameter, σ^2 is the scale parameter,

θ is the skewness parameter and ν is the shape parameter (Abramowitz (1972)).

When $x = \mu$, two cases need to be distinguished: when $\nu \leq 1/2$, the density is singular at μ and when $\nu > 1/2$ the density reduces to:

$$f(x; \mu, \sigma^2, \gamma, \nu) = \frac{\theta(\nu - 1/2)}{\sigma \sqrt{2\pi(1/\nu)^\nu \theta(\nu)}} \left(\frac{2\sigma^2}{\sqrt{2\sigma^2\nu + \theta^2}} \right)^{\nu - \frac{1}{2}} \quad (4)$$

The characteristic function of VG process is given by:

$$\phi_{X(t)}(u) = E[e^{iuX(t)}] = \left(\frac{1}{1 - i\theta\nu u + (\frac{\sigma^2\nu}{2})u^2} \right)^{\frac{t}{\nu}} \quad (5)$$

A sample path of VG was plotted using the parameters $VG(\sigma=0.5, \theta=0.5, \nu=0.5, \mu=0)$ in Figure 1. It can be observed that the path consists of many small jumps as expected.

Fig. 1. Sample path of $VG(\sigma=0.5, \theta=0.5, \nu=0.5, \mu=0)$

2.2. Variance Gamma process as the difference of two Gamma processes

As mentioned by Madan et al. (1998), the VG process can be written as a difference of two increasing independent gamma process since it is a process of finite variation. The first represents the increases in the process and the second represents the decrease in the process. The process can be written as the law of the characteristic function of $VG(\sigma, \nu, \theta)$:

$$\Phi_{VG}(u; \sigma, \nu, \theta) = (1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-1/\nu} \quad (6)$$

This distribution is infinitely divisible and the VG process $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$ can be

defined as the process which starts at zero, has independent and stationary increments. The increment $X_{s+t}^{(VG)} - X_s^{(VG)}$ follows the Variance Gamma $VG(\sigma\sqrt{t}, \nu/t, t\theta)$. VG process can also be defined as a difference of two independent gamma processes as below:

$$X_t^{(VG)} = G_t^{(1)} - G_t^{(2)}$$

where $G^{(1)} = \{G_t^{(1)}, t \geq 0\}$ is an independent gamma process with a shape parameter $a=C$ and scale parameter $b=M$, whereas $G^{(2)} = \{G_t^{(2)}, t \geq 0\}$ is an independent gamma process with parameters the same shape parameter $a=C$ and the scale parameter $b=G$ with

$$C = \frac{1}{\nu} > 0,$$

$$G = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu \right)^{-1} > 0,$$

$$M = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu \right)^{-1} > 0.$$

3. Properties: Parametric estimation and Failure Time distribution

3.1. Parametric estimation

The parametric estimation of the VG process has been the subject of several research papers recently. Cervellera and Tucci (2017) confirms that, it is impossible to replicate the estimation obtained in Madan et al. (1998). In order to validate this latter, the investigation of the computational problems related to finding the maximum likelihood estimator of the parameters was performed. Both R, MATLAB and a non-standard optimization software such as Ezgrad were used. The complexity of log-likelihood function is due to the presence of many local optima and Bessel function of the second kind. A new algorithm (Bee et al. (2018)) was developed based on Nitithumbundit and Chan (2015) corresponding multivariate versions, for maximum likelihood estimation (MLE) of the univariate VG distribution and results were compared with the results obtained from R Variance Gamma (Scott et al. (2018)) and the ghyp packages (Luethi et al. (2013)).

The distribution of the model is flexible enough to accommodate skewness and leptokurtosis but the obtention of MLE is difficult because the probability density function (PDF) is not in closed form and is unbounded for small values of the shape parameter. The log-likelihood function is

given as follows:

$$L(\mu, \theta, \sigma, \nu) = \frac{T}{2} \log\left(\frac{2}{\Pi}\right) + \sum_{t=1}^T \frac{(x_t - \mu)\theta}{\sigma^2} - \sum_{t=1}^T \log(\Gamma(a)\theta) + \sum_{t=1}^T \log(K_{\nu-0.5}\left(\frac{\sqrt{2\sigma^2 + \theta^2}|x_t - \mu|}{\sigma^2}\right)) + \sum_{t=1}^T \left(\mu - \frac{1}{2}\right) \left(\log(|x_t - \mu| - \frac{1}{2}\log(2\sigma^2 + \theta^2))\right) \quad (7)$$

This equation explains why the researchers are confronting difficulties in the treatment and the calculation of the MLE.

