Higher Order Dissipative-Dispersive System and Application

Samer Israwi

► To cite this version:

Samer Israwi. Higher Order Dissipative-Dispersive System and Application. Journal of Partial Differential Equations, 2020, 33 (3), 10.4208/jpde.v33.n3.6 . hal-03168607

HAL Id: hal-03168607

<https://hal.science/hal-03168607>

Submitted on 13 Mar 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Higher Order Dissipative-Dispersive System and Application

ISRAWI Samer*,

*Department of Mathematics, Faculty of Sciences 1 and Laboratory of Mathematics,
Doctoral School of Sciences and Technology, Lebanese University Hadat, Lebanon.*

Received 20 November 2019; Accepted 14 May 2020

Abstract. In this paper, a generalized nonlinear dissipative and dispersive equation with time and space-dependent coefficients is considered. We show that the control of the higher order term is possible by using an adequate weight function to define the energy. The existence and uniqueness of solutions are obtained via a Picard iterative method. As an application to this general Theorem, we prove the well-posedness of the higher order Camassa-Holm type equation as a scalar model which approximates the Euler system with some accuracy for water wave problem.

AMS Subject Classifications: 35C07, 47G30, 35L05, 35A01

Key Words: Cauchy problem, Euler system, Dispersive wave equation, Camassa-Holm equation.

1 Introduction

1.1 Presentation of the problem

In this paper, we study the Cauchy problem for the general nonlinear higher order dissipative-dispersive equation:

$$\left\{ \begin{array}{l} (1 - m\partial_x^2)u_t + a_1(t, x, u)u_x + a_2(t, x, u, u_x)u_{xx} \\ + a_3(t, x, u)u_{xxx} + a_4(t, x)u_{xxxx} + a_5(t, x)u_{xxxxx} = f, \\ \text{for } (t, x) \in (0, T] \times \mathbb{R}, \\ u|_{t=0} = u^0. \end{array} \right. \quad (1.1)$$

*Corresponding author. Email addresses: s.israwi@ul.edu.lb (S. Israwi)

Where $u = u(t, x)$, from $[0, T] \times \mathbb{R}$ into $\mathbb{R}$, is the unknown function of the problem, $m > 0$, a_i , $1 \leq i \leq 5$ and f are real-valued smooth given functions which exact regularities will be pre-cised later. This equation covers several important unidirectional models for the water wave problem at different regimes which take into account the variations of the bottom and the surface tension. We have in view in particular the example of the Camassa-Holm equation which was first derived by Camassa and Holm in [1] (see also [2], [3], [4]), which is more nonlinear than the KdV and BBM equations (see for instance [5] [17], [6, 9], [10], [11], [12], [13], [14], [15], [16]). The presence of the fifth order derivative term is very important, so that the equation describes both nonlinear and dispersive effects as does the Camassa-Holm equation in the case of special tension surface values (see [8, 18] page 230 the Kawahara approximation).

Looking for solutions of (1.1) plays an important and significant role in the study of unidirectional limits for water wave problems with variable depth and topographies. To the best of our knowledge the problem (1.1) has not been analyzed previously. In the present paper, we prove the local well-posedness of the initial value problem (1.1) by a standard Picard iterative scheme and the use of adequate energy estimates under a condition of nondegeneracy of the higher dispersive coefficient a_5 . Therefore we apply this general theorem, to prove the well-posedness of the higher order Camassa-Holm-type equation.

1.2 Notations

In the following, C_0 denotes any nonnegative constant different than zero whose exact expression is of no importance. The notation $a \lesssim b$ means that $a \leq C_0 b$.

We denote by $C(\lambda_1, \lambda_2, \dots)$ a nonnegative constant depending on the parameters $\lambda_1, \lambda_2, \dots$ and whose dependence on the λ_j is always assumed to be nondecreasing.

For any $s \in \mathbb{R}$, we denote $[s]$ the integer part of s .

Let p be any constant with $1 \leq p < \infty$ and denote $L^p = L^p(\mathbb{R})$ the space of all Lebesgue-measurable functions f with the standard norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p} < \infty.$$

The real inner product of any two functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R})$ is denoted by

$$(f_1, f_2) = \int_{\mathbb{R}} f_1(x) f_2(x) dx.$$

The space $L^\infty = L^\infty(\mathbb{R})$ consists of all essentially bounded and Lebesgue-measurable functions f with the norm

$$\|f\|_{L^\infty} = \sup |f(x)| < \infty.$$

We denote by $W^{1,\infty}(\mathbb{R}) = \{f, \text{ s.t. } f, \partial_x f \in L^\infty(\mathbb{R})\}$ endowed with its canonical norm.

For any real constant $s \geq 0$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with the norm $\|f\|_{H^s} = \|\Lambda^s f\|_{L^2} < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1 - \partial_x^2)^{1/2}$. For any two functions $u = u(t, x)$ and $v(t, x)$ defined on $[0, T] \times \mathbb{R}$

with $T > 0$, we denote the inner product, the L^p -norm and especially the L^2 -norm, as well as the Sobolev norm, with respect to the spatial variable x , by $(u, v) = (u(t, \cdot), v(t, \cdot))$, $|u|_{L^p} = |u(t, \cdot)|_{L^p}$, $|u|_{L^2} = |u(t, \cdot)|_{L^2}$ and $|u|_{H^s} = |u(t, \cdot)|_{H^s}$, respectively.

Let E be a given normed space we denote $L^\infty([0, T]; E)$ the space of functions such that $u(t, \cdot)$ is controlled in E , uniformly for $t \in [0, T]$:

$$\|u\|_{L^\infty([0, T]; E)} = \text{ess sup}_{t \in [0, T]} |u(t, \cdot)|_E < \infty.$$

Let E be a given normed space we denote $C([0, T]; E)$ the space of functions such that $u(t, \cdot)$ is controlled in E , uniformly for $t \in [0, T]$:

$$\|u\|_{C([0, T]; E)} = \sup_{t \in [0, T]} |u(t, \cdot)|_E < \infty.$$

Let X be a given space, we denote $C([0, T]; X)$ the space of functions such that $u(t, \cdot)$ is in X .

Finally, $C^k(\mathbb{R}^i)$, $i \geq 1$ denote the space of k -times continuously differentiable functions over $\mathbb{R}^i$.

