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A Panorama on the Minoration of the Mahler Measure: from the Problem of Lehmer to its Reformulations in Topology and Geometry

Jean-Louis Verger-Gaugry

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Foreword: *This text is addressed to readers who are already engaged in their own research in the problem of Lehmer in at least one domain concerned, like graduate students, post-docs and active researchers. Initially the problem of Lehmer appeared in the original paper of D. Lehmer in Number Theory in 1933. The great richness of the problem of Lehmer is reflected today by the fact that the problem of minoration of the Mahler measure can been reformulated in several domains. Many interesting directions of research have appeared in doing so, and are developping in parallel. These domains are evoked in the present article. The reader not accustomed to the problem of Lehmer may find an interest to discover the different reformulations of the Mahler measure and its realizations in topology, geometry,... The aims of this text is to provide an extremely sketchy panorama of a large area of this part of mathematics, now spread over Number Theory, Arithmetic Geometry, and much more. It can be used as a guide to those who want to embark on a more serious study of one of these domains, or develop new analogues in existing theories.*

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The Conjecture of Lehmer, and its refinement in 1965 by Schinzel, the Conjecture of Schinzel-Zassenhaus, amounts to a problem of universal minoration of the Mahler measure, and of the height in higher dimension in Arithmetic Geometry. The objective of this Survey is to review the numerous minorations obtained in these two domains, in particular Dobrowolski's inequality, then to present the analogues of the problem of Lehmer in different contexts with various analogues of the Mahler measure and the height.

The reformulation of the problem of Lehmer in other domains brings to light a certain number of situations generating integer polynomials for which the Problem of Lehmer is asked, and, if a nontrivial lower bound exists to the Mahler measure of these polynomials, the meaning and the realization of the situation of extremality. In several cases Lehmer's number is found to be a nontrivial minorant and is shown to be reached.

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Contents

1. Introduction

The question (called *Problem of Lehmer*) asked by Lehmer in [336] (1933) about the existence of integer univariate polynomials of Mahler measure arbitrarily close to one became a conjecture. Let us recall it:

Problem of Lehmer. *If ϵ is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \ldots + a_r$$

where the a_is are integers, such that the absolute value of the product of those roots of f which lie outside the unit circle, lies between 1 and $1+\epsilon$... Whether or not the problem has a solution for $\epsilon < 0.176$ we do not know.

This problem takes its origin in the search for large prime numbers. In [336] Lehmer introduces an arithmetic process to obtain explicitely large prime numbers from integer polynomials of very small Mahler measure > 1, where "very small" Mahler measures would correspond to "very large" prime numbers. Since then, the strategy of his method has been revisited (see §2.1). Lehmer's Conjecture is addressed to minimal polynomials of algebraic integers. It is stated as follows:

Conjecture 1.1 (Lehmer's Conjecture). *There exists an universal constant $c > 0$ such that the Mahler measure $M(\alpha)$ satisfies $M(\alpha) \geq 1 + c$ for all nonzero algebraic numbers α, not being a root of unity.*

If α is a nonzero algebraic integer, $\mathrm{M}(\alpha) = 1$ if and only if $\alpha = 1$ or is a root of unity by Kronecker's Theorem (1857) [311]. Lehmer's Conjecture asserts a discontinuity of the value of $\mathrm{M}(\alpha)$, $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$, at 1. Then solving Lehmer's Conjecture amounts to a problem of minoration of the Mahler measure $\mathrm{M}(\alpha)$ when the absolute value > 1 of $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$ tends to 1^+.

Lehmer's Conjecture has been extensively studied in Number Theory, e.g. by Amoroso [9], [10], Bertin, Decomps-Guilloux, Grandet-Hugot, Pathiaux-Delefosse and Schreiber [48], Blansky and Montgomery [59], Boyd [76], [77], Cantor and Strauss [114], Dobrowolski [160], Dubickas [172], Langevin [325], Louboutin [351], Mossinghoff, Rhin and Wu [391], Schinzel [463], Smyth [493], [494], Stewart [500], Waldschmidt [526], [527]). It has been extended to Arithmetic Geometry by replacing the Mahler measure M by a suitable notion of "height", and by reformulating the minoration problem to elliptic curves, to Abelian varieties, ..., e.g. by Amoroso and David [18], David and Hindry [137], [139], Hindry and Silverman [261], Laurent [331], Masser [374], Silverman [482].

In his attempt to solve Lehmer's Conjecture in 1965, Schinzel has refined the minoration problem of the Mahler measure $\mathrm{M}(\alpha)$ by replacing the condition "when the absolute value > 1 of $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$ tends to 1^+" by the condition "when the house $\overline{|\alpha|} > 1$ of $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$ tends to 1^+". The following Conjecture has been formulated in [467].

Conjecture 1.2 (Schinzel - Zassenhaus's Conjecture). *Denote by $\mathrm{m}_h(n)$ the minimum of the houses $\overline{|\alpha|}$ of the algebraic integers α of degree n which are not a root* *of unity. There exists a (universal) constant $C > 0$ such that*

$$\mathrm{m}_h(n) \geq 1 + \frac{C}{n}, \qquad n \geq 2. \tag{1.1}$$

The objective of this Survey is first to review a certain number of results in Number Theory (§ 2 and § 3) and in higher dimension in Arithmetic Geometry (§ 4), then to go beyond to various extensions where different reformulations of the minoration problem of Lehmer exist (§ 5). In other domains (§ 6) the minoration problem of Lehmer emphasizes the role played by algebraic integers like Pisot numbers, Salem numbers, Perron numbers and calls for the problem of the realization when a nontrivial minimum is reached. It occurs in some cases that the minimum is realized by Lehmer's number $1.17628\ldots$ (cf (2.7) and (2.8)).

This Panorama/Survey tries to take stock of the problem of minoration of the Mahler measure in all its forms.

Standard notations are reported in the Appendix.

2. Number theory

2.1. Prime numbers, asymptotic expansions, minorations. The search for very large prime numbers has a long history. The method of linear recurrence sequences of numbers (Δ_m), typically satisfying

$$\Delta_{m+n+1} = A_1 \Delta_{m+1} + A_2 \Delta_{m+2} + \ldots + A_n \Delta_{m+n}, \tag{2.1}$$

in which prime numbers can be found, has been investigated from several viewpoints, by many authors [55], [201], [202]: in 1933 Lehmer [336] developped an exhaustive approach from the Pierce numbers [418]

$$\Delta_n = \Delta_n(P) = \prod_{i=1}^{d} (\alpha_i^n - 1) \tag{2.2}$$

of a monic integer polynomial P where α_i are the roots of P. The sequence (A_i) in (2.1) is then the coefficient vector of the integer monic polynomial which is the least common multiple of the $d+1$ polynomials: $P_{(0)}(x) = x - 1$,

$$P_{(1)}(x) = \prod_{i=1}^{d} (x - \alpha_i), P_{(2)}(x) = \prod_{i>j=1}^{d-1} (x - \alpha_i \alpha_j), \ldots, P_{(d)}(x) = x - \alpha_1 \alpha_2 \ldots \alpha_d$$

(Theorem 13 in [336]). Large prime numbers, possibly at a certain power, can be found in the factorizations of $|\Delta_n|$ that have large absolute values (Dubickas [181], Ji and Qin [283] in connection with Iwasawa theory). This can be done fairly quickly if the absolute values $|\Delta_n|$ do not increase too rapidly (slow growth rate). If P has no root on the unit circle, Lehmer proves

$$\lim_{n\to\infty} \frac{\Delta_{n+1}}{\Delta_n} = \mathrm{M}(P). \tag{2.3}$$

Einsiedler, Everest and Ward [193] revisited and extended the results of Lehmer in terms of the dynamics of toral automorphisms ([199], Lind [342]). They considered expansive (no root on $|z| = 1$), ergodic (no α_i is a root of unity) and quasihyperbolic (if P is ergodic but not expansive) polynomials P and number theoretic heuristic arguments for estimating densities of primes in (Δ_n). In the quasihyperbolic case (for instance for irreducible Salem polynomials P), more general than the expansive case considered by Lehmer, (2.3) does not extend but the following more robust convergence law holds [342]:

$$\lim_{n\to\infty} \Delta_n^{1/n} = \mathrm{M}(P). \tag{2.4}$$

If P has a small Mahler measure, $< \Theta$, it is reciprocal by [489] and the quotients Δ_n/Δ_1 are perfect squares for all $n \geq 1$ odd. With $\Gamma_n(P) := \sqrt{\Delta_n/\Delta_1}$ in such cases, they obtain the existence of the limit

$$\lim_{j\to\infty} \frac{j}{\operatorname{Log}\operatorname{Log}\Gamma_{n_j}}, \tag{2.5}$$

(n_j) being a sequence of integers for which Γ_{n_j} is prime, as a consequence of Merten's Theorem. This limit, say E_P, is likely to satisfy the inequality: $E_P \geq 2e^{\gamma}/\operatorname{Log}\mathrm{M}(P)$, where $\gamma = 0.577\ldots$ is the Euler constant. Moreover, by number-fields analogues of the heuristics for Mersenne numbers (Wagstaff, Caldwell), they suggest that the number of prime values of $\Gamma_{n_j}(P)$ with $n_j \leq x$ is approximately

$$\frac{2e^{\gamma}}{\operatorname{Log}\mathrm{M}(P)}\operatorname{Log} x. \tag{2.6}$$

This result shows the interest of having a polynomial P of small Mahler measure to obtain a sequence (Δ_n) associated with P very rich in primes. These authors consider many examples which fit coherently the heuristics. However, the discrepancy function is still obscure and reflects the deep arithmetics of the factorization of the integers $|\Delta_n|$ and of the quantities Γ_n.

More generally, than in the case of (2.1), the Conjecture of Lehmer means that the growth rate of an integer linear recurrence sequence is uniformly bounded from below (§ 7.6 in [202], [200]).

In view of understanding the size of the primes $p \geq 3$ found in (Δ_n) generated by the exhaustive method of the Pierce's numbers, Lehmer, in [337] (1977), established correlations between the Pierce's numbers $|\Delta_n|$ and the prime factors of the first factor of the class number of the cyclotomic fields $\mathbb{Q}(\xi_p)$ (ξ_p is a primitive pth root of unity), using Kummer's formula, the prime factors being sorted out into arithmetic progressions: let $h(p)$ the class number of $\mathbb{Q}(\xi_p)$ and let $h^+(p)$ be the class number of of the real subfield $\mathbb{Q}(\xi_p + \xi_p^{-1})$. Kummer (1851) established that the ratio $h^-(p) = h(p)/h^+(p)$ is an integer, called *relative class number* or *first factor of the class number*, and that p divides $h(p)$ if and only if p divides $h^-(p)$. The factorization and the arithmetics of the large values of $h^-(p)$ is a deep problem [3], [231], [246], [338], related to class field theory in [370], where the validity of Kummer's conjectured asymptotic formula for $h^-(p)$ was reconsidered by Granville [143] [246].

The smallest Mahler measure $\mathrm{M}(\alpha)(> 1)$ known, where α is a nonzero algebraic number which is not a root of unity, is Lehmer's number

$$= 1.17628\ldots, \tag{2.7}$$

the smallest Salem number discovered by Lehmer [336] in 1933 as dominant root of Lehmer's polynomial (2.8).

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1. \tag{2.8}$$

It is the smallest Salem number known [389] [391]. Lehmer discovered other small Salem numbers (cf §3). Small Salem numbers were reinvestigated by Boyd in [74], [76], [78], by Flammang, Grandcolas and Rhin [221]. The search of small Mahler measures was reconsidered by Mossinghoff [387], [388], then, using auxiliary functions, by Mossinghoff, Rhin and Wu [391]. For degrees up to 180, the list of Mossinghoff [389] (2001), with contributions of Boyd, Flammang, Grandcolas, Lisonek, Poulet, Rhin and Sac-Epée [443], Smyth, gives primitive, irreducible, noncyclotomic integer polynomials of degree at most 180 and of Mahler measure less than 1.3; this list is complete for degrees less than 40 [391], and, for Salem numbers, contains the list of the 47 known smallest Salem numbers, all of degree ≤ 44 [221].

Lehmer's Conjecture is true (solved) in the following particular cases.

(1) for the closed set S of Pisot numbers (Salem [451], Siegel [478], Bertin et al [48]),

(2) for the set of algebraic numbers α for which the minimal polynomial P_α is nonreciprocal by Smyth's Theorem (1971) [489], [490], which asserts:

$$\mathrm{M}(\alpha) = \mathrm{M}(P_\alpha) \geq \Theta, \tag{2.9}$$

proved to be an isolated infimum by Smyth [490] (Θ is defined in (7.2)),

(3) for every nonzero algebraic integer $\alpha \in \mathbb{L}$, of degree d, assuming that $\mathbb{L}$ is a totally real algebraic number field, or a CM field (a totally complex quadratic extension of a totally real number field); then Schinzel [463] obtained the minoration

$$\mathrm{M}(\alpha) \geq \big(\frac{1+\sqrt{5}}{2}\big)^{d/2}, \text{from which : } \mathrm{M}(\alpha) \geq ((1+\sqrt{5})/2)^{1/2} = 1.2720\ldots \tag{2.10}$$

Improvments of this lower bound, by Bertin, Rhin, Zaimi and Garza are given in § 3,

(4) for α an algebraic number of degree d such that there exists a prime number $p \leq d \operatorname{Log} d$ that is not ramified in the field $\mathbb{Q}(\alpha)$; then Mignotte [378] [379] showed: $\mathrm{M}(\alpha) \geq 1.2$; by extension, Silverman [483] proved that the Conjecture of Lehmer is true if there exist primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_d$ in $\mathbb{Q}(\alpha)$ satisfying $N\mathfrak{p}_i \leq \sqrt{d \operatorname{Log} d}$,

(5) for any noncyclotomic irreducible polynomial P with all odd coefficients; Borwein, Dobrowolski and Mossinghoff [67] [236] proved (cf Theorem 2.3 and Silverman's Theorem 2.4 for details)

$$\mathrm{M}(P) \geq 5^{1/4} = 1.4953\ldots, \tag{2.11}$$

(6) in terms of the Weil height, Amoroso and David [15] proved that there exists a constant $c > 0$ such that, for all nonzero algebraic number α, of degree d, not being a root of unity, under the assumption that the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois, then

$$h(\alpha) \geq \frac{c}{d}. \tag{2.12}$$

Some minorations are known for some classes of polynomials (Panaitopol [408]). Bazylewicz [40] extended Smyth's Theorem, i.e. the lower bound given by (2.9), to polynomials over Kroneckerian fields K (i.e. for which $K/\mathbb{Q}$ is totally real or is a totally complex nonreal quadratic extension of such fields). Notari [399] and Lloyd-Smith [349] extended such results to algebraic numbers. Lehmer's problem is related to the minoration problem of the discriminant (Bertrand [50]).

Perron numbers, topology and limit points of their subclasses: Mahler measures, Salem numbers. Mahler measures $\{\mathrm{M}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ are Perron numbers, by the work of Adler and Marcus [2] in topological dynamics, as a consequence of the Perron-Frobenius theory. The set $\mathbb{P}$ of Perron numbers is everywhere dense in $[1, +\infty)$ and is important since it contains subcollections which have particular topological properties for which conjectures exist. Let us recall them.

The set $\mathbb{P}$ admits a nonfactorial multiplicative arithmetics [102], [343], [519], for which the restriction of the usual addition $+$ to a given subcollection is not necessarily internal [180]. Salem [452] proved that $S \subset \mathbb{P}$ is closed, and that $S \subset \overline{T}$. Boyd [73] conjectured that

$$S \cup \mathrm{T} \quad \text{is closed} \tag{2.13}$$

and that the first derived set of $S \cup \mathrm{T}$ (cf Appendix for a definition) satifies

$$S = (S \cup \mathrm{T})^{(1)} \tag{2.14}$$

([74], p. 237). This second Conjecture would imply that all Salem numbers $< \Theta$ would also be isolated Salem numbers, not only Lehmer's number. The set of Mahler measures $\{\mathrm{M}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ and the semi-group $\{\mathrm{M}(P) \mid P(X) \in \mathbb{Z}[X]\}$ are strict subsets of $\mathbb{P}$ and are distinct (Boyd [82], Dubickas [177] [178]). The probabilistic distribution and values of Mahler measures were studied by Boyd

[81], [82], [83], Chern and Vaaler [124], Dixon and Dubickas [158], Dubickas [176], Schinzel [465], Sinclair [487]. Boyd [83] has shown that the Perron numbers γ_n which are the dominant roots of the height one (irreducible) trinomials $-1-z+z^n$, $n \geq 4$, are not Mahler measures. The *inverse problem* for the Mahler measure consists in determining whether, or not, a Perron number γ is the Mahler measure $\mathrm{M}(P)$ of an integer polynomial P, and to give formulas for the number

$$\#\{P \in \mathbb{Z}[X] \mid \mathrm{M}(P) = \gamma\} \tag{2.15}$$

of such polynomials with measure γ and given degree (Boyd [81], Dixon and Dubickas [158], Staines [496]). Drungilas and Dubickas [170] and Dubickas [176], [178], proved that the subset of Mahler measures is very rich: namely, for any Perron number β, there exists an integer $n \in \mathbb{N}$ such that $n\beta$ is a Mahler measure, and any real algebraic integer is the difference of two Mahler measures.

Minorations and extremality. Algebraic numbers close to 1 ask many questions [161] and require new methods of investigation, as reported by Amoroso [9]. For α an algebraic integer of degree $d > 1$, not a root of unity, Blansky and Montgomery [59] showed, with multivariate Fourier series,

$$\mathrm{M}(\alpha) > 1 + \frac{1}{52}\frac{1}{d\mathrm{Log}\,(6d)}. \tag{2.16}$$

By a different approach, using an auxiliary function and a proof of transcendence (Thue's method), Stewart [500] obtained the same minoration but with a constant $c \neq 1/52$ instead of $1/52$ [103], [332], [382], [526]. In 1979 a remarkable minoration has been obtained by Dobrowolski [160] who showed

$$\mathrm{M}(\alpha) \;>\; 1 + (1-\epsilon)\left(\frac{\mathrm{Log}\,\mathrm{Log}\,d}{\mathrm{Log}\,d}\right)^3, \quad d > d_1(\epsilon). \tag{2.17}$$

for any nonzero algebraic number α of degree d, with $1-\epsilon$ replaced by $1/1200$ for an effective version (then with $d \geq 2$), in particular for $|\alpha| > 1$ arbitrarily close to 1. The minoration (2.17) was also obtained by Mignotte in [378] [379] but with a constant smaller than $1-\epsilon$. Cantor and Strauss [114], [413], then Louboutin [351], improved the constant $1-\epsilon$: they obtained $2(1+o(1))$, resp. $9/4$ (cf also Rausch [436] and Lloyd-Smith [349]). If α is a nonzero algebraic number of degree $d \geq 2$, Voutier [525] obtained the better effective minorations:

$$\mathrm{M}(\alpha) > 1 + \frac{1}{4}\left(\frac{\mathrm{Log}\,\mathrm{Log}\,d}{\mathrm{Log}\,d}\right)^3 \qquad \text{and} \qquad \mathrm{M}(\alpha) > 1 + \frac{2}{(\mathrm{Log}\,(3d))^3}. \tag{2.18}$$

For sufficiently large degree d, Waldschmidt ([527], Theorem 3.17) showed that the constant $1-\epsilon$ could be replaced in (2.17) by $1/250$ with a transcendence proof which uses an interpolation determinant. It is remarkable that these minorations only depend upon the degree of α and not of the size of the coefficients, i.e. of the (naïve) height of their minimal polynomial. Dobrowolski's proof is a transcendence proof (using Siegel's lemma, extrapolation at finite places) which has been extended to the various Lehmer problems (§ 4).

An algebraic integer α, of degree n, is said *extremal* if $\lceil\overline{\alpha}\rceil = \mathrm{m}_h(n)$. An extremal algebraic integer is not necessarily a Perron number [52].

In 1965 Schinzel and Zassenhaus [467] formulated Conjecture 1.2 and obtained the first result: for $\alpha \neq 0$ being an algebraic integer of degree $n \geq 2$ which is not a root of unity, then

$$\lceil\overline{\alpha}\rceil > 1 + 4^{-(s+2)}, \tag{2.19}$$

where $2s$ is the number of nonreal conjugates of α. For a nonreciprocal algebraic integer α of degree n, Cassels [119] obtained:

$$\lceil\overline{\alpha}\rceil > 1 + \frac{c_2}{n}, \qquad \text{with } c_2 = 0.1; \tag{2.20}$$

Breusch [100] independently showed that $c_2 = \mathrm{Log}\,(1.179\ldots) = 0.165\ldots$ could be taken; Schinzel [463] showed that $c_2 = 0.2$ could also be taken. Finally Smyth [489] improved the minoration (2.20) with $c_2 = \mathrm{Log}\,\Theta = 0.2811\ldots$. On the other hand, Boyd [81] showed that c_2 cannot exceed $\frac{3}{2}\mathrm{Log}\,\Theta = 0.4217\ldots$. In 1997 Dubickas [173] showed that $c_2 = \omega - \epsilon$ with $\omega = 0.3096\ldots$ the smallest root of an equation in the interval $(\mathrm{Log}\,\Theta, +\infty)$, with $\epsilon > 0$, $n_0(\epsilon)$ an effective constant, and for all $n > n_0(\epsilon)$. These two bounds seem to be the best known extremities delimiting the domain of existence of the constant c_2 [164].

The expression of the minorant in (1.1), "in $1/n$", as a function of n, is not far from being optimal, being "in $1/n^2$" at worse in (2.22). Indeed, for nonzero algebraic integers α, Kronecker's Theorem [311] implies that $\lceil\overline{\alpha}\rceil = 1$ if and only if α is a root of unity. The sufficient condition in Kronecker's Theorem was weakened by Blansky and Montgomery [59] who showed that α, with $\deg \alpha = n$, is a root of unity provided

$$\lceil\overline{\alpha}\rceil \leq 1 + \frac{1}{30n^2 \mathrm{Log}\,(6n)}. \tag{2.21}$$

Dobrowolsky [159] sharpened this condition by: if

$$\lceil\overline{\alpha}\rceil < 1 + \frac{\mathrm{Log}\, n}{6n^2}, \tag{2.22}$$

then α is a root of unity. Matveev [375] proved, for α, with $\deg \alpha = n$, not being a root of unity,

$$\lceil\overline{\alpha}\rceil \geq \exp \frac{\mathrm{Log}\,(n + \frac{1}{2})}{n^2}.$$

Rhin and Wu [445] verified Schinzel Zassenhaus's Conjecture up to $n = 28$ and improved Matveev's minoration as:

$$\lceil\overline{\alpha}\rceil \geq \exp \frac{3\mathrm{Log}\,(\frac{n}{3})}{n^2} \qquad 4 \leq n \leq 12, \tag{2.23}$$

and, for $n \geq 13$,

$$\lceil\overline{\alpha}\rceil \geq \exp \frac{3\mathrm{Log}\,(\frac{n}{2})}{n^2}. \tag{2.24}$$

Matveev's minoration is better than Voutier's lower bound [525]

$$m_h(n) \geq \left(1 + \frac{1}{4}\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^3\right)^{1/n} \tag{2.25}$$

for $n \leq 1434$, and Rhin Wu's minoration is better than Voutier's bound for $13 \leq n \leq 6380$. For reciprocal nonzero algebraic integers α, $\deg(\alpha) = n \geq 2$, not being a root of unity, Dobrowolski's lower bound is

$$\overline{|\alpha|} > 1 + (2 - \epsilon)\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^3 \frac{1}{n}, \qquad n \geq n_0(\epsilon), \tag{2.26}$$

where the constant $2 - \epsilon$ could be replaced by $\frac{9}{2} - \epsilon$ (Louboutin [351]), or, better, by $\frac{64}{\pi^2} - \epsilon$ (Dubickas [171]). Callahan, Newman and Sheingorn [108] introduce a weaker version of Schinzel Zassenhaus's Conjecture: given a number field K, they define the *Kronecker constant* of K as the least $\eta_K > 0$ such that $\overline{|\alpha|} \geq 1 + \eta_K$ for all $\alpha \in K$. Under certain assumptions on K, they prove that there exists $c > 0$ such that $\eta_K \geq c/[K : \mathbb{Q}]$.

The sets of extremal algebraic integers are still unknown. In Boyd [79] [533] the following conjectures on *extremality* are formulated:

Conjecture 2.1 (Lind - Boyd). *The smallest Perron number of degree $n \geq 2$ has minimal polynomial*

$$\begin{array}{cl} X^n - X - 1 & \text{if } \; n \not\equiv 3, 5 \mod 6, \\ (X^{n+2} - X^4 - 1)/(X^2 - X + 1) & \text{if } \; n \equiv 3 \mod 6, \\ (X^{n+2} - X^2 - 1)/(X^2 - X + 1) & \text{if } \; n \equiv 5 \mod 6. \end{array}$$

Conjecture 2.2 (Boyd). *(i) If α is extremal, then it is always nonreciprocal,*

(ii) if $n = 3k$, then the extremal α has minimal polynomial

$$X^{3k} + X^{2k} - 1, \qquad \text{or} \quad X^{3k} - X^{2k} - 1, \tag{2.27}$$

(iii) the extremal α of degree n has asymptotically a number of conjugates $\alpha^{(i)}$ outside the closed unit disc equal to

$$\cong \frac{2}{3}\, n, \qquad n \to \infty. \tag{2.28}$$

This asymptotic proportion of $\frac{2}{3}\, n$ would correspond to a fairly regular angular distribution of the complete set of conjugates in a small annulus containing the unit circle, in the sense of the Bilu-Erdős-Turán-Amoroso-Mignotte equidistribution theory [24], [42], [54], [196].