The parametric estimation of the VG in this paper is performed using Variance Gamma and ghyp package in R. In Variance Gamma package, the VGfit function allows the user to employ the Nelder-Mead, the BFGS method (Broyden-Fletcher-Goldfarb-Shanno) or a Newton-type algorithm while the fit.VGuv function of the ghyp package is based on the Nelder-Mead algorithm. Both packages use the Nelder-Mead algorithm as implemented in the optim R function. BFGS is also implemented in optim, whereas the Newton-type algorithm is from the nlm (Non-Linear Minimization) function.

Table 1. RMSE results from fit.VGuv and VGfit methods

Parameters	ghyp	VGfit
VG($\sigma=0.5, \theta=0.5, \nu=0.5, \mu=0$)		
N = 1000, n= 100		
μ	0.003859279	0.013340216
σ	0.0005072501	0.0637157550
θ	0.004379053	0.033325612
ν	0.4397658	0.8360905
N = 10000, n= 100		
μ	0.0002849596	0.0059760528
σ	4.451899e-05	5.009331e-02
θ	0.0003302026	0.0238633483
ν	0.44141475	0.5795967
VG($\sigma=1, \theta=1, \nu=1, \mu=0$)		
N = 1000, n= 100		
μ	0.001631385	0.004462029
σ	0.001749834	0.131206113
θ	0.004413952	0.352547189
ν	0.008584044	0.365553041
N = 10000, n= 100		
μ	0.0002031116	0.0002094777
σ	0.0002094777	0.0001527683
θ	0.0004655316	0.0004709406
ν	0.00054044	0.00105044
VG($\sigma=3, \theta=1.5, \nu=2, \mu=0$)		
N = 1000, n= 100		
μ	1.652700e-05	1.951251e-05
σ	0.1771473	0.4843569
θ	0.1508169	0.7965047
ν	0.1394611	0.2002643
N = 10000, n= 100		
μ	2.826034e-06	7.711633e-06
σ	0.1761572	0.3642330
θ	0.1301680	0.6865433
ν	0.1403962	0.08342571

These two methods are tested on different sets

of data to analyse the robustness of the algorithms. The estimation will be performed with 1000 and 10000 number of samples having a sample length of 10000 using different values of parameters VG($\sigma=0.5, \theta=0.5, \nu=0.5, \mu=0$), VG($\sigma=1, \theta=1, \nu=1, \mu=0$) and VG($\sigma=3, \theta=1.5, \nu=2, \mu=0$). The obtained RMSE (Root Mean Square Error) results were presented in Table 1. It is noted that the estimation of the VG parameters or the log-likelihood function is affected by the addition or removal of observations (Cervellera and Tucci (2017)). It also shows enormous changes in the estimated parameter values which proves the impact of the number of data on the estimation. However, the estimation of the parameter ν using this package did not show big improvement while the number of data increases, and this is due to the presence of the Bessel function in the log-likelihood expression. This is also related to the computational complexity in the estimation of the loglikelihood when $\nu = 0.5$.

The results of the RMSE using the same data and same number of samples show that the ghyp package is considered to be more efficient in the estimation comparing to the Variance Gamma package. It is essential to mention that the estimation depends only on the data and on the starting values. Both methods are efficient, but ghyp package provides good estimation results even though it shows some sensitivity when the value of $\nu = 0.5$. As the work in this study was performed using the simulated data, the estimation results were precise. In the case of the estimation of real data, it is essential to verify the results using statistical tests before the beginning of the prognostics.

