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Information and control theory models of embodied consciousness: toward new statistical tools for data analysis

Rodrick Wallace
The New York State Psychiatric Institute
rodrick.wallace@nyspi.columbia.edu rodrick.wallace@gmail.com

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Abstract

Without invoking panpsychism or identifying consciousness as a weird form of matter, without mind/body dualism, without the *ignis fatuus* of the 'hard problem' and the many other such constructs that haunt contemporary consciousness studies, the asymptotic limit theorems of information and control theories permit development of mathematical models recognizably similar to the empirical pictures Bernard Baars and others have drawn of high-level mental phenomena. The methodology revolves around constructing an iterated Morse Function free energy analog from information source uncertainties associated with sources necessarily 'dual' in a formal sense to cognitive phenomena. This leads to an iterated entropy-analog from which application of the Onsager approximation from nonequilibrium thermodynamics gives large-scale system dynamics. We make application to the dynamics of arousal and distraction as an example. A modified version of the Kadanoff picture of phase transitions in consciousness emerges from the Morse Function itself in a surprisingly standard manner. It should be possible, on the basis of these probability models, to develop new statistical tools for the analysis of empirical data regarding cognition and consciousness.

Key Words: control theory, entropy gradient, free energy, groupoid, information theory, Morse Function, phase transition, symmetry breaking

Nothing in biology makes sense except in light of evolution.

— T. Dobzhansky

1 Introduction

The author, as an undergraduate, had a singular conversation with the mathematician John Kemeny, later President of Dartmouth College. Kemeny noted, with some considerable asperity, that physicists were wont to repeatedly publish rediscovered standard results from probability theory, derived using 'childish methods'. The results Kemeny referred to were, of course, the well-known asymptotic limit theorems of the discipline, the Central Limit Theorem, the Renewal Theorem, and so on.

From Dretske (1994) through Tononi et al. (2016), and so on, authors at various levels of scientific sophistication have repeatedly invoked 'communication theory', 'information theory', and similar terms, in various attempts at formal characterization of consciousness. Most often, however, this is done as statement of a – sometimes elaborate – shibboleth rather than as a usable treatment of the subject based on the asymptotic limit theorems of the discipline.

Dretske (1994), by some contrast, provides a – perhaps even the – fundamental insight:

> Unless there is a statistically reliable channel of communication between [a source and a receiver] . . . no signal can carry semantic information . . . [thus] the channel over which the [semantic] signal arrives [must satisfy] the appropriate statistical constraints of communication theory.

The asymptotic limit theorems of information theory (Cover and Thomas 2006; Khinchin 1957) constrain any and all possible mathematical models of consciousness, and we do well, as Kemeny recognized, to hew closely to them. Embodiment, which is perhaps the most characteristic blindspot afflicting Western consciousness studies, adds another asymptotic limit, the Data Rate Theorem (Nair et al. 2007, described in the Mathematical Appendix), addressing the minimum rate at which control information must be applied to stabilize an inherently unstable system.

Here, we explore ways in which the asymptotic limit theorems of information and control theories can be used to construct models of consciousness that recognizably hew to the picture written by Bernard Baars (1989, 2005) and others as 'global workspace' theory (e.g., Dehaene and Nacche 2001; Dehaene et al., 2011), models that aid in the creation of new – informed – speculation regarding the phenomenon. The ultimate intent of any such models, however, is the creation of new statistical tools for the analysis of observational and experimental data, the only sources of new knowledge, as opposed to new speculation.

Some biological context

Consciousness is an ancient evolutionary adaptation that provides selective advantage to organisms having identifiable neural systems. It involves a rapid, highly tunable, strongly punctuated, 'spotlight' acquisition to attention and response, typically characterized by a time constant of about 100 milliseconds.

Consciousness is not some panpsychic phlogiston, cognitiferous aether, or a new state of matter. The phenomenon gives selective advantage under competition, in spite of the sometimes tenfold metabolic free energy burden that neural structures carry over other tissue forms (Wallace 2012b, and references therein). Consciousness is about an embodied organism interacting with an embedding, and often unfriendly, but highly structured, ecosystem.

That being said, there are, in most organisms, many roughly analogous phenomena, tunable in very much the same sense, but acting far more slowly, and, most critically, having multiple simultaneous 'spotlights':

- Glycan code cellular interactions (milliseconds to hours).
- The immune response (hours to days).
- Tumor control (days to years).
- Wound healing (minutes to about 18 months).
- Gene expression (through the life span).
- Institutional and sociocultural group cognition (seconds to centuries).

Since consciousness is constrained to approximately 100 millisecond response times in neural systems – requiring significant rates of supply of metabolic free energy – higher animals appear to sustain at most a single such 'tunable spotlight'. Slower systems, as they can entertain multiple, interacting, tunable spotlights, are far more complicated (e.g., Wallace 2012a, b). Consciousness is, then, a bare – indeed, stripped-down – version of these richer mechanisms. Such gross simplification is necessary for high-speed response to rapidly shifting patterns of threat and affordance.

Some mathematical context

There are four asymptotic limit theorems of information and control theories that are central to the study of embodied cognition and consciousness:

- The Shannon Coding Theorem (and 'tuning theorem' variants).
- The Source Coding or Shannon-McMillan Theorem.
- The Rate Distortion Theorem (itself a tuning theorem variant).
- The Data Rate Theorem (connecting information and control across inherently unstable systems).

These apply to stationary, ergodic phenomena. 'Stationary' means that the probabilities remain constant in time, and 'ergodic' that time averages converge to phase averages. Here, however, we will be interested in nonergodic systems, more likely to represent real-world phenomena. Derivation of analogous – or more precisely, different but relevant – formal results for such systems is no small matter. The Ergodic Decomposition Theorem gives a Ptolemaic system, not a Keplerian or Newtonian one (Hoyrup 2013). The resulting work-arounds generate different classes of Keplerian 'regression model' statistical tools that might be fitted to data.

As Wallace (2018) describes the matter,

> ...[W]hile every non-ergodic measure has a unique decomposition into ergodic ones, this decomposition is not always computable.

> From another perspective, such expansions – in terms of the usual ergodic decomposition or the groupoid/directed homotopy equivalents – both explain everything and explain nothing, in the sense that almost any real function can be written as a Fourier series or integral that reatins the essential character of the function itself. Sometimes this helps if there are basic underlying periodicities leading to a meaningful spectrum, otherwise not. The analogy is the contrast between the Ptolemaic expansion of planetary orbits in circular components around a fixed Earth vs. the Newtonian/Keplerian gravitational model in terms of ellipses with the Sun at one focus. While the Ptolemaic expansion converges to any required accuracy, it conceals the essential dynamics.

Extending the approach of Wallace (2018), we adapt methods from nonequilibrium thermodynamics to study both nonergodic systems and their ergodic components – when decomposition is exact – in terms of groupoid symmetries associated with equivalence classes of directed homotopy 'meaningful sequences'. The approach is based on recognition of information as a form of free energy rather than as an 'entropy', mathematical form notwithstanding. In the extreme case which will be the starting point individual pathways can be associated with an information source function, but this cannot be represented in terms of a 'Shannon entropy' across a probability distribution.

An equivalence class structure then arises via a metric distance measure – described later in Eq.(32) – for which the high probability meaningful sequences of one kind of 'game' are closer together than for a significantly different 'game', Averaging occurs according to siuch equivalence classes, generating groupoid symmetries. The dynamics are then characterized by symmetry-breaking according to 'temperature' changes that, contrary to Wallace (2018), must be studied from first principles and may incorporate both underlying regulatory mechanisms and the influence of embedding environments. The standard decomposition can, in part, be recovered by noting that larger equivalence classes across which uncertainty measures are constant can be collapsed into single paths on an appropriate quotient manifold.

And, finally, a trick question: What – and under what conditions – is the asymptotic limit theorem satisfied by the 'integrated information' of Tononi et al. (2016)?

Information and free energy

Next, we take a page from Feynman (2000, p. 146) – literally – to argue that information can be viewed as a form of free energy. That page recapitulates what Feynman calls a 'quite subtle' argument by Bennett, showing how to construct an ideal engine to convert the information within a message into work, provided the system is 'ergodic', that is, equating time averages to phase averages. Bennett's machine is displayed in figure 1.

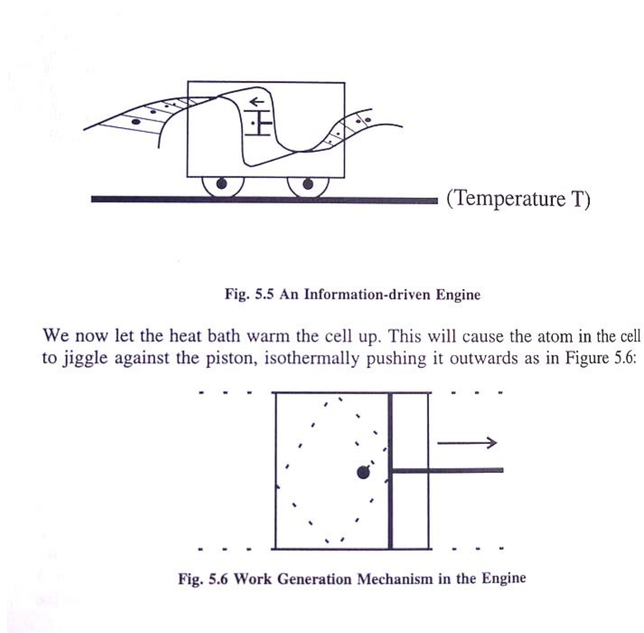

Figure 1: From Feynman (2000). Bennett's ideal machine converting information within a message to work – free energy.

The 'message tape' is fed into the wheeled engine isothermally, generating an average force against the piston, allowing the extraction of work.

There is a more direct way to see this, following the pattern of Wilson (1971). For a physical system of volume V and partition function $\mathcal{Z}(K_1, ..., K_m, V)$, where the K_j are parameters and V is the volume, the free energy can be defined as

$$F(K_1, ..., K_m) = \lim_{V \to \infty} \frac{\log[\mathcal{Z}(K_1, ..., K_m, V)]}{V} \tag{1}$$

For a stationary, ergodic information source, according to the Shannon-McMillan Source Coding Theorem, system paths – messages, in a large sense – can be divided into two sets, one of high probability consonant with underlying grammar and syntax, and one of vanishingly small probability not so consonant (Khinchin 1957).