Structure of coefficients vectors and minorations. The nature of the coefficient vector of an integer polynomial P is linked to the Mahler measure $\mathrm{M}(P)$ and to extremal properties [366]. If some inequalities between coefficients occur, then Brauer [98] proved that P is a Pisot polynomial; in this case Lehmer's problem is solved for this class $\{P\}$. Stankov [498] proved that a real algebraic integer

$\tau > 1$ is a Salem number if and only if its minimal polynomial is reciprocal of even degree ≥ 4 and if there is an integer $n \geq 2$ such that τ^n has minimal polynomial $P_n(x) = a_{0,n} + a_{1,n}x + \ldots + a_{d,n}x^n$ which is also reciprocal of degree d and satisfies the condition

$$|a_{d-1,n}| > \frac{1}{2}(\frac{d}{d-2})(2 + \sum_{k=2}^{d-2} |a_{k,n}|). \tag{2.29}$$

Related to Kronecker's Theorem [311] is the problem of finding necessary and sufficient conditions on the coefficient vector of reciprocal, self-inversive, resp. self-reciprocal polynomials to have all their roots on the unit circle (unimodularity): Lakatos [315] proved that a polynomial $P(x) = \sum_{j=0}^{m} A_j x^j \in \mathbb{R}[x]$ satisfying the conditions $A_{m-j} = A_j$ for $j \leq m$ and

$$|A_m| \geq \sum_{j=0}^{m} |A_j - A_m| \tag{2.30}$$

has all zeroes on the unit circle. Schinzel [466], Kim and Park [299], Kim and Lee [298], Lalin and Smyth [324] obtained generalizations of this result. Suzuki [505] established correlations between this problem and the theory of canonical systems of ordinary linear differential equations. Lakatos and Losonczi [318] [319] proved that, for a self-inversive polynomial $P_m(z) = \sum_{j=0}^{m} A_k z^k \in \mathbb{C}[z]$, $m \geq 1$, the roots of P_m are all on the unit circle if $|A_m| \geq \sum_{k=1}^{m-1} |A_k|$; moreover if this inequality is strict then the zeroes $e^{i\varphi_l}$, $l = 1, \ldots, m$, are simple and can be arranged such that, with $\beta_m = \arg(A_m \left(\overline{A_0}/A_m\right)^{1/2})$,

$$\frac{2((l-1)\pi - \beta_m)}{m} < \varphi_l < \frac{2(l\pi - \beta_m)}{m}. \tag{2.31}$$

In the direction of Salem polynomials, ν-Salem polynomials and more [296] [454], a generalization was obtained by Vieira [522]: if a sufficient condition is satisfied then a self-inversive polynomial has a fixed number of roots on the unit circle. Namely, let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \in \mathbb{C}[z]$, $a_n \neq 0$, be such that $P(z) = \omega\, z^n\, \overline{P}(1/z)$ with $|\omega| = 1$. If the inequality

$$|a_{n-l}| > \frac{1}{2}\left(\frac{n}{n-2l}\right) \sum_{k=0, k\neq l, k\neq n-l}^{n} |a_k|, \qquad l < n/2 \tag{2.32}$$

is satisfied, then $P(z)$ has exactly $n - 2l$ roots on the unit circle and these roots are simple; moreover, if n is even and $l = n/2$, then $P(z)$ has no root on $|z| = 1$ if the inequality $|a_{n/2}| > \sum_{k=0, k\neq n/2}^{n} |a_k|$ is satisfied.

Questions of irreducibility of P as a function of the coefficient vector were studied in [175]. Flammang [219] obtained new inequalities for the Mahler measure $\mathrm{M}(P)$ [528], and Flammang, Rhin and Sac-Epée [222] proved relations between the integer transfinite diameter and polynomials having a small Mahler measure.

The lacunarity of P and the minoration of $\mathrm{M}(P)$ are correlated: when P is a noncyclotomic (sparse) integer polynomial, Dobrowolski, Lawton and Schinzel [164], then Dobrowolski [162], [163], obtained lower bounds of $\mathrm{M}(P)$ as a function of the number k of its nonzero coefficients: e.g. in [163], with $a < 0.785$,

$$\mathrm{M}(P) \geq 1 + \frac{1}{\exp(a 3^{\lfloor (k-2)/4 \rfloor} k^2 \operatorname{Log} k)}, \tag{2.33}$$

and, if P is irreducible,

$$\mathrm{M}(P) \geq 1 + \frac{0.17}{2^{\lfloor k/2 \rfloor} \lfloor k/2 \rfloor !}. \tag{2.34}$$

Dobrowolski, then McKee and Smyth [354] obtained minorants of $\mathrm{M}(P)$ for the reciprocal polynomials $P(z) = z^n D_A(z + 1/z)$ where D_A is the characteristic polynomial of an integer symmetric $n \times n$ matrix A; McKee and Smyth obtained $\mathrm{M}(P) = 1$ or $\mathrm{M}(P) \geq 1.176280\ldots$ (Lehmer's number) solving the problem of Lehmer for the family of such polynomials. Dobrowolski (2008) proved that many totally real integer polynomials P cannot be represented by integer symmetric matrices A, disproving a conjecture of Estes and Guralnick. Dubickas and Konyagin [183] [184], then Dubickas and Sha [185] [186] with the counting problem, studied the number of integer polynomials as a function of their (naïve) height and resp. their Mahler measure. The next two theorems show that Lehmer's Conjecture is true for the set of the algebraic integers which are the roots of polynomials in particular families of monic integer polynomials.

Theorem 2.3 (Borwein, Dobrowolski, Mossinghoff [67]). *Let $m \geq 2$, and let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial of degree D with no cyclotomic factors that satisfies*

$$f(X) \equiv X^D + X^{D-1} + \ldots + X^2 + X + 1 \qquad \bmod m.$$

Then

$$\sum_{f(\alpha)=0} h(\alpha) \geq \frac{D}{D+1} C_m, \tag{2.35}$$

where we may take

$$C_2 = \frac{1}{4} \operatorname{Log} 5 \qquad \textit{and} \quad C_m = \operatorname{Log} \frac{\sqrt{m^2+1}}{2} \quad \textit{for } m \geq 3.$$

Theorem 2.4 (Silverman [486]). *For all $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ with the following property: let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial of degree D such that*

$$f(X) \textit{ is divisible by } X^{n-1} + X^{n-2} + \ldots + X + 1 \quad \textit{in } (\mathbb{Z}/m\mathbb{Z})[X].$$

for some integers $m \geq 2$ and $n \geq \max\{\epsilon D, 2\}$. Suppose further that no root of $f(X)$ is a root of unity. Then

$$\sum_{f(\alpha)=0} h(\alpha) \geq \; C_\epsilon \operatorname{Log} m. \tag{2.36}$$

2.2. Limit points. The set of limit points of $\{\mathrm{M}(P) \mid P(X) \in \mathbb{Z}[X]\}$ is obtained by the following useful Theorem of Boyd and Lawton [79], [80], [333], which correlates Mahler measures of univariate polynomials to Mahler measures of multivariate polynomials:

Theorem 2.5. *let $P(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$) and $\underline{r} = (r_1, r_2, \ldots, r_n)$, $r_i \in \mathbb{N}_{>0}$. Let $P_{\underline{r}}(x) := P(x^{r_1}, x^{r_2}, \ldots, x^{r_n})$. Let*

$$q(\underline{r}) \ := \ \min\{H(\underline{t}) \mid \underline{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}^n, \underline{t} \neq (0, \ldots 0), \ \sum_{j=1}^{n} t_j r_j = 0\},$$

where $H(\underline{t}) = \max\{|t_i| \mid 1 \leq j \leq n\}$. Then

$$\lim_{q(\underline{r}) \to \infty} \mathrm{M}(P_{\underline{r}}) = \mathrm{M}(P). \tag{2.37}$$

This theorem allows the search of small limit points of (univariate) Mahler's measures, by several methods [91]; a more recent method relies upon the (EM) Expectation-Maximization algorithm [357] [194]. The set of limit points of the Salem numbers was investigated either by the "Construction of Salem" [74], [75], or sets of solutions of some equations by Boyd and Parry [92]. Everest [198], then Condon [130], [131], established asymptotic expansions of the ratio $\mathrm{M}(P_{\underline{r}})/\mathrm{M}(P)$. For bivariate polynomials $P(x, y) \in \mathbb{C}[x, y]$ such that P and $\partial P/\partial y$ do not have a common zero on $\mathbb{T} \times \mathbb{C}$, then Condon (Theorem 1 in [131]) establishes the expansion, for k large enough,

$$\mathrm{Log}\left(\frac{\mathrm{M}(P_{\underline{r}})}{\mathrm{M}(P)}\right) = \mathrm{Log}\left(\frac{\mathrm{M}(P(x, x^n))}{\mathrm{M}(P(x, y))}\right) = \sum_{j=2}^{k-1} \frac{c_j}{n^j} \ + O_{P,k}\left(\frac{1}{n^k}\right), \tag{2.38}$$

(n is not the degree of the univariate polynomial $P(x, x^n)$) where the coefficients c_j are values of a quasiperiodic function of n, as finite sums of real and imaginary parts of values of Li_a polylogarithms, $2 \leq a \leq j$, weighted by some rational functions deduced from the derivatives of P, where the sums are taken over algebraic numbers deduced from the intersection of $\mathbb{T}^2$ and the hypersurface of $\mathbb{C}^2$ defined by P (affine zero locus). In particular, if P is an integer polynomial, the coefficients c_j are $\overline{\mathbb{Q}}$-linear combinations of polylogarithms evaluated at algebraic arguments. For instance, for $P(x, y) = -1 + x + y$, $G_n(x) = -1 + x + x^n$, the coefficient $c_2(n)$ in the expansion of $\mathrm{Log}\,(\mathrm{M}(G_n)/\mathrm{M}(P))$, though a priori quasiperiodic, is a periodic function of n modulo 6 which can be directly computed (Theorem 1.3 in [521]), as: for n odd:

$$c_2(n) = \begin{cases} \sqrt{3}\pi/18 = +0.3023\ldots & \text{if } \ n \equiv 1 \text{ or } 3 \pmod 6, \\ -\sqrt{3}\pi/6 = -0.9069\ldots & \text{if } \ n \equiv 5 \pmod 6, \end{cases}$$

for n even:

$$c_2(n) = \begin{cases} -\sqrt{3}\pi/36 = -0.1511\ldots & \text{if } \ n \equiv 0 \text{ or } 4 \pmod 6, \\ +\sqrt{3}\pi/12 = +0.4534\ldots & \text{if } \ n \equiv 2 \pmod 6. \end{cases}$$

For the height one trinomial $1+x+y$, Corollary 2 in [131] gives the coefficients c_j, $j \geq 2$, as linear combinations of polylogaritms evaluated at third roots of unity, with coefficients coming from the Stirling numbers of the first and second kind, i.e. in $\frac{1}{2\pi}\mathbb{Z}[\sqrt{3}]$. The method of Condon also provides the other coefficients c_j, $j \geq 3$, for the trinomial $-1+x+y$ in the same way.

Doche in [165] obtains an alternate method to Boyd-Lawton's Theorem, in the objective of obtaining estimates of the Mahler measures of bivariate polynomials: let $P(y,z) \in \mathbb{C}[y,z]$ be a polynomial such that $\deg_z(P) > 0$, let $\xi_n := e^{\frac{2i\pi}{n}}$ and assume that $P(\xi_n^k, z) \not\equiv 0$ for all n, k. Then

$$\mathrm{M}(P(y,z))^{1/\deg_z(P)} = \lim_{n\to\infty} \mathcal{M}\Big(\prod_{k=1}^{n} P(\xi_n^k, z)\Big) \tag{2.39}$$

(n is not the degree of the univariate polynomial $\prod_{k=1}^{n} P(\xi_n^k, x)$). Doche's and Condon's methods cannot be used for the problem of Lehmer since 1 does not belong to the first derived set of the set of Mahler measures of univariate integer polynomials (assuming true Lehmer's Conjecture).

Limit points of Mahler measures of univariate polynomials are algebraic numbers or transcendental numbers: by (2.39) and Theorem 2.5, they are Mahler measures of multivariate polynomials. The problem of finding a positive lower bound of the set of such limit points of Mahler measures is intimately correlated to the problem of Lehmer [462]. Smyth (1971)[492] found the remarkable identity:

$$\operatorname{Log} \mathrm{M}(1+x+y) = \Lambda \tag{2.40}$$

with

$$\Lambda := \exp\big(\frac{3\sqrt{3}}{4\pi}\mathrm{L}(2,\chi_3)\big) \;=\; \exp\Big(\frac{-1}{\pi}\int_0^{\pi/3} \operatorname{Log}\big(2\sin(\frac{x}{2})\big)dx\Big) \;=\; 1.38135\ldots, \tag{2.41}$$

$L(s,\chi_3) := \sum_{m\geq 1}\frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3, with χ_3 the uniquely specified odd character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to whether $m \equiv 0$, 1 or 2 (mod 3), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol).

The values of logarithmic Mahler measures of multivariate polynomials are sums of special values of different L-functions, often conjecturally [88]; the remarkable conjectural identities discovered by Boyd in [88] (1998), also by Smyth [492] and Ray [437], serve as starting points for further studies, some of them being now proved, e.g. [320], [321], [448], [461], [548].

Indeed, after the publication of [88], Deninger [149] reinterpreted the logarithmic Mahler measures $\operatorname{Log}\mathrm{M}(P)$ of Laurent polynomials $P \in \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$ as topological entropies in the theory of dynamical systems of algebraic origin, with $\mathbb{Z}^n$-actions (Schmidt [469], Chap. V, Theorem 18.1; Lind, Schmidt and Ward [347]). This new approach makes a link with higher K-theory, mixed motives (Deninger [150]), real Deligne cohomology, the Bloch-Beilinson conjectures on special values of L- functions, and Mahler measures. There are two cases: either P

does not vanish on $\mathbb{T}^n$, in which case $\operatorname{Log} \mathrm{M}(P)$ is a Deligne period of the mixed motive over $\mathbb{Q}$ which corresponds to the nonzero symbol $\{P, x_1, \ldots, x_n\}$ (Theorem 2.2 in [149]), or, if P vanishes on $\mathbb{T}^n$, under some assumptions, it is a difference of two Deligne periods of certain mixed motives, equivalently, the difference of two symbols evaluated against topological cycles ("integral K-theory cycles") (Theorem 3.4 in [149], with a motivic reinterpretation in Theorem 4.1 in [149]).

Let $\mathbb{G}_{m,A}^n := \operatorname{Spec}(A[\mathbb{Z}^n])$ be the split n-torus defined over the commutative ring $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. The polynomial $P \not\equiv 0$ defines the irreducible closed subscheme $Z := \operatorname{Spec}(\mathbb{Z}[\mathbb{Z}^n]/(P)) \subset \mathbb{G}_{m,\mathbb{Z}}^n, Z \neq \mathbb{G}_{m,\mathbb{Z}}^n$, For any coherent sheaf $\mathcal{F}$ on $\mathbb{G}_{m,A}^n$, the group $\Gamma(\mathbb{G}_{m,A}^n, \mathcal{F})$ of global sections, equipped with the discrete topology, admits a Pontryagin dual $\Gamma(\mathbb{G}_{m,A}^n, \mathcal{F})^*$ which is a compact group. This compact group endowed with the canonical $\mathbb{Z}^n$-action constitute an arithmetic dynamical system for which the entropy can be defined according to [469], and correlated to the Mahler measure (Theorem 18.1 in [469]); the application to P, $A = \mathbb{Z}$ and $\mathcal{F} = \mathcal{O}_Z$ provides the identity with the entropy: $h(\mathcal{O}_Z) = \operatorname{Log} \mathrm{M}(P)$. The definition

$$\operatorname{Log} \mathrm{M}(P) := \frac{1}{(2i\pi)^n} \int_{\mathbb{T}^n} \operatorname{Log} |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \tag{2.42}$$

corresponds to the integration of a differential form in connection with the cup-product $\operatorname{Log}|P| \cup \operatorname{Log}|x_1| \cup \ldots \cup \operatorname{Log}|x_n|$ in the real Deligne cohomology of $\mathbb{G}_{m,\mathbb{R}}^n \setminus Z_{\mathbb{R}}$. The link between the L-series $L(M,s)$ of a motive M, and its derivatives, Deligne periods, and the Beilinson conjectures, comes from the Conjecture of Deligne-Scholl ([149] Conjecture 2.1). Further, Rodriguez-Villegas [447] studied the conditions of applicability of the conjectures of Bloch-Beilinson for having logarithmic Mahler measures $\operatorname{Log} \mathrm{M}(P)$ expressed as L-series.

The example of $\operatorname{Log} \mathrm{M}(P(x_1, x_2)) = \operatorname{Log} \mathrm{M}((x_1 + x_2)^2 + k)$, with $k \in \mathbb{N}$, is computed in Proposition 19.10 in [469]. For instance, for $k = 3$, we have

$$\operatorname{Log} \mathrm{M}((x_1 + x_2)^2 + 3) = \frac{2}{3} \operatorname{Log} 3 + \frac{\sqrt{3}}{\pi} L(2, \chi_3). \tag{2.43}$$

Deninger shows (ex. [149] p. 275) the cohomological origin of each term: $\frac{\sqrt{3}}{\pi} L(2, \chi_3)$ from the first $\mathcal{M}$-cohomology group $H_{\mathcal{M}}^1(\partial A, \mathbb{Q}(2))$, $\frac{2}{3}\operatorname{Log} 3$ from the second $\mathcal{M}$-cohomology group $H_{\mathcal{M}}^2(Z^{\mathrm{reg}}, \mathbb{Q}(2))$. Bornhorn [65], and later Standfest [497], reinvestigated further the conjectural identities of Boyd [88] in particular the formulas of mixed type, containing several types of L-series. The logarithmic Mahler measure $\operatorname{Log} \mathrm{M}(P)$ is then written

$$= *L'(s_1, \chi) + *L'(E, s_2), \tag{2.44}$$

where χ is a Dirichlet character, $L(s_1, \chi)$ the corresponding Dirichlet series, $L(E, s_2)$ the Hasse-Weil L-function of an elliptic curve $E/\mathbb{Q}$ deduced from P, and s_1, s_2 algebraic numbers. Following Deninger and Rodriguez-Villegas, Lalin [320], [321], introduces techniques for applying Goncharov's constructions of the regulator on polylogarithmic motivic complexes in the objective of computing Mahler measures of multivariate Laurent polynomials. With some three-variable polynomials, whose zero loci define singular $K3$ surfaces, Bertin et al [49] prove that the logarithmic Mahler measure is of the form

$$*L'(g, O) + *L'(\chi, -1) \tag{2.45}$$

where g is the weight 4 newform associated with the $K3$ surface and χ is a quadratic character. Other three-variable Mahler measures are associated with special values of modular and Dirichlet L-series [453]. Some four-variables polynomials define a Calabi-Yau threefold and the logarithmic Mahler measure is of the form

$$*L'(f, O) + *\zeta'(-2) \tag{2.46}$$

where f is a normalized newform deduced from the Dedekind eta function [410]. Multivariable Mahler measures are also related to mirror symmetry and Picard - Fuchs equations in Zhou [547].

In comparison, the limit points of the set S of Pisot numbers were studied by analytical methods by Amara [7]. The set of values

$$\{\operatorname{Log} \mathrm{M}(P) \mid P \in \mathbb{Z}[\mathbb{Z}^n], n \geq 1\} \tag{2.47}$$

is conjecturally (Boyd [80]) a closed subset of $\mathbb{R}$ for the usual topology.

3. Salem numbers, interlacing, association equations, dynamics and dichotomy, partially and totally real algebraic numbers

The set of (positive) Salem numbers is a subcollection of Mahler measures of algebraic integers.

A negative Salem number is by definition the opposite of a Salem number, a negative Pisot number is by definition the opposite of a Pisot number. Negative and (positive) Salem numbers occur in number theory, e.g. for graphs or integer symmetric matrices in [353], [354], [355], and in other domains (cf § 6.4), like Alexander polynomials of links of the variable "$-x$", e.g. in Theorem 6.11 (Hironaka [263]).

The set of Pisot numbers admits the minorant Θ by a result of Siegel [478]. The set S of Pisot numbers is closed (Salem [452]). Its successive derived sets $S^{(i)}$, were extensively studied by Dufresnoy and Pisot [190], and their students (Amara [7], ...), by means of compact families of meromorphic functions, following ideas of Schur. This analytic approach is reported extensively in the book [48]. Salem (1944) (Samet [454], [451]) proved that every Salem number is the quotient of two Pisot numbers. Apart from this result, few relations were known between Salem numbers and Pisot numbers. The set of Pisot numbers is better known than the set of Salem numbers.

To study Salem numbers association equations between Pisot numbers and Salem numbers have been introduced by Boyd ([74] Theorem 4.1), Bertin and Pathiaux-Delefosse ([46], [47] pp 37-46, [48] chapter 6). Association equations are generically written

$$(X^2+1)P_{Salem} = XP_{Pisot}(X) + P^*_{Pisot}(X), \tag{3.1}$$

to investigate the links between infinite collections of Pisot numbers and a given Salem number, of respective minimal polynomials P_{Pisot} and P_{Salem}.

The theory of interlacing of roots on the unit circle is somehow a powerful tool for studying classes of polynomials having special geometry of zeroes of modulus one [314], [316], [317], in particular Salem polynomials. In [46] ([47]) Bertin and Boyd obtained two interlacing theorems, namely Theorem A and Theorem B, turned out to be fruitful with their limit-interlacing versions. McKee and Smyth in [355] obtained new interlacing theorems. Theorem 5.3 in [355] shows that all Pisot numbers are produced by a suitable interlacing condition, supporting the second Conjecture of Boyd, i.e. (2.14). Similarly Theorem 7.3 in [355], using Boyd's association Theorems, shows that all Salem numbers are produced by interlacing and that a classification of Salem numbers can be made.

In [250] Guichard and Verger-Gaugry reconsider the interest of the interlacing Theorems of [46], as potential tools for the study of the limit points of sequences of algebraic integers in neighbourhoods of Salem numbers, as analogues of those of McKee and Smyth. Focusing on Theorem A of [46] they obtain association equations between Salem polynomials (and/or cyclotomic polynomials) and expansive polynomials, generically

$$(z-1)P_{Salem}(z) = zP_{expansive}(z) - P^*_{expansive}(z), \tag{3.2}$$

and deduce rational n-dimensional representations of the neighbourhoods of a Salem number of degree n, using the formalism of Stieltjès continued fractions. These representations are tools to study the limit points of sequences of algebraic numbers in the neighbourhood of a given Salem number, for instance by dynamical methods.

In the same direction association equations between Salem numbers and (generalized) Garsia numbers are obtained by Hare and Panju [255] using the theory of interlacing on the unit circle.

As counterpart, Salem numbers are linked to units: they are given by closed formulas from Stark units in Chinburg [125] [126], exceptional units in Silverman [484]. From [126] they are related to relative regulators of number fields [12], [13], [129], [134], [237].

Dynamics of Salem numbers, Parry Salem numbers. Let $\beta > 1$ and assume $\beta \notin \mathbb{N}$. Let $T_\beta : [0,1] \to [0,1], x \to \{\beta x\}$ the β-transformation. The greedy β-expansion of 1 is by definition denoted by

$$d_\beta(1) = 0.t_1t_2t_3\ldots \qquad \text{and uniquely corresponds to} \qquad 1 = \sum_{i=1}^{+\infty} t_i\beta^{-i}\,, \tag{3.3}$$

where

$$t_1 = \lfloor\beta\rfloor, t_2 = \lfloor\beta\{\beta\}\rfloor = \lfloor\beta T_\beta(1)\rfloor, t_3 = \lfloor\beta\{\beta\{\beta\}\}\rfloor = \lfloor\beta T_\beta^2(1)\rfloor, \ldots \tag{3.4}$$

Multiplying (3.3) by β itself gives the β-expansion of β. The sequence $(t_i)_{i\geq 1}$ is given by the orbit of one $(T_\beta^j(1))_{j\geq 0}$ by

$$T_\beta^0(1) = 1,\ T_\beta^j(1) = \beta^j - t_1\beta^{j-1} - t_2\beta^{j-2} - \ldots - t_j \in \mathbb{Z}[\beta] \cap [0,1] \tag{3.5}$$

for all $j \geq 1$. The digits t_i belong to the finite alphabet $\mathcal{A}_\beta = \{0, 1, \ldots, \lfloor\beta\rfloor\}$. We say that $d_\beta(1)$ is finite if it ends in infinitely many zeros.

