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▶ **To cite this version:**

Eric Kamwa, Vincent Merlin, Faty Mbaye Top. Scoring Run-off Rules, Single-peaked Preferences and Paradoxes of Variable Electorate. 2023. hal-03143741v2

HAL Id: hal-03143741
https://hal.science/hal-03143741v2

Preprint submitted on 27 Oct 2023

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Scoring Run-off Rules, Single-peaked Preferences and Paradoxes of Variable Electorate*

Eric Kamwa† Vincent Merlin‡ Faty Mbaye Top§

Updated: October 26, 2023

Abstract

This paper analyzes the vulnerability of scoring run-off rules to abstention and participation paradoxes when voters preferences are single-peaked. These paradoxes occur when the size of the electorate varies (grows or diminishes). In particular, the Abstention or No-show paradox occurs when a voter is better off by not casting his ballot in the election. First, we show that all the scoring run-off rules that always elect the Condorcet winner on this domain are immune to the different forms of Abstention and Participation paradoxes. Secondly, when these paradoxes are still possible, we compute their likelihood in three-candidate elections under the Impartial Anonymous Culture assumption. We conclude that considering the single-peaked domain drastically reduces, and even sometimes eliminates the impact of No-show paradoxes, for scoring run-off rules.

JEL Codes : D70, D71
Keywords : Paradoxes, Scoring Rules, Run-offs, No-show, Abstention, Participation, Impartial Anonymous Culture, Probability.

*The authors are very grateful to Issofa Moyouwou for the careful reading and the meaningful advices. Thanks also to Michel Le Breton for his helpful remarks and comments on the early draft of this paper. Vincent Merlin acknowledges support from the project ANR-14-CE24-0007-01 CoCoRICo-CoDec. Thanks also to all the participants of the Voting sessions at the 14th Meeting of the Society for Social Choice and Welfare (Seoul), at The 15th Spain-Italy-Netherlands Meeting on Game Theory (Turku University) and at the 2019 Annual Meeting of the Association for Public Economic Theory (Strasbourg).

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1 Introduction

In an election, a voting paradox is a counterintuitive result a ballot can lead to. The social choice literature describes a large number of voting paradoxes. The reader can refer to Nurmi (1999, 1989), Saari (1995) and Felsenthal (2012) for a non exhaustive review of these paradoxes. In general, voting paradoxes are gathered by families or classes. For example, the family of *Condorcet paradoxes* (Condorcet, 1785, Gehrlein, 2006) deals with the selection or not of the *Condorcet winner* and of the *Condorcet loser*. The *Condorcet winner* (resp. *Condorcet loser*) is a candidate that beats (resp. is beaten by) all the other candidates in pairwise comparisons. Borda (1781) and Condorcet (1785) were the first authors to describe such paradoxes, as they depicted voting situations where the Plurality rule may fail to pick the Condorcet winner, and even select the Condorcet loser.

When defining voting paradoxes, most of them assume a fixed size of the electorate. For example, a monotonicity paradox occurs whenever some voters improve the ranking of a candidate in their preferences while its position falls in the social ordering. When the size of the electorate varies *i.e* an increase or a decrease in the number of voters, we enter the realm of paradoxes of variable electorate. The most famous paradox in this family is the *No-show paradox* also called the *Abstention paradox* (AP): it occurs when a voter or a group of voters can get better off by not showing up at the pooling station. A voting method would seem perverse if by abstaining, one could see the victory of a preferred candidate while this wouldn't have been the case if his vote has been casted. Symmetrically, the *Participation* property states that it is always beneficial for a voter (or a group of voters) to cast his ballot. When it is not the case, a *Participation paradox* (PP) occurs; that is, a voter or a group of voters is harmed when joining the voting population. However, widely used voting methods suffer from this default. Doron and Kronick (1977) and Fishburn and Brams (1983) provide examples where these phenomena happen for the Plurality run-off method[1] and three candidates. In this simple case, the Plurality run-off method assumes that in the first run, each voter votes for only one candidate; if none of them obtains more than 50% of the vote, only the top two candidates will go to the second contest that will determine the final winner. Indeed, the No-show paradox may occur with any type of iterative scoring rules (see Smith (1973)). Unfortunately, it has been discovered that the No-show paradox is not limited to scoring run-off rules; for four candidates and more, Moulin (1988) showed that all the *Condorcet consistent*[2] voting rules are sensitive to the No-show paradox.[3] Recently, Zahid (2009) et Laslier (2018) proved that the *Majority Judgement*, a voting rule popularized by Balinski and Laraki (2011) also suffers from the No-Show paradox. Felsenthal and Nurmi (2019a,b) showed that for a No-show paradox to occur under a Condorcet consistent rule

[1] Eliminations methods based upon the Plurality scores are used in many countries to elect local officers, mayors, representatives, presidents, etc. From one country to another, the type of the ballot, the number of people to elect per jurisdiction and the number of elimination rounds vary. These methods have different names such as *Preferential Voting*, *Instant Run-off*, *Two Stage Plurality*, *Alternative Vote*, *Single Transferable Vote*, etc.

[2] A voting rule is said to be *Condorcet consistent* if it always selects the Condorcet winner when he exists.

[3] For recent extensions or refinements of Moulin's theorem, see Brandt et al. (2017), Duddy (2013), Jimeno et al. (2009).

it is not, in general, necessary that a majority top cycle exists in the original profile, while for this paradox to occur when using non Condorcet consistent rules it is, almost always, necessary that a top cycle exists.

Participation is not the only condition that deals with variable electorate in social choice literature. The *Separability* condition states that whenever a population is divided in two sub-populations that elect the same winner, so does the entire population[4]. Smith (1973) also shows that scoring run-off rules does not meet the Separability condition. The same result holds for all the Condorcet consistent voting rules (Young and Levenglick, 1978).

In three-candidate elections with strict rankings of candidates, Lepelley and Merlin (2001) analyzed the No-show paradox for three well known run-off scoring rules: The Plurality run-off, the Antiplurality run-off and the Borda run-off. They focussed on two particular forms of the No-show paradox: the *Positive Abstention Paradox* (PAP) which arises when deleting some voters who rank a loser a_j at the top of their rankings, a_j becomes a winner; and the *Negative Abstention Paradox* (NAP) which occurs when deleting some voters who rank the winner a_j at the bottom, a_j becomes a loser. Lepelley and Merlin (2001) also analyzed two symmetric paradoxes due to an increase of the size of the electorate: the *Positive Participation Paradox* (PPP) and the *Negative Participation Paradox* (NPP). The PPP occurs when adding some voters who rank the winner a_j at the top of their rankings, a_j becomes a loser. The NPP appears when adding some voters who rank a loser a_j at the bottom of their rankings, a_j becomes a winner. Hence, Lepelley and Merlin (2001) have computed the probabilities of occurrence of the PAP, the NAP, the AP, the PPP, and the NPP for these scoring run-off rules on the universal domain for a well-known probability assumption, the *Impartial Anonymous Culture* (IAC) which will be defined later. Notice that Lepelley and Merlin (2001) considered the coalitional version of these paradoxes: a voting situation may lead to an abstention (resp. a participation) paradox if a group of voters (possibly a unique voter) obtains a better result by not casting their ballots (resp. by casting their ballots). Though the limit values when n goes to the infinity remain low (between 4% and 6%), they found that the susceptibility to manipulation by abstention could be huge for small populations. To give some examples, we reach the maximal value with four voters, at 62% with the Borda run-off rule and 38% with the Plurality run-off. For the Antiplurality run-off, the highest propensity to manipulate is as high as 26% with 6 voters.

