PAPR analysis of non-contiguous duplex multicarrier signals

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This paper deals with the peak to average power ratio (PAPR) of duplex multicarrier signals. A simple but accurate expression of the complementary cumulative distribution of the PAPR is suggested, based on both empirical and theoretical analyzes of the mean PAPR of such a continuous duplex signal. To this end, we show that the mean PAPR can be written in function of the frequency distance between the two signals composing the duplex one. The relevance and the accuracy of the suggested analysis are shown through simulations, and a discussion paves the way to further analyzes involving general multiplex signals with different powers and subcarriers numbers.

Introduction: Multicarrier signals (MC) are well known to be prone to high power fluctuations due to the inherent summation of independent information carried on different tones. These power fluctuations can be characterized through the ratio of the maximum instantaneous power and the mean power of the transmitted signal, which leads to the definition of the peak to average power ratio (PAPR). PAPR is a metric that is directly related to the instantaneous power fluctuations of a signal. When considering a power amplifier, the power budget is the ratio of the output power of the amplified signal and the supply power. As the output power is a function of the input power, the efficiency of the power amplifier is generally inversely proportional to the PAPR of the input signal. This is why PAPR reduction methods aim to mitigate the PAPR so as to drive the signal as close as possible to the saturation point of the power amplifier where the efficiency is the highest [1].

While PAPR derivation is well known considering a single-band multicarrier signal [2–4], very few works have been suggested for non-contiguous multiplex of signals [5–7]. Nevertheless, this situation raises in contexts of heterogeneous signals transmission such as cognitive radio or 5G, when several non-contiguous signals, each associated to different users, have to be transmitted and amplified all together. In this context the resulting PAPR should not only depend on the signals themselves (respective number of subcarriers, respective mean powers, number of users) but on the frequency distance between the signals as well. Given these features, any opportunistic spectrum access or frequency resource allocation has to be evaluated on the PAPR criteria.

In [5,6], the PAPR is analyzed only through simulations [5], and through upper bounds of the complementary cumulative distribution functions (CCDFs) [6]. More recently, the authors in [7] derived a thorough study of the PAPR of multiplex signals with different numerology for an application to 5G technology. Although the theoretical expression of the PAPR generally well matches the simulations results, it is not easily tractable in practice. Furthermore, the authors in [7] did not specifically analyze the effect of the frequency gap between signals on the PAPR behavior.

In this paper we suggest handling this PAPR issue by considering the combination of two non contiguous OFDM signals with the same number of carriers and power. A simple and tractable expression of the CCDF of PAPR is proposed, based on both theoretical and empirical original analyzes of the mean PAPR of duplex multicarrier signals. The resulting PAPR is evaluated with the frequency distance between the two spectra, and we show the relevance of the derived results through simulations. In particular, we show that the PAPR of a duplex signal increases when the frequency distance between signal increases, but it reaches an upper bound. Moreover, we discuss the results to provide leads for further studies including more general multiplex of multicarrier signals as in [7]. This PAPR analysis can be useful to any engineer or researcher seeking to rapidly evaluate the PAPR of duplex multicarrier signals without need of sophisticated functions.

The remainder of the paper is organized as follows. The PAPR of the duplex signal is derived in Section after the statement of the problem given in Section . Simulations are provided in Section and a conclusion ends the paper in Section .

Problem Statement: We consider a duplex multicarrier signal $x(t)$ composed of two signals $x_1(t)$ and $x_2(t)$ as

$$x(t) = x_1(t) + x_2(t), \tag{1}$$

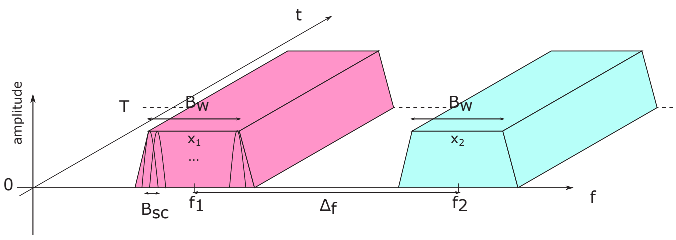

Fig. 1 *Time-frequency representation of duplex of multicarrier signals x_1 and x_2 transmitted at frequencies f_1 and f_2.*