Definition 3.1. *If $d_\beta(1)$ is finite or ultimately periodic (i.e. eventually periodic), then the real number $\beta > 1$ is said to be a Parry number. In particular, a Parry number β is said to be simple if $d_\beta(1)$ is finite.*

Let $\mathbb{P}$ denote the set of Perron numbers. The set $\mathbb{P}$ of Perron numbers contains the subset $\mathbb{P}_P$ of all (simple and nonsimple) Parry numbers by a result of Lind [342] (Blanchard [58], Boyle [94], Denker, Grillenberger and Sigmund [153], Frougny in [350] chap.7). The set $\mathrm{T} \subset \mathbb{P}$ of Salem numbers is separated into two disjoint subsets

$$\mathrm{T} = \big(\mathrm{T} \cap \mathbb{P}_P\big) \ \cup\ \big(\mathrm{T} \setminus \mathrm{T} \cap \mathbb{P}_P\big) \tag{3.6}$$

Salem numbers which are Parry numbers are called *Parry Salem numbers.*

Definition 3.2. *If β is a simple Parry number, with $d_\beta(1) = 0.t_1t_2\ldots t_m$, $t_m \neq 0$, the polynomial*

$$P_{\beta,P}(X) := X^m - t_1X^{m-1} - t_2X^{m-2} - \ldots t_m \tag{3.7}$$

is called the Parry polynomial of β. If β is a Parry number which is not simple, with $d_\beta(1) = 0.t_1t_2\ldots t_m(t_{m+1}t_{m+2}\ldots t_{m+p+1})^\omega$ and not purely periodic (m is $\neq 0$), then

$$\begin{aligned} P_{\beta,P}(X) := X^{m+p+1} - t_1X^{m+p} - t_2X^{m+p-1} - \ldots - t_{m+p}X - t_{m+p+1} \\ - X^m + t_1X^{m-1} + t_2X^{m-2} + \ldots + t_{m-1}X + t_m \end{aligned} \tag{3.8}$$

is the Parry polynomial of β. If β is a nonsimple Parry number such that $d_\beta(1) = 0.(t_1t_2\ldots t_{p+1})^\omega$ is purely periodic (i.e. $m = 0$), then

$$P_{\beta,P}(X) := X^{p+1} - t_1X^p - t_2X^{p-1} - \ldots - t_pX - (1 + t_{p+1}) \tag{3.9}$$

is the Parry polynomial of β. By definition the degree d_P of $P_{\beta,P}(X)$ is respectively m and $p+1$ in the first and third case, and $m+p+1$ in the second case (m is taken minimal).

If β is a Parry number, the Parry polynomial $P_{\beta,P}(X)$, belonging to the ideal $P_\beta(X)\mathbb{Z}[X]$, admits β as simple root and is often not irreducible. For instance Parry polynomials have applications in the geometric and topological properties of Thurston's Master Teapot and the Thurston set (defined in Thurston [511]) for superattracting unimodal continuous self-maps of intervals [99].

The small Salem numbers found by Lehmer in [336], reported below, either given by their minimal polynomial or equivalently by their β-expansion ("dynamization"), are Parry Salem numbers:

$\deg(\beta)$	$\beta = \mathrm{M}(\beta)$	minimal pol. of β	$d_\beta(1)$
4	$1.722\ldots$	$X^4 - X^3 - X^2 - X + 1$	$0.1(100)^\omega$
6	$1.401\ldots$	$X^6 - X^4 - X^3 - X^2 + 1$	$0.1(0^2 10^4)^\omega$
8	$1.2806\ldots$	$X^8 - X^5 - X^4 - X^3 + 1$	$0.1(0^5 10^5 10^7)^\omega$
10	$1.17628\ldots$	$X^{10} + X^9 - X^7 - X^6 - X^5$ $-X^4 - X^3 + X + 1$	$0.1(0^{10}10^{18}10^{12}10^{18}10^{12})^\omega$

Let $\Theta = 1.3247\ldots$ be the dominant root of $X^3 - X - 1$. If $\beta \in (1, \Theta]$ is a real number for which the Rényi β-expansion of unity starts by $d_\beta(1) = 0.10^m 1\ldots$, then the dynamical degree of β is by definition $\mathrm{dyg}(\beta) := m + 2$. The respective dynamical degrees $\mathrm{dyg}(\beta)$ of the last two Salem numbers β are 7 and 12, with Parry polynomials of respective degrees 20 and 75.

The smallest Salem numbers of degree ≤ 44 are all known from the complete list of Mahler measures ≤ 1.3 of Mossinghoff [389] of irreducible monic integer polynomials of degree ≤ 180. Recall that, for $n \geq 5$, θ_n denotes the real root > 0 of the polynomial $-1 + X + X^n$.

dyg	deg	β	$P_{\beta,P}$	$d_\beta(1)$
5	3	$\theta_5^{-1} = 1.324717$	5	$0.10^3 1$
6	18	1.29567	22	$0.1(0^4 10^9 10^6)^\omega$
6	10	1.293485	12	$0.1(0^4 10^6)^\omega$
6	24	1.291741	24 irr.	$0.1(0^4 10^{11} 10^6)^\omega$
6	26	1.286730	30	$0.1(0^4 10^{17} 10^6)^\omega$
6	34	1.285409	38	$0.1(0^4 10^{25} 10^6)^\omega$
6	30	1.285235	45	$0.1(0^4 10^{32} 10^6)^\omega$
6	44	1.285199	66	$0.1(0^4 10^{54} 10^6)^\omega$
6	6	$\theta_6^{-1} = 1.285199$	6 irr.	$0.10^4 1$
7	26	1.285196	44	$0.1(0^5 10^5 10^5 10^5 10^5 10^5 10^7)^\omega$
7	26	1.281691	..	$0.1(0^5 10^5 10^9 10^5 10^{17} 10^7 10^6 10^6 10^7 10^{12})^\omega$
7	8	1.280638	20	$0.1(0^5 10^5 10^7)^\omega$
7	10	1.261230	14	$0.1(0^5 10^7)^\omega$
7	24	1.260103	28	$0.1(0^5 10^{13} 10^7)^\omega$
7	18	1.256221	36	$0.1(0^5 10^{21} 10^7)^\omega$
7	7	$\theta_7^{-1} = 1.255422$	7 irr.	$0.10^5 1$
8	18	1.252775	120	$0.1(0^6 10^6 10^{10} 10^{16} 10^{12} 10^7 10^{12} 0^{16} 10^{10} 10^6 10^8)^\omega$
8	12	1.240726	48	$0.1(0^6 10^{11} 10^7 10^{11} 10^8)^\omega$
8	20	1.232613	41	$0.1(0^6 10^{24} 10^8)^\omega$
8	8	$\theta_8^{-1} = 1.232054$	8 irr.	$0.10^6 1$
9	10	1.216391	18	$0.1(0^7 10^9)^\omega$
9	9	$\theta_9^{-1} = 1.213149$	9 irr.	$0.10^7 1$
10	14	1.200026	20	$0.1(0^8 10^{10})^\omega$
10	10	$\theta_{10}^{-1} = 1.197491$	10 irr.	$0.10^8 1$
11	9	$\theta_{11}^{-1} = 1.184276$	11	$0.10^9 1$
12	10	1.176280	75	Lehmer's number : $0.1(0^{10} 10^{18} 10^{12} 10^{18} 10^{12})^\omega$
12	12	$\theta_{12}^{-1} = 1.172950$	12 irr.	$0.10^{10} 1$

Table 1. Smallest Salem numbers $\beta < 1.3$ of degree ≤ 44, which are Parry numbers, computed from the "list of Mossinghoff" [389]. In Column 1 is reported the dynamical degree of β. Column 4 gives the degree d_P of the Parry polynomial $P_{\beta,P}$ of β; $P_{\beta,P}$ is reducible except if "irr." is mentioned.

Table 1 gives the subcollection of those Salem numbers β which are Parry numbers within the intervals of extremities the Perron numbers θ_n^{-1}, $n = 5, 6, \ldots, 12$. In each interval dyg(β) is constant while the increasing order of the βs corresponds to a certain disparity of the degrees deg(β). The remaining Salem numbers in [389] are very probably nonParry numbers though proofs are not available yet; they are not included in Table 1. Apart from them, the other Salem numbers which exist in the intervals $(\theta_n^{-1}, \theta_{n-1}^{-1})$, $n \geq 6$, if any, should be of (usual) degrees deg > 180.

For some families of algebraic integers, the "dynamization" of the minimal polynomial is known explicitly, the digits being algebraic functions of the coefficients of the minimal polynomials: e.g. for Salem numbers of degree 4 and 6 (Boyd [83] [84]), for Salem numbers of degree 8 (Hichri [257] [258] [259]), for Pisot numbers (Boyd [87], Frougny and Solomyak [229], Bassino [37] in the cubic case, Hare [253], Panju [409] for regular Pisot numbers). Schmidt [468], independently Bertrand-Mathis [51], proved that Pisot numbers are Parry numbers. Many Salem numbers are known to be Parry numbers. For Salem numbers of degree 4 it is the case [85]. For Salem numbers of degree ≥ 6, Boyd [86] gave an heuristic argument and a probabilistic model, for the existence of nonParry Salem numbers as a metric approach of the dichotomy of Salem numbers. This approach, coherent with Thurston's one ([511], p. 11), is in contradiction with the conjecture of Schmidt. Hichri [257] [258] [259] further developped the heuristic approach of Boyd for Salem numbers of degree 8. The Salem numbers of degree ≤ 8 are all greater than $1.280638\ldots$ from [389].

Using the "Construction of Salem", Hare and Tweedle [256] obtain convergent families of Salem numbers, all Parry numbers, having as limit points the limit points of the set S of Pisot numbers in the interval $(1, 2)$ (characterized by Amara [7]). These families of Parry Salem numbers do not contain Salem numbers smaller than Lehmer's number.

Parry numbers are studied from the negative β-shift. The negative β-shift was introduced by Ito and Sadahiro [280] (Liao and Steiner [341], Masakova and Pelantova [369], Nguéma Ndong [395] [396]) and the generalized β-shift by Gora [242] [243] and Thompson [508]), in the general context of iterated interval maps and post-critical finite (PCF) interval maps [384] [511]. Indeed, Kalle [289] showed that nonisomorphisms exist between the β-shift and the negative β-shift, possibly leading to new Parry numbers arising from "negative" Parry numbers (called Ito-Sadahiro numbers in [369], Irrap numbers in [341] reading "Parry" from the right to the left). More generally negative Parry numbers and generalized Parry numbers are defined as poles of the corresponding dynamical zeta functions [395] [396] [509]. Negative Pisot and Salem numbers appear naturally in several domains: as roots of Newman polynomials [254], in association equations with negative Salem polynomials [250], in topology with the Alexander polynomials of pretzel links (§ 6.4), as Coxeter polynomials for Coxeter elements (Hironaka [263]; § 6.2), in studies of numeration with negative bases (Frougny and Lai [228]). Generalizing Solomyak's constructions to the generalized β-shift, Thompson [509] investigates the fractal domains of existence of the conjugates.

Partially and totally real algebraic numbers. For totally real algebraic integers, the basic result is the minoration (3.10) given by Schinzel. Let us recall it. Let $\mathbb{L}$ be a totally real algebraic number field, or a CM field (a totally complex quadratic extension of a totally real number field). Then, for any nonzero algebraic integer $\alpha \in \mathbb{L}$, of degree d, not being a root of unity, Schinzel [463] obtained the minoration

$$\mathrm{M}(\alpha) \geq \theta_2^{-d/2} = \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}. \tag{3.10}$$

More precisely, if $H(X) \in \mathbb{Z}[X]$ is monic with degree d, $H(0) = \pm 1$ and $H(-1)H(1) \neq 0$, and if the zeroes of H are all real, then

$$\mathrm{M}(H) \geq \big(\frac{1+\sqrt{5}}{2}\big)^{d/2} \tag{3.11}$$

with equality if and only if $H(X)$ is a power of $X^2 - X - 1$. Bertin [45] improved Schinzel's minoration (3.11) for the algebraic integers α, of degree d, of norm $N(\alpha)$, which are totally real, as

$$\mathrm{M}(\alpha) \geq \max\big\{\theta_2^{-d/2}, \sqrt{N(\alpha)}\,\theta_2^{-\frac{d}{2|N(\alpha)|^{1/d}}}\big\}. \tag{3.12}$$

The totally real algebraic numbers form a subfield, denoted by $\mathbb{Q}^{tr}$, in $\overline{\mathbb{Q}} \cap \mathbb{R}$. Following [45], the natural extension of a Salem number is a ν-Salem number, intermediate between Salem numbers and totally real algebraic numbers. Let us define a ν-Salem as an algebraic integer α having ν conjugates outside $\{|z| \geq 1\}$ and at least one conjugate $\alpha^{(q)}$ satisfying $|\alpha^{(q)}| = 1$; denote by $2\nu + 2k$ its degree. Such an algebraic integer is totally real in the sense that its conjugates of modulus > 1 are all real, and then

$$\mathrm{M}(\alpha) \geq \theta_2^{-\frac{\nu}{2k/\nu}}. \tag{3.13}$$

Further, extending Pisot numbers, lower bounds of $\mathrm{M}(\alpha)$ were obtained by Zaimi [539] [540] when α is a K-Pisot number. Rhin [442], following Zaimi (cf references in [442]), obtained minorations of $\mathrm{M}(\alpha)$ for totally positive algebraic integers α as functions of the discriminant $\mathrm{disc}(\alpha)$. Let K be an algebraic number field and α an algebraic integer of minimal polynomial R over K; by definition [44] α is K-Pisot number if, for any embedding $\sigma : K \to \mathbb{C}$, $\sigma(R_K)$ admits only one root of modulus > 1 and no root of modulus 1. Denote by Δ the discriminant of K. Lehmer's problem and small discriminants were studied by Mahler (1964), Bertrand [50], Matveev [376], Rhin [442]. For any K-Pisot number α, Zaimi [539], [540], showed

$$\mathrm{M}(\alpha) \geq \frac{\sqrt{\Delta}}{2} \qquad K \text{ quadratic}, \tag{3.14}$$

$$\mathrm{M}(\alpha) \geq \frac{\Delta^{1/4}}{\sqrt{6}} \qquad K \text{ cubic and totally real}. \tag{3.15}$$

Other minorations of totally positive algebraic integers were obtained by Mu and Wu [392]. Denote $\mathbb{Z}^{tr} := \mathbb{Q}^{tr} \cap \mathcal{O}_{\overline{\mathbb{Q}}}$. Because the degree d of the algebraic number commonly appears in the exponent of the lower bounds of the Mahler measure, the (absolute logarithmic) Weil height h is more adapted than the Mahler measure. Schinzel's bound, originally concerned with the algebraic integers in $\mathbb{Z}^{tr}$, reads:

$$\alpha \in \mathbb{Z}^{tr}, \alpha \neq 0, \neq \pm 1 \quad \Rightarrow \quad h(\alpha) \geq h\big(\theta_2^{-1}\big) = \frac{1}{2}\mathrm{Log}\,(\frac{1+\sqrt{5}}{2}) = 0.2406059\ldots$$

Smyth [491], [492], proved that the set

$$\{\exp(h(\alpha)) \mid \alpha \text{ totally real algebraic integer}, \alpha \neq 0, \neq \pm 1\} \tag{3.16}$$

is everywhere dense in $(1.31427\ldots, \infty)$; in other terms

$$\liminf_{\alpha \in \mathbb{Z}^{tr}} h(\alpha) \;\leq\; \mathrm{Log}\,(1.31427\ldots) = 0.27328\ldots \tag{3.17}$$

Flammang [218] completed Smyth's results by showing

$$\liminf_{\alpha \in \mathbb{Z}^{tr}} h(\alpha) \;\geq\; \frac{1}{2}\mathrm{Log}\,(1.720566\ldots) = 0.271327\ldots \tag{3.18}$$

with exactly 6 isolated points in the interval $(0, 0.271327\ldots)$, the smallest one being Schinzel's bound $0.2406059\ldots$ In fact, passing from algebraic integers to algebraic numbers lead to various smaller minorants of $h(\alpha)$: for instance $(\mathrm{Log}\, 5)/12 = 0.134119\ldots$ by Amoroso and Dvornicich [23] for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity, where $\mathbb{L}/\mathbb{Q}$ is an abelian extension of number fields, or, by Ishak, Mossinghoff, Pinner and Wiles [278], for nonzero $\alpha \in \mathbb{Q}(\xi_m)$, not being a root of unity,

(i) $h(\alpha) \geq 0.155097\ldots$, for 3 not dividing m,

(ii) $h(\alpha) \geq 0.166968\ldots$, for 5 not dividing m, unless $\alpha = \alpha_0^{\pm 1}\zeta$, with ζ a root of unity, whence $h(\alpha) \geq (\mathrm{Log}\, 7)/12 = 0.162159\ldots$, α_0 being a root of

$$7X^{12} - 13X^6 + 7,$$

(iii) $h(\alpha) \geq 0.162368\ldots$, for 7 not dividing m.

(cf also [23], [29], [234], [235], [278], [426], for other results). For totally real numbers α, Fili and Miner [213], using results of Favre and Rivera-Letelier [211] on the equidistribution of points of small Weil height, obtained the limit infimum of the height

$$\liminf_{\alpha \in \mathbb{Q}^{tr}} h(\alpha) \;\geq\; \frac{140}{3}\left(\frac{1}{8} - \frac{1}{6\pi}\right)^2 = 0.120786\ldots \tag{3.19}$$

Bombieri and Zannier [63] have recently introduced the concept of "Bogomolov property", by analogy with the "Bogomolov Conjecture". Let us recall it. Assuming a fixed choice of embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, a field $\mathbb{K} \subset \overline{\mathbb{Q}}$ is said to possess the Bogomolov property relative to h is and only if $h(\alpha)$ is zero or bounded from below by a positive constant for all $\alpha \in \mathbb{K}$. The search of small Weil's heights is important [23], [25], Choi [128]. Every number field has the Bogomolov property relative to h by Northcott's theorem [470] [471]. Other fields are known to possess the Bogomolov property: (i) $\mathbb{Q}^{tr}$ [463], (ii) finite extensions of the maximal abelian extensions of number fields [28] [29], (iii) totally p-adic fields [63], i.e. for algebraic numbers all of whose conjugates lie in $\mathbb{Q}_p$, (iv) $\mathbb{Q}(E_{tors})$ for $E/\mathbb{Q}$ an elliptic curve [251].

For partially real algebraic integers, let us recall Garza's lower bound. Garza [234] established the following minoration of the Mahler measure $\mathrm{M}(\alpha)$ for α an algebraic number, different from 0 and ± 1, having a certain proportion of real Galois conjugates: if $\deg(\alpha) = d \geq 1$ and $1 \leq r \leq d$ be the number of real Galois conjugates of α, then

$$\mathrm{M}(\alpha) \geq \left(\frac{2^{1-1/R} + \sqrt{4^{1-1/R} + 4}}{2} \right)^{\frac{r}{2}},$$

where $R := r/d$. An elementary proof of this minoration was given by Höhn [275]. If $r = d$, Garza's bound is Schinzel's bound (3.11) for totally real algebraic integers [276].

Garza's minorant satisfies $\lim_{d \to \infty} 2^{-r/2} \big(2^{1-d/r} + \sqrt{4^{1-d/r} + 4}\big)^{r/2} = 1$, for any r fixed, where the limit 1 is reached "without any discontinuity". In some sense, a better minorant is expected since Garza's lower bound does not take into account the discontinuity at 1 claimed by the Conjecture of Lehmer.

4. Small points and Lehmer problems in higher dimension

The theory of heights [62], [460], [528], is a powerful tool for studying distributions of algebraic numbers, algebraic points on algebraic varieties, and of subvarieties in projective spaces by extension. Points having a small height, or "small points", resp. "small" projective varieties, together with their distribution, have a particular interest in the problem of Lehmer in higher dimension.

In the classical Lehmer problem, the "height" is the Weil height, and Lehmer's Conjecture is expressed by a Lehmer inequality where the minorant is "a function of the degree", i.e. it states that there exists a universal constant $c > 0$ such that

$$h(\alpha) \geq \frac{c}{\deg(\alpha)} \tag{4.1}$$

unless $\alpha = 0$ or is a root of unity. The generalizations of Lehmer's problem are still formulated by a minoration as in (4.1), but in which "α" is replaced by a rational point "P" of a (abelian) variety, or replaced by a variety "V", where "h" is replaced by another height, more suitable, where the degree "$\deg(\alpha)$" may be replaced by the more convenient "*obstruction index*" ("degree of a variety"), where the minorant function of the "degree" may be more sophisticated than the inverse "$\deg(\alpha)^{-1}$". These different minoration forms extend the classical Lehmer's inequality into a Lehmer type inequality. Generalizing Lehmer's problem separates into three different Lehmer problems:

(i) the classical Lehmer problem,

(ii) the relative Lehmer problem,

(ii) Lehmer's problem for subvarieties.

(i) **The classical Lehmer problem**. On $\mathbb{G}_m$, Dobrowolski's and Voutier's minorations, given by (2.17) and (2.18), with "(Log deg(α))^3" at the denominator, were up till now considered as the best general lower bounds, as functions of the degree deg(α). Generalizations to higher dimension (below) have been largely studied: e.g. Amoroso and David [15], [17], [18], Pontreau [424] [425], W. Schmidt [472] for points on $\mathbb{G}_m^n$, Anderson and Masser [30], David [138], Galateau and Mahé [232], Hindry and Silverman [261], Laurent [331], Silverman [482], [483], [485], for elliptic curves, David and Hindry [137], [139], Masser [374] for abelian varieties.

Conjecture 4.1. *(Elliptic Lehmer problem) Let E/K be an elliptic curve over a number field K. There is a positive constant $C(E) > 0$ such that, if $P \in E(\overline{K})$ has infinite order,*

$$\widehat{h}(P) \geq \frac{c(E)}{[K(P):K]}. \tag{4.2}$$

Theorem 4.2 (Laurent [331]). *Let E/K be an elliptic curve with complex multiplication over a number field K. There is a positive constant $c(E/K)$ such that*

$$\widehat{h}(P) \geq \frac{c(E/K)}{D}\left(\frac{\mathrm{Log}\,\mathrm{Log}\,3D}{\mathrm{Log}\,2D}\right)^3 \qquad \text{for all } P \in E(\overline{K}) \setminus E_{\mathrm{tors}} \tag{4.3}$$

where $D = [K(P):K]$.

Masser [371], [372], [374], and David [138], gave estimates of lower bounds of $\widehat{h}(P)$ for elliptic curves and abelian varieties, on families of abelian varieties [373], for P of infinite order. Galateau and Mahé [232] solved the elliptic Lehmer problem in the Galois case, extending Amoroso David's Theorem ([15], and [27] for sharper estimates):

Theorem 4.3 (Galateau - Mahé [232]). *Let E/K be an elliptic curve over a number field K. There is a positive constant $C(E) > 0$ such that, if $P \in E(\overline{K})$ has infinite order and the field extension $K(P)/K$ is Galois,*

$$\widehat{h}(P) \geq \frac{c(E)}{[K(P):K]}. \tag{4.4}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{G}_m^n(\overline{\mathbb{Q}}) \subset \mathbb{P}^n(\overline{\mathbb{Q}})$. The height of α in $\mathbb{G}_m^n(\overline{\mathbb{Q}})$ is defined by $h(\alpha) = h(1 : \alpha)$ the absolute logarithmic height. Let $F_0 \in \mathbb{Q}[x_1, \ldots, x_n]$ be a nonzero polynomial vanishing at α. The *obstruction index* of α is by definition $\deg(F_0)$, denoted by $\delta_{\mathbb{Q}}(\alpha)$.

Conjecture 4.4. *(Multiplicative Lehmer problem) For any integer $n \geq 1$, there exists a real number $c(n) > 0$ such that*

$$h(\alpha) \geq \frac{c(n)}{\delta_{\mathbb{Q}}(\alpha)} \tag{4.5}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ such that $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent.

Small points of subvarieties of algebraic tori were studied by Amoroso [11].

Theorem 4.5 (Amoroso - David). *There exist a positive constant $c(n) > 0$ such that, for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ such that $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent,*

$$h(\alpha) \geq \frac{c(n)}{\delta_{\mathbb{Q}}(\alpha)} \big(\mathrm{Log}\,(3\delta_{\mathbb{Q}}(\alpha))\big)^{-\,\eta(n)} \tag{4.6}$$

with $\eta(n) = (n+1)(n+1)!^n - n$.

As a consequence of the main Theorem in [27] Amoroso and Viada improved the preceding Theorem and proved:

Theorem 4.6 (Amoroso - Viada). *Let $\alpha_1, \ldots, \alpha_n$ be multiplicatively independent algebraic numbers in a number field K of degree $D = [K : \mathbb{Q}]$. Then*

$$h(\alpha_1) \ldots h(\alpha_n) \geq \frac{1}{D} \frac{1}{(1050\, n^5 \mathrm{Log}\,(3D))^{n^2(n+1)^2}}. \tag{4.7}$$

The assumption of being multiplicatively independent was reconsidered in the multiplicative group $\overline{\mathbb{Q}}^{\times}/\mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$ by Vaaler in [517].

Let A/K be an abelian variety over K a number field. Let $\mathcal{L}$ be a line bundle over A. Let V be a subvariety of A defined over K. The degree $\deg_{\mathcal{L}}(V)$ of V relatively to the Cartier divisor D associated with $\mathcal{L}$ is defined by the theory of intersection [432]. In particular, if $P \in A(\overline{K})$, and $V = \overline{\{P\}}$, then $\deg_{\mathcal{L}}(V) = [K(P) : K]$.