Let $N(n)$ be the number of high probability grammatical/syntactic paths of length n. Then the Shannon uncertainty of the information source X can be written as

$$H[X] = \lim_{n \to \infty} \frac{\log[N(n)]}{n} \tag{2}$$

For stationary, ergodic information sources dual to cognitive processes one can construct a dynamic theory based simply on this homology (Wallace 2005, 2012):

• Define an entropy as the Legendre transform $S \equiv -H(\mathbf{K}) + \mathbf{K} \cdot \nabla_{\mathbf{K}} H$ and impose the dynamics of first-order Onsager nonequilibrium thermodynamics, understanding that there is no microreversibility for information transmission, and hence no 'Onsager reciprocal relations'.

• Impose Wilson's (1971) version of the Kadanoff model on H, using 'biological renormalizations' (Wallace 2005).

We are going to ber interested here, however, in nonergodic systems more likely to characterize the real world, that is, systems for which time averages are not given by phase averages. This requires some serious thought.

Basic variables

An embodied cognitive agent is embedded in, acting on, and acted on by, a landscape of imprecision in effect, reaction, and result.

The agent enjoys three essential resource streams. These are, first, the rate at which information can be transmitted between parts of itself, characterized by an information channel capacity $\mathcal{C}$. The second resource stream is the rate at which sensory information about the embedding environment is available, at some rate $\mathcal{H}$. The third is the rate at which metabolic free energy and related 'real' resources can be delivered, $\mathcal{M}$.

The resource rates, and time, will interact, generating a correlation matrix analog $\mathbf{Z}$ of dimension 3. Any n dimensional matrix has n scalar invariants – characteristic numbers that remain the same under certain transformations. These invariants can be found from the standard polynomial relation

$$p(\gamma) = \det[\mathbf{Z} - \gamma \mathbf{I}] = \gamma^n - r_1 \gamma^{n-1} + r_2 \gamma^{n-2} - ... + (-1)^n r_n \tag{3}$$

Here, $\mathbf{I}$ is the n-dimensional identity matrix, det the determinant, and γ a real-valued parameter. The first invariant is usually taken as the matrix trace, and the last as $\pm$ the matrix determinant.

These invariants can be used to build a single scalar index $Z = Z(r_1, ..., r_n)$. The simplest such would be $Z = \mathcal{C} \times \mathcal{H} \times \mathcal{M}$. Scalarization, however, must be appropriate to the system under study at the time of study, and there will almost always be cross-interactions between these rates.

The important point for this analysis is that scalarization permits analysis of a one dimensional system. Expansion of Z into vector form leads to sometimes difficult multidimensional dynamic equations (Wallace 2020a, Section 7.1). See the Mathematical Appendix for an outline.

2 Embodied cognition and its dynamics

Here, we follow closely the developments of Wallace (2018, 2020b).

Embodied entities are built from crosstalking cognitive submodules. These range from individual cells, organs, social groupings, formal institutions, to embedding cultures and other environments. For humans in particular, every scale and level of organization, individuals and their social workgroups are constrained, not only by their own experience and training, but by the culture in which they are embedded and with which they interact.

They are likewise constrained by the the environment in which they operate, including actions and intents of competing and cooperating entities.

Further, there is always structured uncertainty imposed by the large deviations possible within the overall embedding environment.

Thus, a number of factors interact to build a composite information source (Cover and Thomas 2006) representing embodied cognition. These are

• Cognition requires choice that reduces uncertainty and implies the existence of an information source formally 'dual' to that cognition at each scale and level of organization (Atlan and Cohen 1998). The argument is direct and agnostic about representation.

• Cognition requires regulation. As Wallace (2017, Ch.3) puts it,

> Cognition and its regulation ... must be viewed as an interacting gestalt, involving not just an atomized individual, but the individual in a rich context... There can be no cognition without regulation, just as there can be no heartbeat without control of blood pressure, and no multicellularity without control of rogue cell cancers. Cognitive streams must be constrained within regulatory riverbanks.

It is here that the Data Rate Theorem, or an appropriate generalization, becomes manifest: there must be an embedding regulatory information source imposing control information at a rate greater than an inherently unstable cognitive process generates its own 'topological information'. See the Mathematical Appendix for details.

• For humans in particular, embedding culture is also an information source, with analogs to grammar and syntax: within a culture, under particular circumstances, some sequences of behavior are highly probable, and others have vanishingly small probability (Khinchin 1957), a sufficient condition for the development of an equivalence class groupoid symmetry-breaking formalism.

• Spatial and social geographies are similarly structured so as to have incident sequences of very high and very low probability: night follows day, summer's dirt roads are followed by October's impassible mud streams.

• Large deviations, as described by Champagnat et al. (2006) and Dembo and Zeitouni (1998), follow high probability developmental pathways governed by entropy-like laws that imply the existence of another information source.

Embedded and embodied cognition is then characterized by a joint information source uncertainty (Cover and Thomas 2006) as

$$H(\{X_i\}, X_V, X_\Delta) \tag{4}$$

The set $\{X_i\}$ includes the cognitive, regulatory, and embedding cultural information sources of the hierarchical system, X_V is the information source of the embedding environment, that may include the actions and intents of adversaries or collaborators, as well as 'weather'. Finally, X_Δ is the information source of the associated large deviations possible to the system.

The essential point here is that crosstalk between coresident information channels and sources is almost inevitable, a consequence of the information chain rule (Cover and Thomas 2006). For a set of interacting stationary ergodic information sources X_i, $i = 1, 2, ...$, the joint uncertainty of interacting sources and channels is always less than or equal to the sum of the independent uncertainties (Cover and Thomas 2006):

$$H(X_1, X_2, ...) \leq \sum_i H(X_i) \tag{5}$$

Each information source X_i is powered by some corresponding free energy source $\mathcal{M}_i$, and it takes more free energy to isolate information sources and channels than to allow their interaction. Such a 'second law' conundrum confounds much of electrical engineering, particularly in the design and construction of microchips. Such 'second law' problems extend to all scales and levels of organization.

Evolutionary process has taken this 'spandrel', in the sense of Gould and Lewontin (1979), and built whole new cathedrals from it (Wallace 2012).

The next steps are somewhat subtle.

According to popular mathematical canon, there is really no serious work to be done on nonergodic information sources as a consequence of the Ergodic Decomposition Theorem (e.g., Gray 1988 Ch.7) which states that it is possible to factor any nonergodic process into a sufficiently large sum (or generalized integral) of ergodic processes, in the same way that any point on a triangle can be expressed in terms of its extremal fixed point vertexes. As Winkelbauer (1970) put it for information source uncertainty,

Theorem II. *The asymptotic rate of a stationary source μ equals the essential supremum of the entropy rates of its ergodic components:*

$$H(\mu) = ess. \sup_{z \in R[\mu]} H(\mu_z)$$

where the μ_z are ergodic.

Is this really a 'simple' result for dynamic systems that can suffer 'absorbing states'? Individual paths – and small, closely-related, equivalence classes of them – are particularly important in biological phenomena, as opposed to physical process. This is because each path may have a unique consequence for the organism or other entity embedded in a stressful environment. After all, there will only be a single 'meaningful sequence' associated with successful capture by a predator.

Recall, further, that it is possible to approximate any reasonably well-behaved real-valued function over a fixed interval in terms of a Fourier series. Recall that it was, in the geocentric Ptolemaic system, via a sufficient number of epicycles, possible to predict planetary positions to any desired accuracy using such a defacto Fourier Decomposition. The underlying astronomical problem was both considerably simplified and greatly enhanced by the non-geocentric empirical observations of Kepler, explained by Newton, and fully elaborated by Einstein.

The phenomena of cognition and consciousness are considerably more complex than the motion of the planets around the sun, and Keplerian laws must still be found across many different physiological phenomena and organisms. Newton and Einstein are nowhere on the horizon for theories of cognition and consciousness.

Here, we significantly expand the development of Wallace (2018), deriving from first principles, rather than imposing, a 'temperature' measure for nonergodic cognitive systems.

We have, above, reduced the spectrum of resources and their interactions – including internal bandwidth, rates of sensory information, and material/energy supply – in terms of a scalar rate variable Z.

To explore some dynamic processes, we next introduce a first-order linear Onsager approximation abducted from nonequilibrium thermodynamics (de Groot and Mazur 1984).

Here, we invoke an *iterated* free energy Morse Function (Pettini 2007) via a formalized Boltzmann probability expression, in the sense of Feynman (2000). This is done by enumerating high probability developmental pathways available to the system as $j = 1, 2, ...$, allowing definition of a path probability P_j

$$P_j = \frac{\exp[-H_j/g(Z)]}{\sum_k \exp[-H_k/g(Z)]} \tag{6}$$

This formulation, following Khinchin (1957) and Wallace (2018), will apply to nonergodic as well as to stationary ergodic information sources and can be used for systems in which each developmental pathway x_j has its own source uncertainty measure H_{x_j}. This, however, is only defined as a Shannon 'entropy' for an ergodic system (Khinchin 1957).

The temperature $g(Z)$, however, must now be calculated from Onsager-like system dynamics built from the partition function, i.e., from the denominator of Eq.(6).

The system's 'rate of cognition' can then be expressed, as in chemical reaction theory (Laidler 1987), by the probability such that $H_j > H_0$, where H_0 is the lower limit for detection of a signal under embodiment in a varying and noisy environment, or for stability via the Data Rate Theorem, as described in the Mathematical Appendix.

There is an important point implicit here: The 'prime groupoid phase transition'. We are, in essence, imposing a symmetry-breaking version of the Source Coding Theorem, the Shannon-McMillan Theorem, on the system, dividing all possible paths into a small set – an equivalence class – of 'meaningful' sequences consonant with some underlying grammar and syntax – in a large sense – and a very large set of vanishingly small probability paths that are not consonant.

Such an equivalence class partition imposes the first of many groupoid symmetry breaking phase transitions. We will discuss groupoid symmetries in more detail below, but, in essence, we have used a groupoid symmetry-breaking phase change to derive – or to impose – a fundamental information theory asymptotic limit theorem. Other such may emerge 'naturally' from related groupoid symmetry-breaking phenomena, all related to equivalence classes of system developmental pathways.

This result can be seen, from a biological perspective, as in the same ballpark as a much earlier phase transition, the sudden transmission of light across the primordial universe after the first 370,000 years.

In sum, groupoid symmetry-breaking extends the Shannon-McMillan Source Coding Theorem to nonergodic – and possibly non-stationary – information sources. This is a major – if 'trivially obvious' – result to which we will return below. The Mathematical Appendix provides a brief introduction to the standard groupoid algebra (Brown 1992; Cayron 2006; Weinstein 1996). The central matter is that products are not defined for all possible pairs of elements, leading to disjoint orbit partition.