where t belongs to an observation window of duration T, *i.e.* $t \in [0, T]$. Typically, T corresponds to the duration of an OFDM symbol. Moreover, we assume that x_1 and x_2 are non-contiguous in the frequency domain, *i.e.* x_1 is transmitted at frequency f_1 and x_2 at frequency f_2 such that we define the frequency distance $\Delta_f = f_2 - f_1$. Without loss of generality, we arbitrarily consider that $f_2 \geq f_1$. Furthermore, we assume that x_1 and x_2 are composed of the same number of subcarriers denoted by N with a common subcarrier spacing B_{sc}. The total bandwidth of each signal $x_1(t)$ or $x_2(t)$ is then $B_w = NB_{sc}$. The time-frequency representation of x_1 and x_2 is illustrated in Fig. 1.

The sampled version of $x_i(t)$, $i \in \{1, 2\}$, at Nyquist rate t_s, is noted $x_{i,n}$ with $n = 0, 1, .., N-1$. In this case, the observation window can be rewritten as $T = Nt_s$. If N is larger than about ten subcarriers, it can be reasonably assumed that the samples $x_{i,n}$ are independent and identically distributed (iid) and $x_{i,n}$ obey a complex Gaussian distribution, *i.e.* $x_{i,n} \sim \mathbb{CN}(0, \sigma^2)$, where $\sigma^2 = \mathbb{E}\{|x_{i,n}|^2\}$ [8], with $\mathbb{E}\{.\}$ the mathematical expectation. Under this assumption, Van Nee *et al.* suggested an expression of the CCDF of the PAPR of $x_i(t)$, $i = 1, 2$, expressed as [2–4]:

$$CCDF(\lambda) = \mathbb{P}\left(\max_{t \in \ 0,T} \frac{|x_i(t)|^2}{\sigma^2} \geq \lambda \right) \tag{2}$$

$$\approx 1 - (1 - e^{-\lambda})^{\alpha N}, \tag{3}$$

where $\mathbb{P}(E)$ means "probability of the event E", and α is an adjusting parameter that has been empirically set to $\alpha = 2.8$ in [2]. The popularity of the CCDF expression in (3) lies in its simplicity and its good accuracy. In fact, the PAPR of a continuous multicarrier signal can be properly approximated by knowing its subcarriers number only. Therefore, it is relevant to suggest a simple expression of the CCDF of the PAPR of the duplex multicarrier signal $x(t)$ in a Van Nee's fashion. To this end, we empirically set different α values in function of Δ_f thanks to the mean PAPR of the duplex multicarrier signal. The developments are hereby derived. Nevertheless, limitations of Van Nee's model have been stated in [4, 9], such as the lack of theoretical justification. We hereby use this approach for its simplicity, but we will, in turn, fairly discuss the suggested adaptation of (3) to duplex signals in Section .

PAPR Analysis: We assume that the CCDF of the PAPR of a multicarrier duplex signal $x(t)$ in (1) can be expressed as in (3), where the coefficient α needs to be defined. This assumption will be verified through simulations. To do so, on one hand we theoretically derive the mean of the PAPR as a function of α (it must be emphasized that the mean of PAPR has no practical significance, but in this paper, it is used as a trick of arithmetic to derive α). On the other hand, we empirically express the mean of PAPR as a function of the frequency distance Δ_f. From these two developments, we obtain a simple expression of the CCDF of PAPR with a coefficient α as a function of Δ_f.

Proposition 1: We define μ as the mean PAPR of the signal $x(t)$ in (1). For large N value ($N \geq 128$), the mean PAPR can be simply expressed as

$$\mu = \gamma + \ln(\alpha N), \tag{4}$$

where γ is the Euler-Mascheroni constant, $\gamma \approx 0.577$, and $\ln(.)$ is the natural logarithm.