For any closed operator T defined on a Banach space X of functions, the commutator $[T, f]$ is defined by $[T, f]g = T(fg) - fT(g)$ with f, g and fg belonging to the domain of T . The same notation is used for f as an operator mapping the domain of T into itself.

Actually, we admit without proof this lemma that presents some properties for the commutator operator.

1.3 Product and commutator estimates in Sobolev spaces.

Let us recall here some product as well as commutator estimates in Sobolev spaces, used throughout the present paper (see [18]).

Lemma 1.1 (product estimates).

Let $s \geq 0$, one has $\forall f, g \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$, one has

$$|fg|_{H^s} \lesssim |f|_{L^\infty} |g|_{H^s} + |f|_{H^s} |g|_{L^\infty}.$$

If $s \geq s_0 > 1/2$, one deduces thanks to continuous embedding of Sobolev spaces,

$$|fg|_{H^s} \lesssim |f|_{H^s} |g|_{H^s}.$$

More generally, for $s \geq 0$ and $s_0 > 1/2$, one has $\forall f \in H^s(\mathbb{R}) \cap H^{s_0}(\mathbb{R}), g \in H^s(\mathbb{R})$,

$$|fg|_{H^s} \lesssim |f|_{H^{s_0}} |g|_{H^s} + \left\langle |f|_{H^s} |g|_{H^{s_0}} \right\rangle_{s > s_0},$$

Let $F \in C^\infty(\mathbb{R})$ be a smooth function such that $F(0) = 0$. If $g \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $s \geq 0$, one has $F(g) \in H^s(\mathbb{R})$ and

$$|F(g)|_{H^s} \leq C(|g|_{L^\infty}, F) |g|_{H^s}.$$

We know recall commutator estimate, mainly due to the Kato-Ponce [19], and recently improved by Lannes [18] (see Theorems 3 and 6):

Lemma 1.2 (commutator estimates).

For any $s \geq 0$, and $\partial_x f, g \in L^\infty(\mathbb{R}) \cap H^{s-1}(\mathbb{R})$, one has

$$|[\Lambda^s, f]g|_{L^2} \lesssim |\partial_x f|_{H^{s-1}} |g|_{L^\infty} + |\partial_x f|_{L^\infty} |g|_{H^{s-1}}.$$

Thanks to continuous embedding of Sobolev spaces, one has for $s \geq s_0 + 1$, $s_0 > \frac{1}{2}$,

$$|[\Lambda^s, f]g|_{L^2} \lesssim |\partial_x f|_{H^{s-1}} |g|_{H^{s-1}}.$$

More generally, for any $s \geq 0$ and $s_0 > 1/2$, $\partial_x f, g \in H^{s_0}(\mathbb{R}) \cap H^{s-1}(\mathbb{R})$, one has

$$|[\Lambda^s, f]g|_{L^2} \lesssim |\partial_x f|_{H^{s_0}} |g|_{H^{s-1}} + \left\langle |\partial_x f|_{H^{s-1}} |g|_{H^{s_0}} \right\rangle_{s > s_0 + 1}.$$

We conclude this section with the following remark. Also, let us remark these continuous embedding.

Remark 1.1. Let $s > \frac{3}{2}$, then:

- $H^s(\mathbb{R}) \hookrightarrow W^{1,\infty}(\mathbb{R})$
- $H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$
- $H^s(\mathbb{R}) \hookrightarrow H^{s-1}(\mathbb{R})$.

Moreover, we define the following operators for $s > 0$: $\Lambda_m^s = (1 - m\partial_x^2)^{\frac{s}{2}}$ and its inverse Λ_m^{-s} such that:

$$\widehat{\Lambda_m^{-s}(u)} = (1 + m\xi^2)^{-\frac{s}{2}} \hat{u}.$$

Finally, we will study the local well-posedness of the initial value problem (1.1) in $H^s(\mathbb{R})$ endowed with canonical norm.

1.4 Main results

Let us now state our main result:

Theorem 1.1. Let $s > \frac{5}{2}$ and $f \in C([0, T]; H^s(\mathbb{R}))$. We suppose that:

- a_1, a_2, a_3 are smooth mappings such that a_1, a_3 in $C([0, T], C^{[s]+1}(\mathbb{R}^2))$ and a_2 in $C([0, T], C^{[s]+1}(\mathbb{R}^3))$.
- $a_4 \in C([0, T]; H^{s+1}(\mathbb{R}))$, $\partial_t a_4 \in L^\infty(0, T; L^\infty(\mathbb{R}))$,
- $a_5 \in C([0, T], L^\infty(\mathbb{R}))$, $\partial_x a_5 \in C([0, T]; H^{s+2}(\mathbb{R}))$, with $\partial_t a_5 \in L^\infty(0, T; L^\infty(\mathbb{R}))$,
- $F(t, x) := \int_0^x \frac{a_4}{a_5} dy \in C([0, T]; L^\infty(\mathbb{R}))$ and $\partial_t F \in L^\infty(0, T; L^\infty(\mathbb{R}))$,

Assume moreover that there is a positive constant $c_1 > 0$ such that $c_1 \leq |a_5(t, x)| \forall (t, x) \in [0, T] \times \mathbb{R}$. Then for all $u^0 \in H^s(\mathbb{R})$, there exist a time $T^* > 0$ and a unique solution u to (1.1) in $C([0, T^*]; H^s(\mathbb{R}))$.

Remark 1.2. There is no restriction on the signs of the coefficients a_2 and a_4 ; this means that our result handles also the case of the anti-diffusive terms, in which case these terms are controlled by dispersion.