3.2. Failure Time

The system is considered to have failed if the level of degradation reaches the threshold referred to as the failure threshold. In order to avoid failure and an undesirable period of inactivity, it is of great interest to model the evolution of the degradation and to be able to predict the failure.

Considering a process $(X_t)_{t \geq 0}$ with initial value $X_0 = x_0$, the failure time is defined to be the first crossing time of the failure threshold denoted by L and it is defined to be the first hitting time which can be written as follow:

$$t_L^* = \inf \{t \geq 0; X_t \geq L\} \quad (8)$$

It could be easy to obtain the distributional properties of t_L^* , if $(X_t)_{t \geq 0}$ is a Lévy subordinator or a time changed Brownian motion with a path continuous process as the underlying time process (Borodin and Salminen (2012)). But in the case of VG process, the underlying time process is a gamma process which is not path continuous, the situation will become more complicated and challenging. This situation made a lot of discussion

among the researchers. In the articles of Hurd (2009) and Hurd and Kuznetsov (2009), it is mentioned that the usual first hitting time cannot be studied. Thus, the author proposed to calculate the first hitting time of the second kind which can be defined as the first time when the time change process (gamma process) exceeds the failure time of the Brownian motion (B).

Let's consider T^* to be the first passage time of B which is defined as $T^* = \inf \{T : x_0 + \theta t + \sigma W(t) \geq 0\}$. And for a TCBM process like VG process $X_t = B_{G_t}$, the first passage time of the second kind of X_t is defined as $t^* = \inf \{t : G_t \geq T^*\}$ (Hurd and Kuznetsov (2009)). Based on this definition, one can notice that the first hitting time of the second kind shares certain properties with the typical first hitting time and can be implemented in a similar way. The advantage of this new concept is that it can be calculated efficiently in many situations where the normal failure time cannot be computed. For this purpose, it is essential to introduce the equation 9 and equation 10 which are used to calculate the failure time distribution of second kind given by the equation 11. The structure function $p_1^*(x_0; s, x_1)$ for general LSBMs associated with failure time t_L^* is introduced using this formula:

$$p_1^*(x_0; s, x_1) = \frac{\exp(\beta(x_1 - x_0))}{2\pi\nu} \int_{\mathbb{R}} \frac{(1 + i\nu z)^{-s/\nu}}{\sqrt{\beta^2 - 2iz}} \exp(-x_0\sqrt{\beta^2 - 2iz}) \times \left(\exp(|x_1|\sqrt{\beta^2 - 2iz}) - \exp(-|x_1|\sqrt{\beta^2 - 2iz}) \right) Ei\left(-|x_1|(\alpha + \sqrt{\beta^2 - 2iz})\right) - Ei\left(-|x_1|(\alpha - \sqrt{\beta^2 - 2iz})\right) \quad (9)$$

where $\alpha = \sqrt{2/\nu + \beta^2}$ and $Ei(x)$ presents the function of the exponential integral. And in the particular case when $s = 0$:

$$p_1^*(x_0; 0, x_1) = \frac{\exp(\beta(x_1 - x_0) - \alpha(x_0 + |x_1|))}{\nu(x_0 + |x_1|)} \quad (10)$$

The failure time distribution of the second type can be estimated by iterating the following formula 2 to 3 times and it can be written as:

$$p_i^*(x_0; s) = \int_{-\infty}^0 p_1^*(x_0; 0, x_1) dx + \int_0^{+\infty} dy \int_0^s du p_1^*(x_0; u, y) p_{i-1}^*(y; s - u) \quad i \geq 2 \quad (11)$$

It is important to have the numerical calculation of the failure time distribution and in order to achieve that it was crucial to reproduce the work presented in Hurd (2009). The equations used are very complex which made their numerical computation so complicated. To overcome this, a simulation method will be used in order to approximate the distribution of the failure time.

4. Goodness of fit

As mentioned before, a non-monotonic and a non-linear process is recommended. Wiener, B & S and VG processes are used to model the degradation, data calibration and their adequacy is compared. The adequacy of the process is performed with four goodness of fit tests: Kolmogorov-test (KS), Chi-square test (chisq), Anderson-test (AD), Cramer-test (Cramer). The goodness of fit typically summarizes the dissimilarity between observed values and the values expected under the model in question.