Proof: In order to develop the mean of PAPR, we rewrite the CCDF in (3) by using the binomial series expansion of $(1-e^{-\lambda})^{\alpha N}$ as

$$CCDF(\lambda) = 1 - \sum_{k=0}^{+\infty} \binom{\alpha N}{k} (-1)^k e^{-k\lambda} \quad (5)$$

$$= -\sum_{k=1}^{+\infty} \binom{\alpha N}{k} (-1)^k e^{-k\lambda}, \quad (6)$$

where $\binom{\alpha N}{k}$ is the generalized binomial coefficient. The distribution of the PAPR, defined as $f(\lambda) = \frac{\partial(1-CCDF(\lambda))}{\partial \lambda}$, is then obtained as

$$f(\lambda) = -\sum_{k=1}^{+\infty} \binom{\alpha N}{k} (-1)^k k e^{-k\lambda}, \quad (7)$$

hence the mean is derived as

$$\mu = \int_0^{+\infty} \lambda f(\lambda) d\lambda = -\sum_{k=1}^{+\infty} \frac{\binom{\alpha N}{k}(-1)^k}{k}. \quad (8)$$

For large αN value, we have $\binom{\alpha N}{k} \sim \frac{(\alpha N)^k}{k!}$, then (8) yields

$$\mu = -\sum_{k=1}^{+\infty} \frac{(\alpha N)^k (-1)^k}{k k!}. \quad (9)$$

From [10], we know that the series expansion of the exponential integral defined for any $z \in \mathbb{C}\backslash\mathbb{R}^-$ as $E_1(z) = \int_z^{+\infty} \frac{e^{-t}}{t} dt$ leads to

$$E_1(z) = -\gamma - \ln(z) - \sum_{k=1}^{+\infty} \frac{(-1)^k z^k}{k k!}. \quad (10)$$

Since $\lim_{z\to+\infty} E_1(z) = 0$, and reminding that we assume large αN, then the substitution of $z = \alpha N$ into (10) leads to (4), which concludes the proof. ■

It can be noticed that this result is very similar to those in [11,12] where Nyquist-sampled signals are considered. Thus, μ in (4) can be seen as a generalized expression of the mean PAPR. From (4), we notice that the α coefficient can in turn be expressed in function of the mean μ as

$$\alpha = \frac{\exp(\mu - \gamma)}{N}. \quad (11)$$

In order to obtain α relatively to the frequency distance Δ_f, we empirically express μ as a function g of Δ_f such as suggested in Proposition 2. For convenience, and in order to make the expression g independent of the bandwidth B_w, we set $\theta = \frac{\Delta_f}{B_w}$.

Proposition 2: Consider a non-contiguous duplex multicarrier signal $x(t)$ such as defined in (1). Let $\theta \in \mathbb{R}_+$, then the mean PAPR μ of $x(t)$ can be expressed as a function of θ as

$$\mu = g(\theta) \quad (12)$$

$$= \beta - \nu \exp(-q.\theta^c), \quad (13)$$

where β, ν, q, and c are positive parameters that need to be determined. It can be noted that μ in (13) is a strictly increasing function of θ, which is upper bounded by β.

The four parameters can be assessed as follows:

- $\beta = \lim_{\theta\to+\infty} \mu$, even though in practice θ should feature low values due to the limitations of the transmitter in term of bandwidth.
- Once β is obtained, we deduce ν by means of the equality $g(0) = \beta - \nu$, which corresponds to the mean PAPR of a continuous (at least largely oversampled) signal featuring N subcarriers. In practice, this case should not happen since it means that x_1 and x_2 overlap since $f_1 = f_2$, but for the sake of the analysis, it allows us to deduce ν.
- Similarly, q can be deduced thanks to $g(1) = \beta - \nu e^{-q}$, which corresponds to the mean PAPR of a continuous signal featuring $2N$ contiguous subcarriers.

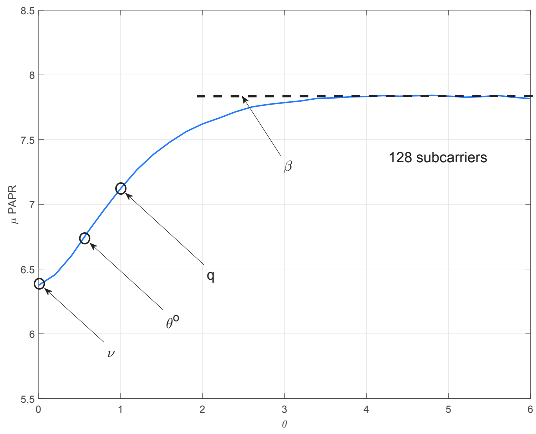

Fig. 2 *Mean PAPR μ as a function of θ, for 128 subcarriers per signal. Markers indicate the values from which the parameters β, ν, q, and θ^o (leading to c) are deduced.*

Table 1: Parameters β, ν, q, and c obtained from simulations.