2 Proof of the Main results

Before we start the proof, we give the following useful lemma:

Lemma 2.1. Let $m > 0, s > 0$ then the linear operator

$\Lambda_m^{-2}: H^{s+2}(\mathbb{R}) \rightarrow H^s(\mathbb{R})$ is well defined, continuous, one-to-one and onto. If we suppose that $u = \Lambda_m^{-2} f$ for $f \in H^s(\mathbb{R})$ then:

$$|u|_{H^{s+2}} \leq \frac{1}{m} |f|_{H^s} \quad \text{if } 0 < m \leq 1 \quad (2.1)$$

$$|u|_{H^{s+2}} \leq |f|_{H^s} \quad \text{if } m \geq 1. \quad (2.2)$$

Moreover,

$$\Lambda^s \Lambda_m^{-2} = \Lambda^{s-2} \Lambda_m^0 = \Lambda_m^0 \Lambda^{s-2},$$

where $\Lambda_m^0: H^s(\mathbb{R}) \rightarrow H^s(\mathbb{R})$ is linear continuous one-to-one and onto operator defined by

$$\widehat{\Lambda_m^0 u}(\xi) = (1 + \xi^2)(1 + m\xi^2)^{-1} \hat{u}(\xi),$$

with

$$|\Lambda_m^0|_{H^s \rightarrow H^s} \leq \max\left(\frac{1}{m}, 1\right), \quad (2.3)$$

$$|(\Lambda_m^0)^{-1}|_{H^s \rightarrow H^s} \leq \max(m, 1). \quad (2.4)$$

Proof. We have $\|\Lambda_m^{-2} f\|_{H^{s+2}} = \|(1 + \xi^2)^{\frac{s}{2}+1} (1 + m\xi^2)^{-1} \hat{f}\|_{L^2}$. If $m \geq 1$, then $1 + m\xi^2 \geq 1 + \xi^2$ and $\frac{1 + \xi^2}{1 + m\xi^2} \leq 1$, therefore

$$\|(1 + \xi^2)^{\frac{s}{2}+1} (1 + m\xi^2)^{-1} \hat{f}\|_{L^2} = \|(1 + \xi^2)^{\frac{s}{2}} (1 + \xi^2)(1 + m\xi^2)^{-1} \hat{f}\|_{L^2} \leq \|(1 + \xi^2)^{\frac{s}{2}} \hat{f}\|_{L^2}.$$

If $0 < m < 1$, we have $\frac{1 + \xi^2}{1 + m\xi^2} = 1 + (1 - m) \frac{\xi^2}{1 + m\xi^2} \leq 1 + \frac{(1 - m)}{m} = \frac{1}{m}$, then

$$\|\Lambda_m^{-2} f\|_{H^{s+2}} \leq \frac{1}{m} \|f\|_{H^s}.$$

Now we have

$$\|\Lambda_m^0 f\|_{H^s} = \|\Lambda^2 \Lambda_m^{-2} f\|_{H^s} = \|\Lambda_m^{-2} f\|_{H^{s+2}} \leq \max(1, \frac{1}{m}) \|f\|_{H^s}.$$

and

$$\|(\Lambda_m^0)^{-1} f\|_{H^s} = \|(1+m\zeta^2)(1+\zeta^2)^{-1}(1+\zeta^2)^{\frac{s}{2}} \hat{f}\|_{L^2}.$$

If $m \geq 1$, then $(1+m\zeta^2)(1+\zeta^2)^{-1} = 1 + (m-1)\frac{\zeta^2}{1+\zeta^2} \leq m$, therefore

$$\|(\Lambda_m^0)^{-1} f\|_{H^s} \leq m \|f\|_{H^s}.$$

If $0 < m < 1$, $(1+m\zeta^2)(1+\zeta^2)^{-1} \leq 1$, then

$$\|(\Lambda_m^0)^{-1} f\|_{H^s} \leq \|f\|_{H^s}.$$

Finally $\|(\Lambda_m^0)^{-1} f\|_{H^s} \leq \max(1, m) \|f\|_{H^s}$.

We will start the proof of Theorem 1.1 by studying a linearized problem associated to (1.1).

2.1 Linear analysis:

For any smooth enough v , we define the “linearized” operator:

$$\mathcal{L}(v, \partial) = \Lambda_m^2 \partial_t + a_1(t, x, v) \partial_x + a_2(t, x, v, v_x) \partial_x^2 + a_3(t, x, v) \partial_x^3 + a_4(t, x) \partial_x^4 + a_5(t, x) \partial_x^5.$$

and the following initial value problem:

$$\begin{cases} \mathcal{L}(v, \partial) u = f, \\ u|_{t=0} = u^0. \end{cases} \quad (2.5)$$

Equation (2.5) is a linear equation which can be solved by a standard method (see [20]) in any time interval in which its coefficients are defined and regular enough. We first establish some precise energy-type estimates of the solution. We define the “energy” norm,

$$E^s(u)^2 = |w \Lambda^s u|_{L^2}^2,$$

where w is a weight function that will be chosen later. For the moment, we just require that there exists two positive numbers w_1, w_2 such that for all (t, x) in $(0, T] \times \mathbb{R}$,

$$w_1 \leq w(t, x) \leq w_2,$$

so that $E^s(u)$ is uniformly equivalent to the standard H^s -norm. Differentiating $\frac{1}{2} e^{-\lambda t} E^s(u)^2$ with respect to time, one gets using (2.5)

$$\frac{1}{2} e^{\lambda t} \partial_t (e^{-\lambda t} E^s(u)^2) = -\frac{\lambda}{2} E^s(u)^2 - (\Lambda_m^0 \Lambda^{s-2} (a_1 u_x), w^2 \Lambda^s u)$$

$$\begin{aligned}
& -(\Lambda_m^0 \Lambda^{s-2}(a_2 u_{xx}), w^2 \Lambda^s u) - (\Lambda_m^0 \Lambda^{s-2}(a_3 u_{xxx}), w^2 \Lambda^s u) \\
& -(\Lambda_m^0 \Lambda^{s-2}(a_4 u_{xxxx}), w^2 \Lambda^s u) - (\Lambda_m^0 \Lambda^{s-2}(a_5 u_{xxxxx}), w^2 \Lambda^s u) \\
& + (\Lambda_m^0 \Lambda^{s-2} f, w^2 \Lambda^s u) + (w w_t \Lambda^s u, \Lambda^s u).
\end{aligned}$$

We now turn to estimating the different terms of the r.h.s of the previous identity by using the needed estimates provided from section 1.3

• Estimate of $(\Lambda^{s-2}(a_1 u_x), \Lambda_m^0 w^2 \Lambda^s u)$. By the Cauchy-Schwarz inequality and the section 1.3 on the composite functions we have

$$\begin{aligned}
|(\Lambda^{s-2}(a_1 u_x), \Lambda_m^0 w^2 \Lambda^s u)| & \leq \frac{1}{m} |a_1(t, x, v) - a_1(t, x, 0)|_{H^{s-2}} |a_1(t, x, 0) u_x|_{H^{s-1}} |w^2 \Lambda^s u|_{L^2} \\
& \leq C(m^{-1}, a_1, \|v\|_{H^s}, |w|_{L^\infty}) E^s(u)^2.
\end{aligned}$$