For any $P \in A(\overline{K})$ the *obstruction index* $\delta_{K,\mathcal{L}}(P)$ of P is now extended as

$$:= \min\{\deg_{\mathcal{L}}(V)^{\frac{1}{\mathrm{codim}(V)}} \mid V_{/K} \text{ subvariety of } A, \text{for which } P \in V(\overline{K})\}. \tag{4.8}$$

Conjecture 4.7 (David - Hindry, 2000 - Abelian Lehmer problem). *Let A/K be an abelian variety over a number field K and $\mathcal{L}$ an ample symmetric line bundle over A. Then there exists a real number $c(A, K, \mathcal{L}) > 0$ such that the canonical height $\widehat{h}_{\mathcal{L}}(P)$ of P satisfies*

$$\widehat{h}_{\mathcal{L}}(P) \geq \frac{c(A, K, \mathcal{L})}{\delta_{K,\mathcal{L}}(P)} \tag{4.9}$$

for every point $P \in A(\overline{K})$ of infinite order modulo every proper abelian subvariety $V_{/K}$ of A. Moreover, if $D = [K(P) : K]$, for any $P \in A(\overline{K})$ not being in the torsion,

$$\widehat{h}_{\mathcal{L}}(P) \geq \frac{c(A, K, \mathcal{L})}{D^{1/g_0}} \tag{4.10}$$

where g_0 is the dimension of the smallest algebraic subgroup containing P.

For any abelian variety A defined over a number field K [260], let us denote, for any integer $n \geq 1$, $K_n := K(A[n])$ the extension generated by the group of the torsion points $A[n]$, so that $K_{\mathrm{tors}} = \cup_{n \geq 1} K(A[n])$.

Theorem 4.8 (Ratazzi [434]). *Let A/K be a CM abelian variety of dimension g over a number field K and $\mathcal{L}$ an symmetric ample line bundle over A. Then there exists a real number $c(A,K,\mathcal{L})>0$ such that, for every point $P \in A(\overline{K})$, the canonical height $\widehat{h}_{\mathcal{L}}(P)$ satisfies either*

$$(i) \qquad \widehat{h}_{\mathcal{L}}(P) \geq \frac{c(A/K,\mathcal{L})}{\delta_{K_n,\mathcal{L}}(P)} \left(\frac{\operatorname{Log}\operatorname{Log} 3\,[K_n:K]\,\delta_{K_n,\mathcal{L}}(P)}{\operatorname{Log} 2\,[K_n:K]\,\delta_{K_n,\mathcal{L}}(P)} \right)^{\eta(g)} \tag{4.11}$$

with $\eta(g) = (2g+5)(g+2)(g+1)!(2g.g!)^g$; or

(ii) the point P belongs to a proper torsion subvariety, $B \subset A_{K_n}$, defined over K_n, having a degree bounded by

$$(\deg_{\mathcal{L}} B)^{1/\operatorname{codim} B} \leq \frac{1}{c(A/K,\mathcal{L})} \delta_{K_n,\mathcal{L}}(P) \left(\operatorname{Log} 2[K_n:K]\delta_{K_n,\mathcal{L}}(P)\right)^{2g+2\eta(g)}. \tag{4.12}$$

(ii) **The relative Lehmer problem**. The generalization of the classical Lehmer problem for subfields $K \subset \overline{\mathbb{Q}}$ is decomposed into two steps:

(ii-i) does there exist a real number $c(K)>0$ such that $h(\alpha) \geq c(K)$ for all $\alpha \in \mathbb{G}_m(K)/\mathbb{G}_m(K)_{\text{tors}}$?

(ii-ii) if (i) is satisfied, does there exist a real number $c'(K)>0$ such that, for all $\alpha \in \mathbb{G}_m(\overline{K})/\mathbb{G}_m(\overline{K})_{\text{tors}}$, $h(\alpha) \geq \frac{c'(K)}{[K(\alpha):K]}$?

If K is a number field, (ii-i) is satisfied by Northcott's Theorem and (ii-ii) amounts to the classical Lehmer problem. If K is an infinite extension of $\mathbb{Q}$ the problem is more difficult. In (ii-i), when the field K is $\mathbb{Q}^{\text{ab}}$, or the abelian closure of a number field, it is usual to speak of the *abelian Lehmer problem*. The abelian Lehmer problem was solved by Amoroso and Dvornicich [23]: they proved that, if $\mathbb{L}/\mathbb{Q}$ is an abelian extension of number fields,

$$h(\alpha) \geq \frac{\operatorname{Log} 5}{12} \tag{4.13}$$

for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity. As for (ii-ii), it is usual to speak of *relative Lehmer problem.* The abelian and the relative Lehmer problems are naturally extended in higher dimension. If G denotes either an abelian variety A/K over a number field K or the n-torus $\mathbb{G}_m^n$, and $K_{\text{tors}} = K(G_{\text{tors}})$, the minorant function of the height is expected to depend upon the "nonabelian part of the degree D", where $D = [K(P):K]$. This "nonabelian part : $D_{\text{tors}} = [K_{\text{tors}}(P) : K_{\text{tors}}]$ of D" is equal to $[K^{\text{ab}}(P):K^{\text{ab}}]$, where K^{ab} is the abelian closure of K (if $G=A$, A is assumed CM).

Given an abelian extension $\mathbb{L}/\mathbb{K}$ of number fields and a nonzero algebraic number α which is not a root of unity, with $D := [\mathbb{L}(\alpha):\mathbb{L}]$, Amoroso and Zannier [28] proved the following result, which makes use of Dobrowolski's minoration and the previous minoration:

$$h(\alpha) \geq \frac{c(\mathbb{K})}{D} \left(\frac{\operatorname{Log}\operatorname{Log} 5D}{\operatorname{Log} 2D} \right)^{13}, \tag{4.14}$$

where $c(\mathbb{K}) > 0$, in the direction of the relative problem. Amoroso and Delsinne [22] computed a lower bound, depending upon the degree and the discriminant of the number field $\mathbb{K}$, for the constant $c(\mathbb{K})$. In 2010, given $\mathbb{K}/\mathbb{Q}$ an extension of algebraic number fields, of degree d, Amoroso and Zannier [29] showed

$$h(\alpha) \geq 3^{-d^2-2d-6} \tag{4.15}$$

for any nonzero algebraic number α which is not a root of unity such that $\mathbb{K}(\alpha)/\mathbb{K}$ is abelian. As a corollary they obtained

$$h(\alpha) \geq 3^{-14} \tag{4.16}$$

for any dihedral extension $\mathbb{L}/\mathbb{Q}$ and any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity. For cyclotomic extensions, they obtained sharper results: (i) if $\mathbb{K}$ is a number field of degree d, there exists an absolute constant $c_2 > 0$ such that, with $\mathbb{L}$ denoting the number field generated by $\mathbb{K}$ and any given root of unity, then

$$h(\alpha) \geq \frac{c_2}{d} \frac{(\mathrm{Log}\,\mathrm{Log}\, 5d)^3}{(\mathrm{Log}\, 2d)^4}, \tag{4.17}$$

for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity; (ii) if $\mathbb{K}$ is a number field of degree d, and α any nonzero algebraic number, not a root of unity, such that $\alpha^n \in \mathbb{K}$ for some integer n under the assumption that $\mathbb{K}(\alpha)/\mathbb{K}$ is an abelian extension, then

$$h(\alpha) \geq \frac{c_3}{d} \frac{(\mathrm{Log}\,\mathrm{Log}\, 5d)^2}{(\mathrm{Log}\, 2d)^4}, \tag{4.18}$$

for some constant $c_3 > 0$.

In higher dimension [137] [432], with $G = A$ an abelian variety over a number field K, the *torsion obstruction index* $\delta^{\mathrm{tors}}_{K,\mathcal{L}}(P)$ of a point P is now defined by

$$:= \min\{\deg_{\mathcal{L}^{\mathrm{tors}}}(V)^{\frac{1}{\mathrm{codim}(V)}} \mid V_{/K_{\mathrm{tors}}} \text{ subvariety of } A_{K_{\mathrm{tors}}}, \text{for which } P \in V(\overline{K})\}. \tag{4.19}$$

Conjecture 4.9 (David). *Let A/K be an abelian variety over a number field K and $\mathcal{L}$ an ample symmetric line bundle over A. Then there exists a real number $c(A, K, \mathcal{L}) > 0$ such that the canonical height $\widehat{h}_{\mathcal{L}}(P)$ satisfies*

$$\widehat{h}_{\mathcal{L}}(P) \geq \frac{c(A, K, \mathcal{L})}{\delta^{\mathrm{tors}}_{K,\mathcal{L}}(P)} \tag{4.20}$$

for every point $P \in A(\overline{K})$ of infinite order modulo every proper abelian subvariety $V_{/K}$ of A.

The analogue of Amoroso and Dvornicich's theorem [23] (abelian Lehmer problem) was obtained by Baker and Silverman for abelian varieties [32], [33], and for elliptic curves by Baker [32], then by Silverman [485]:

Theorem 4.10 (Baker - Silverman). *Let A/K be an abelian variety over a number field K and $\mathcal{L}$ an symmetric ample line bundle over A. Let $\widehat{h}(P) : A(\overline{K}) \to \mathbb{R}$ the associated canonical height. Then there exists a real number $c(A, K, \mathcal{L}) > 0$ such that*

$$\widehat{h}(P) \geq c(A, K, \mathcal{L}) \qquad \text{for all nontorsion points } P \in A(K^{\mathrm{ab}}). \tag{4.21}$$

The proof relies upon Zahrin's theorem on torsion points of abelian varieties deduced from the proof of Faltings's theorem [205] of the Mordell Conjecture.

Theorem 4.11 (Silverman). *Let $K/\mathbb{Q}$ be a number field, let E/K be an elliptic curve, and $\widehat{h} : E(\overline{K}) \to \mathbb{R}$ be the canonical height on E. There is a constant $C(E/K) > 0$ such that every nontorsion point $P \in E(K^{\mathrm{ab}})$ satisfies*

$$\widehat{h}(P) > C(E/K). \tag{4.22}$$

Small points were studied by Carrizosa [117]. Ratazzi [430] obtained the relative version of Amoroso and Zannier's minoration [28]:

Theorem 4.12 (Ratazzi). *Let E/K be an elliptic curve with complex multiplication over a number field K. Then there exists a constant $c(E, K) > 0$ such that*

$$\widehat{h}(P) \geq \frac{c(E,K)}{D}\left(\frac{\operatorname{Log}\operatorname{Log} 5D}{\operatorname{Log} 2D}\right)^{13} \qquad \text{for all nontorsion points } P \in E(\overline{K}), \tag{4.23}$$

where $D = [K^{\mathrm{ab}}(P) : K^{\mathrm{ab}}]$.

In the direction of the relative problem, a better lower bound of the canonical height of a point P in a CM abelian variety A/K in terms of the degree of the field generated by P over $K(A_{\mathrm{tors}})$ was obtained by Carrizosa [116]. For tori Delsinne [148] obtained the following (the obstruction index $\omega_K(\alpha)$ is defined below):

Theorem 4.13 (Delsinne). *Let $n \geq 1$ be an integer. There exist constants $c_1(n)$, $\kappa_1(n)$, $\mu(n)$, $\eta_1(n) > 0$ such that, for any $\alpha \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$ satisfying*

$$h(\alpha) \leq \Big(c_1(n)\omega_{\mathbb{Q}^{\mathrm{ab}}}(\alpha)\big(\operatorname{Log}(3\omega_{\mathbb{Q}^{\mathrm{ab}}}(\alpha))\big)^{\kappa_1(n)}\Big)^{-1}, \tag{4.24}$$

there exists a torsion subvariety B containing α, the degree of B being bounded by

$$(\deg B)^{1/\operatorname{codim} B} \leq c_1(n)\omega_{\mathbb{Q}^{\mathrm{ab}}}(\alpha)^{\eta_1(n)}\big(\operatorname{Log}(3\omega_{\mathbb{Q}^{\mathrm{ab}}}(\alpha))\big)^{\mu(n)}; \tag{4.25}$$

the constants are effective and one can take the following values:

$$c_1(n) = \exp\Big(64nn!(2(n+1)^2(n+1)!)^{2n}\Big), \tag{4.26}$$

$$\kappa_1(n) = 3(2(n+1)^2(n+1)!)^n, \quad \mu(n) = 8n!(2(n+1)^2(n+1)!)^n, \tag{4.27}$$

$$\eta_1(n) = (n-1)!\left(\sum_{i=0}^{n-3}\frac{1}{i!} + 1\right) \tag{4.28}$$

(iii) **Lehmer's problem for subvarieties**. The extension from points to subvarieties has been formulated for nontorsion subvarieties V of the multiplicative group $\mathbb{G}_m^n$ or of an abelian variety A/K over a number field K by David and Philippon [141], [142], and Ratazzi [431], [432]. The natural extension of the minoration problem for the height consists in obtaining the best minoration of *the height* $\widehat{h}_{\mathcal{L}}(V)$, resp. of the essential minimum, as a function of the degree of V or of the *obstruction index of* V. The *obstruction index* $\delta_{K,\mathcal{L}}(V)$ *of* V, resp. $\omega_K(V)$, extends the obstruction index $\delta_{K,\mathcal{L}}(P)$ of a point P [142]. As for the definition of the height of V relatively to an symmetric ample line bundle $\mathcal{L}$, two approaches were followed [432]: one by Philippon [417], another one by Bost, Gillet and Soulé [71], using theorems of Soulé [495] and Zhang [543]. In the second construction Zhang [543] showed how to consider the canonical height (or Néron-Tate height, or normalized height) $\widehat{h}_{\mathcal{L}}(V)$ as a limit of Arakelov heights.

Define the canonical height (say) $\widehat{h}$ on $\mathbb{G}_m^n(\overline{\mathbb{Q}})$ by $\widehat{h}(\alpha_1, \ldots, \alpha_n) = h(\alpha_1) + \ldots + h(\alpha_n)$. For $\theta > 0$, let V be a subvariety of $\mathbb{G}_m^n$ defined over $\overline{\mathbb{Q}}$. For $\theta > 0$, let:

$$V_\theta := \{P \in V(\overline{\mathbb{Q}}) \mid \leq \theta\}, \tag{4.29}$$

and the *essential minimum*

$$\widehat{\mu}^{\text{ess}}(V) := \inf\{\theta > 0 \mid V_\theta \text{ is Zariski dense in } V\}. \tag{4.30}$$

The generalized Bogomolov conjecture for subvarieties of tori asserts that $\widehat{\mu}^{\text{ess}}(V) = 0$ is and only if V is a torsion subvariety. In the case where V is a point, $V = \{P\}$, $\widehat{\mu}^{\text{ess}}(V) = \widehat{h}(P)$. Zhang [542], [543], [544], showed that the minoration problem of $\widehat{\mu}^{\text{ess}}(V)$ is essentially the same problem as finding lower bounds for the canonical height $\widehat{h}(V)$ of V, in the sense of Arakelov theory. Indeed, from his Theorem of the Successive Minima, Zhang proved:

$$\widehat{\mu}^{\text{ess}}(V) \leq \frac{\widehat{h}(V)}{\deg(V)} \leq (\dim(V) + 1)\, \widehat{\mu}^{\text{ess}}(V) \tag{4.31}$$

for V any subvariety of $\mathbb{G}_m^n$ over $\overline{\mathbb{Q}}$. Zhang obtained similar results for subvarieties of abelian varieties. The canonical height $\widehat{h}(V)$ of V is related to the problem of minoration of multivariate Mahler measures by the following: for V being a hypersurface defined by a polynomial $F(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ (having relatively prime integer coefficients), then

$$\widehat{h}(V) = \int_0^1 \ldots \int_0^1 \operatorname{Log} |F(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| dt_1 \ldots dt_n \tag{4.32}$$

is the logarithmic Mahler measure $\operatorname{Log} \mathrm{M}(F)$ of F. Let K be a field of characteristic zero, and let V be a subvariety of $\mathbb{G}_m^n$ defined over $\overline{\mathbb{Q}}$. Define the index of obstruction $\omega_K(V)$ to be the minimum degree of a nonzero polynomial $F \in K[x_1, \ldots, x_n]$ vanishing identically on V. Equivalently, it is the minimum degree of a hypersurface defined over K and containing V. The *higher-dimensional Lehmer Conjecture* takes the following form (i.e. the two following conjectures):

Conjecture 4.14 (Amoroso - David, 1999). *Let V be a subvariety of $\mathbb{G}_m^n$, and assume that V is not contained in any torsion subvariety (i.e., a translate of a proper subgroup by a torsion point). Then there exists a constant $C(n) > 0$ such that*
$$\widehat{\mu}^{\mathrm{ess}}(V) \geq \frac{C(n)}{\omega_{\mathbb{Q}}(V)}. \tag{4.33}$$

A 0-dimensional subvariety $V = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{G}_m^n$ is contained in a torsion subvariety if and only if $\alpha_1, \ldots, \alpha_n$ are multiplicatively dependent.

In a similar way, for $\theta > 0$, V a subvariety of an abelian variety A defined over a number field K, and $\mathcal{L}$ a symmetric ample line bundle on A, we define: $V(\theta, \mathcal{L}) := \{x \in V(K) \mid \widehat{h}_{\mathcal{L}}(\overline{K}) \leq \theta\}$. The *essential minimum* of V is
$$\widehat{\mu}_{\mathcal{L}}^{\mathrm{ess}}(V) := \{\theta > 0 \mid \overline{V(\theta, \mathcal{L})} = V\} \tag{4.34}$$
where $\overline{V(\theta, \mathcal{L})}$ is the adherence of Zariski of $V(\theta, \mathcal{L})$ in A.

Conjecture 4.15 (David - Philippon, 1996). *Let A be an abelian variety defined over a number field K, and $\mathcal{L}$ a symmetric ample line bundle on A. Let V/K be a proper subvariety of A, K-irreducible and such that $V_{\overline{K}}$ is not the union of torsion subvarieties, then*
$$\frac{\widehat{h}_{\mathcal{L}}(V)}{\deg_{\mathcal{L}}(V)} \geq \frac{c(A/K, \mathcal{L})}{(\deg_{\mathcal{L}}(V))^{1/(s-\dim(V))}} \tag{4.35}$$
for some constant $c(A/K, \mathcal{L}) > 0$ depending on A/K and $\mathcal{L}$, where s is the dimension of the smallest algebraic subgroup containing V.

Generalizing (4.6) the *higher dimensional Dobrowolski bound* takes the following form, proved in [15] for $\dim(V) = 0$, in [16] for $\mathrm{codim}(V) = 1$ and in [17] for varieties of arbitrary dimension.

Theorem 4.16 (Amoroso - David). *Let V be a subvariety of $\mathbb{G}_m^n$ defined over $\mathbb{Q}$ of codimension k. Let us assume that V is not contained in any union of proper torsion varieties. Then, there exist two constants $c(n)$ and $\kappa(n) = (k+1)(k+1)!^k - k$ such that*
$$\widehat{\mu}^{\mathrm{ess}}(V) \geq \frac{C(n)}{\omega_{\mathbb{Q}}(V)} \frac{1}{(\mathrm{Log}\, 3\omega_{\mathbb{Q}}(V))^{\kappa(k)}}. \tag{4.36}$$

Amoroso and Viada [26] introduced relevant invariants of a proper projective subvariety $V \subset \mathbb{P}^n$: e.g. $\delta(V)$ defined as the minimal degree δ such that V is, as a set, the intersection of hypersurfaces of degree $\leq \delta$.

Theorem 4.17 (Amoroso - Viada [27]). *Let $V \subset \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible variety of dimension d. Then, for any $\alpha \in V^*(\overline{\mathbb{Q}})$,*
$$h(\alpha) \geq \frac{1}{\delta(V)} \frac{1}{(935\, n^5 \mathrm{Log}\,(n^2\delta(V))^{(d+1)(n+1)^2}}. \tag{4.37}$$

Following the main Theorem1.3 in [27] the essential minimum admits the following lower bound:

Theorem 4.18 (Amoroso - Viada). *Let $V \subset \mathbb{G}_m^n$ be a $\mathbb{Q}$-irreducible variety of dimension k which is not contained in any union of proper torsion varieties. Then,*

$$\widehat{\mu}^{\text{ess}}(V) \geq \frac{1}{\omega_{\mathbb{Q}}(V)} \frac{1}{(935\, n^5 \text{Log}\,(n^2\omega_{\mathbb{Q}}(V))^{k(k+1)(n+1)}}. \tag{4.38}$$

Theorem 4.19 (Ratazzi [432]). *Let A be a CM abelian variety defined over a number field K, and $\mathcal{L}$ a symmetric ample line bundle on A. Let V/K be a proper subvariety of A, K-irreducible and such that $V_{\overline{K}}$ is not the union of torsion subvarieties. Then*

$$\frac{\widehat{h}_{\mathcal{L}}(V)}{\deg_{\mathcal{L}}(V)} \geq \widehat{\mu}_{\mathcal{L}}^{\text{ess}}(V) \geq \frac{c(A/K,\mathcal{L})}{(\deg_{\mathcal{L}}(V))^{1/(n-\dim(V)}} \frac{1}{(\text{Log}\,(2\deg_{\mathcal{L}}(V))^{\kappa(n)}} \tag{4.39}$$

with $\kappa(n) = (2n(n+1)!)^{n+2}$, for some constant $c(A/K,\mathcal{L}) > 0$ depending only on A/K and $\mathcal{L}$.

Ratazzi in [432] obtained more precise minorations of $\widehat{h}_{\mathcal{L}}(V)$ in the case where V is an hypersurface. In [431] Ratazzi proves that the optimal lower bound given by David and Philippon [141] in Conjecture 4.15 is a consequence of a Conjecture of David and Hindry on the abelian Lehmer problem.

On the way of proving the relative abelian Lehmer Conjecture, Carrizosa [116] [118] obtained a lower bound of the canonical height of a point P in a CM abelian variety A/K defined over a number field K in terms of the degree of the field generated by P over $K(A_{\text{tors}})$. As Corollary of Theorem 4.13, with the same constants, Delsinne obtained the relative result:

Theorem 4.20 (Delsinne). *Let V be a subvariety of $\mathbb{G}_m^n$ which is not contained in any proper algebraic subgroup of $\mathbb{G}_m^n$. Then*

$$\widehat{\mu}^{\text{ess}}(V) \geq \Big(c_3(n)\omega_{\mathbb{Q}^{\text{ab}}}(V)(\text{Log}\,(3\omega_{\mathbb{Q}^{\text{ab}}}(V)))^{\kappa_1(n)}\Big)^{-1} \tag{4.40}$$

with $c_3(n) = c_1(n)(\dim(V)+1)$.

Concomitantly to the Lehmer problems, the geometry of the distribution of the small points, their Galois orbits, the limit equidistribution of conjugates on some subvarieties, the theorems of finiteness, were investigated, e.g. in Amoroso and David [19] [20] Bilu [54], Bombieri [61], Burgos Gil, Philippon, Rivera-Letelier and Sombra [106], Chambert-Loir [120], D'Andrea, Galligo, Narváez-Clauss and Sombra [135] [136], Favre and Rivera-Letelier [211], Habegger [251], Hughes and Nikeghbali [277], Litcanu [348], Petsche [415] [416], Pritsker [429], Ratazzi and Ullmo [435], Rémond [439], Rumely [450], Szpiro, Ullmo and Zhang [506], Zhang [543], [544].

The proof of Dobrowolski in [160] has been revisited and generalized, e.g. by Amoroso and David [14], Carrizosa [118], Laurent [331], Meyer [377], Ratazzi [433]. It is a keystone to the above-mentionned minoration problems.

5. Analogues of the Mahler measure and Lehmer's problem

Several generalizations and analogues of the Mahler measure were introduced, for which the analogue of the problem of Lehmer holds, or not.

The **Zhang-Zagier height** $\mathcal{H}(\alpha)$ of an algebraic number α is defined as $\mathcal{H}(\alpha) = \mathcal{M}(\alpha)\mathcal{M}(1-\alpha)$. After Zhang [542] and Zagier [536] [537], if α is an algebraic number different from the roots of $(z^2-z)(z^2-z+1)$, then

$$\mathcal{H}(\alpha) \geq \sqrt{\frac{1+\sqrt{5}}{2}} = 1.2720196\ldots \tag{5.1}$$

Doche [165], [166], using (2.39), obtains the following better minorant: if α is an algebraic number different from the roots of $(z^2-z)(z^2-z+1)\Phi_{10}(z)\Phi_{10}(1-z)$, then

$$\mathcal{H}(\alpha) \geq 1.2817770214 =: \eta, \tag{5.2}$$

and the smallest limit point of $\{\mathcal{H}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ lies in $[1.2817770214, 1.289735]$.

Dresden [169] introduced a **generalization of the Zhang-Zagier height**: given G a subgroup of $PSL(2,\overline{\mathbb{Q}})$, the G-orbit height of $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is

$$h_G(\alpha) := \sum_{g\in G} h(g\alpha). \tag{5.3}$$

For G the cyclic group generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Dresden finds, for $\alpha \neq 0, \neq 1$ not being a primitive sixth root of unity,

$$h(\alpha) + h(\frac{1}{1-\alpha}) + h(\frac{1}{\alpha}) \geq 0.42179\ldots \tag{5.4}$$

with equality for α any root of $(X^2-X+1)^3 - X^2(X-1)^2$; otherwise, $h_G(\alpha) = 0$.

The *G-invariant Lehmer problem* is stated as follows in van Ittersum ([281] p. 146): given G a finite subgroup of $PSL(2,\mathbb{Q})$, does there exist a positive constant $D = D_G > 0$ such that

$$h_G(\alpha) = 0 \quad \text{or} \quad h_G(\alpha) \geq D, \qquad \text{for all } \alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})? \tag{5.5}$$

If G is trivial this constant D does not exist [536]. Denote by Orb_G the set of all orbits of the action of G on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathrm{Orb}_{G,unit} := \{Y \in \mathrm{Orb}_G \mid$ for all $\alpha \in Y, \alpha = 0$ or $|\alpha| = 1\}$. Dresden's result [169] was generalized in [281]: van Ittersum [281] proved the G-invariant Lehmer problem under the assumption on G that $\mathrm{Orb}_{G,unit}$ is finite.