Related papers are more optimistic about the occurrence of these phenomena. Recently, Brandt et al. (2021) studied the susceptibility to manipulation by abstention for several Condorcet consistent rules under different probability assumptions (including IAC). By focusing on individual manipulability only, they derived extremely low values (inferior to 2%) for 3 alternatives, even for small populations. Also, from the PrefLib library, which gathers data of real elections, they only find two situations out of 315 where the No-show Paradox can occur for one given Condorcet consistent rule, namely, Black's rule[5]. However, in scenarios with 10 to 50 voters having to choose among 10 to 30 candidates, the likelihood of the ab-

[4]The same principle is known as the *Consistency* condition (Young, 1974, 1975) or the *Reinforcement* axiom (Moulin, 1988, Myerson, 1995) elsewhere in the literature.

[5]Black's rule Black (1958) selects the Condorcet winner whenever it exists; otherwise, it picks the Borda winner.

stention paradox can rise drastically (almost 28% for the Maximin[6] rule with a population of 30 voters and 30 candidates). Courtin et al. (2010) studied the violation of the Reinforcement axiom for scoring run-off rules. They also obtain extremely low values under the IAC assumption. However, their main scenario deals with the merging of two parliaments into a unique chamber, and not with the incentives a voter (or a small group) faces when he has to decide whether he should vote. All in all, the fact that abstention and participation could be part of a strategic behavior cannot be neglected for some voting scenarios. This is particularly true when we use scoring run-off rules in small groups.

Hence, the purpose of this paper is to question the robustness of Lepelley and Merlin's results when we restrict the domain of preferences to that of single-peaked ones (Black, 1948, 1958). The single-peakedness assumption is a famous condition in public economics literature. It assumes that for certain collective decision issues, not all the preferences are present in the society. More precisely, candidates can be ordered on a left-right axis, such that for each voter, his utility is decreasing with the distance to his preferred alternative along this axis. With three candidates, this restriction implies that one of them, the centrist candidate, is never ranked last by the voters. We know from social choice literature that imposing single-peaked preferences can have various effects on both the existence and the likelihood of voting paradoxes, such as: -*eliminating the majority cycles* (Black, 1948); *reconciling the Condorcet and Borda approach for some rules* (Lepelley and Vidu, 2000, Smith, 1973)- *opening the possibility to design strategy proof rules* (Moulin, 1980) -*increasing the occurrence of some paradoxes* as in the case of the Borda paradox (Lepelley, 1993) or -*reducing the likelihood of some paradoxes* such as the monotonicity paradoxes (Lepelley et al., 1995).

Given the number of democracies (e.g. Australia, France, Ireland, Senegal) or institutions (e.g. the Olympic Committee for the choice of the host city, the Academy of Motion Pictures Arts and Sciences for the Oscars, etc.) using elimination processes to take decisions, examining the behavior of all the scoring run-off rules when that voters' preferences are single-peaked is of particular interest.

After a presentation of the notation and definitions (Section 2), our characterization results of Section 3 help us to identify all the scoring run-off rules vulnerable to each of the four paradoxes of variable electorate (PAP, NAP, PPP and NPP). In particular, we are able to take advantage of the characterization of the scoring run-off rules that satisfy the Condorcet criteria on the single-peaked domain proposed by Lepelley and Vidu (2000) to derive our necessary and sufficient conditions for a paradox to occur. For rules likely to produce a paradox in three-candidate elections, we assess the occurrence in Section 4 using the assumption of the Impartial and Anonymous Culture. From this data, we are able to assess to the likelihood of the Participation Paradox (PP) and of the Abstention Paradox (AP). Due the progress of computation techniques, we also evaluate the probability of some events that Lepelley and Merlin (2001) were not able to derive. We also compare our figures to results that have been obtained in the literature regarding the monotonicity paradoxes and manipulation probabilities under the single-peaked domain. Section 5 concludes.

[6]The Maximin rule picks the alternative which worse score in pairwise comparisons against the other candidates is maximal.

2 Notation and definitions

2.1 Preferences

Let N be a set of n voters with $n \geq 2$ and $A = \{a_1, a_2, \dots a_m\}$ the set of m candidates. When we will consider only three candidates, we will assume that $A = \{a, b, c\}$. It is assumed each voter ranks all the candidates in A a strict manner without ties. His preference is depicted by a linear ordering P_i on A. The preference profile $\pi = (P_1, P_2, \dots, P_i, \dots, P_n)$ gives for each voter i among the n voters, his strict ranking P_i. The universal domain $D(A)^U$, is the set of all the possible preference profiles on A, whatever the size of population is. When we will consider a subset of alternatives $B \subset A$, we will assume that the preference of individual i is the linear ordering P_i restricted to B, $P_i|_B$.

In the three-candidate case, a voting situation $\tilde{n} = (n_1, n_2, n_3, n_4, n_5, n_6)$ indicates for the six possible types of strict rankings on $A = \{a, b, c\}$, the number of voters of each type t such that $\sum_{t=1}^{6} n_t = n$. Table 1 describes the voting situation on $A = \{a, b, c\}$ where abc means that n_1 voters rank a before b and b before c.

Table 1: Possible types of preferences on $A = \{a, b, c\}$

$abc:$	$acb:$	$cab:$	$cba:$	$bca:$	$bac:$
n_1	n_2	n_3	n_4	n_5	n_6

The single-peakedness assumption describes a situation where all the voters agree to position in the same way all the candidate on a common left-right axis (Black, 1948, 1958). For the sake of simplicity, we will assume that this axis is described by the linear ranking:

$$a_1 \bar{P} a_2 \bar{P} a_3 \dots \bar{P}_{a_m}$$

Candidate a_1 is the most leftist candidate, a_2 the second most leftist, etc... and a_m is the most rightist candidate. Moreover, it is assumed that each voter has a unique preferred candidate, and that is, his preference is strictly decreasing as we move away from it along the left-right axis depicted by $\bar{P}$. The domain of singled-peaked preferences on A is denoted by $D(A)^{SP}$. Sen (1966) shows that the single-peakedness assumptions is equivalent to consider that on each triplet of candidates, one of them is never ranked last. For three candidates case, assuming c, the centrist candidate, is never ranked last, we obtain the ordering $a\bar{P}c\bar{P}b$ and the preferences depicted in Table 2.

Table 2: single-peaked preferences on $A = \{a, b, c\}$

$acb:$	$cab:$	$cba:$	$bca:$
n_2	n_3	n_4	n_5

For a linear ordering P, the rank of alternative x is defined by $r(P,x) = \#\{y \in A : yPx\} + 1$. We recall some well known results regarding single-peaked preferences (see for example Lepelley and Vidu (2000)).

Lemma 1. *For any profile in $D(A)^{SP}$, the j^{th} option in $\bar{P}$ cannot be ranked after the rank $(m+1-j)$ in any individual preference if $j \leq m/2$; it cannot be ranked after rank j if $j > m$. As a consequence, only extreme alternatives a_1 and a_m can be ranked last in individual preferences.*

2.2 Voting Rules

Definition 1. For any $B \subset A$, a Social Choice Function f is a mapping from $D(B)^U$ into B.