• Estimate of $(\Lambda^{s-2}(a_2 u_{xx}), \Lambda_m^0 w^2 \Lambda^s u)$. Similarly as the above estimation, we have

$$|(\Lambda^{s-2}(a_2 u_{xx}), \Lambda_m^0 w^2 \Lambda^s u)| \leq C(m^{-1}, a_2, \|v\|_{H^s}, |w|_{L^\infty}) E^s(u)^2.$$

• Estimate of $(\Lambda^{s-2}(a_3 u_{xxx}), \Lambda_m^0 w^2 \Lambda^s u)$. Since we have more than s derivative on u , we remark that one can write:

$$a_3 u_{xxx} = \partial_x^2(a_3 \partial_x u) - \partial_x^2 a_3 \partial_x u - 2a_3 \partial_x^2 u,$$

then

$$\Lambda^{s-2}(a_3 u_{xxx}) = \Lambda^{s-2}(\partial_x^2(a_3 \partial_x u)) - \Lambda^{s-2}(\partial_x^2 a_3 \partial_x u) - 2\Lambda^{s-2}(\partial_x a_3 \partial_x^2 u).$$

Now use the identity $\Lambda^2 = 1 - \partial_x^2$ to get that

$$\Lambda^{s-2}(\partial_x^2(a_3 \partial_x u)) = \Lambda^{s-2}((1 - \Lambda^2)(a_3 \partial_x u)) = \Lambda^{s-2}(a_3 \partial_x u) - \Lambda^s(a_3 \partial_x u) = \Lambda^{s-2}(a_3 \partial_x u) - [\Lambda^s, a_3] \partial_x u - a_3 \Lambda^s \partial_x u,$$

then we obtain:

$$\begin{aligned}
(\Lambda^{s-2}(a_3 u_{xxx}), \Lambda_m^0 w^2 \Lambda^s u) & = (\Lambda^{s-2}(a_3 \partial_x u), \Lambda_m^0 w^2 \Lambda^s u) - ([\Lambda^s, a_3] \partial_x u, \Lambda_m^0 w^2 \Lambda^s u) \\
& - (a_3 \Lambda^s \partial_x u, \Lambda_m^0 w^2 \Lambda^s u) - (\Lambda^{s-2}(\partial_x^2 a_3 \partial_x u), \Lambda_m^0 w^2 \Lambda^s u) - 2(\Lambda^{s-2}(\partial_x a_3 \partial_x^2 u), \Lambda_m^0 w^2 \Lambda^s u).
\end{aligned}$$

By integration by parts, the third term of the last equality becomes:

$$(a_3 \Lambda^s \partial_x u, \Lambda_m^0 w^2 \Lambda^s u) = -\frac{1}{2}(\partial_x(\Lambda_m^0 w^2 a_3), (\Lambda^s u)^2),$$

Now by Cauchy Schwarz we have:

$$\begin{aligned}
|(\Lambda^{s-2}(a_3 u_{xxx}), \Lambda_m^0 w^2 \Lambda^s u)| &\leq \frac{1}{m} (\|a_3 \partial_x u\|_{H^{s-2}} E^s(u) + \|\partial_x a_3\|_{H^{s-1}} \|\partial_x u\|_{H^{s-1}} E^s(u) \\
&\quad + \|w^2 a_3\|_{W^{1,\infty}} E^s(u)^2 \\
&\quad + \|\partial_x^2 a_3 \partial_x u\|_{H^{s-2}} E^s(u) + \|a_3 \partial_x^2 u\|_{H^{s-2}} E^s(u)) \\
&\leq C(m^{-1}, a_3, \|v\|_{H^s}, \|w\|_{W^{1,\infty}}) E^s(u)^2.
\end{aligned}$$

• Estimate of $([\Lambda^{s-2}, a_4] \partial_x^4 u, \Lambda_m^0 w^2 \Lambda^s u) + (a_4 \Lambda^{s-2} \partial_x^4 u, \Lambda_m^0 w^2 \Lambda^s u)$:

$$a_4 \Lambda^{s-2} \partial_x^4 u = a_4 \Lambda^{s-2} (1 - \Lambda^2) \partial_x^2 u = a_4 (\Lambda^{s-2} - \Lambda^s) \partial_x^2 u = a_4 \Lambda^{s-2} \partial_x^2 u - a_4 \Lambda^s \partial_x^2 u,$$

then:

$$(a_4 \Lambda^{s-2} \partial_x^4 u, \Lambda_m^0 w^2 \Lambda^s u) = (a_4 \Lambda^{s-2} \partial_x^2 u, \Lambda_m^0 w^2 \Lambda^s u) - (a_4 \Lambda^s \partial_x^2 u, \Lambda_m^0 w^2 \Lambda^s u)$$

By Cauchy Schwarz, the first term of the last equality is controlled by:

$$|(a_4 \Lambda^{s-2} \partial_x^2 u, \Lambda_m^0 w^2 \Lambda^s u)| \leq \frac{1}{m} |a_4 \Lambda^{s-2} \partial_x^2 u|_{L^2} E^s(u) \leq C(m^{-1}, |a_4|_{L^\infty}) E^s(u)^2.$$

$$(a_4 \Lambda^s \partial_x^2 u, \Lambda_m^0 w^2 \Lambda^s u) = -(a_4 \Lambda_m^0 w^2, (\partial_x \Lambda^s u)^2) + Q_1, \text{ where}$$

$$|Q_1| \leq C(m, s, |w|_{W^{1,\infty}}, |\partial_x a_4|_{L^\infty}) E^s(u)^2.$$

Now, using the first order Poisson brackets : (see [21] for more details)

$$\{\Lambda^{s-2}, a_4\}_1 = -(s-2) \partial_x (a_4) \Lambda^{s-2} \partial_x,$$

we get:

$$([\Lambda^{s-2}, a_4] \partial_x^4 u, \Lambda_m^0 w^2 \Lambda^s u) = (s-2) (\partial_x (a_4) \Lambda^s \partial_x u, \Lambda_m^0 w^2 \Lambda^s u) + Q_2,$$