The iterated free energy Morse Function F is defined by the relation

$$\exp[-F/g(Z)] \equiv \sum_k \exp[-H_k/g(Z)] = h(g(Z)) \tag{7}$$

F is a Morse Function subject to symmetry-breaking transitions as $g(Z)$ varies (Pettini 2007; Matsumoto 1997). See the Mathematical Appendix for an outline of Morse Function formalism.

We reiterate that these symmetries are not those associated with simple physical phase transitions represented by standard group structures. Cognitive phase change involves punctuated transitions between equivalence classes of high probability signal sequences, represented as groupoids. As Tateishi et al. (2013) put it, if experimental data can be grouped into equivalence classes compatible with an algebraic structure, a groupoid approach can capture the symmetries of the system in a way not be possible with group theory, for example in the analysis of neural network dynamics. Deeper delvings into similar matters can be found in Schreiber and Skoda (2010).

In this work, groupoid symmetries are driven by the directed homotopy induced by failure of local time reversibility for information systems. This is because palindromes have vanishingly small probability. In English, ' the ' has meaning in context while ' eht ' has vanishingly low probability. The ubiquitous mathematical trope 'except on a set of measure zero' implies a fundamental symmetry breaking.

Particularly complicated cognitive systems may require even more general structures, for example, small categories and/or semigroupoids for analogs to the standard symmetry-breaking dynamics of physical systems.

These more general symmetry-breaking phase changes represent extension of the Data Rate Theorem (DRT) to cognitive systems. Again, the DRT states that the rate at which externally-supplied control information must be imposed on an inherently unstable system to stabilize it must exceed the rate at which that system generates its own 'topological information'. The model is of steering a vehicle on a rough, twisting roadway at night. The headlight/steering/driver complex must impose control information at a rate greater than the 'twistiness/roughness' of the road imposes its own information on the vehicle.

There may, then, be many phase analogs available to a cognitive system as $g(Z)$ varies, rather than just the 'on/off' of stability implied by the DRT itself. We will make something of this in a following section that generalizes 'renormalization' analysis of phase transition.

Dynamic equations can be derived from from Eq(6) by invoking a first order Onsager approximation in the gradient of an entropy measure constructed from the iterated free energy Morse Function F via the Legendre transform

$$S(Z) \equiv -F(Z) + ZdF/dZ \tag{8}$$

After some development,

$$\begin{aligned}
&\exp[-F/g(Z)] = \\
&\sum_k \exp[-H_k/g(Z)] \equiv h(g(Z)) \\
&F(Z) = -\log(h(g(Z))g(Z) \\
g(Z) = -&\frac{F(Z)}{RootOf\left(\mathrm{e}^X - h(-F(Z)/X)\right)} \\
&\partial Z/\partial t \approx \mu \partial S/\partial Z = f(Z)
\end{aligned} \tag{9}$$

where the *RootOf* construct defines a generalized Lambert W-function in the sense of Maignan and Scott (2016), Mezo and Keady (2015), and Scott et al. (2006).

The last expression in Eq.(9) represents imposition of the entropy gradient formalism of Onsager nonequilibrium thermodynamics (de Groot and Mazur 1984), for which $f(Z)$ is the 'adaptation function', the fundamental rate at which the system adjusts to changes in Z. Again, for information transmission there is no 'temporal microreversibility', so that there can be no Onsager Reciprocal Relations in these models.

After some further work,

$$\begin{aligned}
&f(Z) = Zd^2F/dZ^2 \\
&F(Z) = \int\int \frac{f(Z)}{Z} dZdZ + C_1 Z + C_2 \\
-Z\int \frac{f(Z)}{Z}dZ - \log(h(g(Z)))g(Z) - C_1 Z + &\int (f(Z)dZ + C_2 = 0
\end{aligned} \tag{10}$$

Again, taking $F = -\log(h(g(Z))g(Z)$, with h determined by underlying internal structure, leads to expressing g in terms of a generalized Lambert W-function, suggesting an underlying formal network structure (Newman 2010).

Specification of any two of f, g, h, in theory, allows calculation of the third. Note, however, that h is fixed by the internal structure of the larger system, and the 'adaptation rate' f is imposed by externalities. In addition, the 'boundary conditions' C_1, C_2 are likewise externally-imposed, also structuring the temperature-analog $g(Z)$. Indeed, the 'temperature' $g(Z)$ might well be viewed as itself an order parameter.

We assume here that embodied cognitive systems can be characterized by the scalar parameter Z, mixing material resource/energy supply with internal and external flows of information under time constraint. There may be more than one such composite irreducible entity driving system dynamics. More explicitly, it may be necessary to replace the scalar Z with some $m \leq n$-dimensional vector having a number of independent – even orthogonal – components accounting for considerable portions of the total variance in the rate of supply of essential resources. The dynamic equations can then be reexpressed in a more complicated vector form (Wallace 2020a, Section 7.1). See the Mathematical Appendix for an outline.

In a similar way, it may be necessary to introduce nonlinear or higher order Onsager models. An introduction to these matters can be found in Wallace (2021a), involving, for example, expressions of the form

$$S = -F + \sum_j a^j Z^j d^j F/dZ^j$$
$$\partial Z/\partial t \approx \sum_j b_j d^j S/dZ^j = f(Z)$$
$$\partial Z/\partial t \approx dS/dZ \times d^2S/dZ^2 + ... = f(Z) \quad (11)$$

leading to formal algebraic power series treatments in the sense of Jackson et al. (2017). These considerations take us beyond the simplest '$Y = mX + b$' regression model analogs.

The dynamics are driven at rates determined by the adaptation function $f(Z)$. We can ask more detailed questions regarding what happens at critical points defined in terms of the 'temperature' variate $g(Z)$ through the abduction of another approach from physical theory.

3 Toward a more general theory of embodied cognition

We have, for cultural and historical reasons, focused largely on rapid, single-workspace phenomena of neural consciousness having the 100ms time constant. If we enthrone the Data Rate Theorem rather than rate of cognition, we can say something regarding embodied cognition as a (or even the) more fundamental gestalt. The argument – a significant condensation of Wallace (2021a) – is surprisingly simple, and seems independent of such niceties as ergodicity of information sources.

Recall the Data Rate Theorem – e.g., figure 11 in the Mathematical Appendix. Stabilization of an inherently unstable control system engaged in some fundamental task requires that control information be delivered at a rate greater than the rate at which the unstable system generates its own 'topological information', say H_0. We do not specify 'H_0', except as a scalar entity that may indeed change with time.

Most simply, we characterize H as the (scalar) rate at which external regulatory mechanisms provide such control information, and (yet again) define a Boltzmann pseudoprobability

$$dP(H) \equiv \frac{\exp[-H/g(Z)]dH}{\int_0^\infty \exp[-H/g(Z)]dH} \tag{12}$$

where $g(Z)$ and Z are as described above, supposing that the regulatory-action process is itself composed of subcomponents that interact with each other and with an embedding environment through both information exchanges and the use of 'materiel' in various forms.

We define iterated free energy and entropy analogs as above from the 'partition function' denominator of Eq.(12), obtaining the usual relations

$$\begin{aligned}
\exp[-F/g(Z)] \equiv \int_0^\infty \exp[-H/g(Z)]dH = g(Z) \\
g(Z) = \frac{-F(Z)}{W(n, -F(Z))} \\
S \equiv -F + ZdF/dZ \\
\partial S/\partial t \propto dS/dZ = f(Z) \\
f(Z) = Zd^2F/dZ^2 \\
L(Z) = \frac{\int_{H_0}^\infty \exp[-H/g(Z)]dH}{\int_0^\infty \exp[-H/g(Z)]dH} = \exp[-H_0/g(Z)]
\end{aligned} \tag{13}$$

where, again, $W(n, x)$ is the appropriate Lambert W-function and $L(Z)$ the cognition rate. A more convoluted line of argument leada to the 'generalized Lambert W-functions' of Eq.(9).

Embodiment – directly implying interaction with the embedding world – appears to impose a certain draconian or procrustian simplicity, as Charles Darwin, Alfred Russel Wallace, and many others have noted.

4 Examples

We have assembled enough tools for a simple application, recognizing the basic punctuated transition in consciousness – on/off – as being roughly analogous to the Data Rate Theorem in that there is a critical value of information flow rate H_0 for full function. That is, allowing a continuous approximation to the sum in the second expression of Eq.(9), we assume a minimum necessary critical limit H_0 and can write

$$\begin{aligned}\exp[-F/g(Z)] = \int_{-H_0}^{\infty} \exp[-(H_0+x)/g(Z)]dx = g(Z)\\ F = -\log[g(Z)]g(Z)\\ g(Z) = \frac{-F(Z)}{W(n, -F(Z))}\end{aligned} \tag{14}$$

where $W(n,x)$ is the Lambert W-function of order n that satisfies the relation $W(n,x)\exp[W(n,x)] = x$. It is real-valued only for orders $n = 0, -1$ over respective ranges $-\exp[-1] < x < \infty$ and $-\exp[-1] < x < 0$.

The appearance of the Lambert W-function is a distinct red flag, implying the possibility of re-envisioning and reconstructing the underlying problem in terms of a more fundamental, if substantially more abstract, submodular network. See Newman (2010) for general arguments and examples, and Yi et al. (2011) for an application to neural networks with time delays. Recall that the fraction of nodes in the 'giant component' within a random network of N nodes can be described in terms of the probability of contact between nodes p as $\{W(0, -Np\exp[-Np]) + Np\}/Np$. This expression is highly punctuated in the variable Np, leading, as discussed in the next chapter, to an elementary model of the accession to consciousness in a tunable linked system of cognitive submodules.

Eq.(14) leads to an expression for the cognition rate – in an argument abducted from chemical reaction theory (Laidler 1987) – as

$$L(Z) = \frac{\int_{H_0}^{\infty} \exp[-x/g(Z)]dx}{\int_{-H_0}^{\infty} \exp[-(H_0+x)/g(Z)]dx} = \exp[-H_0/g(Z)] \tag{15}$$

Thus, in Eq.(9), we find $h(g(Z)) = g(Z)$ and can carry out an explicit calculation for g in terms of $f(Z)$ from Eq.(10), giving

$$g(Z) = \frac{-C_1 Z - Z\int \frac{f(Z)}{Z}dZ + C_2 + \int f(Z)dZ}{W(-C_1 Z - Z\int \frac{f(Z)}{Z}dZ + C_2 + \int f(Z)dZ)} \tag{16}$$

where, again, $dZ/dt = f(Z(t))$ defines the adaptation function f, and $W(x)$ is the Lambert W-function, taken here of of order 0 and real-valued for $-\exp[-1] < x < \infty$.