For each set of candidates proposed to the voters, the social choice function will select a unique winner. It may be necessary to use tie breaking mechanisms in the decision process. In this paper, we will not define any particular tie breaking rule. In case of a tie between any two alternatives x and y, we will always favor the choice of the alternative that lead to a paradox and/or a problem, in order to tackle the worst case scenario. In that sense, our results will not depend upon the choice of any particular tie breaking rule.

Condorcet Principle

Given a profile π (resp. a voting situation $\tilde{n}(\pi)$), we will write $aM(\pi)b$ (resp. $aM(\tilde{n})b$) to say that candidate a is majority preferred to b *i.e.* the number of voters who rank a before b is strictly greater than that of those who rank b before a. When there is no risk of confusion, we will simply write aMb.

A *Condorcet Winner* (CW) is an alternative that is able to defeat any other candidate in pairwise comparisons. Candidate a is a CW at profile π whenever:

$$aM(\pi)x,\ \forall x \in A,\ x \neq a.$$

By definition, a CW is unique. A *Weak Condorcet Winner* (WCW) is an alternative which is never beaten in pairwise comparisons. Candidate a is a WCW at profile π if there is no $x \in A$, $x \neq a$, such as $xM(\pi)a$. We know from Black (1948) that the CW winner always exists on the single-peaked domain when n is odd; when n is even, the set of WCW is non empty too, but may contain up to m alternatives[7]. We denote by $C(\pi)$, or simply C, the set of WCW at profile π.

[7]Consider the profile where half of the population has the preference $a_1 P_i a_2 \dots P_i a_m$ and the other half, exactly the opposite preference.

Scoring rules and scoring run-off rules

Given a set B of p alternatives, a scoring vector is a vector $v^p = (v_1^p, v_2^p, \ldots, v_j^p, \ldots, v_p^p) \in \mathbb{R}^p$ such as $w_1^p = 1$, $w_m^p = 0$ and $w_j^p \geq w_{j+1}^p$. Given a scoring vector v^p, voters give points to candidates in B according to their rankings in preferences. A scoring rule f_v is a mapping from $D(B)^U$ into A, which picks the alternative with the highest total score[8]. The most famous scoring rules are the *Plurality rule* ($v_P^p = (1, 0, \ldots, 0)$), the *Antiplurality rule* ($v_A^p = (1, 1, \ldots, 1, 0)$) and the *Borda count* ($v_B^p = (1, \frac{m-2}{m-1}, \ldots, \frac{m-j}{m-1} \ldots, \frac{1}{m-1}, 0)$).

For the three-candidate case, one point is given to a candidate each time he is ranked first, s point ($0 \leq s \leq 1$) when he is ranked second and 0 when he is last. This defines $v_s = (1, s, 0)$. When $s = 0$, we get the *Plurality rule*; $s = 1$ defines the *Antiplurality rule*; when $s = \frac{1}{2}$ we recover the *Borda count*.

A set $v^\star = \{v^2, v^3, \ldots, v^p, \ldots, v^m\}$ is a family of scoring vectors; it can be used to define a scoring run-off rule $f_{v^\star}$. A scoring run-off rule eliminate progressively the candidates on the basis of the scores they obtained with a given scoring vector. When p alternatives are in contention, the alternative which obtained the lowest score with f_{v^p} is removed from the set of alternatives (again, a tie breaking rule might be used in case of a tie for the last position). Alternatives are eliminated one by one till the final duel ($v^2 = (1, 0)$). The most famous members of this family are the Plurality run-off rule (Hare, 1859), the Antiplurality run-off rule (Coombs, 1954) and the Borda run-off rule (Baldwin, 1926). Notice that our definition does not encompass the cases where several alternatives are eliminated in block, as it is the case for two stages Plurality run-off which directly selects the top two candidates for the final run; the original rule suggested by Nanson (1883) that eliminates at each stage the alternatives which Borda scores are below the average and the rule suggested by Kim and Roush (1996) that also eliminates at each stage the alternatives which Antiplurality scores are below the average.

In the three candidate case, the one with the lowest score is eliminated after a first run and the last run is a majority contest involving the two other candidates. Given the single peaked preferences of Table 2 and $v_s = (1, s, 0)$, the scores $w_s(a)$, $w_s(b)$ and $w_s(c)$ of each of the candidate in A are as follows:

$$\begin{aligned} w_s(a) &= n_2 + sn_3 \\ w_s(b) &= sn_4 + n_5 \\ w_s(c) &= n_3 + n_4 + s(n_2 + n_5) \end{aligned} \tag{1}$$

We can now focus on the characterization of the paradoxical voting situations. To do so, we will first use some results from Lepelley and Vidu (2000) in order to identify situations where the abstention and participation paradoxes cannot appear.

[8] A tie-breaking rule might be applied if needed

3 Paradoxical voting situations under the single-peaked domain: Characterization results

3.1 General Results

In order to specify which scoring runoff rules suffer from the Participation and Abstention paradoxes, we will connect this issue with previous results by Lepelley and Vidu (2000); under the single-peaked domain, they proposed necessary and sufficient conditions for the scoring run-off rules to always select the Condorcet winner when he exists.

Proposition 1. *(Lepelley and Vidu, 2000) Let $f_{v^\star}$ be a scoring run-off rule defined on $D(A)^{SP}$. If the family $v^\star$ is such that for all $p = 2, \dots m$, $v^p_{p-1} + v^p_{\lceil \frac{p}{2} \rceil} \geq 1$, then $f_{v^\star}$ will always select the Condorcet winner.*

Proposition 2. *(Lepelley and Vidu, 2000) Let $f_{v^\star}$ be a scoring run-off rule defined on $D(A)^{SP}$. If $f_{v^\star}$ always selects the Condorcet winner at any profile, it the case that, for all p:*

- $v^p_{\frac{p}{2}} + v^p_{\frac{p}{2}+1} \geq \min_{j \in \{1,\dots,\frac{p}{2}-1\}} \left(v^p_{p-j+1} + v^p_j\right)$ *if p is even*
- $v^p_{\frac{p+1}{2}} \geq \min_{j \in \{1,\dots,\frac{p-1}{2}\}} \left(\frac{v^p_{p-j+1} + v^p_j}{2}\right)$ *if p is odd.*

First, the necessary and sufficient conditions they propose are equivalent for $m = 3$ to $v^3_2 = s \geq \frac{1}{2}$) and for $m = 4$ to $v^4_2 + v^4_3 \geq 1$. For these cases, both the Antiplurality run-off and the Borda run-off rules satisfy the necessary and sufficient conditions.

Dealing with variable electorates means that the number of voters who show up may be even or odd. And, if the the number of voter is even, we may not have a unique Condorcet winner. Hence, we need first to extend Proposition 1 to the case of Weak Condorcet Winners (WCW) by showing that if the necessary condition is met, the scoring run-off rules will always select a WCW.

Theorem 1. *Let $f_{v^\star}$ be a scoring run-off rule defined on $D(A)^{SP}$. If the family $v^\star$ is such that for all $p = 2, \dots m$, $v^p_{p-1} + v^p_{\lceil \frac{p}{2} \rceil} \geq 1$, then $f_{v^\star}$ will always select a WCW.*

The proof is very similar to the one proposed by Lepelley and Vidu (2000), as we just need to slightly adapt it to the case of an even number of voters.