The **(logarithmic) metric Mahler measure** $\widehat{m} : \mathcal{G} \to [0,\infty)$ was introduced by Dubickas and Smyth in [188], [189], where

$$\mathcal{G} := \overline{\mathbb{Q}}^{\times} / \mathrm{Tor}(\overline{\mathbb{Q}}^{\times}) \tag{5.6}$$

is the $\mathbb{Q}$-vector space of algebraic numbers modulo torsion, written multiplicatively. For $\underline{\alpha} \in \mathcal{G}$ it is defined by

$$\widehat{m}(\underline{\alpha}) := \inf\Big\{\sum_{n=1}^{N} \operatorname{Log} \mathrm{M}(\alpha_n) \mid N \in \mathbb{N},\ \alpha_n \in \overline{\mathbb{Q}}^{\times},\ \alpha = \prod_{n=1}^{N} \alpha_n\Big\} \tag{5.7}$$

where the infimum is taken over all possible ways of writing any representative α of $\underline{\alpha}$ as a product of other algebraic numbers. The construction may be applied to any height function [189] and is extremal in the sense that any other function $g : \mathcal{G} \to [0, \infty)$ satisfying

(i) $g(\underline{\alpha}) \leq \widehat{m}(\underline{\alpha})$ for any $\underline{\alpha} \in \mathcal{G}$,

(ii) $g(\underline{\alpha}\,\underline{\beta}^{-1}) \leq g(\underline{\alpha}) + g(\underline{\beta})$ for any $\underline{\alpha}, \underline{\beta} \in \mathcal{G}$ (triangle inequality),

is smaller than $\widehat{m}$.

The structure of the completion of $\mathcal{G}$, as a Banach space over the field $\mathbb{R}$ of real numbers, endowed with the norm deduced from the Weil height has been studied by Allcock and Vaaler [6]. Indeed, by construction, the Weil height satisfies: for any $\alpha \in \overline{\mathbb{Q}}^{\times}$ and any root of unity ζ, $h(\alpha) = h(\zeta\alpha)$, so that h extends to $h : \mathcal{G} \to \infty$ with the properties:

(i) $h(\underline{\alpha}) = 0$ if and only if $\underline{\alpha}$ is the identity element $\underline{1}$ in $\mathcal{G}$,

(ii) $h(\underline{\alpha}) = h(\underline{\alpha}^{-1})$ for all $\underline{\alpha} \in \mathcal{G}$,

(iii) $h(\underline{\alpha}\,\underline{\beta}) \leq h(\underline{\alpha}) + h(\underline{\beta})$ for all $\underline{\alpha}, \underline{\beta} \in \mathcal{G}$.

These conditions imply that the map $(\underline{\alpha}, \underline{\beta}) \to h(\underline{\alpha}\,\underline{\beta}^{-1})$ is a metric on the quotient group $\mathcal{G}$, on which the $\mathbb{Q}$-action is defined by $(r/s, \underline{\alpha}) \to \underline{\alpha}^{r/s}$ by the roots of the polynomials $z^s - (\zeta\alpha)^r = 0$ for any $\alpha \in \overline{\mathbb{Q}}^{\times}$ and any root ζ of unity. With the usual absolute value $|\cdot|$ on $\mathbb{Q}$, $h(\alpha^{r/s}) = |\frac{r}{s}| h(\alpha)$, and h is a norm on the $\mathbb{Q}$-vector space $\mathcal{G}$.

Let Y denote the totally disconnected, locally compact, Hausdorff space of all places y of $\overline{\mathbb{Q}}$. Let $\mathcal{B}$ be the Borel σ-algebra of Y. For any number field $k \subset \overline{\mathbb{Q}}$ such that $k/\mathbb{Q}$ is Galois and any place v of k, denote $Y(k, v) := \{y \in Y \mid y|v\}$ so that

$$Y = \bigsqcup_{\text{allplaces } v \text{ of } k} Y(k, v) \qquad \text{(disjoint union).} \tag{5.8}$$

Let λ be the unique regular measure on $\mathcal{B}$, positive on open sets, finite on compact sets, which satisfies:

$$(i) \quad \lambda(Y(k, v)) = \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \quad \text{for any Galois } k/\mathbb{Q}, \text{ any place } v \text{ of } k, \tag{5.9}$$

(ii) $\lambda(\tau E) = \lambda(E)$ for all $\tau \in \mathrm{Aut}(\overline{\mathbb{Q}}/k)$ and $E \in \mathcal{B}$. Allcock and Vaaler [6] proved that the (not surjective) map

$$f : \mathcal{G} \to L^1(Y, \mathcal{B}, \lambda), \quad \alpha \to f_\alpha \text{ given by } \ f_\alpha(y) := \operatorname{Log} \|\alpha\|_y \tag{5.10}$$

is a linear isometry of norm $2h$, i.e.

$$f_{\alpha\beta}(y) = f_\alpha(y) + f_\beta(y), \quad f_{\alpha^{r/s}}(y) = (r/s) f_\alpha(y),$$
$$\int_Y |f_\alpha(y)| d\lambda(y) = 2h(\alpha),$$

with the property:

$$\int_Y f_\alpha(y) d\lambda(y) = 0.$$

Denote by $\mathcal{F} := f(\mathcal{G})$ the image of $\mathcal{G}$ in $L^1(Y, \mathcal{B}, \lambda)$ and

$$\chi := \{F \in L^1(Y, \mathcal{B}, \lambda) \mid \int_Y F(y) d\lambda(y) = 0\}$$

the co-dimension one linear subspace of $L^1(Y, \mathcal{B}, \lambda)$. They proved that $\mathcal{F}$ is dense in χ ([6] Theorem 1), i.e. that χ is the completion of $(\mathcal{G}, h)$, up to isometry. They also proved that, for any real $1 < p < \infty$, $\mathcal{F}$ is dense in $L^p(Y, \mathcal{B}, \lambda)$ ([6] Theorem 2), and $\mathcal{F}$ is dense in the Banach space $\mathcal{C}_0(Y)$ of continuous real valued functions on Y which vanish at infinity, equipped with the sup-norm ([6] Theorem 3).

Fili and Miner [214] proved that the space $\mathcal{F}$ admits linear operators canonically associated to the Mahler measure and to the L^p norms on Y. They introduced norms, called **Mahler p-norms**, from orthogonal decompositions of $\mathcal{F}$, and, in this context, obtained extended formulations, called *L^p Lehmer Conjectures*, of the Lehmer Conjecture and the Conjecture of Schinzel-Zassenhaus. Namely, let $\mathcal{K}$ be the set of finite extensions of $\mathbb{Q}$ and

$$\mathcal{K}^G := \{K \in \mathcal{K} \mid \sigma(K) = K \text{ for all } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}.$$

For each $K \in \mathcal{K}$, denote by $V_K := \{f_\alpha \mid \alpha \in K^\times/\mathrm{Tor}(K^\times)\}$ the $\mathbb{Q}$-vector subspace of $\mathcal{F}$ constituted by the nonzero elements of K modulo torsion, and, for $n \geq 0$, $V^{(n)} := \sum_{K \in \mathcal{K}, [K:\mathbb{Q}] \leq n} V_K$. Denote by $\langle f, g \rangle = \int_Y f(y) g(y) d\lambda(y)$ the inner product on $\mathcal{F}$.

Theorem 5.1 (Fili - Miner). *(i) There exist projection operators $T_K : \mathcal{F} \to \mathcal{F}$ for each $K \in \mathcal{K}^G$ such that $T_K(\mathcal{F}) \subset V_K$, $T_K(\mathcal{F}) \perp T_L(\mathcal{F})$ for all $K, L \in \mathcal{K}^G$, $K \neq L$, with respect to the inner product on $\mathcal{F}$, and*

$$\mathcal{F} = \bigoplus_{K \in \mathcal{K}^G} T_K(\mathcal{F}), \tag{5.11}$$

(ii) for all $n \geq 1$, there exist projections $T^{(n)} : \mathcal{F} \to \mathcal{F}$ such that $T^{(n)}(\mathcal{F}) \subset V^{(n)}$, $T^{(m)}(\mathcal{F}) \perp T^{(n)}(\mathcal{F})$ for all $m \neq n$, and

$$\mathcal{F} = \bigoplus_{K \in \mathcal{K}^G} T^{(n)}(\mathcal{F}), \tag{5.12}$$

(iii) for every $K \in \mathcal{K}^G$ and $n \geq 0$, the projections T_K and $T^{(n)}$ commute.

Now, for any $\alpha \in \overline{\mathbb{Q}}^{\times}$ and any real number $1 \leq p \leq \infty$, let $h_p(\alpha) := \|f_\alpha\|_p$ (recalling that $h_1(\alpha) = 2h(\alpha)$).

Conjecture 5.2 (Fili - Miner (L^p Lehmer Conjectures)). *For any real number $1 \leq p \leq \infty$, there exists a real constant $c_p > 0$ such that*

$$(*_p) \qquad m_p(\alpha) := \deg_{\mathbb{Q}}(\alpha)\, h_p(\alpha) \geq c_p \qquad \textit{for all } \alpha \in \overline{\mathbb{Q}} \setminus \mathrm{Tor}(\overline{\mathbb{Q}}). \tag{5.13}$$

For $p = 1$ Conjecture 5.2 is exactly the classical Lehmer Conjecture. Moreover, Fili and Miner ([214], Proposition 4.1) proved that, for $p = \infty$, Conjecture 5.2 is exactly the classical Conjecture of Schinzel-Zassenhaus.

The operator $M : \mathcal{F} \to \mathcal{F}, f \to \sum_{n=1}^{\infty} nT^{(n)} f$ is well-defined, unbounded, invertible, and is always a finite sum. The norm $f \to \|Mf\|_p$ is called the Mahler p-norm on $\mathcal{F}$. For any $f \in \mathcal{F}$, let $d(f) := \min\{\deg_{\mathbb{Q}}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}^{\times}, f_\alpha = f\}$ be the smallest degree possible in the class of f. For any $f \in \mathcal{F}$, the *minimal field*, denoted by K_f, is defined to be the minimal element of the set $\{K \in \mathcal{K} \mid f \in V_K\}$. Let $\delta(f) = [K_f : \mathbb{Q}]$. The P_K operators on $\mathcal{F}$ are defined from the T_K operators as: $P_K := \sum_{F \in \mathcal{K}^G, F \subset K} T_F$. An element $f \in \mathcal{F}$ is said to be *Lehmer irreducible* (or *representable*) if $\delta(f) = d(f)$. The set of Lehmer irreducible elements of $\mathcal{F}$ is denoted by $\mathcal{L}$. An element $f \in \mathcal{F}$ is said to be *projection irreducible* if $P_H(f) = 0$ for all propers subfields H of K_f. The set of projection irreducible elements of $\mathcal{F}$ is denoted by $\mathcal{P}$. Let $\mathcal{U} = \{f \in \mathcal{F} \mid \mathrm{supp}_Y(f) \subset Y(\mathbb{Q}, \infty)\}$ be the subset of algebraic units.

Theorem 5.3 (Fili - Miner). *For every real number $1 \leq p \leq \infty$, the L^p Lehmer Conjecture $(*_p)$ holds if and only the following minoration on the Mahler p-norms holds*

$$(**_p) \qquad \|\sum_{n=1}^{\infty} nT^{(n)} f_\alpha\|_p \geq c_p \qquad \textit{for all } f_\alpha \in \mathcal{L} \cap \mathcal{P} \cap \mathcal{U},\ f_\alpha \neq 0. \tag{5.14}$$

Further, for $1 \leq p \leq q \leq \infty$, if $(**_p)$ holds, then $(**_q)$ also holds.

An element $f_\beta \in \mathcal{F}$ is said to be a *Pisot number*, resp. a *Salem number*, if it has a representative $\beta \in \overline{\mathbb{Q}}^{\times}$ which is a Pisot number, resp. a Salem number. Fili and Miner ([214], Prop. 4.2 and Prop. 4.3) proved that every Pisot number and every Salem number is Lehmer irreducible, moreover that every Salem number is also projection irreducible. A *surd* is an element $f \in \mathcal{F}$ such that $\delta(f) = 1$, i.e. for which $K_f = \mathbb{Q}$ and $\|Mf\|_p = \|f\|_p = h_p(f)$; a surd is projection irreducible.

The **t-metric Mahler measure**, was introduced by Samuels [282], [456], [458]. For $t \geq 1$, the t-metric Mahler measure is defined by

$$\mathrm{M}_t(\alpha) := \inf\Big\{\Big(\sum_{n=1}^{N} (\mathrm{Log}\, \mathrm{M}(\alpha_n))^t\Big)^{1/t} \mid N \in \mathbb{N},\ \alpha_n \in \overline{\mathbb{Q}}^{\times},\ \alpha = \prod_{n=1}^{N} \alpha_n\Big\} \tag{5.15}$$

and, by extension, for $t = \infty$, by

$$\mathrm{M}_\infty(\alpha) := \inf\Big\{\max_{1 \leq n \leq N}\big\{\mathrm{Log}\, \mathrm{M}(\alpha_n)\big\} \mid N \in \mathbb{N},\ \alpha_n \in \overline{\mathbb{Q}}^{\times},\ \alpha = \prod_{n=1}^{N} \alpha_n\Big\}. \tag{5.16}$$

For $t = 1$, M_1 is the metric Mahler measure introduced in [188]. These functions satisfy an analogue of the triangle inequality [282], and the map $(\alpha, \beta) \to \mathrm{M}_t(\alpha\beta^{-1})$ defines a metric on $\mathcal{G} := \overline{\mathbb{Q}}^{\times}/\mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$ which induces the discrete topology if and only if Lehmer's Conjecture is true. For $t \in [1, \infty]$ and $\alpha \in \overline{\mathbb{Q}}$ we say that the infimum in $\mathrm{M}_t(\alpha)$ is attained by $\alpha_1, \ldots, \alpha_n$ if the equality case holds: i.e.,

- for $1 \leq t < \infty$, if $\mathrm{M}_t(\alpha) = \left(\sum_{n=1}^{N}(\mathrm{Log}\,\mathrm{M}(\alpha_n))^t\right)^{1/t}$ and,
- for $t = \infty$, if $\mathrm{M}_\infty(\alpha) = \max_{1\leq n\leq N}\{\mathrm{Log}\,\mathrm{M}(\alpha_n)\}$.

For $\alpha \in \overline{\mathbb{Q}}$, denote by $\mathbb{K}_\alpha$ the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$, and let

$$\mathrm{Rad}(\mathbb{K}_\alpha) := \{\beta \in \overline{\mathbb{Q}} \mid \beta^m \in \mathbb{K}_\alpha \text{ for some } m \in \mathbb{N}\}.$$

Following a conjecture of Dubickas and Smyth [188], Samuels [456] [457] proved that the infimum of $\mathrm{M}_t(\alpha)$ is attained in $\mathrm{Rad}(\mathbb{K}_\alpha)$. Whether this infimum is attained in proper subsets of $\overline{\mathbb{Q}}$ leads to many open questions ([282], Question 1.5), though Jankauskas and Samuels proved some results for certain cases of decompositions of rational numbers in prime numbers ([282], Theorem 1.3, Theorem 1.4). In particular for $\alpha \in \mathbb{Q}$, they proved that the infimum of $\mathrm{M}_t(\alpha)$ may be attained using only rational points.

The p-metric, resp. the t-metric, constructions of Fili and Miner [213] and Jankauskas and Samuels [282] are of different nature, though they are esentially the same for $p = 1$. Fili and Miner [213] studied the minimality of the Mahler measure by several norms, related to the metric Mahler measure introduced in [188], using results of de la Masa and Friedman [368] on heights of algebraic numbers modulo multiplicative group actions. Fili and Miner [213] introduced an infinite collection $(h_t)_t$ of vector space norms on $\mathcal{G}$, called *L^t Weil heights*, $t \in [1, \infty]$, which satisfy extremality properties, and *minimal logarithmic L^t Mahler measures* $(m_t)_t$ from $(h_t)_t$. By definition, for $\mathbb{K}$ a number field, $\Sigma_{\mathbb{K}}$ its set of places and $||.||_\nu$ the absolute value on $\mathbb{K}$ extending the usual p-adic absolute value on $\mathbb{Q}$ if ν is finite or the usual archimedean absolute value if ν is infinite, for $1 \leq t < \infty$ real,

$$h_t(\alpha) := \Big(\sum_{\nu\in\Sigma_{\mathbb{K}}} \frac{[\mathbb{K}_\nu : \mathbb{Q}_\nu]}{[\mathbb{K} : \mathbb{Q}]}.\,|\mathrm{Log}\,||\alpha||_\nu|^t\Big)^{1/t}, \qquad \alpha \in \mathbb{K}^\times \tag{5.17}$$

for which $2h = h_1$ [6], and

$$h_\infty(\alpha) := \sup_{\nu\in\Sigma_{\mathbb{K}}} |\mathrm{Log}\,||\alpha||_\nu|\,, \qquad \alpha \in \mathbb{K}^\times. \tag{5.18}$$

They reformulated Lehmer's Conjecture in this context (with $1 \leq t < \infty$). Because h_∞ serves as a generalization of the (logarithmic) house of an algebraic integer, they also reformulated the Conjecture of Schinzel and Zassenhaus. In [282] Jankauskas and Samuels investigate the t-metric Mahler measures of surds and rational numbers.

The **ultrametric Mahler measure** was introduced by Fili and Samuels [217], [457], to give a projective height of $\mathcal{G}$, which satisfies the strong triangle inequality.

The ultrametric Mahler measure induces the discrete topology on $\mathcal{G}$ if and only if Lehmer's Conjecture is true.

Two p**-adic Mahler measures** are introduced by Besser and Deninger in [53] in view of developping natural analogues of the classical logarithmic Mahler measures of Laurent polynomials, following Deninger [149]. The p-adic analogue of Deligne cohomology is now Besser's modified syntomic cohomology, but with the same symbols in the algebraic K-theory groups. For one p-adic Mahler measure the authors show that there is *no analogue* of Lehmer's problem.

Generalized Mahler measures, higher Mahler measures and multiple k-higher Mahler measures were introduced by Gon and Oyanagi [240], resp. Kurokawa, Lalín and Ochiai [312] and reveal deep connections between zeta functions, polylogarithms, multiple L- functions (Sasaki [459]) and multiple sine functions. For any $n \geq 1$, given $P_1, \ldots, P_s \in \mathbb{C}[x_1, \ldots, x_n]$ (not necessarily distinct) nonzero polynomials, the **generalized Mahler measure** is defined by $m_{\max}(P_1, \ldots, P_s) :=$

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \max\{\operatorname{Log}|P_1(x_1, \ldots, x_n)|, \ldots, \operatorname{Log}|P_s(x_1, \ldots, x_n)|\} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}, \tag{5.19}$$

the **multiple Mahler measure** by $m(P_1, \ldots, P_s) :=$

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \operatorname{Log}|P_1(x_1, \ldots, x_n)| \ldots \operatorname{Log}|P_s(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}, \tag{5.20}$$

the k**-higher Mahler measure** of P by $m_k(P) := m(P, \ldots, P) =$

$$\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \operatorname{Log}^k|P_1(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}, \tag{5.21}$$

The k-higher Mahler measures are deeply related to the zeta Mahler measures, and their derivatives, introduced by Akatsuka [5]. The *problem of Lehmer* for k-higher Mahler measures is considered by Lalín and Sinha in [323]. Asymptotic formulas of $m_k(P)$, with k, are given in [57] and [323], for some families of polynomials P. Analogues of Boyd-Lawton's Theorem are studied in Issa and Lalín [279]. By analogy with Deninger's approach, the motivic reinterpretation of the values of k-higher Mahler measures in terms of Deligne cohomology is given by Lalín in [322].

The logarithmic Mahler measure m_G **over a compact abelian group** G is introduced by Lind [345]. The group is equipped with the normalized Haar measure μ. By Pontryagin's duality the dual group $\widehat{G}$ (characters) is discrete and the class of functions f to be considered is $\mathbb{Z}[\widehat{G}]$. For $f \in \mathbb{Z}[\widehat{G}]$

$$\mathrm{m}_G(f) = \int_G \operatorname{Log}|f| d\mu \tag{5.22}$$

generalizes

$$\mathrm{m}(f) = \int_0^1 \operatorname{Log}|f(e^{2i\pi t})| dt \text{ for } f \in \mathbb{Z}[x^{\pm 1}]. \tag{5.23}$$

The *Lehmer constant of* G is then defined by

$$\lambda(G) := \inf\{\mathrm{m}_G(f) \mid f \in \mathbb{Z}[\widehat{G}], \mathrm{m}_G(f) > 0\}. \tag{5.24}$$

The author considers several groups G (connected, finite) and the *problem of Lehmer* in each case. The classical Lehmer's problem asks whether $\lambda(\mathbb{T}) = 0$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $n \geq 2$, denote by $\rho(n)$ the smallest prime number that does not divide n. Lind proves that $\lambda(G) = \lambda(\mathbb{T})$ for any nontrivial connected compact abelian group, and $\lambda(\mathbb{Z}/n\mathbb{Z}) \leq \frac{\mathrm{Log}\,\rho(n)}{n}$ for $n \geq 2$. This Lehmer's constant has been named *Lind-Lehmer's constant* more recently. It is known in some cases [422]. Kaiblinger [288] obtained results on $\lambda(G)$ for finite cyclic groups G of cardinality not divisible by 420; Pigno and Pinner [419] solved the case $|G| = 420$. De Silva and Pinner [154], [155], made progress on noncyclic finite abelian groups $G = \mathbb{Z}_p^n$, then Pigno, Pinner and Vipismakul [524], [420], on general p-groups $G_p = \mathbb{Z}_{p^{l_1}} \times \ldots \times \mathbb{Z}_{p^{l_n}}$ and $G = \mathbb{Z}_m \times G_p$ for m not divisible by p.

An areal analogue of Mahler's measure is reported by Pritsker [427], linked to Hardy and Bergman normed spaces of functions on the unit disk.

Lehmer's problems in positive characteristic and Drinfeld modules: let $k = \mathbb{F}_q(T)$ be the fraction field of the ring $\mathbb{F}_q[T]$ of polynomials with coefficients in the finite field $\mathbb{F}_q$ (p is a prime number and q a power of p). Let $k_\infty = \mathbb{F}_q((1/T))$ be the completion of k for the $1/T$-adic valuation v. The valuation, still denoted by v, is extended to the algebraic closure $\overline{k}$ of k, resp. $\overline{k_\infty}$ of k_∞. The degree $\deg(x)$ of $x \in k_\infty$ is equal to the integer-valued $-v(x)$, with the convention $\deg(0) = -\infty$. Let t denote a formal variable. By definition a t-module of dimension N and rank d on $\overline{k}$ is given by the additive group $(\mathbb{G}_a)^N$ and an injective ring homomorphism $\Phi : \mathbb{F}_q[T] \to \mathrm{End}(\mathbb{G}_a)^N$ which satisfies:

$$\Phi(t) = a_0 F^0 + \ldots + a_d F^d, \tag{5.25}$$

where F is the Frobenius endomorphism on $(\mathbb{G}_a)^N$ and $a_0, a_1, \ldots, a_d$ are $N \times N$ matrices with coefficients in $\overline{k}$. In [151] Denis constructed a canonical height $\widehat{h} = \widehat{h}_\Phi$ on t-modules for which a_d is invertible, from the Weil height. Denis formulated Lehmer's problem for t-modules as follows, in two steps: (i) for $\alpha \in (\overline{k})^n$, defined over a field of degree $\leq \delta$, not in the torsion of the t-module, does there exist $c(\delta) = c_{a_0,\ldots,a_d,N,d,q,F}(\delta) > 0$ such that $\widehat{h}(\alpha) \geq c(\delta)$?; (ii) if (i) is satisfied, on a Drinfeld module of rank d, does there exist $c > 0$ such that, for any α not belonging to the torsion,

$$\widehat{h}(\alpha) \geq \frac{c}{\delta}? \tag{5.26}$$

The second problem is the extension of the classical Lehmer problem [407]. Denis partially solved Lehmer's problem ([151] Theorem 2) in the case of Carlitz modules, i.e. with $N = 1$ and $d = 1$ for which $\Phi(T)(x) = Tx + x^q$. He obtained the following minoration which is an analogue of Laurent's Theorem 4.2 for CM elliptic curves (elliptic Lehmer problem) and Dobrowolski's inequality (2.17):

Theorem 5.4 (Denis). *There exists a real number $\eta > 0$ such that, for any α belonging to the regular separable closure of k, not to the torsion, of degree $\leq \delta$, the minoration holds:*

$$\widehat{h}(\alpha) \geq \eta \frac{1}{\delta} \left(\frac{\operatorname{Log} \operatorname{Log} \delta}{\operatorname{Log} \delta} \right)^3 \tag{5.27}$$

(the real number η is effective and computable from q).