It is important to recognize that, in higher animals, metabolic free energy is provided by the hydrolysis of adenosine triphosphate (ATP) to diphosphate (ADP). The free energy available from this reaction is significant, ranging between 30-60 KJ/mol, dependent on embedding physiological details. The energy supplied by this process can be equivalent to thousands of degrees K, suggesting that neural tissues, which typically consume metabolic free energy at a rate ten times that of more ordinary tissues, can indeed be driven to operate at very high 'reaction rates'.

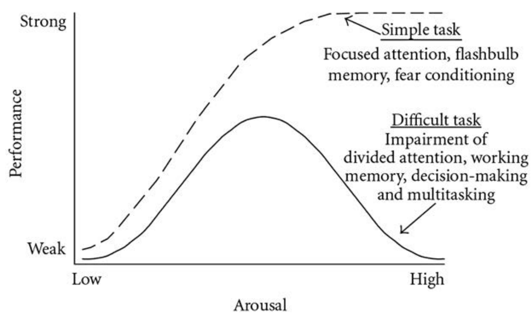

Figure 2: From Diamond et al. (2007). Canonical forms of the Yerkes-Dodson law, for simple and difficult tasks.

Arousal

Following Wallace (2021b), the Yerkes-Dodson law (Diamond et al. 2007) relates complex task performance to arousal for individual animals under experimental conditions. Figure 2, as adapted from Diamond et al. (2007), shows that, depending on the difficulty of the task, there can be either a 'topping out' or an inverted-U pattern for performance vs. arousal.

Typically, $f(Z)$, the adaptation function, might be seen as taking an 'exponential' form, i.e., $f(Z(t)) = \beta - \alpha Z(t)$.

In figure 3, we plot two expressions for $L(Z)$, the cognition rate, from Eq.(15), letting $H_0 = 1$. Here, $\alpha = 0.1$ and $\alpha = 1.0$, while the arousal index, β, varies. $C_1 = -3$ for both, while C_2 takes the values -18 and -3. Here, the lighter curve, corresponding to $\alpha = 1.0$, represents the easier task.

With proper manipulation of parameters and boundary conditions, the formalism produces something remarkably like the Yerkes-Dodson law.

Figure 4, by contrast, examines the efficiency of cognition, $L(Z)/Z$, as a function of the arousal β. Hard problems are are, in this model, clearly far more demanding of resources than simple ones, suggesting that it is more efficient to break up a hard problem into a series of simpler ones, or, perhaps more to the point, to a parallel set of them.

Distraction

Conscious effort and attention is not only affected by arousal, but by distraction, and it is not difficult to explore the dynamics of cognition rate $L(Z)$ using formalism available from the standard theory of stochastic differential equations (Protter 2005).

This requires expanding the relation $dZ/dt = f(Z(t)) = \beta - \alpha Z(t)$ as

$$dZ_t = (\beta - \alpha Z_t)dt + \sigma Z_t dB_t \tag{17}$$

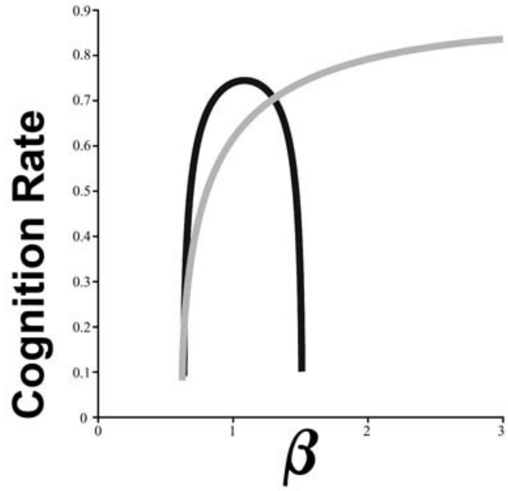

Figure 3: From Wallace (2021b). Cognition rates $L(Z)$ vs. β as an arousal index for simple and difficult tasks. Here, $H_0 = 1$, taking $\alpha = 0.1$ for the difficult and $\alpha = 1.0$, for the easier task. $C_1 = -3$ for both, while C_2 takes the values -18 and -3. The lighter curve, having $\alpha = 1.0$, represents the easier task.

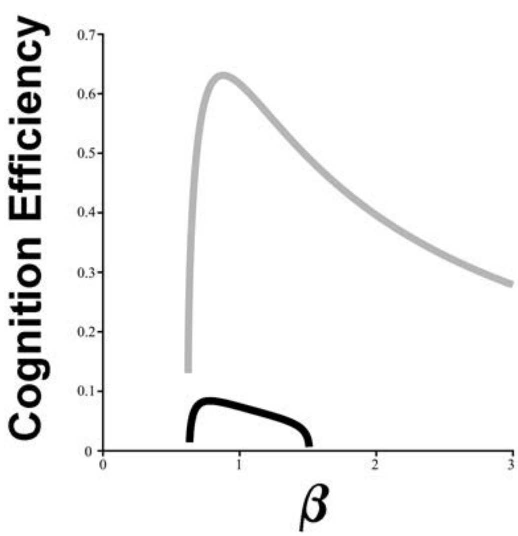

Figure 4: Cognition efficiency $L(Z)/Z$ vs. arousal for the examples of figure 3. It is, in this model, far more efficient to convert a hard problem into a series or parallel set of simpler ones.

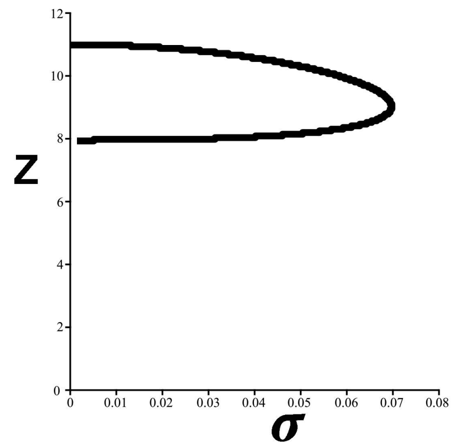

Figure 5: Application of the Ito Chain Rule to the cognition rate $L(Z)$, with $f(Z) = 1.1 - 0.1Z$. The value of β has been taken as the peak for the more difficult problem of figure 3. We find the solution set $\{\sigma, Z\}$ for the nss relation $< dL_t >= 0$ using numerical methods. Instability begins at a much lower value of σ than 0.447, the limit for variance in Z as driven by simple volatility for $\alpha = 0.1$. By contrast, L faces the possibility of a bifurcation instability for any $\sigma > 0$, and fully collapses under the burden of distraction if $\sigma > 0.07$.

where the second term represents a standard model of 'volatility', with σ the magnitude of the distracting 'noise' dB_t, taken here as ordinary flat-spectrum Brownian white noise. 'Colored' noise is possible, at the expense of mathematical complication (Protter 2005).

Using the Ito Chain Rule on Z_t^2, (Protter 2005), 'it is easy to show' that, for a nonequilibrium steady state (nss), the variance of Z_t for the exponential model is

$$< Z^2 > - < Z >^2 = \left(\frac{\beta}{\alpha - \sigma^2/2} \right)^2 - \left(\frac{\beta}{\alpha} \right)^2 \tag{18}$$

For the difficult task of figure 3, $\alpha = 0.1$ and, independent of arousal β, variance in Z explodes if the noise burden $\sigma > \sqrt{0.2} \approx 0.447$.

What happens to the cognition rate of Eq.(15) under noise/distraction measured by σ? Again, it is possible to apply the Ito Chain Rule to $L(Z)$ as it was to Z^2. We study the more difficult problem of figure 3, having $\alpha = 0.1$, but fix $\beta = 1.1$, i.e., at the peak value of the cognition rate L. The resulting relation at nss – the solution set $\{\sigma, Z\}$ to the equation $< dL_t >= 0$ – is literally too long to write on this page, but can be solved numerically via the *implicitplot* function of the computer algebra program Maple 2020, giving figure 5.

The cognition rate $L(Z)$ is very highly sensitive to the magnitude of distraction σ in this model. While, for the more difficult problem of figure 3, Z itself becomes unstable if $\sigma > 0.447$, L faces the possibility of a bifurcation instability for any $\sigma > 0$, and collapses entirely if $\sigma > 0.07$.

Distraction and arousal under fixed delay

Imposition of a fixed, discrete, delay δt on the adaptation function $f(Z) = \beta - \alpha Z(t)$, so that

$$dZ(t) = \beta - \alpha Z(t - \delta t) \tag{19}$$

gives a delay-differential equation. In general, these are most difficult to analyze. Here, however, we can directly impose a solution to Eq.(19) as

$$Z_s(t) \equiv \frac{\beta}{\alpha}(1 - \exp[st]) \tag{20}$$

Application of Eq.(19) to this expression permits an explicit solution for s:

$$s = \frac{W(n, -\alpha\delta t)}{\delta t} \tag{21}$$

where, again, W is the Lambert W-function of order n.

Recall that, only for $n = 0$, is it real-valued, for $-\exp[-1] < x < \infty$, and for $n = -1$, real-valued for $-\exp[-1] < x < 0$. Lambert W-functions of all other orders are complex-valued.

$\alpha \times \delta t$, the product of a rate and a time, is a dimensionless number driving system dynamics. Yi et al. (2011) extend the general method to multidimensional systems, using the matrix Lambert W-function. Figure 6 shows the real and complex values of the $n = -1$ branch of Eq.(21), the fundamental form for what we will do here, as functions of the product $\alpha\delta t$.

There are two critical values for the (dimensionless) index $\alpha\delta t$, derived from the appearance of the Lambert W-function, here taken of order -1. The first is at the point where the complex component becomes nonzero, i.e., when $\alpha\delta t > \exp[-1]$. This signifies onset of dying oscillatory dynamics. The second critical value is the value of $\alpha\delta t$ at which the real component of s becomes greater than zero, triggering explosive growth in oscillations. Note that the periodicity, determined by the magnitude of the complex component, changes as $\alpha\delta t$ increases beyond the first critical point.

Eq.(20) implies

$$dZ_s/dt = s(Z_s - \beta/\alpha) \equiv f(Z_s) \tag{22}$$

permitting, as above, analysis of the resource delivery system's properties under stochastic fog, via the stochastic differential equation

$$dZ_t^s = s(Z_t^s - \beta/\alpha)dt + \sigma Z_t^s dB_t \tag{23}$$

Again, the second term represents volatility under conditions of Brownian noise.