parameter	N	value
β	$\{128, 256, 512\}$	$\{7.83, 8.6, 9.36\}$
ν	$\{128, 256, 512\}$	$\{1.45, 1.49, 1.5\}$
q	$\{128, 256, 512\}$	0.7
c	$\{128, 256, 512\}$	1.5

- Let us note $\theta^o = \sup_{\theta\in\mathbb{R}_+} \frac{\partial g(\theta)}{\partial \theta}$, then c can be deduced by solving $\frac{\partial^2 g(\theta^o)}{\partial \theta^2} = 0$. In fact, a unique solution θ^o exists, leading to a unique solution to $\frac{\partial^2 g(\theta^o)}{\partial \theta^2} = 0$ as well. Alternatively, the parameter c can be empirically set to fit the behavior of μ in function of θ obtained through simulations.

To summarize our approach, we suggest estimating the CCDF of PAPR of duplex multicarrier signal $x(t)$ in (1) by using a simple Van Nee-like expression as in (3) where the coefficient α must be defined. To this end, we theoretically express α as a function of the mean PAPR μ in (11) on one hand. On the other hand, we empirically derive an expression of μ in (13) as a function of the frequency distance Δ_f (or equivalently θ) between the two multicarrier signals that compose the duplex. Then, substituting the empirical mean μ into (11) finally leads to the simple expression of CCDF of PAPR (3).

Simulations: This section aims to evaluate the relevance of the previous developments through simulations. Both signals x_1 and x_2 are composed of N subcarriers with $N \in \{128, 256, 512\}$, carrying elements that are randomly taken from a 16-QAM constellation. We assume that x_1 and x_2 are synchronized in time, even though time synchronization should not have influence on the PAPR value, according to "remark 2" in [7]. A large oversampling rate of $R = 32$ is considered in order to approximate continuous signals, and at least to respect the condition $\Delta_f << \frac{R}{t_s}$ for reasonable Δ_f range. When this condition does not hold, some simulations artifacts appears when $\Delta_f > \frac{R}{4t_s}$ (*i.e.* when Δ_f is too close to the sampling frequency), which are not further described in this paper. All the simulations results have been averaged over 10000 independent runs.

Fig. 2 shows the mean PAPR μ (linear scale) versus θ for a duplex signal $x(t)$ featuring 128 subcarriers per signal $x_i(t)$, $i = 1, 2$. The behavior of μ has been obtained by simulation, and allows us to deduce the four parameters β, ν, q, and θ^o (leading to c), such as highlighted by the different markers. The corresponding values have been reported in Table 1, for $N = 128$, 256, and 512. We observe from Fig. 2 that μ is strictly increasing with θ, and reaches an upper bound for $\theta \geq 4$.

It is worth noticing from Table 1, that β and ν depend on θ, whereas q and c are constant. Furthermore, ν tends to 1.5 when N increases. We deduce that, for N large (typically $N \geq 512$) it is sufficient to assess β for $\theta \geq 4$ to estimate μ for any θ, which further simplifies the PAPR estimation. Otherwise, further simulations have shown that both β and ν must be set when $N < 512$.

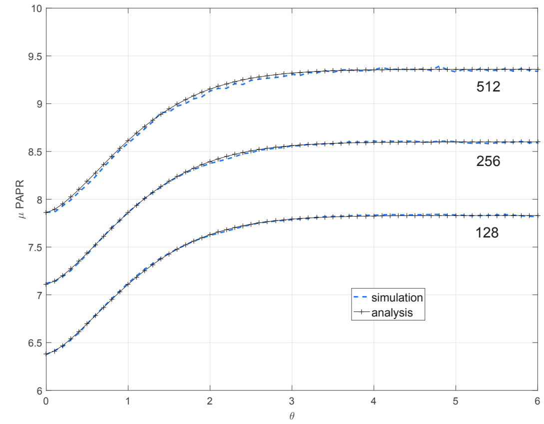

Fig. 3 *Mean PAPR μ as a function of θ, for 128, 256, and 512 subcarriers per signal. Comparison of simulations and analysis (13).*