The three processes are tested in order to determine the most flexible process which can cover all the fluctuations and changes of the system based on simulated data. Data are simulated respectively from VG, Wiener, B & S. The goodness of fit test results are presented in Table 2. The probability of error (significance level) is set as $p_{value} = 0.05$ i.e. confidence of 95% and the results are analysed and presented in Table 2.

In this study, data are generated from the three different processes (VG, Wiener and B & S) and the parameters are estimated. Later, the data are regenerated and the goodness of fit tests is used to verify to which model these data fit the most.

Table 2. Results of goodness of fit : VG, Wiener, B & S Data

Process	KS	Chisq	Cramer	AD
Data from VG process				
VG	0.2408	0.2419	0.2255	0.2466
Wiener	0.00692	0.026457	5.967e-03	0.003193
B.S	4.218e-04	0.0247	0.023113	0.006120
Data from Wiener process				
VG	0.00692	0.3876	0.2517	0.3165
Wiener	0.2439	0.086457	5.714 e-03	0.1045
B.S	9.158e-04	6.643e-03	5.967e-03	0.003193
Data from B & S process				
VG	0.2452	0.4309	0.3750	0.7362
Wiener	4.71e-05	3.94e-06	6.59e-08	4.51e-06
B.S	0.375	0.2183	0.1632	0.0853

It is noted that, the goodness of fit tests applied on the data generated from the VG process rejected both Wiener and B & S processes. In the case of data generated from Wiener process, VG was accepted with 95% confidence according to the results of chisq, Cramer and AD tests but it was rejected by KS test. On the other hand, B & S is rejected by all the four tests. According to the goodness of fit tests, the VG process can fit to data originated from B & S process, however the Wiener process was rejected by all the tests. It can be concluded that the VG is the most flexible model able to fit to the different type of data Table 2.

The data were generated from the three different processes (VG, Wiener, BS) with the same

number of samples $N=10000$. Consequently, the assumption that the size of the samples will lead to reject the null hypothesis for all the processes can not be taken in consideration, since they accepted the results for VG process and another process in each scenario.

It can be mentioned that VG process is able to fit several types of data generated from different processes due to its flexibility. It is due to its representation as a random change of the time in a Brownian motion, following a gamma process. This time change allows to reflect the random speedups and slowdowns in real time.

5. Prognostics on simulated data

It is always valuable to evaluate a parametric model of the FT distribution since it provides the possibility to integrate the evolution of the state of the system at the inspection time. In other word, the expression of the FT distribution will be updated at every inspection based on the collected information. First, the histogram of the FT realisation is studied graphically in order to analyse if there is a famous distribution able to model the FT distribution. Afterward, a goodness of fit test is carried out to check if the candidate distribution fit the failure data.

It is known that the exponential distribution and the lognormal distribution are right skewed. The Weibull distribution can be symmetric, right skewed or left skewed. The Inverse Gaussian distribution which has several properties analogous to a Gaussian distribution were also tested. Other theoretical distribution, such as Gamma and beta will be evaluated. It can be noticed that all the histograms (Figure 2) obtained are right-skewed where most of the sample values are clustered on the right side of the histogram. The results obtained from the goodness of fit tests are presented in Table 3 and a probability error is set as $p_{value} = 0.05$. It is noted that all the four tests reject the distribution candidates.

Table 3. Test results for the fit of the FT distribution to different laws

Law	KS	Chi-squared	Cramer	AD
Weibull	3.765e-08	1.2987e-10	9.754e-06	1.9678e-07
log-normal	1.34 e-06	3.475 e-02	2.76 e-01	7.7141e-12
Gamma	8.745e-05	6.547e-10	4.798e-09	0.004024
Beta	8.374 e-09	5.475 e-02	7.76 e-05	8.9678e-07
Exponential	0.002498	0.0376	0.02682	7.245e-05
Gaussian	8.9678e-07	8.856e-26	6.658e-07	14.374 e-05