Where

$$|Q_2| \leq C(m, s, |w|_{W^{2,\infty}}, |a_4|_{H^{s+1}}) E^s(u)^2.$$

Now, by integration by parts we have:

$$(s-2) (\partial_x (a_4) \Lambda^s \partial_x u, \Lambda_m^0 w^2 \Lambda^s u) = -\frac{(s-2)}{2} (\partial_x (\partial_x (a_4) \Lambda_m^0 w^2) \Lambda^s u, \Lambda^s u),$$

then

$$|([\Lambda^{s-2}, a_4] \partial_x^4 u, \Lambda_m^0 w^2 \Lambda^s u)| \leq C(m, s, |w|_{W^{2,\infty}}, |a_4|_{H^{s+1}}) E^s(u)^2.$$

• Estimate of $([\Lambda^{s-2}, a_5] \partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u) + (a_5 \Lambda^{s-2} \partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u)$:

$$a_5 \Lambda^{s-2} \partial_x^5 u = a_5 \Lambda^{s-2} (1 - \Lambda^2) \partial_x^3 u = a_5 \Lambda^{s-2} \partial_x^3 u - a_5 \Lambda^s \partial_x^3 u = a_5 \Lambda^{s-2} \partial_x u - a_5 \Lambda^s \partial_x u - a_5 \Lambda^s \partial_x^3 u.$$

Therefore,

$$(a_5 \Lambda^{s-2} \partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u) = (a_5 \Lambda^{s-2} \partial_x u, \Lambda_m^0 w^2 \Lambda^s u) - (a_5 \Lambda^s \partial_x u, \Lambda_m^0 w^2 \Lambda^s u) - (a_5 \Lambda^s \partial_x^3 u, \Lambda_m^0 w^2 \Lambda^s u).$$

The first two terms can be easily controlled by $E^s(u)^2$ as above. Now,

$$\begin{aligned} (a_5 \partial_x^3 \Lambda^s u, \Lambda_m^0 w^2 \Lambda^s u) &= -\frac{1}{2} (\partial_x^3 (a_5 \Lambda_m^0 w^2) \Lambda^s u, \Lambda^s u) - \frac{3}{2} (\partial_x^2 (w^2 \Lambda_m^0 a_5) \Lambda^s \partial_x u, \Lambda^s u) \\ &\quad - \frac{3}{2} (\partial_x (\Lambda_m^0 w^2 a_5) \Lambda^s u, \Lambda^s \partial_x^2 u). \end{aligned}$$

By integration by parts, we obtain

$$-\frac{3}{2} (\partial_x (\Lambda_m^0 w^2 a_5) \Lambda^s u, \Lambda^s \partial_x^2 u) = \frac{3}{2} (\partial_x^2 (\Lambda_m^0 w^2 a_5) \Lambda^s u, \Lambda^s \partial_x u) + \frac{3}{2} (\partial_x (a_5 \Lambda_m^0 w^2), (\Lambda^s \partial_x u)^2).$$

Now:

$$[\Lambda^{s-2}, a_5] \partial_x^5 u = \{\Lambda^{s-2}, a_5\}_2 \partial_x^5 u + Q_3 \partial_x^5 u,$$

where $\{\cdot, \cdot\}_2$ stands for the second order Poisson brackets,

$$\{\Lambda^{s-2}, a_5\}_2 = -(s-2) \partial_x (a_5) \Lambda^{s-4} \partial_x + \frac{1}{2} [(s-2) \partial_x^2 (a_5) \Lambda^{s-4} - (s-4)(s-2) \partial_x^2 (a_5) \Lambda^{s-6} \partial_x^2]$$

and Q_3 is an operator of order $s-5$ that can be controlled by the general commutator estimates (see [21]). We thus get

$$|(Q_3 \partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u)| \leq C(m, |\partial_x a_5|_{H^{s+1}}) E^s(u)^2.$$

We now use the fact that $H^1(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$ to get

$$\begin{aligned} &|([s \partial_x^2 (a_5) \Lambda^{s-4} - (s-4)(s-2) \partial_x^2 (a_5) \Lambda^{s-6} \partial_x^2] \partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u)| \\ &\leq C(m, s, |\partial_x a_5|_{H^{s+1}}, |w|_{W^{1,\infty}}) E^s(u)^2. \end{aligned}$$

This leads to the expression

$$([\Lambda^{s-2}, a_5] \partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u) = -(s-2) (\partial_x (a_5) \Lambda^s \partial_x^2 u, \Lambda_m^0 w^2 \Lambda^s u) + Q_4,$$

where $|Q_4| \leq C(m, s, |w|_{W^{1,\infty}}, |a_5|_{H^{s+1}}) E^s(u)^2$. Remarking now, by integration by parts

$$\begin{aligned} -(s-2) (\partial_x (a_5) \Lambda^s \partial_x^2 u, \Lambda_m^0 w^2 \Lambda^s u) &= (s-2) (\partial_x (\partial_x (a_5) \Lambda_m^0 w^2) \Lambda^s \partial_x u, \Lambda^s u) \\ &\quad + (s-2) (\partial_x (a_3) \Lambda_m^0 w^2, (\Lambda^s \partial_x u)^2). \end{aligned} \quad (2.6)$$

We now choose w such that

$$-(s-2)(\partial_x(a_5)\Lambda_m^0 w^2, (\Lambda^s \partial_x u)^2) + \frac{3}{2}(\partial_x(a_5\Lambda_m^0 w^2), (\Lambda^s \partial_x u)^2) + (a_4\Lambda_m^0 w^2, (\partial_x \Lambda^s u)^2) = 0; \quad (2.7)$$

therefore, if we take $w = (\Lambda_m^0)^{-1} \left(|a_5|^{\left(\frac{2s-7}{6}\right)} \exp\left(-\frac{1}{3} \int_0^x \frac{a_4}{a_5} dy\right) \right)$ we easily obtain (2.7). Finally, one has

$$\begin{aligned} & ([\Lambda^{s-2}, a_5] \partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u) + (a_5 \partial_x^5 \Lambda^{s-2} u, \Lambda_m^0 w^2 \Lambda^s u) \\ &= Q_4 + (s-2)(\partial_x(\partial_x(a_5)\Lambda_m^0 w^2) \Lambda^s \partial_x u, \Lambda^s u) - \frac{1}{2}(\partial_x^3(a_5\Lambda_m^0 w^2) \Lambda^s u, \Lambda^s u) \\ &\quad - \frac{3}{2}(\partial_x^2(a_5\Lambda_m^0 w^2) \Lambda^s \partial_x u, \Lambda^s u) + \frac{3}{2}(\partial_x^2(a_5\Lambda_m^0 w^2) \Lambda^s \partial_x u, \Lambda^s u); \end{aligned}$$

therefore,

$$|([\Lambda^{s-2}, a_5] \partial_x^5 u, \Lambda_m^0 w^2 \Lambda^s u) + (a_5 \partial_x^5 \Lambda^{s-2} u, \Lambda_m^0 w^2 \Lambda^s u)| \leq C(s, m, |\partial_x a_5|_{H^{s+1}}) E^s(u)^2.$$