Proof. Assume that $\forall p = 2, \dots m$, $v^p_{p-1} + v^p_{\lceil \frac{p}{2} \rceil} \geq 1$. Let B the set of alternatives still in competition at any stage of the elimination process and $C \subset B$ the set of WCW. Without loss of generality, we can relabel the alternatives in B from a_1 to a_p.

Case 1) Suppose $a_1 \in C$ (or symmetrically $a_p \in C$). It comes that a_1 is ranked first by at least half of the population, because the only ordering in the single-peaked domain where a_1 is ranked before a_2 is $a_1 P a_2 P a_3 \dots a_{p-1} P a_p$. Hence, the score of a_1 is at least $\frac{n}{2}$.

If a_1 is ranked last for the other half of the voters (which means that n is even), his score is exactly $\frac{n}{2}$. If the score of a_p is lower, a_1 won't be eliminated at this stage. Otherwise, the score of a_p will exactly be $\frac{n}{2}$ too. This is only possible if the other half of the population has the preference $a_p P a_{p-1} \bar{P} a_{p-2} \dots a_2 P a_1$. In that case, $C = B$: all the alternatives still in competition are WCW, and whoever is eliminated at this stage, a WCW will be elected at the end.

Case 2) Suppose that $a_1 \notin C$ and $a_p \notin C$. From Lemma 1, we can deduce that the median option $a_{\lceil \frac{n}{2} \rceil}$ along the axis $a_1 \bar{P} a_2 \dots \bar{P} a_p$ is always ranked before rank $\lceil \frac{p}{2} \rceil$ in all the individual orderings. Hence, an element a_c of C must appear before this rank for at least half of the voters. Otherwise, a_c could not be a WCW. Moreover, as a_c is not one of the extreme options (a_1 or a_p), it cannot be ranked last. Hence, his score is at least equal to:

$$\frac{n+1}{2} v^p_{\lceil \frac{p}{2} \rceil} + \frac{n-1}{2} v^p_{p-1} = \frac{n-1}{2} \left(v^p_{\lceil \frac{p}{2} \rceil} + + v^p_{p-1} \right) + v^p_{\lceil \frac{p}{2} \rceil} \quad \text{if } n \text{ is odd,}$$

and to

$$\frac{n}{2} \left(v^p_{\lceil \frac{p}{2} \rceil} + v^p_{p-1} \right) \quad \text{if } n \text{ is even.}$$

If $v^p_{\lceil \frac{p}{2} \rceil} + v^p_{p-1} \geq 1$ and n is odd, the score of a_c is superior to $\frac{n-1}{2}$. Moreover, one of the extreme alternatives, a_1 or a_p, has a score inferior of equal to $\frac{n-1}{2}$ (as one has to be ranked last by more than half of the voters). Hence, a_c is not eliminated. If $v^p_{\lceil \frac{p}{2} \rceil} + v^p_{p-1} \geq 1$ and n is even, the score of a_c is superior or equal to $\frac{n}{2}$. The maximal score an extreme alternative can get is also $\frac{n}{2}$. There is only one case for which a_1 and a_p have both this score: when half of the population has the preference $a_1 P a_2 \dots a_{p-1} P a_p$ and the other half, the opposite preference. In that case, $C = B$, and in any case, a CWC is elected at the end. Otherwise, one of them as a lower score and a_c is not eliminated. Thus, in any case, we are sure that if $v^p_{\lceil \frac{p}{2} \rceil} + v^p_{p-1} \geq 1$, $f_{v^\star}(\pi) \in C(\pi)$. □

Theorem 2. *Let f be a Social Welfare Function on $D(A)^{SP}$. If f always select a member of $C(\pi)$ at any profile in $D(A)^{sp}$, it cannot suffer from any Abstention nor Participation paradox.*

Proof. First, we only consider the case of one individual leaving or entering the population, as each change in the population is a sequence of individual entries or leaves.

Assume that $f_{v^\star}$ always select a WCW, and let $f_{v^\star}(\pi) = a$. Assume that a participation paradox occurs, that is, a new voter with aP_ib joins population to form the profile π' and $f_{v^\star}(\pi') = b$. But as $f_{v^\star}$ always select a WCW, it cannot select b. Either b was tied with a, and it is now dominated by b in π', or b was already dominated, and it remains dominated. In any case, we have $aM(\pi')b$ and $b \notin C$, which is a contradiction.

The same reasoning holds for the abstention paradox. Assume that $f_{v^\star}(\pi) = a$, and that a voter i with bP_ia leaves the population. But b cannot be elected as long as $f_{v^\star}$ always select a WCW. If b was a WCW, being tied with a, it is now dominated and cannot be selected. □

3.2 Specific results related to three-candidate elections

Note that Theorems 1 and 2 are valid for any number of alternatives. We adapt these results to the three-candidate case. First of all, let us recall that the *Positive Abstention Paradox* (PAP) occurs when some voters with a loser ranked last abstain, this loser becomes the winner. Symmetrically, the *Positive Participation Paradox* (PPP) describes a situation under which by joining the electorate, some voters with the winner ranked first in their preferences, can turn this winner into a loser. The *Negative Abstention Paradox* (NAP) appears after some voters with the winner ranked last abstain and this winner becomes a loser ; for the *Negative Participation Paradox* (NPP), some voters with a loser ranked last can turn this candidate into a winner by joining the electorate.

Theorem 3. *Given a voting situation with three candidates and single-peaked preferences, the Positive Abstention Paradox nor the Positive Participation Paradox can occur with the Plurality run off and with all the scoring run-off located between the Borda run-off and the Antiplurality run-off. Nonetheless, for all* $s \in]0\ \frac{1}{2}[$*, there exists some voting situations* $\tilde{n}$ *under which these paradoxes can happen.*

The sufficient part is just the consequence of Theorems 1 and 2 for $s \in [\frac{1}{2}, 1]$. Smith (1973) prove that the Plurality run-off does not suffer from PPP and PAP. The necessary part is proved via examples, provided in Lemmata 2 and 3 (See Appendix 1).

Theorem 4 characterizes in three-candidate elections, all the scoring run off rules sensitive to the Negative Participation Paradox and to the Negative Abstention Paradox under the single-peaked domain.

Theorem 4. *With three candidates and single-peaked preferences, the Negative Participation Paradox and the Negative Abstention Paradox never occur for all the scoring run-off rules located between the Borda run-off and the Antiplurality run-off* i.e. *for all* $s \in [\frac{1}{2}\ 1]$*. On the other hand, each of these paradoxes can happen for any* $s \in [0\ \frac{1}{2}[$ *at some profiles.*

Again, the sufficient part is just the consequence of Theorems 1 and 2. The necessary part is proved via examples, provided in Lemmata 4 and 5 (See Appendix 2).

As for three candidates, an Abstention (resp. Participation) paradox can only be an instance of PAP or a NAP (resp. PPP or NPP), Table 3 summarizes our results by displaying for each of the paradoxes, the rules (the values of s) for which they can appear.