According to Jardine et al. (2006) and Ahmadzadeh and Lundberg (2014), it is essential to calculate the residual life expectancy of the system. The period of useful life of a system is defined as the period when the system normally

operates under well-defined conditions from a well-defined date that ends with a failure of the system. The main objective of the prognostics is to provide information for good decisions as well as risk measures for the monitored systems. To assure a better prognostics, different type of metrics must be considered. The main one is the RUL (Remaining Useful Life) or the residual time before failure like TTF (Time To Failure). The accuracy in the prediction of the RUL or TTF of the defected components is significant to the prognostics of the system and it will assist the operators to replace the components at the appropriate time. The failure of the system is signaled by the passage of the degradation level through its predefined failure threshold. The failure time defined in the equation 8 is used to give a more analytical definition of the RUL. In this case, the calculation of the failure time is performed conditionally to the state of the system at the present time (time of the last inspection t_i), which can introduce the RUL as the first exceeding time of indicator $X(t)$ to the threshold L follows:

$$t_{t_i}^* := \inf \{h \geq 0; X(t_i + h) \geq L \mid X(t_i) < L\} \quad (12)$$

Table 4. Positioning of the threshold time compared to the FT

	10%	Median	90%	Mean	Time
First threshold					
(a)	320	413	479	327.4	385
Second threshold					
(b)	1650	1870	2315	1575	1774
Third threshold					
(c)	1570	1870	2370	1751	1630

The degradation of the system is modelled by a VG path and the threshold is settled at a fixed time. Using the simulation method, the time of crossing this threshold is obtained and its histogram (Figure 2) shows its distribution. The location of the real time of the failure referred by Time compared to the mean, median and 10% quantile and 90% quantile of the series of time of exceeding the threshold was calculated and presented in Table 4. It is noticed that the distribution of FT is right skewed as mentioned before. From Figure 2, the time of crossing the threshold is always located in the middle of the histogram.

At $t=0$, $N=10000$ samples were generated and the failure threshold was set. Here, the analysis and results were based on a number n chosen samples ($n < N$) from the 10000 samples. The evolution of the FT density was followed for one chosen sample. At $t=0$, the histogram of the FT were plotted using the 10000 samples and the real FT of the chosen path was compared to the mean, median and the confidence interval related to the histogram of FT. The results are considered to be

Fig. 2. Histogram of FT of (a) first, (b) second and (c) third threshold

satisfactory, when the first passage time is located in the confidence interval. Same work was also reproduced to the same chosen path at time $T = t_{insp}$ and the new location of the FT were obtained and compared to the new histogram. This work was reproduced with n number of samples and similar satisfactory results were found. Based on these results, one can affirm that the real FT is always located in the confidence interval, which can lead to affirm that VG can offers a good prognostics.

6. Conclusion

In this paper, the VG process is proposed as a degradation model. An analysis of its parameters and properties is carried out in order to justify its use in degradation modelling. The estimation of the VG parameters is obtained using two R packages and the complexity of the Log-likelihood function is discussed. The evaluation of the two packages is insured by the calculation of the RMSE. As a degradation model, it was important to study the distribution of its failure time. Due to the difficulty in determining the distribution of FT analytically, the simulation method is proposed as an alternative. The data generated from different processes: VG, Wiener, B & S are used to calibrate the choice of the model and based on a comparative study, most of the goodness of fit tests accepted VG as the appropriate model. Since it is important to find the distribution of the FT, simulations are used to define it and goodness of fit tests is used to fit it to some distributions. Unfortunately, all the distributions used were not able to fit according to all goodness of fit tests to the distribution of the FT. The concept of FT is introduced into the definition of RUL which is important in suggesting the prognostics. The

prognostics results showed that in most of the cases VG was able to provide a good prognostics about the system. Although this work was based on simulated data, it should be possible to extend the approach to more complex real data collected from complex systems. The model accuracy can be further improved by using deeper research on each parameter effect.

Acknowledgement

The authors would like to acknowledge the valuable financial support of the European Regional Development Fund (FEDER) and the Departmental Council of Aube, France during this research.

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