• Estimate of $(w_t \Lambda^{s-2} u, \Lambda_m^0 w \Lambda^s u)$: Using the Cauchy-Schwarz inequality we obtain

$$|(w_t \Lambda^s u, w \Lambda^s u)| \leq C(m, |w_t|_{L^\infty}, |w|_{L^\infty}) E^s(u)^2.$$

Gathering the information provided by the above estimates, since one has

$$|(\Lambda^{s-2} f, \Lambda_m^0 w^2 \Lambda^s u)| \leq \frac{1}{m} E^s(f) E^s(u).$$

If we assemble the previous estimates and using Gronwall's lemma we obtain the following estimate:

$$e^{\lambda t} \partial_t (e^{-\lambda t} E^s(u)^2) \leq (C(E^s(v)) - \lambda) E^s(u)^2 + 2E^s(f) E^s(u).$$

Taking $\lambda = \lambda_T$ large enough (how large depends on $\sup_{t \in [0, T]} C(E^s(v(t)))$ for the first term of the right hand side of the above inequality to be negative for all $t \in [0, T]$, we deduce that

$$E^s(u(t)) \leq e^{\lambda_T t} E^s(u^0) + 2 \int_0^t e^{\lambda_T(t-t')} E^s(f(t')) dt'.$$

2.2 Proof of the theorem:

Thanks to this energy estimate, we classically conclude (see e.g. [22]) the existence of a time

$$T^* = T^*(E^s(u^0)) > 0,$$

and a unique solution $u \in C([0, T^*]; H^s(\mathbb{R})) \cap C^1([0, T^*]; H^{s-3}(\mathbb{R}))$ to (1.1) as the limit of the iterative scheme

$$u_0 = u^0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \begin{cases} \mathcal{L}(u^n, \partial) u^{n+1} = f, \\ u^n|_{t=0} = u^0. \end{cases}$$

Corollary 2.1. Under the same conditions of the Theorem 1.1 with $f = a_0(t, x)u$ and $a_0 \in L^\infty(0, T, H^{s-2}(\mathbb{R}))$ one gets the same result.

Proof. One has by using the Cauchy-Schwarz inequality and the fact that $H^{s-2}(\mathbb{R})$ is an algebra space ($s-2 > \frac{1}{2}$), the following inequality

$$|(\Lambda^{s-2}(a_0 u), \Lambda_m^0 w^2 \Lambda^s u)| \leq \frac{1}{m} |a_0|_{H^{s-2}} E^s(u)^2.$$

Then the result is done. $\square$

3 Application to a scalar water wave equation.

3.1 Reformulation of the Euler model

Consider a domain Ω_t occupied by an irrotational, incompressible fluid. Let Γ_t be the free surface and Σ be the bottom, that is

$$\Omega_t = \{(x, y) \in \mathbb{R}^2; b(x) < y < h_0 + \zeta(x, t)\},$$

$$\Gamma_t = \{(x, y) \in \mathbb{R}^2; y = h_0 + \zeta(x, t)\},$$

$$\Sigma = \{(x, y) \in \mathbb{R}^2; y = b(x)\}.$$

where h_0 is the mean depth of the fluid. Here, b is a given function, ζ is the unknown wave profile, $v = (v_1, v_2)$ is the velocity and p is the pressure. By Euler, we have the following equations

$$\begin{cases} \rho(v_t + (v \cdot \nabla)v) + \nabla p = -\rho(0, g), \\ \nabla \cdot v = 0, \nabla^\perp \cdot v = 0 \quad \text{in } \Omega_t, \end{cases}$$

where ρ is the constant density and g is the gravitational constant and $\nabla^\perp = (-\partial_y, \partial_x)$. The boundary conditions on the free surface are given by

$$p = p_0 - \sigma H, (\partial_t + v \cdot \nabla)(y - \zeta(x, t)) = 0 \quad \text{in } \Gamma_t,$$

where p_0 is the atmospheric pressure, σ is the surface tension and H is the curvature of the free surface. It is assumed that p_0 is a constant and σ is a positive constant. In our parametrization of the free surface the curvature H at the point $(x, h_0 + \zeta(x, t))$ is written as

$$H(x, t) = ((1 + (\zeta_x(x, t))^2)^{-1/2} \zeta_{xx}(x, t))_x.$$

The boundary condition at the bottom is given by

$$v \cdot n_- = 0,$$

where n_- is a normal vector to Σ . We introduce the non-dimensional parameters

$$\delta = \frac{h_0}{\lambda}, \quad \epsilon = \frac{a}{h_0}, \quad \beta = \frac{b_0}{h_0} \quad \text{and} \quad \alpha = \frac{\lambda}{\lambda_b},$$

where λ and λ_b are respectively the typical wave lengths of the free surface and the bottom, a is the typical amplitude of the free surface and b_0 is the order of amplitude of the variations of the bottom topography. We consider here the asymptotic behavior

$$\delta^4 = \epsilon.$$

Making the scaling

$$\begin{cases} x = \lambda \tilde{x}, & y = h_0 \tilde{y}, & t = \frac{\lambda}{\sqrt{gh}} \tilde{t}, & v_1 = \frac{a}{h} \sqrt{gh_0} \tilde{v}_1, \\ v_2 = \frac{a}{h_0} \sqrt{gh_0} \tilde{v}_2, & p = \tilde{p}_0 + \rho gh_0 \tilde{p}, & \zeta = a \tilde{\zeta}, & b = b_0 \tilde{b} \end{cases}$$