Table 3: single-peaked preferences and existence of paradoxes for scoring run-off rules

Paradoxes					
PPP	NPP	PP	NAP	PAP	ABS
$s \in]0,\ \frac{1}{2}[$	$s \in [0,\ \frac{1}{2}[$	$s \in [0,\ \frac{1}{2}[$	$s \in [0,\ \frac{1}{2}[$	$s \in]0,\ \frac{1}{2}[$	$s \in [0,\ \frac{1}{2}[$

Table 4: The six types of basic voting situation

Case:	Initial Scores	Winner	PPP	PAP	NPP	NAP
1	$w_s(a) \geq w_s(b) \geq w_s(c)$	a	Yes, for c (Lemma 2)	No	No	Yes, for c (Lemma 5)
2	$w_s(a) \geq w_s(b) \geq w_s(c)$	b	Yes, for c (Lemma 6)	No	No	Yes, for c (Lemma 7)
3	$w_s(a) \geq w_s(c) \geq w_s(b)$	a	No	No	No	No
4	$w_s(a) \geq w_s(c) \geq w_s(b)$	c	No	Yes, for a (Lemma 3)	Yes, for a (Lemma 4)	No
5	$w_s(c) \geq w_s(a) \geq w_s(b)$	a	No	No	No	No
6	$w_s(c) \geq w_s(a) \geq w_s(b)$	c	No	No	No	No

From Table 3, we learn that in three-candidate elections, none of the scoring run-off rules located between the Borda run-off and the Antiplurality run-off is vulnerable to variable electorate paradoxes when preferences are single-peaked. However, notice that the Plurality run-off, remains vulnerable to the NPP and to the NAP. Before turning to Section 4 where we will evaluate how often the run-off rules located between $s = 0$ and $s = \frac{1}{2}$ are vulnerable to these variable electorate paradoxes, let us characterize all the situations where the paradoxes can happen.

Lemmata 2 to 5 (see Appendices) enable us to confirm that some rules do suffer from participation and abstention paradoxes on the single peaked domain. However, if we want to evaluate the likelihood of these paradoxes, we need to numerate precisely all the voting situations that may lead to a paradox.

Theorem 5. *On the single peaked domain for three alternatives, Table 4 describes all the possible occurrences of the PPP, PAP, NPP and NAP for any* $s \in]0, \frac{1}{2}[$. *As the situations leading to a PPP or NPP are disjoint, the likelihood of the PP is the sum of the likelihood of the PPP and NPP. Similarly, as the situations leading to a PAP and a NAP are disjoint, the likelihood of an AP is the sum of the likelihood of the PAP and NAP.*

Proof. Due to the specific role played by the centrist alternative (candidate c) on the single-peaked domain, there are indeed 6 "basic" types of voting situations (instead of just 2 for the universal domain). They are described in Table 4. Also according to Lepelley and Merlin (2001, Prop. 5), in three-candidate elections (under the universal domain), the NAP and the PAP situations are exclusive. So the likelihood of the No-show paradox is just equal to the sum of the PAP and the NAP. Obviously the same result holds on a restricted domain, such as the single peaked domain. However, Lepelley and Merlin (2001) displayed a voting situation where one can end with the PPP as with the NPP simultaneously from the same original profile. Hence, we need to further explore the 6 cases to see whether the PPP and

NPP situations are exclusive. This enumeration will also pave the way for the probability computations we will derive in the next section.

Case 1) We have $w_s(a) \geq w_s(b) \geq w_s(c)$ for the initial profile, and a wins against b in the runoff. A PPP may occur if we add acb types, c goes to the runoff and then beats a (see Lemma 2 for an example). A PAP could occur if, by removing a bca (resp cba or cab), b (resp. c) would get elected. This is impossible here, as b (resp. c) would loose ground to a and c (resp. b) in the first round. As c is never last in $D(A)^{SP}$, a NPP could only occur by adding acb or cab voters. But adding any of this preference type cannot help b to win. However, a NAP can occur: by removing a bca voter, c could reach the run-off and win against a (see Lemma 5 for an example).

Case 2) We have $w_s(a) \geq w_s(b) \geq w_s(c)$ for the initial profile, and b wins against a in the runoff. By adding bca voters, it could that both b and c get more vote than a, and that ultimately, c beat b, leading to a PPP. This is a tricky situation, as both b and c need extra support to eliminate a in the first round. However, we provide an example in Appendix 3 (Lemma 6), which starts from a situation where all the scores are equal and lead to this result for any $s \in]0, \frac{1}{2}] \cap \mathbb{Q}$. A PAP could occur if, by deleting acb voters, a would win. This obviously not possible. Removing cba or cab voters cannot either help c. For a NPP to occur, adding cba voters should enable c to reach the second stage, and then a could beat c. However, for a to beat c, we need to have $n_2 \geq \frac{n}{2}$ initially, which is not possible, as b initially beats a. However, a NAP can occur: removing acb voters could lead to a duel between b and c that c could win. Lemma 7 in Appendix 3 provides such an example for any $s \in]0, \frac{1}{2}] \cap \mathbb{Q}$; again, it uses the fact all the scores are initially equals.

Case 3) We have $w_s(a) \geq w_s(c) \geq w_s(b)$ and a wins against c in the runoff. Obviously, a PPP cannot occur, nor a NPP. Removing a preference with c (resp b) first cannot make c win (resp. b), and we cannot have a PAP. Deleting a preference with a last cannot make him lose either (as a beats c, we are sure that $n_2 > \frac{n}{2}$ initially; so, there is no NAP .

Case 4) We have initially $w_s(a) \geq w_s(c) \geq w_s(b)$ and c beats a. Adding cba could lead to a PPP if both c and b go to the runoff. As b wins, we should have had initially $n_5 > \frac{n}{2}$. Hence, his score should have been superior to $\frac{n}{2}$. But as $s < \frac{1}{2}$, the sum of the points distributed by all the voters is inferior to $\frac{3}{2}n$. This is a contradiction, as in this case, b's initial score would have been superior to the average, and he would not have been last. A PPP cannot happen. For a PAP to happen, removing acb could lead to a duel between a and b if c becomes eliminated first; see Lemma 3 for an example. Obviously, a NAP cannot occur as c is never last. A NPP can happen if, by adding bca voter, b could dislodge c for the runoff, and next, a would win (see Lemma 4 for an example).

Case 5) We have initially $w_s(c) \geq w_s(a) \geq w_s(b)$ and a beats c. A PPP cannot happen as the only possibility is to add acb voters. By deleting cab voters, candidate a could loses points and be eliminated first and then c would emerge as a final winner, leading to a PAP. But, as a beats c initially, we are sure that $n_2 \geq n/2$, and by removing a cab voter, the score of a diminishes by s, which is also inferior to the average diminution. Hence, a cannot be eliminated and he still beats b and c in the second round. For a NPP to occur, adding voters with b last should help him; this is no possible here either. For a NAP to occur, deleting

voters with a last should hurt a. This could be possible if by deleting cba voters, we could end up in a duel between a and b that b could win. But initially, we had $n_2 > n/2$ as a was the winner against c. This is always the case after deleting cba voters, and b cannot win.

Case 6) We have initially $w_s(c) \geq w_s(a) \geq w_s(b)$ and c beats a. A PPP could happen, if, by adding a cba preference, b would reach the second stage and beat c. However, this would mean that initially, $n_5 > n/2$, and that b would not be last, a contradiction. A PAP cannot happen : removing bca won't help b and removing acb won't help a. Adding preferences with a or b last will not help them to win, the NPP is not possible either. A NAP cannot happen, as c is never ranked last.