Grandet-Hugot in [244] studied analogues of Pisot and Salem numbers in fields of formal series: $x \in k_\infty$ is a Salem number if it is algebraic on k, $\deg(x) > 0$, and all its conjugates satisfy: $\deg(x_i) \leq 0$. In this context Denis ([152] Theorem 1) proved the fact that there is no Salem number too close to 1, namely:

Theorem 5.5 (Denis). *Let $\alpha \in \overline{k}_\infty$ having at least one conjugate in k_∞. If α does not belong to the torsion, is of degree D on k, then*

$$\widehat{h}(\alpha) \geq \frac{1}{q\,D} \tag{5.28}$$

Extending the previous results, Denis ([152] Theorem 3) solved Lehmer's problem for the following infinite family of t-modules:

Theorem 5.6 (Denis). *Let $\Phi(t) = a_0 F^0 + a_1 F + \ldots a_{d-1} F^{d-1} + F^d$ be a t-module of dimension 1 such that $a_i \in k_\infty \cap \overline{k}$, $0 \leq i \leq d-1$, is integral over $\mathbb{F}_q[T]$. Then there exists a real number $c_\Phi > 0$ depending only upon Φ, such that, if α is an algebraic element of k_∞, not in the torsion, of degree D on k, then*

$$\widehat{h}_\Phi(\alpha) \geq \frac{c_\Phi}{D} \tag{5.29}$$

The abelian Lehmer problem for Drinfeld modules was solved by David and Pacheco [140] using Denis's construction of the canonical height (with $A = \mathbb{F}_q[T]$):

Theorem 5.7 (David - Pacheco). *Let K/k be a finite extension, $\overline{K}$ an algebraic closure of K, and K^{ab} the largest abelian extension of K in $\overline{K}$. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of rank ≥ 1. Then there exists $c = c(\phi, K) > 0$ which depends only upon ϕ and K such that, for any $\alpha \in K^{\mathrm{ab}}$, not being in the torsion,*

$$\widehat{h}_\Phi(\alpha) \geq c. \tag{5.30}$$

In [239] Ghioca investigates statements, for Drinfeld modules of generic characteristic, which would imply that the classical Lehmer problem for Drinfeld modules is true. In [238] Ghioca obtained several Lehmer type inequalities for the height of nontorsion points of Drinfeld modules. Using them, as consequence of Theorem 5.8 below, Ghioca proved several Mordell-Weil type structure theorems for Drinfeld modules over certain infinitely generated fields (the definitions of the terms can be found in [238]):

Theorem 5.8 (Ghioca). *Let $K/\mathbb{F}_q$ be a field extension, and $\phi : A \to K\{\tau\}$ be a Drinfeld module. Let L/K be a finite field extension. Let t be a non-constant element of A and assume that $\phi_t = \sum_{i=0}^{r} a_i \tau^i$ is monic. Let U be a good set of* *valuations on L and let $C(U)$ be the field of constants with respect to U. Let S be the finite set of valuations $v \in U$ such that ϕ has bad reduction at v. The degree of the valuation v is denoted by $d(v)$. Let $x \in L$.*

a) If S is empty, then either $x \in C(U)$ or there exists $v \in U$ such that

$$\widehat{h}_{U,v}(x) \geq d(v), \tag{5.31}$$

b) If S is not empty, then either $x \in \phi_{tors}$, or there exists $v \in U$ such that

$$\widehat{h}_{U,v}(x) > \frac{d(v)}{q^{2r+r^2 N_\phi |S|}} \geq \frac{d(v)}{q^{r\left(2+(r^2+r)|S|\right)}}. \tag{5.32}$$

Moreover, if S is not empty and $x \in \phi_{tors}$, then there exists a polynomial $b(t) \in \mathbb{F}_q(t)$ of degree at most $rN_\phi|S|$ such that $\phi_{b(t)}(x) = 0$.

Let K be a finitely generated field extension of $\mathbb{F}_q$, and K^{alg} an algebraic closure of K. Ghioca [239] developped global heights associated to a Drinfeld module $\phi : A \to K\{\tau\}$ and, for each divisor v, local heights $\widehat{h}_v : K^{alg} \to \mathbb{R}^+$ associated to ϕ. For Drinfeld modules of finite characteristic Ghioca [239] obtained Lehmer type inequalities with the local heights, extending the classical Lehmer problem:

Theorem 5.9 (Ghioca). *For $\phi : A \to K\{\tau\}$ a Drinfeld module of finite characteristic, there exist two positive constants C and r depending only on ϕ such that if $x \in K^{alg}$ and v is a place of $K(x)$ for which $\widehat{h}_v(x) > 0$, then*

$$\widehat{h}_v(x) \geq \frac{C}{d^r} \tag{5.33}$$

where $d = [K(x) : K]$.

Bauchère [38] generalized David Pacheco's Theorem 5.7 to Drinfeld modules having complex multiplications, proving the abelian Lehmer problem in this context:

Theorem 5.10 (Bauchère). *Let ϕ be a A-Drinfeld module defined over $\overline{k}$ having complex multiplications. Let K/k be a finite field extension, L/K a Galois extension (finite or infinite) with Galois group $G =$Gal(L/K). Let H be a subgroup of the center of G and $E \subset L$ the subfield fixed by H. Let d_0 be an integer. We assume that there exists a finite place v of K such that $[E_w : K_v] \leq d_0$ for every place w of E, $v|w$. Then there exists a constant $c_0 = c_0(\phi) > 0$ such that, for any $\alpha \in L$, not belonging to the torsion for ϕ,*

$$\widehat{h}_\phi(\alpha) \geq \frac{1}{q^{c_0\, d(v)\, d_0^2\, [K:k]}}. \tag{5.34}$$

Theorem 5.10 is the analogue of a result obtained by Amoroso, David and Zannier [21] for the multiplicative group. Theorem 5.10 is particularly interesting when L/K is infinite. Bauchère [38] deduced special minorations of the heights $\widehat{h}_\phi(\alpha)$ in two Corollaries, for $L = K^{ab}$, and in the case where the subgroup H is trivial.

6. In other domains

The conjectural discontinuity of the Mahler measure $\mathrm{M}(\alpha), \alpha \in \overline{\mathbb{Q}}$, at 1 has consequences in different domains of mathematics. It is linked to the notions of "smallest complexity", "smallest growth rate", "smallest geometrical dilatation", "smallest geodesics", "smallest Salem number" or "smallest topological entropy" (Hironaka [266]). We will keep an interdisciplinary viewpoint as in the recent survey [494] by C. Smyth and refer the reader to [237], [494]; we only mention below a few more or less new results. The smallest Mahler measures, or smallest Salem numbers, correspond to peculiar geometrical constructions in their respective domains.

6.1. Coxeter polynomials, graphs, Salem trees. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a simple graph with set of enumerated vertices $\Gamma_0 = \{v_1, \ldots, v_n\}$, Γ_1 being the set of edges where $(v_i, v_j) \in \Gamma_1$ if there is an edge connecting the vertices v_i and v_j. The adjacency matrix of Γ is $\mathrm{Ad}_\Gamma := [a_{ij}] \in M_n(\mathbb{Z})$ where $a_{ij} = 1$, if $(v_i, v_j) \in \Gamma_1$ and $a_{ij} = 0$ otherwise. Assume that Γ is a tree. Denote by W_Γ the Weyl group of Γ, generated by the reflections $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $\Phi_\Gamma := \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_n \in W_\Gamma$ the Coxeter transformation of Γ. The Coxeter polynomial of Γ is the characteristic polynomial of the Coxeter transformation $\Phi_\Gamma : \mathbb{R}^n \to \mathbb{R}^n$:

$$\mathrm{cox}_\Gamma(x) := \det(x \cdot \mathrm{Id}_n - \Phi_\Gamma) \in \mathbb{Z}[x]. \tag{6.1}$$

Coxeter (1934) showed remarkable properties of the roots of the Coxeter polynomials. Coxeter polynomials were extensively studied for Γ any simply laced Dynkin diagram $\mathbb{A}_n, \mathbb{D}_n$ and $\mathbb{E}_n$. For $\Gamma = \mathbb{E}_n$, Gross, Hironaka and McMullen [248] have obtained the factorization of Coxeter polynomials $\mathrm{cox}_\Gamma(x)$ as products of cyclotomic polynomials and irreducible Salem polynomials. In particular,

$$\mathrm{cox}_{\mathbb{E}_{10}}(x) := x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 \tag{6.2}$$

is Lehmer's polynomial. A tree $\mathcal{T}$ is said to be *cyclotomic*, resp. *a Salem tree*, if $\mathrm{cox}_\mathcal{T}(x)$ is a product of cyclotomic polynomials, resp. the product of cyclotomic polynomials by an irreducible Salem polynomial. Such objects generalize $\mathbb{E}_n$ as far as their Coxeter polynomials remains of the same form. Evripidou [204], following Lakatos [313], [314], [316], [317] and [248], obtained structure theorems and formulations for the Coxeter polynomials of families of Salem trees, for the spectral radii of the respective Coxeter transformations. Lehmer's problem asks whether there exists a Salem tree of minimal Salem number; what would be its decomposition?

The Mahler measure $\mathrm{M}(G)$ of a finite graph G, with n vertices, is introduced in McKee and Smyth [353]. If $D_G(z)$ is the characteristic polynomial of G, then the reciprocal integer polynomial associated with G is $z^n D_G(z+1/z)$. The Mahler measure of this later polynomial is the Mahler measure $\mathrm{M}(G)$ of G; explicitely,

$$\mathrm{M}(G) = \prod_{D_G(\chi)=0, |\chi|>2} \frac{1}{2}(|\chi| + \sqrt{\chi^2 - 4}). \tag{6.3}$$

Cooley, McKee and Smyth [132] [353] [354] [355] studied Lehmer's problem from various constructions of finite graphs. They prove ([132] Theorem 1 and Figures 1 to 3) that every connected non-bipartite graph that has Mahler measure smaller than the golden mean $1.618\ldots$ is one of the following type: (i) an odd cycle, (ii) a kite graph, (iii) a balloon graph, or (iv) one of the eight sporadic examples $Sp_a, \ldots, Sp_h$.

6.2. Growth series of groups, Coxeter groups, Coxeter systems. Let G be an infinite group. Assume that G admits a finite generating set S. Define the length of an element g in $G = (G,S)$ to be the least nonnegative integer n such that g can be expressed as a product of n elements from $S \cup S^{-1}$. For every nonnegative integer n let $N_S(n)$ be the number of elements in G with length n. Following Milnor [383] the growth series of the group (G,S) is by definition

$$f(x) = \sum_{n=1}^{\infty} N_S(n) x^n, \qquad \text{for which } N_S(n) \leq (2|S|)^n. \tag{6.4}$$

The asymptotic *growth rate* of $G = (G,S)$, finite and ≥ 1, is by definition

$$\limsup_{n\to\infty} (N_S(n))^{1/n}, \tag{6.5}$$

its inverse, positive, being the radius of convergence of $f(x)$. A Coxeter group G, with S being a finite generating set for G, is a group where every element of S has order two and all the other defining relators for G are of the form $(g\,h)^{m(g,h)} = 1_G$ where $m(g,h) = m(h,g)$ and $m(g,h) \geq 2$. Steinberg [499] and Bourbaki [72] showed that the growth series of a Coxeter group is a rational function. Salem numbers, Pisot numbers and Perron numbers occur as roots of the polynomials at the denominator (here the definition of a Salem number is often extended to quadratic Pisot numbers, conveniently and abusively).

Let us consider a hyperbolic cocompact Coxeter group G with generating set of reflections S acting in low dimensions $n \geq 2$.

Case $n = 2$: for the Coxeter reflection groups $G_{p_1,\ldots,p_d}$, with p_i any positive integer, of presentation $G_{p_1,\ldots,p_d} := (g_1, \ldots, g_d \mid (g_i)^2 = 1, (g_i g_{i+1})^{p_i} = 1)$, the denominator $\Delta_{p_1,\ldots,p_d}(x)$ of the growth series $f(x)$ of $G_{p_1,\ldots,p_d}$ is explicitely given by the following theorem [109].

Theorem 6.1 (Cannon - Wagreich [110], Floyd - Plotnick [224], Parry [411]).

$$\Delta_{p_1,\ldots,p_d}(x) = [p_1][p_2]\ldots[p_d](x-d+1) + \sum_{i=1}^{d} [p_1]\ldots\widehat{[p_i]}\ldots[p_d]. \tag{6.6}$$

The polynomial $\Delta_{p_1,\ldots,p_d}(x)$ is either a product of cyclotomic polynomials or a product of cyclotomic polynomials times an irreducible Salem polynomial. The Salem polynomial occurs if and only if $G_{p_1,\ldots,p_d}$ is hyperbolic, that is,

$$\frac{1}{p_1} + \ldots + \frac{1}{p_d} \;<\; d-2. \tag{6.7}$$

Then hyperbolic Coxeter reflection groups have Salem numbers as asymptotic growth rates. Such Salem numbers form a subclass of the set of Salem numbers. Lehmer's polynomial is $\Delta_{2,3,7}(x)$, denominator of the growth series of the (2,3,7)-hyperbolic triangle group (Takeuchi [507]). The *Construction of Salem* [452], [74], for establishing the existence of sequences of Salem numbers converging to a given Pisot number, on the left and on the right, admits an analogue in terms of geometric convergence for the fundamental domains of cocompact planar hyperbolic Coxeter groups. Using the Construction of Salem Parry [411] gives a new proof of Theorem 6.1. Bartholdi and Ceccherini-Silberstein [36] studied the Salem numbers which arise from some hyperbolic graphs. Hironaka [237] solves the problem of Lehmer for the subclass of Salem numbers occuring as such asymptotic growth rates:

Theorem 6.2 (Hironaka [263]). *Lehmer's number is the smallest Salem number occuring as dominant roots of $\Delta_{p_1,\ldots,p_d}$ polynomials for any $(p_1,\ldots,p_d)$, p_i being positive integers.*

Case $n = 3$: Parry [411] extends its 2-dimensional approach to every hyperbolic cocompact reflection Coxeter group on $\mathbb{H}^3$ generated by reflections whose fundamental domain is a bounded polyhedron (not just tetrahedron). Parry's approach is based on the properties of anti-reciprocal rational functions with Salem numbers. Kolpakov [306] provides a generalization to the three-dimensional case, by establishing a metric convergence of fundamental domains for cocompact hyperbolic Coxeter groups with finite-volume limiting polyhedron; for instance, the compact polyhedra $\mathcal{P}(n) \subset \mathbb{H}^3$ of type $< 2, 2, n, 2, 2 >$ converging, as $n \to \infty$, to a polyhedron $\mathcal{P}_\infty$ with a single four-valent ideal vertex. In this context, Kolpakov investigates the growth series of Coxeter groups acting on $\mathbb{H}^n$, $n \geq 3$ and their limit properties. The growth rates of ideal Coxeter polyhedra in $\mathbb{H}^3$ was studied by Nonaka and Kellerhals [398].

Case $n \geq 4$: the growth rates of cocompact hyperbolic Coxeter groups are not Salem numbers anymore. Kellerhals and Perren [295], §3 Example 2, show this fact with the example of the compact right-angled 120-cell in $\mathbb{H}^4$.

Lehmer's problem asks about the geometry of the poles of the growth rates of hyperbolic Coxeter groups acting on $\mathbb{H}^n$, and structure theorems about such groups having denominators of growth series of minimal Mahler measure.

Conjecture 6.3 (Kellerhals - Perren). *Let G be a Coxeter group acting cocompactly on $\mathbb{H}^n$ with natural generating set S and growth series $f_S(x)$. Then,*

(a) for n even, $f_S(x)$ has precisely $\frac{n}{2}$ poles $0 < x_1 < \ldots < x_{\frac{n}{2}} < 1$ in the open unit interval $(0,1)$;

(b) for n odd, $f_S(x)$ has precisely the pole 1 and $\frac{n-1}{2}$ poles $0 < x_1 < \ldots < x_{\frac{n-1}{2}} < 1$ in the interval $(0,1)$.

In both cases, the poles are simple, and the non-real poles of $f_S(x)$ are contained in the annulus of radii x_m and x_m^{-1} for some $m \in \{1,\ldots,\lfloor \frac{n}{2} \rfloor\}$.

Theorem 6.4 (Kellerhals - Perren). *Let G be a Lannér group, an Esselmann group or a Kaplinskaya group, respectively, acting with natural generating set S on $\mathbb{H}^4$. Then,*

(1) *the growth series* $f_S(x)$ *of* G *is a quotient of relatively prime, monic and reciprocal polynomials of equal degree over the integers,*

(2) *the growth series* $f_S(x)$ *of* G *possesses four distinct positive real poles appearing in pairs* (x_1, x_1^{-1}) *and* (x_2, x_2^{-1}) *with* $x_1 < x_2 < 1 < x_2^{-1} < x_1^{-1}$*; these poles are simple,*

(3) *the growth rate* $\tau = x_1^{-1}$ *is a Perron number,*

(4) *the non-real poles of* $f_S(x)$ *are contained in an annulus of radii* x_2 *,* x_2^{-1} *around the unit circle,*

(5) *the growth series* $f_S(x)$ *of the Kaplinskaya group* G_{66} *with graph* K_{66} *has four distinct negative and four distinct positive simple real poles; for* $G \neq G_{66}$*,* $f_S(x)$ *has no negative pole.*

Kellerhals and Kolpakov [294] (2014) prove that the simplex group $(3,5,3)$ has the smallest growth rate among all cocompact hyperbolic Coxeter groups on $\mathbb{H}^3$, and that it is, as such, unique. The growth rate is the Salem number $\tau' = 1.35098\ldots$ of minimal polynomial

$$X^{10} - X^9 - X^6 + X^5 - X^4 - X + 1. \tag{6.8}$$

Their approach provides a different proof for the analog situation in $\mathbb{H}^2$ where Hironaka [263] identified Lehmer's number as the minimal growth rate among all cocompact planar hyperbolic Coxeter groups and showed that it is (uniquely) achieved by the Coxeter triangle group $(3,7)$.

After Meyerhoff who proved that among all cusped hyperbolic 3- orbifolds the quotient of $\mathbb{H}^3$ by the tetrahedral Coxeter group $(3,3,6)$ has minimal volume, Kellerhals [293] (2013) proves that the group $(3,3,6)$ has smallest growth rate among all non- cocompact cofinite hyperbolic Coxeter groups, and that it is as such unique. This result extends to three dimensions some work of Floyd [223] who showed that the Coxeter triangle group $(3,\infty)$ has minimal growth rate among all non-cocompact cofinite planar hyperbolic Coxeter groups. In contrast to Floyd's result, the growth rate of the tetrahedral group $(3,3,6)$ is not a Pisot number. The following Theorem completes the picture of growth rate minimality for cofinite hyperbolic Coxeter groups in three dimensions.

Theorem 6.5 (Kellerhals). *Among all hyperbolic Coxeter groups with non-compact fundamental polyhedron of finite volume in* $\mathbb{H}^3$*, the tetrahedral group* $(3,3,6)$ *has minimal growth rate, and as such the group is unique.*

In [308] Komori and Umemoto, for three-dimensional non-compact hyperbolic Coxeter groups of finite covolume, show that the growth rate of a three-dimensional generalized simplex reflection group is a Perron number. In [309] Komori and Yukita show that the growth rates of ideal Coxeter groups in hyperbolic 3-space are also Perron numbers ; a Coxeter polytope P is called ideal if all vertices of P are located on the ideal boundary of hyperbolic space. They prove that the growth rate τ of an ideal Coxeter polytope with n facets in $\mathbb{H}^n$ satisfies $n-3 \leq \tau \leq n-1$. The smallest growth rates occur only if $n \leq 4$. They prove that the minimum of the growth rates is $0.492432^{-1} \approx 2.03074$, which is uniquely realized by the ideal Coxeter simplex with p = q = s = 2. In [515] Umemoto shows that infinitely many 2-Salem numbers can be realized as the growth rates of cocompact Coxeter groups in $\mathbb{H}^4$; the Coxeter polytopes are here constructed by successive gluing of Coxeter polytopes, which are called Coxeter dominoes [516]. The algebraic integers having a fixed number of conjugates outside the closed unit disk were studied by Bertin [45], Kerada [296], Samet [454], Zaimi [539], [540], in particular 2-Salem numbers in [296] to which Umemoto refers. In [541] Zehrt and Zehrt-Liebendörfer construct infinitely many growth series of cocompact hyperbolic Coxeter groups in $\mathbb{H}^4$, whose denominator polynomials have the same distribution of roots as 2-Salem polynomials; their Coxeter polytopes are the Coxeter garlands built by the compact truncated Coxeter simplex described by the Coxeter graph on the left side of Figure 1 in [541]. Lehmer's problem asks about the minimality of the houses of the 2-Salem numbers involved in these constructions. In [398] Kellerhals and Nonaka prove that the growth rates of three-dimensional Coxeter groups (Γ, S) given by ideal Coxeter polyhedra of finite volume are Perron numbers.

A Coxeter system (W, S) is a Coxeter group W with a finite generating set S; the permuted products $s_{\sigma(1)} s_{\sigma(2)} \ldots s_{\sigma(n)}$, $\sigma \in S_n$, are the *Coxeter elements* of (W, S). The element $w \in W$ is said to be *essential* if it is not conjugate into any subgroup $W_I \subset W$ generated by a proper subset $I \subset S$. The Coxeter group (W, S) acts naturally by reflections on $V \equiv \mathbb{R}^S$. Let $\lambda(w)$ be the spectral radius of $w|V$. When $\lambda(w) > 1$ it is also an eigenvalue of w. MacMullen [362] proves the three following results.

Theorem 6.6 (MacMullen). *Let (W, S) be a Coxeter system and let $w \in W$ be essential. Then*

$$\lambda(w) \geq \inf_{S_n} \lambda(s_{\sigma(1)} s_{\sigma(2)} \ldots s_{\sigma(n)}). \tag{6.9}$$

Let $\alpha(W, S)$ be the dominant eigenvalue of the adjacency matrix (A_{ij}) of (W, S), defined by $A_{ij} = 2\cos(\pi/m_{ij})$ for $i \neq j$ and $A_{ii} = 0$. Let $\beta(W, S)$ be the largest root of the equation $\beta + \beta^{-1} + 2 = \alpha(W, S)^2$ provided $\alpha(W, S) \geq 2$. If $\alpha(W, S) < 2$ then we put: $\beta(W, S) = 1$. Then $\lambda(w) = \beta(W, S)$ for all bicolored Coxeter element.

Theorem 6.7 (McMullen). *For any Coxeter system (W, S), we have*

$$\inf_{S_n} \lambda(s_{\sigma(1)} s_{\sigma(2)} \ldots s_{\sigma(n)}) \geq \beta(W, S). \tag{6.10}$$

Theorem 6.8 (McMullen). *There are 38 minimal hyperbolic Coxeter systems (W, S), and among these the infimum* $\inf \beta(W, S)$ *is Lehmer's number.*

Lehmer's problem is solved in this context. The quantity $\beta(W, S)$ can be viewed as a measure (not in logarithmic terms) of the complexity of a Coxeter system. Denote by $Y_{a,b,c}$ the Coxeter system whose diagram is a tree with 3 branches of lenghts a, b and c, joined by a single node. MacMullen [362] shows that the smallest Salem numbers of respective degrees 6, 8 and 10 coincide with $\lambda(w)$ for the Coxeter elements of $Y_{3,3,4}$, $Y_{2,4,5}$ and $Y_{2,3,7}$ respectively; in particular

Lehmer's number is $\lambda(w)$ for the Coxeter elements of $Y_{2,3,7}$. MacMullen shows that the set of irreducible Coxeter systems with $\beta(W,S) < \Theta$ consists exactly of $Y_{2,4,5}$ and $Y_{2,3,n}$, $n \geq 7$. He shows that the infimum of $\beta(W,S)$ over all high-rank Coxeter systems coincides with Θ. There are 6 Salem numbers < 1.3 that arise as eigenvalues in Coxeter groups, five of them arising from the Coxeter elements of $Y_{2,3,n}$, $7 \leq n \leq 11$.

6.3. Mapping classes: small stretch factors. We refer to Birman [56], Farb and Margalit [209] and Hironaka [268]. If S is a surface the *mapping class group* of S, denoted by $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation-preserving diffeomorphisms of S (that restrict to the identity on ∂S if $\partial S \neq \emptyset$). An irreducible mapping class is an isotopy class of homeomorphisms f of a compact oriented surface S to itself so that no power preserves a nontrivial subsurface. The classification of Nielsen-Thuston states that a mapping class $[f] \in \mathrm{Mod}(S)$ is either periodic, reducible or pseudo-Anosov [209], [210]. In the periodic case, the situation is "analogous to roots of unity" in Lehmer's problem. The minoration problem of the Mahler measure finds its analogue in the minoration of the dilatation factors of the pseudo-Anosovs. We refer to a mapping class $[f]$ by one of its representive f.

Let S_g be a closed, orientable surface of genus $g \geq 2$ and $\mathrm{Mod}(S_g)$ its mapping class group. For any Pseudo-Anosov element $f \in \mathrm{Mod}(S_g)$, and any integer $0 \leq k \leq 2g$, let

(i) $\kappa(f)$ be the dimension of the subspace of $H_1(S_g, \mathbb{R})$ fixed by f (for which $0 \leq \kappa(f) \leq 2g$),

(ii) $h(f) = \mathrm{Log}\,(\lambda(f))$ be the entropy of f, as logarithm of the *stretch factor* $\lambda(f) > 1$ (or *dilatation*; the dilatation measures the dynamical complexity),

(iii)
$$L(k,g) := \min\{h(f) \mid f : S_g \to S_g \text{ and } \kappa(f) \geq k\}. \tag{6.11}$$

Thurston [210], [510], noticed that the set of stretch factors for pseudo-Anosov elements of $\mathrm{Mod}(S_g)$ is closed and discrete in $\mathbb{R}$, and proved that any dilatation factor $\lambda(f) > 1$ is a Perron number, with $\lambda(f) + \lambda(f)^{-1}$ an algebraic integer of degree $\leq 4g-3$. The Perron number $\lambda(f)$ is the growth rate of lengths of curves under iteration (of any representant) of f, in any metric on S_g. These stretch factors appear as the length spectrum of the moduli space of genus g Riemann surface.