By dropping the tilde sign, Euler system and the boundary conditions on the free surface becomes

$$\begin{cases} \epsilon v_{1t} + \epsilon^2 (v_1 v_{1x} + v_2 v_{1y}) + p_x = 0 & \text{in } \Omega_t^\epsilon, \\ \epsilon^{3/2} v_{2t} + \epsilon^{5/2} (v_1 v_{2x} + v_2 v_{2y}) + p_y + 1 = 0 & \text{in } \Omega_t^\epsilon, \\ v_{1x} + v_{2y} = 0 & \text{in } \Omega_t^\epsilon, \\ v_{1y} - \epsilon^{1/2} v_{2x} = 0 & \text{in } \Omega_t^\epsilon, \\ p = -\frac{\sigma}{\rho gh} \epsilon^{3/2} [(1 + \epsilon^{5/2} \zeta_x^2)^{-1/2} \zeta_x]_x & \text{in } \Gamma_t^\epsilon, \\ \eta_t + \epsilon v_1 \zeta_x - v_2 = 0 & \text{in } \Gamma_t^\epsilon. \end{cases}$$

Where,

$$\Omega_t^\epsilon = \{(x, y) \in \mathbb{R}^2; \beta b(x) < y < 1 + \epsilon \zeta(x, t)\},$$

and

$$\Gamma_t^\epsilon = \{(x, y) \in \mathbb{R}^2; y = 1 + \epsilon \zeta(x, t)\}.$$

The boundary condition at the bottom becomes (by dropping the tilde sign)

$$b_0 \frac{1}{\lambda} b' \frac{a}{h} \sqrt{gh_0} v_1 - \frac{a}{\lambda} \sqrt{gh_0} v_2 = 0, \quad \text{in } \Sigma^\beta$$

where b' is the derivative function of the bottom function b and

$$\Sigma^\beta = \{(x, y) \in \mathbb{R}^2; y = \beta b(x)\}.$$

Then, we get

$$\beta b' v_1 - v_2 = 0, \quad \text{in } \Sigma^\beta.$$

Let us consider the following scaling $b(x) = b^{(\alpha)}(\alpha x)$. we obtain then,

$$\beta \alpha \partial_x b^{(\alpha)}(\alpha \cdot) v_1 - v_2 = 0, \quad \text{in } \Sigma^\beta.$$

We denote $u(x, t) = v(x, 1 + \epsilon \zeta(x, t), t)$. Reformulating the above equations as a problem on the above surface, we obtain

$$\begin{cases} u_{1t} + \zeta_x + \epsilon u_1 u_{1x} + \epsilon^{3/2} \zeta_x (u_{2t} + \epsilon u_2 u_{2x}) = \mu \epsilon^{1/2} [(1 + \epsilon^{5/2} \zeta_x^2)^{-1/2} \zeta_x]_{xx} & \text{in } \Gamma_t^\epsilon \\ \zeta_t + \epsilon u_1 \zeta_x - u_2 = 0 & \text{in } \Gamma_t^\epsilon, \end{cases} \quad (3.1)$$

where $\mu = \frac{\sigma}{\rho g h}$ and u_2 is determined by solving the following elliptic problem

$$\begin{cases} v_{1x} + v_{2y} = 0; v_{1y} - \epsilon^{1/2} v_{2x} = 0 & \text{in } \Omega_t^\epsilon, \\ v_1 = u_1 & \text{in } \Gamma^\epsilon, \\ \beta \alpha \partial_x b^{(\alpha)} v_1 - v_2 = 0, & \text{in } \Sigma^\beta. \end{cases} \quad (3.2)$$

From now on and for the sake of simplicity, we denote $b^{(\alpha)}$ by b and we always assume that there exists $a^* > 0$ such that

$$a = 1 - \beta b \geq a^*,$$

and we assume also that $\beta \alpha = O(\epsilon)$ to get

$$a_x = O(\beta \alpha) = O(\epsilon).$$

3.2 Regularized CH-type equation

The Camassa-Holm equation was first derived by using asymptotic expansions directly in the Hamiltonian for Euler's equations for inviscid incompressible flow in the shallow water regime. It was thereby shown to be bi-Hamiltonian and integrable by the inverse scattering transform in the work of Camassa and Holm [1]. We use here the BBM trick (from Benjamin-Bona-Mahony [5]) to get a regularized Camassa-Holm type equation in the hope that helps us in proving its well-posedness by using our main Theorem in this article, we consider the higher order CH-type equation (3.3) derived from the systems (3.1) and (3.2) (see [23, 24] for the details of derivation) as a scalar model which approximates the above Euler system with accuracy $O(\epsilon^{\frac{3}{2}})$, the point is the following: first of all

we derive an asymptotic model of the above Euler system like a fifth order Boussinesq system on (ζ, u_1) and then we split it into two scalar models on ζ and u_1 similar to the following equation up to $O(\epsilon^{\frac{3}{2}})$

$$u_t + \sqrt{a}u_x + \frac{1}{4} \frac{a_x}{\sqrt{a}}u + \frac{3}{2}\epsilon \frac{uu_x}{\sqrt{a}} + \epsilon^{1/2} \frac{a^2\sqrt{a}}{2}u_{xxx} \left[\frac{1}{3} - \frac{\mu}{a^2} \right] + \epsilon a_5^* u_{xxxxx} = 0, \quad (3.3)$$

where

$$a_5^* = \frac{a^4\sqrt{a}}{2} \left(\frac{19}{180} - \frac{\mu}{6a^2} - \frac{\mu^2}{4a^4} \right).$$

Remark that

$$u_t = -\sqrt{a}u_x - \epsilon^{1/2} \frac{a^2\sqrt{a}}{2} \left[\frac{1}{3} - \frac{\mu}{a^2} \right] u_{xxx} + O(\epsilon),$$

Then, by differentiating twice with respect to x , we get

$$u_{txx} = -\sqrt{a}u_{xxx} - \epsilon^{1/2} \frac{a^2\sqrt{a}}{2} \left[\frac{1}{3} - \frac{\mu}{a^2} \right] u_{xxxxx} + O(\epsilon),$$

Now, equation (3.3) becomes

$$\begin{aligned} u_t - \epsilon^{1/2} u_{txx} + \sqrt{a}u_x + \frac{1}{4} \frac{a_x}{\sqrt{a}}u + \frac{3}{2}\epsilon \frac{uu_x}{\sqrt{a}} + \epsilon^{1/2} \frac{a^2\sqrt{a}}{2}u_{xxx} \left[\frac{1}{3} - \frac{\mu}{a^2} \right] \\ - \epsilon^{1/2} \sqrt{a}u_{xxx} + \epsilon \left(-\frac{a^2\sqrt{a}}{2} \left[\frac{1}{3} - \frac{\mu}{a^2} \right] + a_5^* \right) u_{xxxxx} = O(\epsilon^{3/2}). \end{aligned}$$