□

4 The likelihood of the paradoxes under IAC

Before evaluating the likelihood of the paradoxes, we first need to define the necessary and sufficient conditions that characterize each of these paradoxes.

4.1 Necessary and sufficient conditions

Propositions 3 to 4 respectively characterize all the voting situations in three-candidate elections with single-peaked preferences in which the NAP, the PAP, the NPP and the PPP are likely to occur given s. We skip the proofs of these propositions since they follow the same scheme as those of Lepelley and Merlin (2001); nonetheless, they are available upon request.

Proposition 3. *Consider a voting situation with single-peaked preferences described by Table 2. If a beats b in the runoff, PPP occurs in favor of c (Lemma 2) for all run-off rule $s \in]0; \frac{1}{2}[$ if and only if:*

$$\begin{cases} (s-1)n_2 + (1-s)n_3 + n_4 + sn_5 \leq 0 & (A.1) \\ sn_2 + n_3 + (1-s)n_4 + (s-1)n_5 \leq 0 & (A.2) \\ -n_2 - n_3 + n_4 + n_5 \leq 0 & (A.3) \\ -(1+s)n_3 - n_4 + (1-2s)n_5 \leq 0 & (A.4) \end{cases}$$

In Proposition 3, the first three conditions respectively describe $w_s(a) \geq w_s(c)$, $w_s(b) \geq w_s(c)$ and aMb; the last condition guarantees that the paradox occurs. Notice that Lemmata 2 and 6 are symmetric; so, Proposition 3 can be used to describe both Cases 1 and 2 of Table 4 for the PPP.

Proposition 4. *Consider a voting situation with single-peaked preferences described by Table 2. If a beats b in the runoff, the NAP occurs in favor of c (Lemma 5)for all run-off rule*

$s \in [0; \frac{1}{2}[$ *if and only if*

$$\begin{cases} (s-1)n_2 + (1-s)n_3 + n_4 + sn_5 \leq 0 & (B.1) \\ sn_2 + n_3 + (1-s)n_4 + (s-1)n_5 \leq 0 & (B.2) \\ -n_2 - n_3 + n_4 + n_5 \leq 0 & (B.3) \\ (1-2s)n_2 + (s-2)n_3 + 2(s-1)n_4 \leq 0 & (B.4) \end{cases}$$

Note that for the NAP, Proposition 4 covers Case 1 of Table 4 and it can be used to characterize Case 2 since Lemmata 5 and 7 are symmetric. In this proposition, the first three conditions respectively describe $w_s(a) \geq w_s(c)$, $w_s(b) \geq w_s(c)$ and aMb; the last condition ensures that the paradox can happen.

Proposition 5. *Consider a voting situation with single-peaked preferences described by Table 2. If c beats a in the runoff, the PAP occurs in favor of a (Lemma 3) for all run-off rule* $s \in]0; \frac{1}{2}[$ *if and only if*

$$\begin{cases} -n_2 - sn_3 + sn_4 + n_5 \leq 0 & (C.1) \\ -sn_2 - n_3 + (s-1)n_4 + (1-s)n_5 \leq 0 & (C.2) \\ n_2 - n_3 - n_4 - n_5 \leq 0 & (C.3) \\ (1-s^2)n_3 + (1-s+s^2)n_4 + (2s-1)n_5 \leq 0 & (C.4) \\ (1-s)n_3 + n_4 + (2s-1)n_5 \leq 0 & (C.5) \end{cases}$$

In Proposition 5, the first three conditions respectively describe $w_s(a) \geq w_s(b)$, $w_s(c) \geq w_s(b)$ and cMa; the two last inequalities are technical conditions guaranteeing that the paradox occurs.

Proposition 6. *Consider a voting situation with single-peaked preferences described by Table 2. If c beats a in the runoff, the NPP occurs in favor of a (Lemma 4) for all run-off rule* $s \in [0; \frac{1}{2}[$ *if and only if*

$$\begin{cases} -sn_2 - n_3 + (s-1)n_4 + (1-s)n_5 \leq 0 & (D.1) \\ n_2 - n_3 - n_4 - n_5 \leq 0 & (D.2) \\ (2s-1)n_2 + (1-s+s^2)n_3 + (1-s^2)n_4 \leq 0 & (D.3) \\ (2s-1)n_2 + sn_3 + 2(1-s)n_4 \leq 0 & (D.4) \end{cases}$$

The first two conditions of Proposition 6, respectively describe $w_s(c) \geq w_s(b)$ and cMa; the two last inequalities are technical conditions guaranteeing that the paradox occurs.

We are now ready to evaluate the likelihood of the different paradoxes we presented; before doing so, let us present the probability model we will use, namely, the Impartial Anonymous Culture assumption.

4.2 The Impartial Anonymous Culture

The Impartial Anonymous Culture (IAC) is one of the most used assumptions in the social choice theory for the computation of the likelihood of voting events (see for example,

Gehrlein and Lepelley (2010, 2017)). Under IAC, it is assumed that each voting situation is equiprobable. When no restriction is made on voters' preferences (universal domain), the total number of the possible voting situations is given by the $m! - 1$ degree polynomial $C^n_{n-m!-1}$ with n the number of voters and m the number of candidates. For three-candidate election and single-peaked preferences, the total number of the possible voting situations is given by the following three degree polynomial[9] :

$$\frac{(n+1)(n+2)(n+3)}{6}$$

The probability of a given event obeys to the following ratio :

$$\frac{\text{total number of voting situations where the event is likely to happen}}{\text{total number of possible voting situations}}$$

Many techniques have been suggested in the literature for the determination of the total number of voting situations where the event is likely to happen. According to Ehrhart (1962, 1967), this amounts to count the number of vectors with integer components contained in a polytope[10]. This author showed that this number is a specific function in n (here, the size of the electorate); for an appropriate value of n, this specific function is a polynomial in n which depends of the congruence that appears due to the form of the considered polytope. In this way, diverse algorithms have been developed for performing the counting of the vectors with integer components given a polytope. We can refer to the works of Barvinok (1994), Barvinok and Pommersheim (1999), Clauss and Loechner (1998), Haung and Chua (2000), Lepelley et al. (2008), Wilson and Pritchard (2007). The most used algorithm in the literature is that of Barvinok (1994). For more on this algorithm and its extensions, see the works of Clauss and Loechner (1998), Verdoolaege et al. (2004). In this paper, we use the *parameterized Barvinok's algorithm* developed by Verdoolaege et al. (2004) for small electorates. The Parameterized Barvinok's algorithm covers the limitations of the algorithm proposed by Clauss and Loechner (1998) and it extends the Barvinok's algorithm by considering parametric polytopes with any numbers of parameters. It generalizes the Barvinok's algorithm by considering validity domains with the ability to handle periodic numbers. For more details on the Parameterized Barvinok's algorithm, we refer the reader to works of Verdoolaege et al. (2004). Due to space constraints, we are not going to report all the polynomials we obtained; they are available upon request. For large electorates, we use the technique suggested by Cervone et al. (2005) and report the results in Appendix 4.

[9] See Lepelley (1995).

[10] A polytope is a geometric object with flat sides; it may exist in any general number of dimensions n as an n-dimensional polytope or n-polytope.