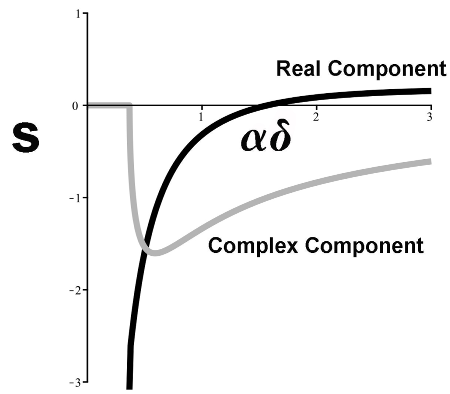

Figure 6: s from Eq.(21), taking $n = -1$, real-valued only for $-exp[-1] < -\alpha\delta t < 0$. There are two critical values for the dimensionless index $\alpha\delta t$, derived from the appearance of the Lambert W-function. The first is at the point where the complex component becomes nonzero, representing the onset of dying oscillatory dynamics when $\alpha\delta t > \exp[-1]$. The second is the point at which the real component of s becomes greater than zero, implying explosive growth in oscillations. Periodicity, determined by the magnitude of the complex component, changes as $\alpha\delta t$ increases beyond the first critical point.

Applying the Ito Chain Rule to Z_s^2 gives the variance as

$$< Z_s^2 > - < Z_s >^2 = \left(\frac{s\beta/\alpha}{s + \sigma^2/2} \right)^2 - \left(\frac{\alpha}{\beta} \right)^2 \tag{24}$$

Again, the system becomes grossly unstable if $\alpha\delta t > \exp[-1]$. Thus delay – δt – can greatly exacerbate inherent stochastic instabilities.

However, even if s is real-valued and negative, sufficient noise, measured by $\sigma^2/2$, also triggers explosive instability.

Recall the expressions for $g(Z)$ and $L(Z)$ from Eqs.(3.2) and (3.3), assuming a delayed adaptation function, so that $\partial Z_s/\partial t = f(Z_s(t)) = s(Z_s(t) - \beta/\alpha)$, where s is from Eq.(21), so that $Z_s(t) \to \beta/\alpha$.

Numerical exploration finds instability in L can be imposed by even very small delays δt.

We construct another Yerkes-Dodson 'arousal' analysis, showing cognition rate as a function of β, letting $Z = \beta/\alpha$ and taking appropriate values for other parameters. This is done in figure 7, where, for the stable – real-valued only – solutions, we set $\alpha = 1, C_1 = C_2 = 3$ and plot $L(\beta)$ for $\delta t = 0.04$, 0.3, $\exp[-1]$, taking the Lambert W-function of order -1 for s in Eq.(21) and 0 in the expression for $g(Z_s)$. As the fixed delay δt increases, the 'inverted-U' of institutional cognition progressively collapses in a manner consistent with Delayed Auditory Feedback studies in which an artificially-induced delay of about 175ms between speech and hearing triggers extreme stress.

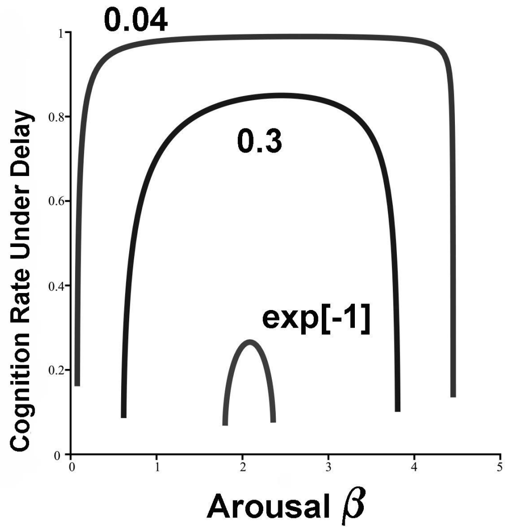

Figure 7: A Yerkes-Dodson 'arousal' analysis for cognition rate, setting $n = -1$ in the expression for s and $n = 0$ in that for g, with $Z = \beta/\alpha$, $\alpha = 1$, $C_1 = C_2 = 3$. Here, $\delta t = 0.04$, $0.3, \exp[-1]$. Increasing fixed delay collapses cognitive function.

Two-mode dynamics

Next, we examine what is perhaps the simplest possible example, a two-mode nonergodic system for which the high probability meaningful sequences are assumed to fall into two sets, each taken to be of the same size N, having source uncertainties $H_{\pm} = H_0 \pm \delta$ for a fixed $\delta > 0$. The larger value represents the 'on' mode, and the smaller the 'off'. Here, we – in effect – invoke the Ergodic Decomposition Theorem, as the $H_{\pm}$ represent ergodic 'extreme points' across the full nonergodic regime.

Some thought gives

$$
\begin{aligned}
\exp[-F/g(Z)] &= N \exp[-H_0/g(Z)] 2\cosh(\delta/g(Z)) \\
F &= -\log[2N\cosh(\delta/g(Z))]g(Z) + H_0 \\
L(Z) &= \frac{\exp[-\delta/g(Z)]}{2\cosh(\delta/g(Z))}
\end{aligned}
\tag{25}
$$

where $L(Z)$ is again the cognition rate.

Again, we impose a first-order Onsager model, taking $S \equiv -F + ZdF/dZ$ and $\partial Z/\partial t \propto dS/dZ = f(Z)$.

Approximating the resulting relations to fourth order in δ gives, surprisingly, a second order expression as

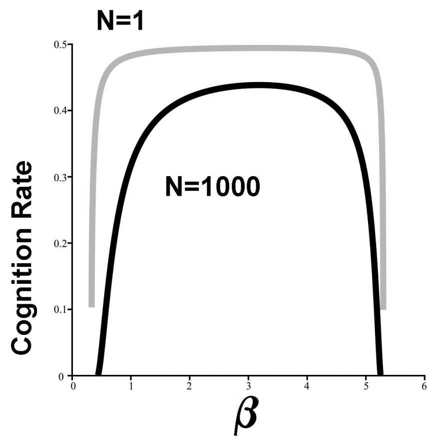

Figure 8: Yerkes-Dodson arousal plot approximation, showing cognition rate L vs. β for the adaptation function $f(Z) = \beta - \alpha Z$. Here, $N = 1, 1000$, $\delta = 0.1, \alpha = 1, Z = \beta$ and the boundary conditions are $C_1 = -2, C_1 = -1$.

$$-Z\left(\ln(2)+\ln(N)\right)\left(\frac{d^2}{dZ^2}g(Z)\right)+Z\left(\frac{\frac{d^2}{dZ^2}g(Z)}{2g(Z)^2}-\frac{\left(\frac{d}{dZ}g(Z)\right)^2}{g(Z)^3}\right)\delta^2\approx f(Z) \tag{26}$$

We will again take $f(Z) = \beta - \alpha Z$. The resulting equation can be explicitly solved for $g(Z)$, leading to a distractingly complicated expression for the cognition rate which we omit for clarity.

Figure 8, however, shows the resulting – and highly approximate – Yerkes-Dodson arousal relations, i.e., the $L(\beta)$, for two different values of N. Here, $\alpha = 1, Z = \beta, N = 1000, 1, \delta = 0.1$, with the necessary two boundary conditions as $C_1 = -2, C_2 = -1$.

Because of the simple structure – only two modes – we have been able to explicitly derive the h-function relation of Eqs.(7) and(9).

Multi-mode dynamics

Suppose there a many, say N, possible H-values, clustered around a base value H_0. Then

$$\exp[-F/g(Z)] = \sum_{j=1}^{N} \exp[-(H_0+\delta_j)/g(Z)] =$$

$$\exp[-H_0/g(Z)]\left(\sum_{j=1}^{N}\exp[-\delta_j/g(Z)]\right) \tag{27}$$

We can estimate the sum in δ_j using the approximation

$$\exp[-\delta_j/g(Z)] \approx 1 - \delta_j/g(Z) + \frac{1}{2}\frac{\delta_j^2}{g(Z)^2} \tag{28}$$

Then

$$\exp[-F/g(Z)] \approx \exp[-H_0/g(Z)]\left(N - \frac{\sum_j \delta_j}{g(Z)} + \frac{\sum_j \delta_j^2}{2g(Z)^2}\right) \tag{29}$$

so that

$$F(Z) \approx -\log\left(N + \frac{\Delta}{2g(Z)^2}\right) g(Z) + H_0 \tag{30}$$

where $\Delta \equiv \sum_j \delta_j^2$ and we have adjusted H_0 so that $\sum_j \delta_j = 0$.

We again introduce an entropy-analog as $S = -F + ZdF/dZ$, and impose Onsager dynamics as $\partial S/\partial t \propto dS/dZ = f(Z)$.

Taking $f(Z) = \beta - \alpha Z$, and assuming $\Delta/2g(Z)^2 \gg N$ leads to

$$g(Z) = \frac{2\ln(Z)Z\beta - \alpha Z^2 + 2C_1 Z - 2\beta Z - 2C_2}{4W\left(n, -\frac{\sqrt{\frac{(2\ln(Z)Z\beta - \alpha Z^2 + 2C_1 Z - 2\beta Z - 2C_2)^2}{\Delta}}\sqrt{2}}{4}\right)} \tag{31}$$

where, again, $W(n,x)$ is the Lambert W-function of order n.

Recall that $W(n,x)$ is real-valued only for $n = 0, -1$ and only over limited ranges in x, suggesting punctuated phase transitions in this expression, driven by synnetry-breaking in the groupoids defining the real-value ranges of x.

The model leading to Eq.(31) is obviously fragile, but, assuming a Data Rate Theorem limit H_1 for a cognition rate defined as $L \propto \exp[-H_1/g(Z)]$, proper choice of boundary conditions does indeed produce the ubiquitous inverted-U. In figure 9, $H_1 = 1, n = -1, \Delta = 1000, \alpha = 1, C_1 = -3, C_2 = 1$, assuming $Z = \beta/\alpha$. The approximation fails for large β.

5 Phase transitions

The possible appearance of the Lambert W-function in the argument above – for the simple case $h(g(Z)) = g(Z)$ – is a warning. The fraction of nodes within the

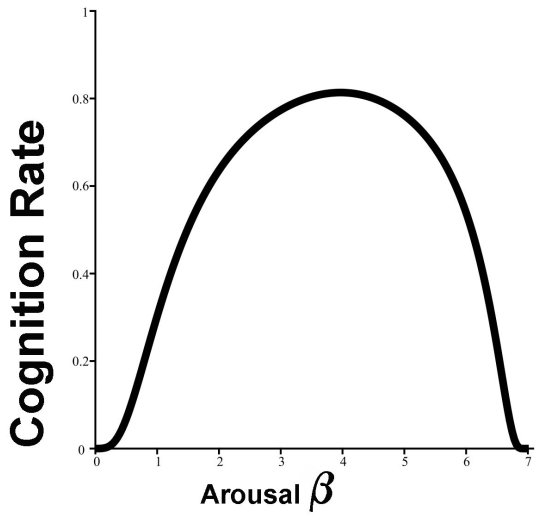

Figure 9: Cognition rate example for Eq.(31). Here, $L \propto \exp[-H_1/g(Z)]$, setting $H_1 = 1, n = -1, \Delta = 1000, \alpha = 1, C_1 = -3, C_2 = 1, Z = \beta/\alpha$. The approximation is fragile and fails at larger β.