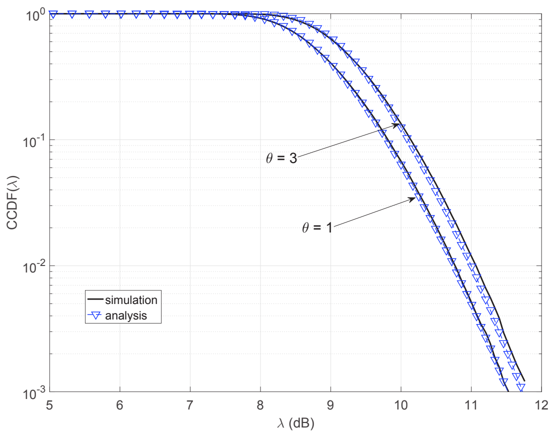

Fig. 4 *$CCDF(\lambda)$ versus λ (dB) for $N = 256$, $\theta = 1, 3$. Comparison between simulations and analysis.*

In Fig. 3, we compare the mean PAPR μ versus θ for 128, 256, and 512 subcarriers per signal, obtained through simulations and from (13) using the parameters in Table 1. It can be observed that the empirical μ values in function of θ perfectly match the trajectories obtained through simulations, therefore validating (13). For applications, we deduce from results in Fig. 3 that it is preferable to transmit the signals x_1 and x_2 in contiguous bands (*i.e.* $\theta = 1$), when possible, as it minimizes the PAPR. However, in practice, other parameters should be considered to properly design the system, such as the synchronization between signals (synchronization mismatch between x_1 and x_2 would lead to interference), or the possible intermodulation distortions that may cause disruptions in both x_1 and x_2, but this is not dealt with in this paper.

Fig. 4 compares the trajectories of $CCDF(\lambda)$ versus λ (dB) obtained through simulations with that obtained with (3). $N = 256$ subcarriers are considered, and $\theta = 1, 3$. The corresponding α values are 5.68 and 11.45, respectively. It can be seen that the curve obtained from analysis for $\theta = 1$ matches that obtained through simulations. Moreover, it should be noticed that $\alpha/2 = 2.82$, which almost corresponds to the empirical value $\alpha = 2.8$ defined in [2] by Van Nee. Note that we find $\alpha/2 = 2.7$ and $\alpha = 3.02$ for $N = 128$, and $N = 512$, respectively. We then deduce that the suggested approach allows to adapt α value according N in Van Nee model (*i.e.* $\theta = 1$). It can be seen for $\theta = 3$ that the results obtained from analysis slightly underestimate the CCDF of PAPR obtained through simulations (of about 0.05 dB). The suggested analysis is nevertheless an accurate approximation of the CCDF of PAPR regarding the simplicity of the expression.

Discussion: Developments and simulations results have shown that the PAPR of duplex multicarrier signals can be simply but accurately approximated in a Van Nee's fashion where the α coefficient is function of the frequency distance Δ_f between the signals x_1 and x_2. However, we voluntary limited the analysis to the case where x_1 and x_2 are equipowered and share the same number of subcarriers. A similar but more general study case could then be carried out in case the two signals do not share the same parameters. In this case, the four parameters β, ν, q, and c should be tuned with respect to the different powers and subcarriers numbers of x_1 and x_2. To go further, the present analysis could even be extended to derive simple expressions of the CCDF of PAPR of multiplex multicarrier signals such as considered in [7, 9]. In that case, a more general mean PAPR expression $\mu = g(\theta_1, \theta_2, .., \theta_{M-1})$ should be considered, where M is number of signals composing the multiplex one, and θ_i are the corresponding frequency distances.

In addition, it must be emphasized that the function g in (13) is empirically derived. Although Fig. 3 shows that g accurately matches simulations, an analytical derivation of g should be investigated as well. Other studies could be undertaken from the present paper, such as the analyze of the instantaneous signal envelope $|x(t)|$ instead of $|x(t)|^2$, and its adaptation to duplex signals featuring low subcarriers numbers (typically $N \leq 64$) by using results in [12].

Conclusion: This paper highlights the dependency between the frequency distance between signals composing a non-contiguous duplex signal and the resulting PAPR. By deriving the mean PAPR of such kind of signal, we came to the conclusions that (i) the lowest PAPR value of the duplex is given when the two signals are contiguous in frequency, corollarily (ii) the larger the frequency gap the larger the PAPR, but (iii) the mean PAPR is upper bounded for large frequency distance between the signals that compose the duplex. To go further, this study could be enriched by considering more than two users of unequal powers and different subcarriers numbers.

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