Penner [414] proved that the asymptotic behaviour $L(0,g) \asymp 1/g$ holds. With $k = 2g$, Farb, Leininger and Margalit [207] proved $L(2g,g) \asymp 1$. For the other values of k, since $L(0,g) \leq L(k,g) \leq L(2g,g)$, the following inequalities hold, from [1] [303] [263] [414],

$$\frac{\mathrm{Log}\,2}{6}\left(\frac{1}{2g-2}\right) \leq L(k,g) \leq \mathrm{Log}\,(62). \tag{6.12}$$

For $k = 0$ and $g = 1$, $L(0, 1) = \text{Log}\left(\frac{3+\sqrt{5}}{2}\right)$ for $\mathbb{T}^2$. For $k = 0$ and $g = 2$, Cho and Ham [127], [328], [546], proved $L(0, 2) \approx 0.5435\ldots$, as logarithm of the largest root of the Salem polynomial $X^4 - X^3 - X^2 - X + 1$; these authors showed that this minimum dilatation is given by Zhirov in [546], and realized by Pseudo-Anosov 5-braids in [252]. In [4] Agol, Leininger and Margalit improved the upper bound to: $(2g-2)L(0, g) < \text{Log}\,(\theta_2^{-4})$ for all $g \geq 2$, where θ_2^{-1} is the golden mean, and proved the main theorem:

$$L(k, g) \asymp \frac{k+1}{g}, \qquad g \geq 2, \quad 0 \leq k \leq 2g. \tag{6.13}$$

Arnoux-Yoccoz's Theorem [31] states that, for $g \geq 2$, for any $C \geq 1$, there are only finitely many conjugacy classes in $\text{Mod}(S_g)$ of pseudo-Anosov mapping classes with stretch factors at most C.

Minimal dilatation problem: what are the values of $L(k, g)$, except $L(0, g)$ for $g = 1, 2$ already determined? i.e. what are the minima $\delta_g := \exp(L(0, g))$, $g \geq 3$?

Lower bounds of the entropy are difficult to establish: e.g. Penner [414], Tsai [513] on punctured surfaces, Boissy and Lanneau [60] on translation surfaces that belong to a hyperelliptic component, Hironaka and Kin [273]. Then Kin [300], [303], [301], formulated several questions and conjectures on the minimal dilatation problem and its realizations. Bauer [39] studied upper bounds of the least dilatations, and Minakawa [385] gave examples of pseudo-Anosovs with small dilatations. Farb, Leininger and Margalit [208] obtained a universal finiteness theorem for the set of all small dilatation pseudo-Anosov homeomorphisms $\phi : S \to S$, ranging over all surfaces S. The following questions were posed by in [360] and [206].

Asymptotic behaviour:

(i) Does $\lim_{g\to\infty} g\ L(0, g)$ exist? What is its value?

(ii) Is the sequence $\{\delta_g\}_{g\geq 2}$ (strictly) monotone decreasing?

Kin, Kojima and Takasawa [301], for monodromies of fibrations on manifolds obtained from the magic 3-manifold N by Dehn filling three cusps with some restriction, proved $\lim_{g\to\infty} g\ L(0, g) = \text{Log}\left(\frac{3+\sqrt{5}}{2}\right)$; they also formulated limit conjectures for the asymptotic behaviour relative to compact surfaces of genus g with n boundary components.

A pseudo-Anosov mapping class $[f]$ is said to be *orientable* if the invariant (un)-stable foliation of a pseudo-Anosov homeomorphism $f \in [f]$ is orientable. Let $\lambda_H(f)$ be the spectral radius of the action of f on $H_1(S_g, \mathbb{R})$. It is the *homological stretch factor* of f. The inequality

$$\lambda_H(f) \leq \lambda(f) \tag{6.14}$$

holds and equality occurs iff the invariant foliations for f are orientable. Stretch factors obey some constraints [476]:

(i) $\deg(\lambda(f)) \geq 2$,

(ii) $\deg(\lambda(f)) \leq 6g - 6$,

(iii) if $\deg(\lambda(f)) > 3g - 3$, then $\deg(\lambda(f))$ is even.

Shin [476] formulates the following questions.

Algebraicity of the stretch factors.

(i) Which real numbers can be the stretch factors, the homological stretch factors?

(ii) What degrees of stretch factors can occur on S_g?

Let us define a mapping class $f_{g,k}$ by

$$f_{g,k} = (T_{c_g})^k \big(T_{d_g} \dots T_{c_2} T_{d_2} T_{c_1} T_{d_1}\big) \in \mathrm{Mod}(S_g), \tag{6.15}$$

where c_i and d_i are simple closed curves on S_g as in Figure 1 in [476], and T_c the Dehn twist about c.

Theorem 6.9 (Shin). *For each $g \geq 2, k \geq 3$, $f_{g,k}$ is a pseudo-Anosov mapping class which satisfies: (i) $\lambda(f_{g,k}) = \lambda_H(f_{g,k})$, (ii) $f_{g,k}$ is a Salem number, (iii) $\lim_{g\to\infty} f_{g,k} = k - 1$, where the minimal polynomial of $\lambda(f_{g,k})$ is the irreducible Salem polynomial*

$$t^{2g} - (k-2)\Big(\sum_{j=1}^{2g-1} t^j\Big) + 1, \qquad \textit{of degree } 2g. \tag{6.16}$$

Shin [476] deduces that, for each $1 \leq h \leq g/2$, there exists a pseudo-Anosov mapping class $\widetilde{f_h} \in \mathrm{Mod}(S_g)$ such that $\deg(\widetilde{f_h}) = 2h$, with $\lambda(\widetilde{f_h})$ a Salem number. He conjectures that, on S_g, there exists a pseudo-Anosov mapping class with a stretch factor of degree d for any even $1 \leq d \leq 6g-6$. He proves that the conjecture is true for $g = 2$ to 5. Shin and Strenner [477] prove that the Perron numbers which are the stretch factors of pseudo-Anosov mapping classes arising from Penner's construction [414] have conjugates which do not belong to the unit circle. In §3 in [477] they ask several questions about the geometry of the Galois conjugates of stretch factors, around the unit circle, obtained by several constructions: by Hironaka [267], by Dunfeld and Tiozzo, by Lanneau and Thiffeault [328], [329], by Shin [476]. For $S_{g,n}$ being an orientable surface with genus g and n marked points, Tsai [513] proves that the infimum of stretch factors is of the order of $(\mathrm{Log}\, n)/n$ for $g \geq 2$ whereas it is of the order of $1/n$ for $g = 0$ and $g = 1$; Tsai asks the question about the asymptotic behaviour of this infimum of dilatation factors in the (g,n)-plane. For some subcollections of mapping classes, by generalizing Penner's construction and by comparing the smallness of dilatation factors with trivial homological dilatation, Hironaka [270] concludes to the existence of subfamilies of pseudoanosovs which have asymptotically small dilatation factors.

In the context of $\mathbb{Z}^n$-actions on compact abelian groups (Proposition 17.2 and Theorem 18.1 in Schmidt [469]) the topological entropy is equal to the logarithm of the Mahler measure. If we assume that the stretch factors are Mahler measures $\mathrm{M}(\alpha)$ of algebraic numbers α (which are Perron numbers by Adler and Marcus [2]), then we arrive at a contradiction since Penner [414] showed that $L(0, g) \asymp \frac{1}{g}$ for surfaces of genus g. Indeed, it suffices to increase the genus g to find pseudo-Anosov elements of $\mathrm{Mod}(S_g)$ with dilatation factors arbitrarily close to 1, while Theorem **??** states that a discontinuity should exist. As a consequence of [414], [513] and of Theorem **??** (ex-Lehmer Conjecture) we deduce the following claims:

(1) **Assertion 1**: *The stretch factors of the pseudo-Anosov elements of* $\mathrm{Mod}(S_g)$ *are Perron numbers which are not Mahler measures of algebraic numbers as soon as* g *is large enough.*

(2) **Assertion 2**: *The stretch factors of the pseudo-Anosov elements of* $\mathrm{Mod}(S_{g,n})$*, where* $S_{g,n}$ *is an orientable surface with fixed genus* g *and* n *marked points, are Perron numbers which are not Mahler measures of algebraic numbers as soon as* n *is large enough.*

Let S be a connected finite type oriented surface. Leininger [339] considers subgroups of $\mathrm{Mod}(S)$ generated by two positive multi-twists; a multi-twist is a product of positive Dehn twists about disjoint essential simple closed curves. Given A and B two isotopy classes of essential 1-manifolds on S, we denote by T_A, resp. T_B, the positive multi-twist which is the product of positive Dehn twists about the components of A, resp. of B.

Theorem 6.10 (Leininger). *Any pseudo-Anosov element* $f \in \langle T_A, T_B \rangle$ *has a stretch factor which satisfies:*

$$\lambda(f) \geq \lambda_L \text{ (Lehmer's number).} \tag{6.17}$$

The realization occurs when S has genus 5 (with at most one marked point), $A = A_{Lehmer}, B = B_{Lehmer}$ given by Figure 1 in [339] up to homeomorphism, and f conjugate to $(T_A T_B)^{\pm 1}$. Leininger's Theorem 6.10 is strikingly reminiscent of McMullen's Theorem 6.8. The following questions are formulated in §9.1 in [339]:

- **Q1**: Which Salem numbers occur as dilatation factors of pseudo-Anosov automorphisms?
- **Q2**: Is there some topological condition on a pseudo-Anosov which guarantees that its dilatation factor is a Salem number?
- **Q3** (Lehmer's problem for dilatation factors): Is there an $\epsilon > 1$ such that if f is a pseudo-Anosov automorphism in a finite co-area Teichmüller disk stabilizer, then $\lambda(f) \geq \epsilon$?

Since dilatations factors of pseudo-Anosovs are Perron numbers and not necessarily Mahler measures of algebraic numbers (cf Assertions 1 and 2 above), Leininger's Theorem 6.10 and McMullen's Theorem 6.8 are addressed to the set of Salem numbers and suggest that Lehmer's number is actually the smallest Salem number in this set; meaning first that Lehmer's Conjecture is true for Salem numbers.

Let

$$f_{k,l}(t) := t^{2k} - t^{k+l} - t^k - t^{k-l} + 1, \tag{6.18}$$

$$\text{resp.} \qquad f_{x,y,z}(t) := t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1, \tag{6.19}$$

and denote $\lambda_{(k,l)} > 1$, resp. $\lambda_{(x,y,z)} > 1$, its dominant root.

Related to the minimization problem is the one for orientable pseudo-Anosovs. The minimal dilatation factor for orientable pseudo-Anosov elements of $\mathrm{Mod}(S_g)$ is denoted by δ_g^+. The minimal dilatation problem for δ_g^+ is open in general. For $g = 2$, Zhirov [546] obtains $\delta_2^+ = \delta_2$. For $g = 1$, $\delta_1^+ = \delta_1$ holds. From [267], [328], $\delta_g < \delta_g^+$ for $g = 4, 6, 8$. Hironaka [267] obtains:

(i) $\delta_g \leq \lambda_{(g+1,3)}$ if $g \equiv 0, 1, 3, 4 \pmod 6$ and $g \geq 3$,

(ii) $\delta_g \leq \lambda_{(g+1,1)}$ if $g \equiv 2, 5 \pmod 6$ and $g \geq 5$,

(iii) $\limsup_{g\to\infty} g \mathrm{Log}\, \delta_g \leq \mathrm{Log}\left(\frac{3+\sqrt{5}}{2}\right)$.

Kin and Takasawa [305] complement and improve these inequalities. They show:

(i) $\delta_g \leq \lambda_{(g+2,1)}$ if $g \equiv 0, 1, 5, 6 \pmod{10}$ and $g \geq 5$,

(ii) $\delta_g \leq \lambda_{(g+2,2)}$ if $g \equiv 7, 9 \pmod{10}$ and $g \geq 7$;

for $g \equiv 2, 4 \pmod{10}$, under the assumption $g+2 \not\equiv 0 \pmod{4641}$, then they prove:

(i) $\delta_g \leq \lambda_{(g+2,3)}$ if $\gcd(g+2, 3) = 1$,

(ii) $\delta_g \leq \lambda_{(g+2,7)}$ if $3|(g+2)$ and $\gcd(g+2, 7) = 1$,

(iii) $\delta_g \leq \lambda_{(g+2,13)}$ if $21|(g+2)$ and $\gcd(g+2, 13) = 1$,

(iv) $\delta_g \leq \lambda_{(g+2,17)}$ if $273|(g+2)$ and $\gcd(g+2, 17) = 1$.

For $g = 8$ and 13 they obtain the sharper upper bounds: $\delta_8 \leq \lambda_{(18,17,7)} < \lambda_{(9,1)}$ and $\delta_{13} \leq \lambda_{(27,21,8)} < \lambda_{(14,3)}$. For orientable pseudo-Anosovs, Lannneau and Thiffeault [328] obtain $\delta_3^+ = \lambda_{(3,1)}$, $\delta_4^+ = \lambda_{(4,1)}$, and the following lower bounds for $g = 6$ to 8: (i) $\delta_6^+ \geq \lambda_{(6,1)}$, (ii) $\delta_7^+ \geq \lambda_{(9,2)}$ and $\delta_8^+ \geq \lambda_{(8,1)}$. Hironaka [267] gives the upper bounds:

(i) $\delta_g^+ \leq \lambda_{(g+1,3)}$ if $g \equiv 1, 3 \pmod 6$,

(ii) $\delta_g^+ \leq \lambda_{(g,1)}$ if $g \equiv 2, 4 \pmod 6$,

(iii) $\delta_g^+ \leq \lambda_{(g+1,1)}$ if $g \equiv 5 \pmod 6$.

Hironaka obtains the asymptotics:

$$\limsup_{g\to\infty, g\not\equiv 0 (\mathrm{mod} 6)} g \,\mathrm{Log}\, \delta_g^+ \leq \mathrm{Log}\left(\frac{3+\sqrt{5}}{2}\right) \tag{6.20}$$

and, from [328], the equality: $\delta_8^+ = \lambda_{(8,1)}$. Kin and Takasawa [305] give the better upper bounds:

(i) $\delta_g^+ \leq \lambda_{(g+2,2)}$ if $g \equiv 7, 9 \pmod{10}$ and $g \geq 7$,

(ii) $\delta_g^+ \leq \lambda_{(g+2,4)}$ if $g \equiv 1, 5 \pmod{10}$ and $g \geq 5$.

Moreover they prove: $\delta_7^+ = \lambda_{(9,2)}$ (Aaber and Dunfeld [1] obtain it independently) and $\delta_5 < \delta_5^+$.

The realization of the minimal dilatations is a basic question, with the uniqueness problem, considered by many authors: associated with least volume [1], [301], braids [127], [252], [273], [303], [304], [329], monodromies [207], [301], [305], Coxeter graphs and Coxeter elements [339], [267], [476], quotient families of mapping classes [272], self-intersecting curves [168], homology of mapping tori [4]. There exists several constructions of small dilatation families, e.g. by Hironaka [269], [271], McMullen [360], Dehornoy [146], with Lorenz knots.

6.4. Knots, links, Alexander polynomials, homology growth, Jones polynomials, lenticularity of zeroes, lacunarity. Constructions of Alexander polynomials of knots and links are given in [292], [393], [449], [473], [514]. Silver and Williams in [480] (reported in [494] § 4.2 for an overview) investigate the Mahler measures of various Alexander polynomials of oriented links with d components in a homology 3-sphere; they obtain theorems on limits of Mahler measures and Mahler measures of derivatives of d-variate Mahler measures by performing $1/q$ surgery ($q \in \mathbb{N}$) on the dth component, allowing $q \to \infty$. In particular they consider the topological realizability of the small Mahler measures and limit Mahler measures on various examples.

For Pretzel links Hironaka ([263], [264], [266], [237] p. 308) solves the minimization problem for the subclass of Salem numbers defined in Theorem 6.1 by

Theorem 6.11 (Hironaka [263]). *Let $p_1, \ldots, p_d$ positive integers. Then the Alexander polynomial of the $(p_1, \ldots, p_d, -1, \ldots, -1)$-pretzel link (Coxeter link), where the number of -1's is $d-2$, with respect to a suitable orientation of its components, is*

$$\Delta_{p_1,\ldots,p_d}(-x). \tag{6.21}$$

Lehmer's polynomial of the variable "$-x$" is the Alexander polynomial of the $(-2,3,7)$-pretzel knot and the $(-2,3,7)$-pretzel knot is equivalent to the $(2,3,7,-1)$- pretzel knot. Theorem 6.2 follows from Theorem 6.11. The Mahler measure of the $(2,3,7,-1)$- pretzel knot is the minimum of the set of Mahler measures of Alexander polynomials of (suitably oriented) $(p_1, \ldots, p_d, -1, \ldots, -1)$-pretzel links, over all d in $2\mathbb{N}+1$.

It is natural to find counterparts of Lehmer's problem in Topology where several polynomial invariants [225], [226], [287], [493] § 14.6, were associated to knots, links and braids, for which the notions of convergence and "limit" can be defined (as in [121], [145], [265], [479]) in addition to Alexander polynomials. Indeed a Theorem of Seifert [473] asserts that (i) any monic reciprocal integer polynomial $P(x)$, (ii) which satisfies $|P(1)| = 1$, is the Alexander polynomial of (at least) one knot, and conversely; Burde [105] extended it to fibered knots (cf Hironaka [265]). A Theorem of Kanenobu [290] asserts that any reciprocal monic integer polynomial $P(x)$ is, up to multiples of $x-1$, the Alexander polynomial of a fibered link. Let us recall that infinitely many knots may possess the same polynomial invariants (Morton [386], Kanenobu [291]).

Periodic homology and exponential growth. The r-fold cyclic covering $X_r(K)$ of a knot $K \subset \mathbb{S}^3$ admits topological invariants, i.e. homology groups $H_1(X_r(K), \mathbb{Z})$, which are also invariants of the knot K. To K is associated a sequence of Alexander polynomials $(\Delta_i), i \geq 1$, in a single variable, such that $\Delta_{i+1} | \Delta_i$. Likewise, to an oriented link with d components is associated a sequence of Alexander polynomials in d variables. In both cases, the first Alexander polynomial of the sequence is usually called the Alexander polynomial of the knot K, resp. of the link. For a knot K Gordon [245] proved that the first Alexander invariant $\lambda_1(t) = \Delta_1(t)/\Delta_2(t)$ satisfies the following equivalence :

$$\lambda_1(t)|(t^n - 1) \quad \Longleftrightarrow \quad H_1(X_r(K), \mathbb{Z}) \;\cong\; H_1(X_{r+n}(K), \mathbb{Z}) \;\text{ for all }\; r. \tag{6.22}$$

The equivalence (6.22) is an analogue of Kronecker's Theorem. Gordon used the Pierce numbers of the Alexander polynomial of K, for which a linear recurrence is expected as in [336], [193]. Gordon obtained periods which are not prime powers and how to find all of them for knots of a given genus.

Theorem 6.12 (Gordon). *There exists a knot K of genus g for which $H_1(X_r(K), \mathbb{Z})$ has proper period n if and only if $n = 1$, or $n = \operatorname{lcm}\{m_i \mid i = 1, 2, \ldots, r\}$, where the m_i's are all distinct, each has at least two distinct prime factors, and $\sum_{i=1}^{r} \Phi(m_i) \leq 2g$.*

Departing from "Kronecker's Theorem" Gordon conjectured that when some zero of $\Delta_1(t)$ is not a root of unity, then the order of $H_1(X_r(K), \mathbb{Z})$ grows exponentially with r. This conjecture was proved by Riley [446], with p-adic methods, and González-Acuña and Short [241]. Both used the Gel'fond-Baker theory of linear forms in the logarithms of algebraic numbers. Silver and Williams [479] extended the conjecture of Gordon and its proof for knots, where the "finite order of $H_1(X_r(K), \mathbb{Z})$" is replaced by the "order of the torsion subgroup of $H_1(X_r(K), \mathbb{Z})$", and for links in $\mathbb{S}^3$. They identified the torsion subgroups with the connected components of periodic points in a dynamical system of algebraic origin [469], connected the limit with the logarithmic Mahler measure (for any finite-index subgroup $\Lambda \subset \mathbb{Z}^d$, the number of such connected components is denoted by b_Λ and $\langle \Lambda \rangle := \{|v| \mid v \in \Lambda \setminus \{0\}\}$ is the norm of the smallest nonzero vector of Λ; cf [479] for the definitions):

Theorem 6.13 (Silver-Williams [479]). *Let $l = l_1 \cup \ldots \cup l_d$ be an oriented link of d components having nonzero Alexander polynomial Δ, in d variables. Then*

$$\limsup_{\langle \Lambda \rangle \to \infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \operatorname{Log} b_\Lambda = \operatorname{Log} \mathrm{M}(\Delta) \tag{6.23}$$

where "$\limsup$" is replaced by "$\lim$" if $d = 1$.

Let M be a finitely generated module over $\mathbb{Z}[\mathbb{Z}^n]$ and $\widehat{M}$ its (compact) Pontryagin dual. For any subgroup $\Lambda \subset \mathbb{Z}^n$ of finite index, let b_Λ be the number of connected components of the set of elements of $\widehat{M}$ fixed by actions of the elements of Λ. Le ([334], Theorem 1) proved a conjecture of K. Schmidt [468] on the growth of the number b_Λ; as a consequence Le generalized ([334], Theorem 2) Silver Williams's Theorem 6.13 on the growth of the homology torsion of finite abelian covering of link complements, with the logarithmic Mahler measure of the first nonzero Alexander polynomial of the link. In each case, since the growth is expressed by the logarithmic Mahler measure of the (first nonzero) Alexander polynomial, Lehmer's problem amounts to establishing a universal minorant > 0 of the exponential base. For non-split links in $\mathbb{S}^3$, that is in the nonabelian covering case, Le [334] generalized Theorem 6.13 using the L^2-torsion, i.e. the hyperbolic volume in the rhs part of (6.23) instead of the logarithmic Mahler measure of the 0th Alexander polynomial; in such a case the minimality of Mahler measures would find its origin in the minimality of hyperbolic volumes [335].

The growth of the homology torsion depends upon the (nonzero) logarithmic Mahler measure of the Alexander polynomial(s) of a knot or a link. Hence the geometry of zeroes of Alexander polynomials is important for the minoration of the homology growth [237]. At this step, let us briefly mention the importance of other studies on the roots of Alexander polynomials: (i) monodromies and dynamics of surface homeomorphisms [274], [449], (ii) knot groups: factorization and divisibility [393], (iii) knot groups: orderability (Perron Rolfsen), (iv) statistical models (Lin Wang).

Applying solenoidal dynamical systems theory to knot theory enabled Noguchi [397] to prove that the dominant coefficient a_n of the Alexander polynomial $\Delta_K(t) = \sum_{i=0}^{n} a_i t^i, a_0 a_n \neq 0$, of a knot K, α_i being the zeroes (counted with multiplicities) of $\Delta_K(t)$, satisfies ($|\cdot|_p$ is the p-adic norm normalized by $|p|_p = 1/p$ on $\mathbb{Q}_p$):

$$\operatorname{Log}|a_n| = \sum_{p<\infty} \sum_{|\alpha_i|_p>1} \operatorname{Log}|\alpha_i|_p \tag{6.24}$$

He proved that the distribution of zeroes measures a "distance" of the Alexander module from being finitely generated as a $\mathbb{Z}$-module, and that the growth of order of the first homology of the r-fold cyclic covering $X_r(K)$ branched over K is related to the zeroes by

$$\lim_{r\to\infty, |H_1(\cdot)|\neq 0} \frac{\operatorname{Log}|H_1(X_r(K);\mathbb{Z})|}{r} = \sum_{p\leq\infty} \sum_{|\alpha_i|_p>1} \operatorname{Log}|\alpha_i|_p. \tag{6.25}$$

Therefore the leading coefficient of $\Delta_K(t)$ is closely related to the homology growth and the p-adic norms of the zeroes α_i.

A link, or a knot, is said to be alternating if it admits a diagram where (along every component) the strands are passed under-over. In 2002 Hoste (Hirasawa and Murasugi [262]) stated the following conjecture: *let K be an alternating knot and $\Delta_K(t)$ its Alexander polynomial. If α is a zero of $\Delta_K(t)$, then $\Re e(\alpha) > -1$.*

Hoste's Conjecture is proved in some cases: cf [274], [352], [502], [503]. The problem of the geometry and the boundedness of zeroes of the (knot and link) Alexander polynomials is difficult and related to two other conjectures on the coefficients of these polynomials, namely the Fox's trapezoidal Conjecture and the Log-concavity Conjecture [310], [502], [503].