'giant component' of a random network of N nodes – here, taken as interacting information sources dual to unconscious cognitive processes – can be described in terms of the probability of contact between nodes, p, as (Newman 2010)

$$\frac{W(0, -Np\exp[-Np]) + Np}{Np} \tag{32}$$

giving the results of figure 10. Note the threshold for onset of a giant component in the random network case.

An important feature here is the topological tunability of the threshold dynamics implied by the two limiting cases, the star-of-stars-of-stars vs. the random network.

Lambert W-functions thus appear to suggest existence of an underlying formal network structure. For our purposes here – neural structures – we can envision the underlying abstract network to be a set of information sources dual to unconscious cognitive phenomena within the brain. These become linked by 'Np' crosstalk, in the context of a tunable topology that shifts somewhere between the two limits of the figure.

See figure 6 of Dehaene and Changeux (2011) for something similar.

Previous sections have abducted results from nonequilibrium thermodynamics to consciousness theory, applicable to nonergodic, as well as ergodic, models of cognition. Here, we abduct the Kadanoff renormalization treatment of physical phase transitions (e.g., Wilson 1971, Wallace 2005, 2012b), applying it to a reduced version of the iterated 'free energy' Morse Function of Eq.(7), expanding

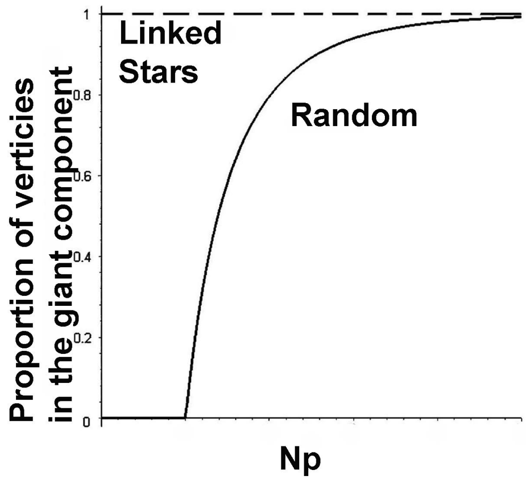

Figure 10: Proportion of N interacting information sources dual to unconscious cognitive processes that are entrained into a 'giant component' global broadcast as a function of the probability of contact p for random and stars-of-stars-of-stars topologies. Tuning topologies determines the threshold for 'ignition' to global broadcast.

the approach of Wallace (2021a).

Although a more general argument can be made, representing embodied consciousness *an sich*, for the sake of familiarity, we project down on to the subsystem dominated by $\mathcal{C}$, the internal system bandwidth, envisioning a number of internal cognitive submodules as connected into a topologically identifiable network having a variable average number of fixed-strength crosstalk linkages between components. The mutual information measure of crosstalk can continuously change, and it becomes then possible to conduct a parameterized renormalization in a now-standard manner (Wilson 1971, Wallace 2005).

The internal modular network linked by information exchange has a topology depending on the magnitude of interaction. We define an interaction parameter, a real number $\omega > 0$, and examine structures characterized in terms of linkages set to zero if crosstalk is less than ω, and renormalized to 1 if greater than or equal to ω. Each ω defines, in turn, a network 'giant component' (Spenser 2010), linked by information exchange greater than or equal to it.

Now invert the argument: a given topology of interacting submodules making up a giant component will, in turn, define some critical value ω_C such that network elements interacting by information exchange at a rate less than that value will be excluded from that component, will be locked out and not 'consciously' perceived.

ω is a tunable, syntactically dependent, detection limit depending on the instantaneous topology of the giant component of linked cognitive submodules defining, by that linkage, a 'global broadcast'.

For 'slow' systems (Wallace 2012b) – immune response, gene expression, institutional process – as opposed to the 100 ms time constant of higher animal consciousness, there can be many such 'global workspace' spotlights acting simultaneously. Such multiple global broadcasts, indexed by the set $\Omega = \{\omega_1, \omega_2, ...\}$, lessen the likelihood of inattentional blindness to critical signals, both internal and external. The immune system, for example, engages simultaneously in pathogen and malignancy attack, neuroimmuno dialog, and routine tissue maintenance (Cohen 2000).

Assuming it possible to scalarize the set Ω in something like the manner of Z, built from Eq.(3), we work with a single, real-value ω, and can model the dynamics of a multiple tunable workspace system.

Recall the definition of the iterated free energy F from Eq.(7), now focused within and characterized by ω. The essential idea is to invoke a 'length' r on the network of internal interacting information sources. r will be more fully defined below. We follow the renormalization methodology of Wilson (1971) as described in Wallace (2005), although other approaches are clearly possible. That is, there is no unique renormalization symmetry.

The central idea is to invoke a 'clumping' transformation under an 'external field strength' that can be, in the limit, set to zero. For clumps of size R, given a field of strength J,

$$\begin{aligned} F[\omega(R), J(R)] &= \mathcal{F}(R) F[\omega(1), J(1)] \\ \chi[\omega(R), J(R)] &= \frac{\chi[\omega(1), J(1)]}{R} \end{aligned} \tag{33}$$

χ represents a correlation length across the linked information sources.

$\mathcal{F}(R)$ is a 'biological' renormalization relation that can take such forms as R^δ, $m\log(R) + 1$, $\exp[m(R-1)/R]$, and so on, so long as $\mathcal{F}(1) = 1$ and is otherwise monotonic increasing. Physical theory is restricted to $\mathcal{F}(R) = R^3$.

Surprisingly, after some tedious algebra, the standard Wilson (1971) renormalization phase transition calculation drops right out for the extended relations, described first in Wallace (2005) and summarized in the Mathematical Appendix.

There remains a problem. Just what is the metric r? In this, we follow Wallace (2012b).

First, impose a topology on the system of interacting information sources such that, near a particular 'language' A associated with some source uncertainty measure H, there is an open set U of closely similar languages $\hat{A}$ such that the set $A, \hat{A} \in U$.

Since the information sources are sufficiently similar, for all pairs of languages $A, \hat{A}$ in U it is possible to

• Create an embedding alphabet which includes all symbols allowed to both.

• Define an information-theoretic distortion measure in the extended joint alphabet between any high probability (i.e, properly grammatical and syntactical) paths in A and $\hat{A}$, written as $d(Ax, \hat{A}x)$. The different languages do not interact in this approximation.

• Define the metric on U as

$$r(A, \hat{A}) \equiv |\int_{A,\hat{A}} d(Ax, \hat{A}x) - \int_{A,A} d(Ax, A\hat{x})| \tag{34}$$

where Ax and $\hat{A}x$ are paths in the languages $A, \hat{A}$ respectively, d is the distortion measure, and the second term is a 'self-distance' for the language A such that $r(A, A) = 0, r(A, \hat{A}) > 0, A \neq \hat{A}$.

Some thought shows this version of r is sufficient, if somewhat counterintuitive. A more formal approach can be found in Glazebrook and Wallace (2009).

Extension of the Wilson technique to a fully-embodied consciousness model seems straightforward. However, since the dynamics of the embedded condition are so highly variable, there will be no unique solution, although there may well be equivalence classes of solutions, defining yet more goupoids in the sense of Tatcishi et al. (2013).

Indeed, groupoids may rear their heads at a more fundamental level: Wilson's renormalization semigroup, in the cognitive circumstance of discrete equivalence classes of developmental pathways, might well require generalization as a renormalization semigroupoid, e.g., a disjoint union of different renormalization semigroups across a nested or otherwise linked set of information sources and/or iterated free energy constructs dual to cognitive modules. Something roughly analogous has been postulated for 'spin foam' gravity models (Oeckl 2003).

A little algebra: the transitive decomposition

Following the arguments of Wallace (2018), we have, in effect, studied equivalence classes of directed homotopy developmental paths – the $\{x_1, ...x_n, ...\}$ – associated with nonergodic cognitive systems defined in terms of single-path source uncertainties. These require imposition of structure in terms of the metric r of Eq.(34), leading to groupoid symmetry-breaking transitions driven by changes in the temperature analog $g(Z)$. There can be an intermediate case under circumstances in which the standare ergodic decomposition of a stationary process is both reasonable and computable – no small constraint. Then there is an obvious natural directed homotopy partition in terms of the transitive components of the path-equivalence class groupoid. It appears that this decomposition is equivalent to, and maps on, the ergodic decomposition of the overall stationary cognitive process. It then becomes possible to derine a constant source uncertainty on each transitive subcomponent, fully indexed by the embedding groupoid. This was done earkuer for the two-mode example.

That is, each ergodic/transitive groupoid component of the ergodic decomposition recovers a constant value of the source uncertainty dual to cognition, presumably given by standard 'Shannon entropy' expression. Since it is possible to view the components themselves as constituting single paths in an appropriate quotient space, this leads to the previous 'nonergodic' developments.

As Wallace (2018) notes, the argument tends toward Mackey's theory of 'virtual groups', i.e., 'ergodic groupoids' (Hahn 1978; Mackey 1963; Series 1977).

A complication emerges through imposition of a double symmetry involving metric r-defined equivalence classes on this quotient space. That is, there are different possible strategies for any two teams playing the same game. In sum, however, groupoid symmetry-breaking in the iterated free energy construct of Eq.(7) will still be driven by changes in $g(Z)$ and/or ω.

6 Discussion and conclusions

As a remark above implies – that for information dynamics there is no microreversibility, and hence no 'Onsager Reciprocal Relations' – cognitive phenomena are likely to be far different from physical processes, although undoubtedly constrained by them. The 'prime groupoid phase transition' is both 'obvious' and unexpected. The 'Kadanoff Picture' of phase transition in cognition is similarly plagued with 'biological renormalizations' that may, in fact, be tunable (Wallace 2005). Underlying this is the matter of 'fundamental symmetries' and 'symmetry-breaking' in cognition. These 'symmetries' will, particularly for the nonergodic cognitive phenomena likely to dominate real-world dynamics, involve equivalence classes of dynamic paths, the long $x_j = \{x_1^j, ..., x_n^j, ...\}$ discussed above. Equivalence class properties can be expressed in terms of groupoids, essentially groups for which there is not necessarily a product defined between all element pairs. The symmetry-breaking of phase transitions familiar from physical theory then becomes a matter of transitions between groupoid structures.