4.3 The probabilities of the paradoxes

4.3.1 The case of small electorates

Let us denote by $\hat{P}^{\text{IAC}}_{\text{NAP}}(s,m,n)$, $\hat{P}^{\text{IAC}}_{\text{PAP}}(s,m,n)$, $\hat{P}^{\text{IAC}}_{\text{PPP}}(s,m,n)$ and $\hat{P}^{\text{IAC}}_{\text{NPP}}(s,m,n)$ respectively the probability under IAC that the NAP, the PAP, the PPP and the NPP occurs for a given scoring run-off s with m candidates and n voters with single-peaked preferences. Our results concerning small electorates ($n = 3,4,5,\ldots,33$) are provided in Table 5 for the PPP and the PAP and Table 6 for the NAP and the NPP.

Table 5: Probabilities $\hat{P}^{\text{IAC}}_{\text{PAP}}(s,3,n)$ and $\hat{P}^{\text{IAC}}_{\text{PPP}}(s,3,n)$

	$\hat{P}^{\text{IAC}}_{\text{PAP}}(s,3,n)$				$\hat{P}^{\text{IAC}}_{\text{PPP}}(s,3,n)$			
n	$s=0.1$	$s=0.2$	s=0.3	$s=0.4$	$s=0.1$	$s=0.2$	$s=0.3$	$s=0.4$
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0.03572	0	0
6	0	0.02381	0	0	0	0.02381	0	0
7	0	0	0	0	0	0	0.01667	0
8	0	0.01212	0.01212	0	0	0.01212	0	0
9	0	0.00909	0	0	0	0.00909	0	0
10	0	0.00699	0	0	0	0.00699	0	0
11	0	0	0	0.00549	0.01098	0.00549	0	0.00549
12	0	0.00439	0.00439	0.00879	0.00879	0.00439	0.00879	0.00879
13	0.00357	0.00357	0.00714	0	0.00714	0.00714	0.00714	0.00357
14	0.00882	0.00588	0.00883	0.00294	0.01176	0.00883	0.00588	0.00294
15	0.00491	0.00735	0	0	0.00491	0.00491	0.00245	0.00245
16	0.00619	0.01032	0.00206	0.00206	0.00413	0.00619	0.00206	0.00206
17	0.00877	0.00175	0.00175	0	0.00351	0.00526	0.00175	0
18	0.01053	0.00301	0.00301	0.00151	0.00301	0.00752	0.00451	0.00151
19	0.00259	0.00259	0.00389	0.00129	0	0.00519	0.00389	0.00129
20	0.00226	0.00452	0.00452	0.00226	0	0.00565	0.00452	0.00113
21	0.00197	0.00494	0.00197	0.00198	0.00197	0.00494	0.00296	0.00099
22	0.00174	0.00608	0.00348	0.00261	0.00261	0.00608	0.00261	0.00174
23	0	0.00307	0.00231	0.00154	0.00231	0.00462	0.00307	0.00154
24	0	0.00478	0.00342	0.00205	0.00274	0.00547	0.00342	0.00274
25	0	0.00427	0.00244	0.00061	0.00427	0.00488	0.00244	0.00122
26	0.00055	0.00493	0.00274	0.00109	0.00438	0.00547	0.00383	0.00164
27	0.00148	0.00296	0.00246	0.00049	0.00345	0.00493	0.00394	0.00148
28	0.00267	0.00401	0.00356	0.00134	0.00489	0.00489	0.00445	0.00178
29	0.00202	0.00282	0.00202	0.00081	0.00363	0.00444	0.00323	0.00121
30	0.00257	0.00366	0.00256	0.00147	0.00293	0.00513	0.00329	0.00147
31	0.00334	0.00367	0.00201	0.00134	0.00301	0.00468	0.00267	0.00101
32	0.00397	0.00489	0.00275	0.00183	0.00367	0.00519	0.00367	0.00122
33	0.00252	0.00336	0.00252	0.00112	0.00252	0.00448	0.00308	0.00084
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	0.00247	0.00342	0.00273	0.00119	0.00307	0.00444	0.00307	0.00119
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
100	0.00201	0.00305	0.00236	0.00092	0.00287	0.00404	0.00288	0.00091

We notice from Table 5 that given $s \in]0;\frac{1}{2}[$, the likelihood of the PPP tends to decrease as the size of the electorate increases; also, given the size of the electorate, the likelihood of the PPP tends to decreases as s increases. Concerning the PAP, we notice that the PAP does not occurs for for certain rules when the population size remains low (up to $n \leq 25$) ; for a

given value of s, it is difficult to describe the evolution of the probability when the size of the electorate varies. As $\hat{P}^{\text{IAC}}_{\text{PAP}}(0.1,3,n) < \hat{P}^{\text{IAC}}_{\text{PAP}}(0.2,3,n) \geq \hat{P}^{\text{IAC}}_{\text{PAP}}(0.3,3,n) > \hat{P}^{\text{IAC}}_{\text{PAP}}(0.4,3,n)$, this leads us to think that given n, the likelihood of the PAP reaches its maximum for $0.2 \leq s \leq 0.3$.

Concerning the NPP and the PPP, the changes in the probabilities do not show a stable trend when the number of voters is below 16 regardless of $s \in [0, \frac{1}{2}[$. With more than 15 voters, it comes that the probabilities of the NPP and the NAP tends to decrease as n increases regardless of s. For a given size of the electorate, the likelihood of the NPP and the NAP tends to decreases as s increases. So, the likelihoods of these paradoxes tend to be maximized by the Plurality run-off. Comparing our results of the NPP and the NAP for the Plurality run-off to those of Lepelley and Merlin (2001), it comes that when we move from the universal domain to the single-peaked one, the probabilities are almost divided by two thirds or even by two according to the number of voters.

From a global perspective, under the single-peaked domain and for small electorates, a paradox of participation is more likely to occur than its equivalent in abstention regardless of s. This pattern is in line with what Lepelley and Merlin (2001) got under the universal domain.

Table 6: Probabilities $\hat{P}^{\text{IAC}}_{\text{NPP}}(s,3,n)$ and $\hat{P}^{\text{IAC}}_{\text{NAP}}(s,3,n)$