Let's take now the following higher order CH-type equation:

$$\begin{aligned} (1 - \epsilon^{1/2} \partial_x^2) u_t + \sqrt{a}u_x + \frac{1}{4} \frac{a_x}{\sqrt{a}}u + \frac{3}{2}\epsilon \frac{uu_x}{\sqrt{a}} + \epsilon^{1/2} \frac{a^2\sqrt{a}}{2}u_{xxx} \left[\frac{1}{3} - \frac{\mu}{a^2} \right] \\ - \epsilon^{1/2} \sqrt{a}u_{xxx} + \epsilon \left(-\frac{a^2\sqrt{a}}{2} \left[\frac{1}{3} - \frac{\mu}{a^2} \right] + a_5^* \right) u_{xxxxx} = 0. \end{aligned}$$

Set

$$m = \epsilon^{1/2},$$

and

$$a_0 = -\frac{1}{4} \frac{a_x}{\sqrt{a}}, \quad f = a_0 u,$$

$$a_1 = \sqrt{a} + \frac{3}{2}\epsilon \frac{u}{\sqrt{a}},$$

$$a_2=0, \quad a_3=\epsilon^{1/2}\frac{a^2\sqrt{a}}{2}\left[\frac{1}{3}-\frac{\mu}{a^2}\right]-\epsilon^{1/2}\sqrt{a},$$

$$a_4=0 \text{ and } a_5=-\frac{a^2\sqrt{a}}{2}\left[\frac{1}{3}-\frac{\mu}{a^2}\right]+\frac{a^4\sqrt{a}}{2}\left(\frac{19}{180}-\frac{\mu}{6a^2}-\frac{\mu^2}{4a^4}\right).$$

This latter formulation of CH-type equation is of the form (1.1), then the result that for $s > \frac{5}{2}$, $b \in H^\infty(\mathbb{R})$ and for all $u^0 \in H^s(\mathbb{R})$, there exist a time $T^* > 0$ and a unique solution u to this CH-type equation in $C([0, T^*]; H^s(\mathbb{R}))$ follows from Corollary 2.1, provided that all the assumptions are easily satisfied.

Acknowledgements

The author want to thank the referee for his (her) careful reading of the proofs and for his (her) suggestions and comments which improved the content of the paper.

References

- [1] R. Camassa and D. Holm, , *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. 71 (1993), 1661-1664.
- [2] S. Israwi, *Variable depth KDV equations and generalizations to more nonlinear regimes*, M2AN volume 44, (2010), 374-390.
- [3] Lili Fan, Hongjun Gao, *Local well-posedness and blow-up of solutions for wave equations on shallow water with periodic depth*. Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 07, pp. 1-17.
- [4] Lili Fan, Hongjun Gao, *Local well-posedness and persistence properties for the variable depth KDV general equations in Besov space $B_{2,1}^{3/2}$* . Differential Integral Equations Volume 29, Number 3/4 (2016), 241-268.
- [5] T. B. Benjamin, J. L. Bona and J. J. Mahoney, *Model equations for long waves in nonlinear dispersive systems*, Phil. Trans. Roy. Soc. London A 227 (1972), 47-78.
- [6] S. Israwi and R. Talhouk, *Local well-posedness of a nonlinear KdV-type equation*, Comptes Rendus Mathematique, 351, 23-24, (2013), 895-899.
- [7] S Israwi, H Kalisch, *Approximate conservation laws in the KdV equation* Physics Letters A, 2019.
- [8] S Israwi, A Mourad *An explicit solution with correctors for the Green?Naghdi equations* Mediterranean Journal of Mathematics 11 (2), 519-532. Springer.
- [9] B. Khorbatly, S. Israwi, *A Conditional Local existence result for the Generalized nonlinear Kawahara equation*, MMAS, Wiley and sons, 1-19, 2020.
- [10] T. Akhunov, *A sharp condition for the well-posedness of the linear KdV-type equation*. Proc. AMS. 2013.
- [11] W. Craig, T. Kappeler, W. Strauss, *Gain of regularity for equations of KdV type*. Annales de l'Institut Henri Poincaré (C) Analyse non linéaire, 9 (2) (1992), 147-186.

- [12] F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*. Springer Science, Business Media (2009).
- [13] Bo Tian, Yi-Tian Gao, *Variable-coefficient balancing-act method and variable-coefficient KdV equation from fluid dynamics and plasma physics*, Eur. Phys. J. B. , **22** (2001), 351-360.
- [14] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal. 192, 2197-2207 (2006).
- [15] D. Ionescu-Kruse, *Variationnal derivation of the Camassa-Holm shallow water equation*, J. Nonlinear Math. Phys. 14, no 3, pp. 303-312 (2007).
- [16] Johnson, *Camassa-Holm, Korteweg-de Vries and related models for water waves*, J. Fluid Mech. 455, 63-82 (2002)
- [17] S. Israwi, H. Kalisch, *Approximate conservation laws in the KdV equation* Physics Letters A, 2019.
- [18] D.Lannes. *The water waves problem: mathematical analysis and asymptotics*. Mathematical Surveys and Monographs (AMS), 2013.
- [19] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891-907, 1988.
- [20] M.E.Taylor. *Partial Differential Equations II Qualitative Studies of Linear Equations*, Applied Mathematical Sciences, 116 (2011).
- [21] D. Lannes, *Sharp Estimates for pseudo-differential operators with symbols of limited smoothness and commutators*, J. Funct. Anal., 232 (2006), 495-539.
- [22] S. Alinhac, P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*. Savoirs Actuels. InterEditions, Paris; Editions du Centre National de la Recherche Scientifique (CNRS), Meudon, 1991. 190 pp.
- [23] S. Israwi, R.Talhouk and I. Zaiter *Capillary effects and the variable depth Kawahara approximation*. In preparation.
- [24] Iguchi. T, *A long wave approximation for capillary-gravity waves and the Kawahara equation*. Bull. Inst. Math. Acad. Sin. (N.S.) 2 (2008), no. 2, 179-220.