That is, symmetry-breaking in cognition should be considered as fundamental to the study of cognitive process – including consciousness – as it is to physical theory. The symmetries are, however, much different than those familiar from physical theory. One may, perhaps in lifting the requirement that the systems be stationary, be driven to even more general symmetry structures than groupoids, for example, small categories and semigroupoids, in the context of dynamics characterized in terms of formal algebraic power series. This work remains to be done.

To reiterate, 'except on a set of measure zero' implies some primordial symmetry breaking.

There is support for this perspective in the literature. Recall the 'information theory chain rule' from Eq.(5) (Cover and Thomas 2006). For two stationary, ergodic information sources X_1 and X_2, the joint uncertainty must be less than the sum of their independent uncertainties:

$$H(X_1) + H(X_2) \geq H(X_1, X_2) \tag{35}$$

Let G be any finite group and G_1, G_2 be subgroups of G. Take $|G|$ to be the order of the group, i.e., the number of elements. The intersection $G_1 \cap G_2$ is also a subgroup and a group inequality can be derived analogous to Eq.(35):

$$\log\left(\frac{|G|}{|G_1|}\right) + \log\left(\frac{|G|}{|G_2|}\right) \geq \log\left(\frac{|G|}{|G_1 \cap G_2|}\right) \tag{36}$$

Yeung (2008) assigns a probability via a pseudorandom variate related to a group G as $\Pr[\mathrm{X} = \mathrm{a}] = 1/|G|$, allowing construction of a group-characterized information source. Yeung (2008) establishes a one-to-one correspondence between unconstrained information inequalities, extensions of Eq.(35), and finite group inequalities. That is, unconstrained inequalities can be proved by techniques in group theory, and many group-theoretic inequalities can be proven by methods of information theory.

We suggest here, in a similar manner, that nonergodic information sources and their dynamics are intimately associated with groupoid algebras. Less regular information processes may require even more general algebraic structures.

We have, then, outlined a mathematical treatment of embodied consciousness – really, the only kind evolution can give us – that, while abducting (ultimately, nonlinear) nonequilibrium thermodynamics and Kadanoff theory, remains true to the asymptotic limit theorems of information and control theories. The underlying example for this is the abduction of classical mechanics into both quantum theory and general relativity, albeit in markedly different directions. The observation of Feynman (2000) and many others that information is a form of free energy permits these abductions, in the context of new, iterative, Morse Theory free energy and entropy analogs. Application of these methods to psychopathologies can be found in Wallace (2016, 2017), and to failure of artificial intelligence under stress in Wallace (2020a).

This work differs significantly from the earlier analyses by Wallace (2005, 2012), who studied similar dynamics, but focused on stationary ergodic source uncertainties dual to cognitive processes. Here, an iterated free energy Morse Function is defined through Eq.(7) for nonergodic systems, permitting greater latitude in modeling dynamic behavior. This iterated 'free energy' approach differs from Friston's free energy formalism (e.g., Bogacz 2017) by avoiding a fundamental contradiction, i.e., not invoking minimization of free energy measures for neural systems that actually require rates of metabolic free energy supply an order of magnitude greater than for other kinds of tissue. The argument here that most parallels Friston's regards efficiency of cognition, as in figure 4, suggesting an evolutionary necessity for highly parallel address of difficult cognitive problems.

Further, the underlying perspective of this line of research differs from Integrated Information Theory (IIT) by actually hewing closely to the asymptotic limit theorems of both control and information theories, and by explicit recognition that consciousness, like immune function and insect wings, is an evolutionary adaptation specific to organisms, and not a general property to be assigned across physical systems. While it may be possible to construct computing machines having any number of rapidly-tunable neural global workspace analogs, there is no panpsychic aether.

Indeed, consciousness in higher animals appears as a necessarily stripped-down, greatly simplified, high-speed example of much slower, but far more general, processes – like immune function and gene expression – that entertain multiple, simultaneous tunable spotlight 'global workspaces'. All such have emerged through evolutionary exaptations of the inevitable crosstalk afflicting information processes through a kind of 'second law' leakage necessarily associated with information as a form of free energy.

This perspective represents a fundamental reorientation in consciousness studies, stripping the subject of various deep, culturally-driven, social constructs.

In sum, without identifying consciousness as a weird, new, form of matter, without mind/body dualism, without the *ignis fatuus* of the 'hard problem', 'qualia', and like conceits, the asymptotic limit theorems of information and control theories permit construction of models recognizably similar to the empirical pictures Bernard Baars and others have drawn of high level mental phenomena. That being said, we are constrained by the warning of the mathematical ecologist E.C. Pielou (1977), that the purpose of mathematical models is new speculation, not new knowledge, which can only arise from observation and experiment.

Most particularly, then, the probability models outlined here should be seen as analogs to such 'models' as the Central Limit and associated asymptotic theorems that are the foundations of statistical tools including t-tests, regression equations, and so on. Such tools, among other uses, provide important benchmarks against which to compare experimental and observational results, and new knowledge is as likely to come from their failures as from their successes.

Following Wallace (2017, Section 7.7), we speculate further that such tools might well aid in the understanding and treatment of the many pathologies afflicting cognitive process at and across the various scales and levels of organization that characterize the living state.

7 Mathematical Appendix

Groupoids

We following Brown (1992). Consider a directed line segment in one component, written as the source on the left and the target on the right.

$$\bullet \longrightarrow \bullet$$

Two such arrows can be composed to give a product **ab** if and only if the target of **a** is the same as the source of **b**

$$\bullet \xrightarrow{\mathbf{a}} \bullet \xrightarrow{\mathbf{b}} \bullet$$

Brown puts it this way,

> One imposes the geometrically obvious notions of associativity, left and right identities, and inverses. Thus a groupoid is often

> thought of as a group with many identities, and the reason why this is possible is that the product **ab** is not always defined.
>
> We now know that this apparently anodyne relaxation of the rules has profound consequences... [since] the algebraic structure of product is here linked to a geometric structure, namely that of arrows with source and target, which mathematicians call a *directed graph*.

Cayron (2006) elaborates this:

> A group defines a structure of actions without explicitly presenting the objects on which these actions are applied. Indeed, the actions of the group G applied to the identity element e implicitly define the objects of the set G by ge = g; in other terms, in a group, actions and objects are two isomorphic entities. A groupoid enlarges the notion of group by explicitly introducing, in addition to the actions, the objects on which the actions are applied. By this approach, many identities may exist (they correspond to the actions that leave an object invariant).

Stewart (2007) describes something of the underlying mechanics by which symmetry changes in general may be expressed:

> Spontaneous symmetry-breaking is a common mechanism for pattern formation in many areas of science. It occurs in a symmetric dynamical system when a solution of the equations has a smaller symmetry group than the equations themselves... This typically happens when a fully symmetric solution becomes unstable and branches with less symmetry bifurcate.

It is of particular importance that equivalence class decompositions permit construction of groupoids in a highly natural manner.

Weinstein (1996) and Golubitsky and Stewart (2006) provide more details on groupoids and on the relation between groupoids and bifurcations.

An essential point is that, since there are no necessary products between groupoid elements, 'orbits', in the usual sense, disjointly partition groupoids into 'transitive' subcomponents.

The Data Rate Theorem

Real-world environments are inherently unstable. Organisms (and organizations), to survive, must exert a considerable measure of control over them. These control efforts range from immediate responses to changing patterns of threat and affordance, through niche construction, and, in higher animals, elaborate, highly persistent, social and sociocultural structures. Such necessity of control can, in some measure, be represented by a powerful asymptotic limit theorem of probability theory different from, but as fundamental as, the Central Limit

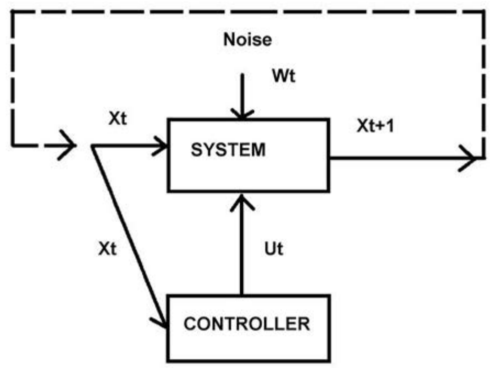

Figure 11: The reduced model of an inherently unstable system stabilized by a control signal U_t.

Theorem. It is called the Data Rate Theorem, first derived as an extension of the Bode Integral Theorem of signal theory.

Consider a reduced model of a control system as follows:

For the inherently unstable system of figure 11, assume an initial n-dimensional vector of system parameters at time t, as x_t. The system state at time $t+1$ is then – near a presumed nonequilibrium steady state – determined by the first-order relation

$$x_{t+1} = \mathbf{A}x_t + \mathbf{B}u_t + W_t \tag{37}$$

In this approximation, $\mathbf{A}$ and $\mathbf{B}$ are taken as fixed n-dimensional square matrices. u_t is a vector of control information, and W_t is an n-dimensional vector of Brownian white noise.

According to the DRT, if H is a rate of control information sufficient to stabilize an inherently unstable control system, then it must be greater than a minimum, H_0,

$$H > H_0 \equiv \log[|\det[\mathbf{A}^m]|] \tag{38}$$

where det is the determinant of the subcomponent $\mathbf{A}^m$ – with $m \leq n$ – of the matrix $\mathbf{A}$ having eigenvalues ≥ 1. H_0 is defined as the rate at which the unstable system generates 'topological information' on its own.

If this inequality is violated, stability fails.

A somewhat different derivation of the DRT, via the 'convexity' inherent to the Rate Distortion function, is also possible (e.g., Wallace 2020a, Section 14.3, 2020b, Section 5).

Morse Theory

Morse theory examines relations between analytic behavior of a function – the location and character of its critical points – and the underlying topology of the manifold on which the function is defined. We are interested in a number of such functions, for example a 'free energy' constructed from information source uncertainties on a parameter space and 'second order' iterations involving parameter manifolds determining critical behavior. These can be reformulated from a Morse theory perspective. Here we follow closely Pettini (2007).

The essential idea of Morse theory is to examine an n-dimensional manifold M as decomposed into level sets of some function $f : M \to \mathbf{R}$ where $\mathbf{R}$ is the set of real numbers. The a-level set of f is defined as

$$f^{-1}(a) = \{x \in M : f(x) = a\},$$

the set of all points in M with $f(x) = a$. If M is compact, then the whole manifold can be decomposed into such slices in a canonical fashion between two limits, defined by the minimum and maximum of f on M. Let the part of M below a be defined as

$$M_a = f^{-1}(-\infty, a] = \{x \in M : f(x) \leq a\}.$$

These sets describe the whole manifold as a varies between the minimum and maximum of f.