In his studies of Lorenz knots [144], [145], [146], Dehornoy obtained the following much more precise statement on the geometry of the zero locus (g is the smallest genus of a surface spanning the knot; the braid index b is the smallest number of strands of a braid whose closure is the knot):

Theorem 6.14 (Dehornoy [145]). *Let K be a Lorenz knot. Let g denote its genus and b its braid index. Then the zeroes of the Alexander polynomial of K lie in the annulus*

$$\{z \in \mathbb{C} \mid (2g)^{-4/(b-1)} \leq |z| \leq (2g)^{4/(b-1)}\}. \tag{6.26}$$

The Alexander polynomial of a Lorenz knot reflects an intermediate step between signatures and genus [145]. A certain proportion of zeroes lie on the unit circle and are controlled by the ω-signatures (Gambaudo and Ghys, cited in [145]). The other zeroes lie within a certain distance from the unit circle and are controlled by the house of the Alexander polynomial, which is the modulus of the largest zero. The problem of the minimality of the house of this Alexander polynomial is reminiscent of the Schinzel-Zassenhaus Conjecture if it were expressed as a function of its degree. For Lorenz knots this house is interpreted as follows: it is the growth rate of the associated homological monodromy (for details, cf [145] § 2). Figure 3.3 in [145] shows two examples of Lorenz knots, with respective braid index and genus $(b, g) = (40, 100)$ and $= (100, 625)$; interestingly, the distribution of zeroes within the annulus (6.26) appears angularly fairly regular (in the sense of Bilu's Theorem [54]) but exhibit lenticuli of zeroes in the angular sector $\arg(z) \in [\pi - \pi/2, \pi + \pi/2]$. Such lenticuli do exist for integer polynomials of small Mahler measure, of the variable "$-x$", and are shown to be at the origin of the minoration of the Mahler measure in the problem of Lehmer in the present note. Though Dehornoy did not publish (yet) further on the Mahler measures of the Alexander polynomials of Lorenz knots, in particular in the way (b, g) tends to infinity, it is very probable that such polynomials are good candidates for giving small Mahler measures together with a topological interpretation of the houses. The above examples suggest that the Alexander polynomials of Lorenz knots are not Salem polynomials, though no proof seems to exist.

Before Le [334], [335], Boyd and Rodriguez-Villegas [89] [90] [93] studied the connections between the Mahler measure of the *A-polynomial of a knot* and the hyperbolic volume of its complement. A-polynomials were introduced in hyperbolic geometry by Cooper et al [133] (are not Alexander polynomials, though "A" is used in homage to Alexander). The irreducible factors of A-polynomials have (logarithmic) Mahler measures which are shown to be finite sums of Bloch-Wigner dilogarithms [233], [538], of algebraic numbers. The values of such dilogarithms are related to Chinburg's Conjecture. Several examples are taken by the authors to investigate Chinburg's Conjecture and its generalization refered to as Boyd's question (cf also Ray [437]). Chinburg's Conjecture [90] is stated as follows: for each negative discriminant $-f$ there exists a polynomial $P = P_f \in \mathbb{Z}[x, y]$ and a nonzero rational number r_f such that

$$\operatorname{Log} \mathrm{M}(P) = r_f \frac{f\sqrt{f}}{4\pi} L(2, \chi_f). \tag{6.27}$$

Boyd's question is stated as follows: for every number field F having a number of complex embeddings equal to 1 (i.e. $r_2 = 1$), does there exist a polynomial $P = P_F \in \mathbb{Z}[x, y]$ and a rational number r_F such that $\operatorname{Log} \mathrm{M}(P) = r_F Z_F$?, where ζ_F is the Dedeking zeta function of F and

$$Z_F = \frac{3\,|\mathrm{disc}(F)|^{3/2}\,\zeta_F(2)}{2^{2n-3}\,\pi^{2n-1}}; \tag{6.28}$$

the starting point being (Smyth [492]): for $f = 3$, $\operatorname{Log} \mathrm{M}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(2, \chi_3)$.

Jones polynomials of knots and links, lacunarity in coefficient vectors. Let L be a hyperbolic link and, for $m \geq 1$, denote by L_m the link obtained from L by adding m full twists on n strands [449], [121]. By Thurston's hyperbolic Dehn surgery, the volume $\mathrm{Vol}(\mathbb{S}^3 \setminus L_m)$ converges to $\mathrm{Vol}(\mathbb{S}^3 \setminus (L \cup U))$, as m tends to infinity, where U is an unknot encircling n strands of L such that L_m is obtained from L by a $-1/m$ surgery on U. More generally, let $\underline{m} = (1, m_1, \ldots, m_s)$, for $s \geq 1$, and $L_{\underline{m}} := L_{m_1,\ldots,m_s}$ the multi-twisted link obtained from a link diagram L by a $-1/m_i$ surgery on an unknot U_i, for $i = 1, \ldots, s$. In the following theorem convergence of Mahler measures has to be taken in the sense of the Boyd Lawton's Theorem 2.5.

Theorem 6.15 (Champanerkar - Kofman). *(i) The Mahler measure* $\mathrm{M}(V_{L_m}(t))$ *of the Jones polynomial of* L_m *converges to the Mahler measure of a 2-variable polynomial, as* m *tends to infinity;*

(ii) the Mahler measure $\mathrm{M}(V_{L_{\underline{m}}}(t))$ *of the Jones polynomial of* $L_{\underline{m}}$ *converges to the Mahler measure of a* $(s+1)$*-variable polynomial, as* $\underline{m}$ *tends to infinity.*

In [121] Theorem 2.4, Champanerkar and Kofman [121] extended Theorem 6.15 to the convergence of the Mahler measures of colored Jones polynomials $J_N(L_m, t)$ and $J_N(L_{\underline{m}}, t)$ for fixed N, as m, resp. $\underline{m}$, tends to infinity; here coloring means by the N-dimensional irreducible representation of $SL_2(\mathbb{C})$ with the normalization of $J_2(L_m, t)$ as $J_2(L_m, t) = (t^{1/2} + t^{-1/2}) V_{L_m}(t)$, resp. for $\underline{m}$. They proved that the limit $\lim_{m\to\infty} \mathrm{M}(J_N(L_m, t))$, resp. for $\underline{m}$, is the Mahler measure of a multivariate polynomial. What smallness of limit Mahler measures can be reached by this construction, and what are the corresponding geometrical realizations?

In [121] (Theorem 2.5 and Corollary 3.2) Champanerkar and Kofman obtain the following theorem.

Theorem 6.16 (Champanerkar - Kofman). *Let* $N \geq 1$ *be a fixed integer. With the above notations,*

(i) let $\{\gamma_{i,m}\}$ *be the set of distinct roots of the Jones polynomial* $J_N(L_m, t)$*. Then* $\liminf_{\kappa\to\infty} \#\{\gamma_{i,m} \mid m \leq \kappa\} = \infty$*, and for any* $\epsilon > 0$*, there exists an integer* q_ϵ *such that the number of such roots satisfies*

$$\#\{\gamma_{i,m} \mid ||\gamma_{i,m}| - 1| \geq \epsilon\} < q_\epsilon, \tag{6.29}$$

(ii) for m *sufficiently large, the coefficient vector of the Jones polynomial* $J_N(L_m, t)$ *has nonzero fixed blocks of integer digits separated by gaps (blocks of zeroes) whose length increases as* m *tends to infinity.*

In addition to the relative limitation of the multiplicities of the roots, Theorem 6.16 means that, in the annulus $1 - \epsilon < |z| - 1 < 1 + \epsilon$, the clustering of the roots occurs, up to q_ϵ of them (densification), and is associated with a moderate lacunarity ("gappiness" in the sense of [518]) of the Jones polynomials which increases with m. This Theorem has been extended to other Jones polynomials by these authors [121] and followed previous experimental observations. From Theorem 6.15 and Theorem 6.16 it is likely that such Jones polynomials lead to very small multivariate Mahler measures, at least are good candidates.

Other families of Jones polynomials, their zeroes and their limit distributions, were investigated, for which interesting limit Mahler measures may be expected: e.g. Chang and Shrock [122], Wu and Wang [532], Jin and Zhang [284], [285], [286], related to models in statistical physics. The moderate lacunarities occurring in the coefficient vectors of Jones polynomials were studied by Franks and Williams [225] in the context of polynomial invariants associated with braids, knot and links which generalize Alexander polynomials and Jones polynomials [225], [226], [287], [393].

6.5. Arithmetic Hyperbolic Geometry. Leininger's constructions in [339] give the dilatation factors of pseudo-Anosovs as spectral radii of hyperbolic elements in some Fuchsian groups. The minimality of the Salem numbers as dilatation factors is defined in a more general context (Neuman and Reid [394], Maclachlan and Reid [358], Ghate and Hironaka [237] p. 303).

Theorem 6.17 (Neuman - Reid). *The Salem numbers are precisely the spectral radii of hyperbolic elements of arithmetic Fuchsian groups derived from quaternion algebras.*

Arithmetic hyperbolic groups are arithmetic groups of isometries of hyperbolic n-space $\mathbb{H}^n$. Vinberg and Shvartsman [523] p. 217 have defined the large subclass of the arithmetic hyperbolic groups of the simplest type, in terms of an admissible quadratic form over a totally real number field K. This subclass includes all arithmetic hyperbolic groups in even dimensions, infinitely many wide-commensurability classes of hyperbolic groups in all dimensions [356], and all non-cocompact arithmetic hyperbolic groups in all dimensions. Isometries of $\mathbb{H}^n$ are either elliptic, parabolic or hyperbolic. An isometry $\gamma \in \mathbb{H}^n$ is hyperbolic if and only if there is a unique geodesic L in $\mathbb{H}^n$, called the *axis* of γ, along which γ acts as a translation by a positive distance $l(\gamma)$ called the *translation length* of γ.

The following theorems generalize previous results of Neumann and Reid [394] in dimension 2 and 3 and show the important role played by the smallest Salem numbers:

Theorem 6.18 (Emery- Ratcliffe - Tschantz [195]). *Let Γ be an arithmetic group of isometries of $\mathbb{H}^n$, $n \geq 2$, of the simplest type defined over a totally real algebraic number K. Let $\Gamma^{(2)}$ be the subgroup of Γ of finite index generated by the squares of the elements of Γ. Let γ be a hyperbolic element of Γ, and let $\lambda = e^{l(\gamma)}$. If n is even or $\gamma \in \Gamma^{(2)}$, then λ is a Salem number such that $K \subset \mathbb{Q}(\lambda + \lambda^{-1})$ and $\deg_K(\lambda) \leq n + 1$.*

Conversely, if $\lambda \in T$*,* K *is a subfield of* $\mathbb{Q}(\lambda+\lambda^{-1})$ *and* n *such that* $\deg_K(\lambda) \leq n+1$*, then there exists an arithmetic group* Γ *of isometries of* $\mathbb{H}^n$ *of the simplest type defined over* K *and a hyperbolic element* $\gamma \in \Gamma$ *such that* $\lambda = e^{l(\gamma)}$*.*

Theorem 6.19 (Emery - Ratcliffe - Tschantz [195]). *Let* Γ *be an arithmetic group of isometries of* $\mathbb{H}^n$*,* $n \geq 2$ *odd, of the simplest type defined over a totally real algebraic number* K*. Let* $\Gamma^{(2)}$ *be the subgroup of* Γ *of finite index generated by the squares of the elements of* Γ*. Let* γ *be a hyperbolic element of* Γ*, and let* $\lambda = e^{l(\gamma)}$*. Then* λ *is a Salem number which is square-rootable over* K*.*

Conversely, if $\lambda \in T$*,* K *is a subfield of* $\mathbb{Q}(\lambda+\lambda^{-1})$ *and* n *an odd positive integer such that* $\deg_K(\lambda) \leq n+1$*, and* λ *is square-rootable over* K*, then there exists an arithmetic group* Γ *of isometries of* $\mathbb{H}^n$ *of the simplest type defined over* K *and a hyperbolic element* $\gamma \in \Gamma$ *such that* $\sqrt{\lambda} = e^{l(\gamma)}$*.*

6.6. Salem numbers and Dynamics of Automorphisms of Complex Compact Surfaces. Let X be a compact Kähler variety and f an automorphism of X. The automorphism f induces an invertible linear map f^* on $H^*(X,\mathbb{C})$, resp. $H^*(X,\mathbb{R})$, $H^*(X,\mathbb{Z})$, which preserves the Hodge decomposition, the intersection form, the Kähler cone. Iterating f provides a dynamical system to which real algebraic integers ≥ 1 are associated. The greatest eigenvalue of the action of f on $H^*(X,\mathbb{C})$ is usually called the *maximal dynamical degree* of f. This terminology is the same as the one used for the β-shift in the present note, but the notions are different. The maximal dynamical degree of f is denoted by $\lambda(f)$; it is related to the topological entropy $h_{top}(f)$ of f by $\operatorname{Log}\lambda(f) = h_{top}(f)$ by a Theorem of Gromov and Yomdin [247][534]. Saying that an automorphism is of positive entropy is equivalent to saying that its maximal dynamical degree is > 1. In particular if X is a surface the characteristic polynomial of f^* on $H^2(X,\mathbb{Z})$ is a (not necessarily irreducible) Salem polynomial (McMullen [361]); the maximal dynamical degree $\lambda(f)$ of f is the spectral radius of f^* on $H^{1,1}(X)$ and is a Salem number. Salem numbers are deeply linked to the geometry of the surface. Among all complex compact surfaces [35], Cantat [111] [112] showed that, if X is a complex compact surface for which there exists an automorphism of X having a positive entropy, then there exists a birational morphism from X to a torus, a $K3$ surface, a surface of Enriques, or the projective plane. Therefore it suffices to consider complex tori (Oguiso and Truong [405], Reschke [440] 2017), Enriques surfaces (Oguiso [401], [403]), and $K3$ surfaces (Gross and McMullen [249], McMullen [361], Oguiso [402], Shimada [475]) if X is not rational.

The restriction to compact Kähler surfaces is justified by the fact that the topological entropy of all automorphisms vanishes on compact complex surfaces which are not Kähler (Cantat [112]). The existence of an automorphism of positive entropy is a deep question [95], [96], [113], [197], [361], [404], [406].

On each type of surface, what are the Salem numbers which appear? In this context the problem of Lehmer can be formulated by asking what are the minimal Salem numbers which occur, per type of surface, and the corresponding geometrical realizations.

In [359] McMullen gives a general construction of $K3$ surface automorphisms f from unramified Salem numbers, such that, for every such automorphism f, the topological entropy $\operatorname{Log} \lambda(f)$ is positive, together with a criterion for the resulting automorphism to have a Siegel disk (domains on which f acts by an irrational rotation). The Salem polynomials involved, of the respective dynamical degrees $\lambda(f)$, have degree 22, trace -1 and are associated to an even unimodular lattice of signature $(3, 19)$ on which f acts as an isometry, by the Theorem of Torelli. The surface is non-projective to carry a Siegel disk.

McMullen [363] (Theorem A.1) proved that Lehmer's number (denoted by λ_{10}) is the smallest Salem number that can appear as dynamical degree of an automorphism of a complex compact surface:

$$h(f) \geq \operatorname{Log} \lambda_{10} = 0.162357\ldots. \tag{6.30}$$

He gave a geometrical realization of Lehmer's number in [363] on a rational surface (cf also Bedford and Kim [41]), in [364] on a nonprojective $K3$ surface, in [365] on a projective $K3$ surface. On the contrary Oguiso [402] proved that Lehmer's number cannot be realized on an Enriques surface. In [365] McMullen proved that the value $\operatorname{Log} \lambda_d$ arises as the entropy of an automorphism of a complex projective $K3$ surface if

$$d = 2, 4, 6, 8, 10 \text{ or } 18, \text{but not if } d = 14, 16 \text{ or } d \geq 20. \tag{6.31}$$

Brandhorst and González-Alonso [97] completed the above "realizability" list with the value $d = 12$ (Theorem 1.2 in [365]).

For projective surfaces, the degree of the Salem number is bounded by the rank of the Néron-Severi group; for $K3$ surfaces in characteristic zero it is at most 20, due to Hodge theory. In positive characteristic the rank 22 is possible (case of supersingular $K3$ surfaces) [96] [535]. Therefore all such Salem numbers, when less than 1.3, are listed in Mossinghoff's list in [389], the list being complete up to degree 44.

Reschke [440], [441], gave a necessary and sufficient condition for a Salem number to be realized as dynamical degree of an automorphism of a complex torus, with degrees 2, 4 or 6; moreover he investigated the relations between the values of the Salem numbers and the corresponding geometry and projectiveness of the tori. Zhao [545] extended the method of Reschke for tori endowed with real structures, showing that it suffices to consider real abelian surfaces. Zhao classified such real abelian surfaces into 8 types according to the number of connected components and the simplicity of the underlying complex abelian surface. For each type the set of Salem numbers which can be realized by real automorphisms is determined. Zhao [545] proved that Lehmer's number cannot be realized by a real $K3$ surface.

Dolgachev [167] investigated automorphisms on Enriques surfaces of dynamical degrees > 1 which are small Salem numbers, of small degree 2 to 10 (Salem numbers of degree 2 are quadratic Pisot numbers). The method does not allow to conclude on the minimality of the Salem numbers. The author uses the lower semi-continuity properties of the dynamical degree of an automorphism g of an algebraic surface S when (S, g) varies in an algebraic family.

In positive characteristic Brandhorst and González-Alonso [97] proved that the values $\operatorname{Log} \lambda_d$ arise as the entropy of an automorphism of a supersingular $K3$ surface over a field of characteristic $p = 5$ if and only if $d \leq 22$ is even and $d \neq 18$, giving in their Appendix B the list of Salem numbers λ_d of degree d and respective minimal polynomials. They develop a strategy to characterize the minimal Salem polynomials, in particular their cyclotomic factors, for various realizations in supersingular $K3$ surfaces having Artin invariants σ ranging from 1 to 7, in characteristic 5. Yu [535] studied the maximal degrees of the Salem numbers arising from automorphisms of $K3$ surfaces, defined over an algebraically closed field of characteristic p, in terms of the elliptic fibrations having infinite automorphism groups, and Artin invariants.

Oguiso and Truong [405] Dinh, Nguyen and Truong [156], [157], investigated the structure of compact Kähler manifolds, in dimension ≥ 3, from the point of view of establishing relations between non-trivial invariant meromorphic fibrations, pseudo-automorphisms f and the dynamical degrees $\lambda_k(f)$. Lehmer's problem can be formulated by asking when the first dynamical degree $\lambda_1(f)$ is a Salem number, what minimal value for $\lambda_1(f)$ can be reached and what are the possible geometrical realizations for the minimal ones.

7. Appendix - Standard notations

The notations used by several authors in the above sections can be found directly in the corresponding articles and books which are quoted in the text. We just report the standard notations in the following. obtained

Let $P(X) \in \mathbb{Z}[X]$, $m = \deg(P) \geq 1$. The *reciprocal polynomial* of $P(X)$ is $P^*(X) = X^m P(\frac{1}{X})$. The polynomial P is reciprocal if $P^*(X) = P(X)$. When it is monic, the polynomial P is said *unramified* if $|P(1)| = |P(-1)| = 1$. If $P(X) = a_0 \prod_{j=1}^{m}(X - \alpha_j) = a_0 X^m + a_1 X^{m-1} + \ldots + a_m$, with $a_i \in \mathbb{C}$, $a_0 a_m \neq 0$, and roots α_j, the *Mahler measure* of P is

$$\mathrm{M}(P) := |a_0| \prod_{j=1}^{m} \max\{1, |\alpha_j|\}. \tag{7.1}$$

The absolute Mahler measure of P is $\mathrm{M}(P)^{1/\deg(P)}$, denoted by $\mathcal{M}(P)$. The Mahler measure of an algebraic number α is the Mahler of its minimal polynomial P_α: $\mathrm{M}(\alpha) := \mathrm{M}(P_\alpha)$. For any algebraic number α the house $\overline{|\alpha|}$ of α is the maximum modulus of its conjugates, including α itself; by Jensen's formula the Weil height $h(\alpha)$ of α is $\operatorname{Log} \mathrm{M}(\alpha)/\deg(\alpha)$. By its very definition, $\mathrm{M}(PQ) = \mathrm{M}(P)\mathrm{M}(Q)$ (multiplicativity).

A *Perron number* is either 1 or a real algebraic integer $\theta > 1$ such that the Galois conjugates $\theta^{(i)}, i \neq 0$, of $\theta^{(0)} := \theta$ satisfy: $|\theta^{(i)}| < \theta$. Denote by $\mathbb{P}$ the set of

Perron numbers. A *Pisot number* is a Perron number > 1 for which $|\theta^{(i)}| < 1$ for all $i \neq 0$. The smallest Pisot number is denoted by

$$\Theta = 1.3247\ldots, \text{ dominant root of } X^3 - X - 1. \tag{7.2}$$

A Salem number is an algebraic integer $\beta > 1$ such that its Galois conjugates $\beta^{(i)}$ satisfy: $|\beta^{(i)}| \leq 1$ for all $i = 1, 2, \ldots, m-1$, with $m = \deg(\beta) \geq 1$, $\beta^{(0)} = \beta$ and at least one conjugate $\beta^{(i)}, i \neq 0$, on the unit circle. All the Galois conjugates of a Salem number β lie on the unit circle, by pairs of complex conjugates, except $1/\beta$ which lies in the open interval $(0, 1)$. Salem numbers are of even degree $m \geq 4$. The set of Pisot numbers, resp. Salem numbers, is denoted by S, resp. by T. If $\tau \in$ S or T, then $\mathrm{M}(\tau) = \tau$. A j-Salem number [296] [454], $j \geq 1$, is an algebraic integer α such that $|\alpha| > 1$ and α has $j-1$ conjugate roots $\alpha^{(q)}$ different from α, satisfying $|\alpha^{(q)}| > 1$, while the other conjugate roots ω satisfy $|\omega| \leq 1$ and at least one of them is on the unit circle. We call the minimal polynomial of a j-Salem number a j-Salem polynomial. Salem numbers are 1-Salem numbers. A Salem number is said *unramified* if its minimal polynomial is unramified. We say that two Salem numbers λ and μ are *commensurable* if there exists positive integers k and l such that $\lambda^k = \mu^l$. Commensurability is an equivalence relation on T. Let $\lambda \in T$, K a subfield of $\mathbb{Q}(\lambda + \lambda^{-1})$, and $P_{\lambda,K}$ the minimal polynomial of λ over K; we say that λ is *square-rootable* over K if there exists a totally positive element $\alpha \in K$ and a monic reciprocal polynomial $q(x)$, whose even degree coefficients are in K and odd degree coefficients are in $\sqrt{\alpha}K$ such that $q(x)q(-x) = P_{\lambda,K}(x^2)$. A Garsia number is an algebraic integer of norm ± 2 such that all of the roots of its minimal polynomial are strictly greater than 1 in absolute value [255].

The set of algebraic numbers, resp. algebraic integers, in $\mathbb{C}$, is denoted by $\overline{\mathbb{Q}}$, resp. $\mathcal{O}_{\overline{\mathbb{Q}}}$. The nth cyclotomic polynomial is denoted by $\Phi_n(z)$. For any postive integer n, let $[n] := 1 + x + x^2 + \ldots + x^{n-1}$. The (naïve) height of a polynomial P is the maximum of the absolute value of the coefficients of P. Let A be a countable subset of the line; the *first derived set* $A^{(1)}$ of A is the set of the limit points of nonstationary infinite sequences of elements of A; the *k-th derived set* $A^{(k)}$ of A is the first derived set of $A^{(k-1)}$, $k \geq 2$.

For $x > 0$, $\lfloor x \rfloor$, $\{x\}$ and $\lceil x \rceil$ denotes respectively the integer part, resp. the fractional part, resp. the smallest integer greater than or equal to x. For $\beta > 1$ any real number, the map $T_\beta : [0,1] \to [0,1], x \to \{\beta x\}$ denotes the β-transformation. With $T_\beta^0 := T_\beta$, its iterates are denoted by $T_\beta^{(j)} := T_\beta(T_\beta^{j-1})$ for $j \geq 1$. A real number $\beta > 1$ is a Parry number if the sequence $(T_\beta^{(j)}(1))_{j \geq 1}$ is eventually periodic; a Parry number is called simple if in particular $T_\beta^{(q)}(1) = 0$ for some integer $q \geq 1$. The set of Parry numbers is denoted by $\mathbb{P}_P$. The terminology chosen by Parry in [412] has changed: β-numbers are now called Parry numbers, in honor of W. Parry.

The Mahler measure of a nonzero polynomial $P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is defined by

$$\mathrm{M}(P) := \exp\left(\frac{1}{(2i\pi)^n}\int_{\mathbb{T}^n} \mathrm{Log}\,|P(x_1,\ldots,x_n)|\frac{dx_1}{x_1}\ldots\frac{dx_n}{x_n}\right),$$

where $\mathbb{T}^n = \{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid |z_1| = \ldots = |z_n| = 1\}$ is the unit torus in dimension n. If $n = 1$, by Jensen's formula, it is given by (7.1). A function $f : \mathbb{R} \to \mathbb{R}$ is said quasiperiodic if it is the sum of finitely many periodic continuous functions. The function, defined for $k \geq 2$, $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$, $|z| \leq 1$, is the kth-polylogarithm function [233], [340], [538]. For $x > 0$, $\mathrm{Log}^+ x$ denotes $\max\{0, \mathrm{Log}\, x\}$. Let $\mathcal{F}$ be an infinite subset of the set of nonzero algebraic numbers which are not a root of unity; we say that the *Conjecture of Lehmer is true for* $\mathcal{F}$ if there exists a constant $c_{\mathcal{F}} > 0$ such that $\mathrm{M}(\alpha) \geq 1 + c_{\mathcal{F}}$ for all $\alpha \in \mathcal{F}$.

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