Morse functions are defined as a particular set of smooth functions $f : M \to \mathbf{R}$ as follows. Suppose a function f has a critical point x_c, so that the derivative $df(x_c) = 0$, with critical value $f(x_c)$. Then f is a Morse function if its critical points are nondegenerate in the sense that the Hessian matrix of second derivatives at x_c, whose elements, in terms of local coordinates are

$$H_{i,j} = \partial^2 f / \partial x^i \partial x^j,$$

has rank n, which means that it has only nonzero eigenvalues, so that there are no lines or surfaces of critical points and, ultimately, critical points are isolated.

The index of the critical point is the number of negative eigenvalues of H at x_c.

A level set $f^{-1}(a)$ of f is called a critical level if a is a critical value of f, that is, if there is at least one critical point $x_c \in f^{-1}(a)$.

Again following Pettini (2007), the essential results of Morse theory are:

1. If an interval $[a, b]$ contains no critical values of f, then the topology of $f^{-1}[a, v]$ does not change for any $v \in (a, b]$. Importantly, the result is valid even if f is not a Morse function, but only a smooth function.

2. If the interval $[a, b]$ contains critical values, the topology of $f^{-1}[a, v]$ changes in a manner determined by the properties of the matrix H at the critical points.

3. If $f : M \to \mathbf{R}$ is a Morse function, the set of all the critical points of f is a discrete subset of M, i.e. critical points are isolated. This is Sard's Theorem.

4. If $f : M \to \mathbf{R}$ is a Morse function, with M compact, then on a finite interval $[a, b] \subset \mathbf{R}$, there is only a finite number of critical points p of f such that $f(p) \in [a, b]$. The set of critical values of f is a discrete set of $\mathbf{R}$.

5. For any differentiable manifold M, the set of Morse functions on M is an open dense set in the set of real functions of M of differentiability class r for $0 \leq r \leq \infty$.

6. Some topological invariants of M, that is, quantities that are the same for all the manifolds that have the same topology as M, can be estimated and sometimes computed exactly once all the critical points of f are known: Let the Morse numbers $\mu_i (i = 1, ..., m)$ of a function f on M be the number of critical points of f of index i, (the number of negative eigenvalues of H). The Euler characteristic of the complicated manifold M can be expressed as the alternating sum of the Morse numbers of any Morse function on M,

$$\chi = \sum_{i=0}^{m} (-1)^i \mu_i.$$

The Euler characteristic reduces, in the case of a simple polyhedron, to

$$\chi = V - E + F$$

where V, E, and F are the numbers of vertices, edges, and faces in the polyhedron.

7. Another important theorem states that, if the interval $[a, b]$ contains a critical value of f with a single critical point x_c, then the topology of the set M_b defined above differs from that of M_a in a way which is determined by the index, i, of the critical point. Then M_b is homeomorphic to the manifold obtained from attaching to M_a an i-handle, i.e., the direct product of an i-disk and an $(m - i)$-disk.

Again, Pettini (2007) contains both mathematical details and further references. See, for example, Matusmoto (1997).

Higher dimensional resource systems

We assumed that resource delivery is sufficiently characterized by a single scalar parameter Z, mixing material resource/energy supply with internal and external flows of information. Real world conditions will likely be far more complicated. Invoking a perspective analogous to Principal Component Analysis, there may be several independent pure or composite entities irreducibly driving system dynamics. It may then be necessary to replace the scalar Z by an n-dimensional vector $\mathbf{Z}$ having orthogonal components that together account for much of the total variance – in a sense – of the rate of supply of essential resources. The dynamic equations (9) (and/or Eq.(11)) must then be represented in vector form:

$$F(\mathbf{Z}) = -\log\left(h(g(\mathbf{Z}))\right) g(\mathbf{Z})$$

$$
\begin{aligned}
S &= -F + \mathbf{Z} \cdot \nabla_{\mathbf{Z}} F \\
\partial \mathbf{Z}/\partial t &\approx \hat{\mu} \cdot \nabla_{\mathbf{Z}} S = f(\mathbf{Z}) \\
-\nabla_{\mathbf{Z}} F + \nabla_{\mathbf{Z}}(\mathbf{Z} \cdot \nabla_{\mathbf{Z}} F) &= \\
\hat{\mu}^{-1} \cdot f(\mathbf{Z}) &\equiv f^*(\mathbf{Z}) \\
\big((\partial^2 F/\partial z_i \partial z_j)\big) \cdot \mathbf{Z} &= f^*(\mathbf{Z}) \\
\big((\partial^2 F/\partial z_i \partial z_j)\big) |_{\mathbf{Z}_{nss}} \cdot \mathbf{Z_{nss}} &= \mathbf{0}
\end{aligned} \tag{39}
$$

F, g, h, and S are, again, scalar functions, but $\hat{\mu}$ is an n-dimensional square matrix of diffusion coefficients. The matrix $((\partial F/\partial z_i \partial z_j))$ is the obvious n-dimensional square matrix of second partial derivatives, and $f(\mathbf{Z})$ is a vector function. The last relation imposes a nonequilibrium steady state condition, i.e. $f^*(\mathbf{Z}_{nss}) = \mathbf{0}$.

Biological renormalizations

Here, we adapt the renormalization scheme of Wallace (2005), focused on a stationary, ergodic, information source H, to the generalized free energy associated with nonergodic cognition.

Equation (33) states that the information source and the correlation length, the degree of coherence on the underlying network, scale under renormalization clustering in chunks of size R as

$$
F[\omega(R), J(R)] = \mathcal{F}(R) F[\omega(1), J(1)]
$$

$$
\chi[\omega(R), J(R)] R = \chi[\omega(1), J(1)]
$$

with $F(1) = 1$.

Differentiating these two equations with respect to R, so that the right hand sides are zero, and solving for $d\omega(R)/dR$ and $dJ(R)/dR$ gives, after some manipulation,

$$
\begin{aligned}
d\omega_R/dR &= u_1 d\log(\mathcal{F})/dR + u_2/R \\
dJ_R/dR &= v_1 J_R d\log(\mathcal{F})/dR + \frac{v_2}{R} J_R
\end{aligned} \tag{40}
$$

The $u_i, v_i, i = 1, 2$ are functions of $\omega(R), J(R)$, but not explicitly of R itself.

We expand these equations about the critical value $\omega_R = \omega_C$ and about $J_R = 0$, obtaining

$$
\begin{aligned}
d\omega_R/dR &= (\omega_R - \omega_C) y d\log(\mathcal{F})/dR + (\omega_R - \omega_C) z/R \\
dJ_R/dR &= w J_R d\log(\mathcal{F})/dR + x J_R/R
\end{aligned} \tag{41}
$$

The terms $y = du_1/d\omega_R|_{\omega_R = \omega_C}, z = du_2/d\omega_R|_{\omega_R = \omega_C}, w = v_1(\omega_C, 0), x = v_2(\omega_C, 0)$ are constants.

Solving the first of these equations gives

$$
\omega_R = \omega_C + (\omega - \omega_C) R^z \mathcal{F}(R)^y \tag{42}
$$

again remembering that $\omega_1 = \omega, J_1 = J, \mathcal{F}(1) = 1$.

Wilson's (1971) essential trick is to iterate on this relation, which is supposed to converge rapidly near the critical point, assuming that for ω_R near ω_C, we have

$$\omega_C/2 \approx \omega_C + (\omega - \omega_C)R^z \mathcal{F}(R)^y \tag{43}$$

We iterate in two steps, first solving this for $\mathcal{F}(R)$ in terms of known values, and then solving for R, finding a value R_C that we then substitute into the first of equations (33) to obtain an expression for $F[\omega, 0]$ in terms of known functions and parameter values.

The first step gives the general result

$$\mathcal{F}(R_C) \approx \frac{[\omega_C/(\omega_C - \omega)]^{1/y}}{2^{1/y} R_C^{z/y}} \tag{44}$$

Solving this for R_C and substituting into the first expression of equation (31) gives, as a first iteration of a far more general procedure (Shirkov and Kovalev 2001), the result

$$F[\omega, 0] \approx \frac{F[\omega_C/2, 0]}{\mathcal{F}(R_C)} = \frac{F_0}{\mathcal{F}(R_C)}$$
$$\chi(\omega, 0) \approx \chi(\omega_C/2, 0)R_C = \chi_0 R_C \tag{45}$$

giving the essential relationships.

Note that a power law of the form $\mathcal{F}(R) = R^m, m = 3$, which is the direct physical analog, may not be biologically reasonable, since it says that 'language richness', in a general sense, can grow very rapidly as a function of increased network size. Such rapid growth is simply not observed in cognitive process.

Taking the biologically realistic example of non-integral 'fractal' exponential growth,

$$\mathcal{F}(R) = R^\delta \tag{46}$$

where $\delta > 0$ is a real number which may be quite small, equation we can be solve for R_C, obtaining

$$R_C = \frac{[\omega_C/(\omega_C - \omega)]^{[1/(\delta y + z)]}}{2^{1/(\delta y + z)}} \tag{47}$$

for ω near ω_C. Note that, for a given value of y, one might characterize the relation $\alpha \equiv \delta y + z =$ constant as a 'tunable universality class relation' in the sense of Albert and Barabasi (2002).

Substituting this value for R_C back gives a complex expression for F, having three parameters: δ, y, z.

A more biologically interesting choice for $\mathcal{F}(R)$ is a logarithmic curve that 'tops out', for example

$$\mathcal{F}(R) = m \log(R) + 1 \tag{48}$$

Again $\mathcal{F}(1) = 1$.

Using a computer algebra program to solve for R_C gives

$$R_C = [\frac{Q}{W[0, Q\exp(z/my)]}]^{y/z} \tag{49}$$

where

$$Q \equiv (z/my)2^{-1/y}[\omega_C/(\omega_C - \omega)]^{1/y}$$

Again, W(n, x) is the Lambert W-function of order n.

An asymptotic relation for $\mathcal{F}(R)$ would be of particular biological interest, implying that 'language richness' increases to a limiting value with population growth. Taking

$$\mathcal{F}(R) = \exp[m(R-1)/R] \tag{50}$$

gives a system which begins at 1 when $R = 1$, and approaches the asymptotic limit $\exp(m)$ as $R \to \infty$. Computer algebra finds

$$R_C = \frac{my/z}{W[0, A]} \tag{51}$$

where

$$A \equiv (my/z)\exp(my/z)[2^{1/y}[\omega_C/(\omega_C - \omega)]^{-1/y}]^{y/z}$$

These developments suggest the possibility of taking the theory significantly beyond arguments by abduction from simple physical models.

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The author declares there are no competing interests, financial or otherwise.

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