	$\hat{P}^{\text{IAC}}_{\text{NPP}}(s,3,n)$					$\hat{P}^{\text{IAC}}_{\text{NAP}}(s,3,n)$				
n	$s=0$	$s=0.1$	$s=0.2$	s=0.3	$s=0.4$	$s=0$	$s=0.1$	$s=0.2$	$s=0.3$	$s=0.4$
3	0.09999	0	0	0	0	0.10000	0	0	0	0
4	0.22857	0.05714	0.05714	0.05714	0	0.17143	0	0	0	0
5	0.07143	0	0	0.03571	0	0.03571	0.03571	0.03571	0	0
6	0.14286	0.04762	0.07143	0.02381	0	0.07143	0.02381	0.04762	0	0
7	0.06667	0.03333	0.03333	0	0	0.05000	0	0.01667	0.01667	0
8	0.13333	0.04848	0.04848	0.02424	0.01212	0.07273	0.02424	0.02424	0.01212	0
9	0.07273	0.01818	0.01818	0.00909	0.00909	0.04545	0.02727	0.00909	0.00909	0
10	0.10490	0.05594	0.02797	0.02098	0.00699	0.05594	0.02098	0.01399	0.00699	0
11	0.06593	0.03297	0.01099	0.01099	0.00549	0.03846	0.01648	0.01648	0	0.00549
12	0.10989	0.05275	0.03956	0.01758	0.00879	0.06154	0.02637	0.01758	0.00879	0.00879
13	0.06429	0.03571	0.02500	0.01071	0	0.03929	0.02143	0.01429	0.00714	0.00357
14	0.09118	0.05588	0.03529	0.01765	0.00294	0.04412	0.02941	0.01765	0.01176	0.00294
15	0.06618	0.03676	0.02451	0.00735	0	0.03922	0.02206	0.01225	0.00735	0.00245
16	0.09288	0.05160	0.03509	0.01445	0.00619	0.05160	0.02477	0.01858	0.00826	0.00413
17	0.06316	0.03860	0.01754	0.00877	0.00351	0.03509	0.01930	0.01404	0.00526	0.00175
18	0.08571	0.04812	0.03008	0.01353	0.00602	0.04361	0.02707	0.01654	0.00752	0.00301
19	0.06234	0.02987	0.02208	0.00909	0.00260	0.03636	0.02078	0.01169	0.00519	0.00130
20	0.08470	0.04404	0.03162	0.01355	0.00339	0.04404	0.02146	0.01468	0.00565	0.00113
21	0.06324	0.03261	0.02273	0.00988	0.00198	0.03557	0.01680	0.01186	0.00395	0.00099
22	0.07913	0.04261	0.02957	0.01304	0.00261	0.04087	0.02174	0.01565	0.00609	0.00174
23	0.06154	0.03077	0.02077	0.00846	0.00154	0.03385	0.01923	0.01308	0.00615	0.00154
24	0.08068	0.04376	0.03009	0.01299	0.00342	0.04239	0.02120	0.01573	0.00752	0.00274
25	0.06105	0.03358	0.02137	0.00916	0.00183	0.03419	0.01770	0.01221	0.00549	0.00122
26	0.07553	0.04379	0.02791	0.01314	0.00328	0.03777	0.02189	0.01423	0.00657	0.00164
27	0.06158	0.03498	0.02167	0.00985	0.00246	0.03399	0.01872	0.01182	0.00542	0.00148
28	0.07608	0.04494	0.02848	0.01246	0.00356	0.04004	0.02225	0.01379	0.00756	0.00222
29	0.06048	0.03589	0.02137	0.00847	0.00202	0.03266	0.01895	0.01210	0.00524	0.00161
30	0.07368	0.04509	0.02749	0.01173	0.00257	0.03739	0.02126	0.01430	0.00623	0.00220
31	0.06016	0.03676	0.02206	0.00902	0.00134	0.03309	0.01872	0.01203	0.00468	0.00134
32	0.07334	0.04431	0.02811	0.01161	0.00214	0.03759	0.02200	0.01436	0.00611	0.00183
33	0.06050	0.03585	0.02157	0.00868	0.00252	0.03277	0.01933	0.01204	0.00504	0.00140
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	0.06616	0.04183	0.02518	0.01076	0.00231	0.03329	0.02083	0.01281	0.00581	0.00145
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
100	0.06107	0.04017	0.02376	0.00983	0.00192	0.03084	0.01988	0.01188	0.00515	0.00113

4.3.2 The case of large electorates

Under the universal domain, Lepelley and Merlin (2001) were able to derive the likelihood of the abstention paradox under IAC as the size of the electorate tends to infinity only for the Plurality run-off, the Antiplurality run-off and the Borda run-off. The results for the Borda run-off were obtained via simulations. The recent developments in computational techniques enable us to compute on the universal domain, the exact values of the limiting probabilities for each of the paradoxes for $s = 0(0.1)1$ under IAC. The results are provided in Table 7 where we also provide the limiting probabilities under the single-peaked domain in order to envisage possible comparisons.

Table 7: Limiting probabilities under the Universal domain (U) and the Single-peaked domain (SP)

		$s=0$	$s=0.1$	$s=0.2$	$s=0.3$	$s=0.4$	$s=0.5$	$s=0.6$	$s=0.7$	$s=0.8$	$s=0.9$	$s=1$
PAP	U	0	0.00263	0.00509	0.00713	0.00864	0.01042	0.01442	0.01997	0.02665	0.03425	0.04253
	SP	0	0.00185	0.00264	0.00213	0.00071	0	0	0	0	0	0
NAP	U	0.04079	0.03554	0.02994	0.02417	0.01853	0.01389	0.01181	0.00994	0.00739	0.00404	0
	SP	0.02778	0.01894	0.01094	0.00459	0.00085	0	0	0	0	0	0
AP	U	0.04079	0.03818	0.03502	0.03129	0.02717	0.02431	0.02623	0.02991	0.03405	0.03829	0.04253
	SP	0.02778	0.02080	0.01358	0.00672	0.00156	0	0	0	0	0	0
PPP	U	0	0.00395	0.00759	0.01043	0.01224	0.01389	0.01703	0.02095	0.02582	0.03162	0.03822
	SP	0	0.00266	0.00364	0.00262	0.00068	0	0	0	0	0	0
NPP	U	0.07292	0.06127	0.04873	0.03651	0.02648	0.02083	0.01927	0.01679	0.01244	0.00666	0
	SP	0.05556	0.03848	0.02202	0.00887	0.00151	0	0	0	0	0	0
PP	SP	0.05556	0.04114	0.02567	0.01149	0.00219	0	0	0	0	0	0

Comparing the figures of Table 7, we observe that on the single peaked domain, the limiting probabilities of the paradoxes are drastically reduced. Thus, single-peakedness can help in reducing the likelihood of the Participation and Abstention paradoxes. This conclusion is also in line with the findings of Lepelley et al. (1995) concerning two paradoxes of monotonicity: the *more-is-less paradox* (M1) and the *less-is-more paradox* (M2). The paradox M1 appears when a winner can become a loser after he has been lifted up, *ceteris paribus*, in one or more voters's rankings. The paradox M2 appears when a loser can become a winner after his position has been degraded, *ceteris paribus*, in one or more voters's rankings. According to the findings of Lepelley et al. (1995), in three-candidate elections, when we go from the universal domain to the single-peaked one, the likelihoods of M1 and M2 are considerably reduced. Table 8 recalls some results of Lepelley et al. (1995) for the Plurality runoff and the Antiplurality runoff when the size of the electorate tends to the infinity.

Table 8: Limiting probabilities of paradoxes M1 and M2 for the Plurality runoff and the Antiplurality runoff under the universal domain and under the single-peaked domain

	Plurality runoff		Antiplurality runoff	
Domain	$M1$	$M2$	$M1$	$M2$
Universal	0.04514	0.01968	0.05556	0.06481
Single-peaked	0.01736	0	0	0.04629

5 Conclusion

In this paper, we have shown that, in three-candidate elections, when we go from the universal domain to the single-peaked domain, none of the paradoxes of variable electorate appear with all the scoring runoff located between the Borda runoff and the Antiplurality runoff. Also, we found that, among the well known scoring runoff, only the Plurality runoff is sensitive to

two of the four paradoxes of variable electorate: the Negative Participation Paradox and the Negative Abstention Paradox. The evaluation of the likelihood of the abstention paradox in three-candidate elections with s varies allowed us to conclude after a comparison with the results obtained by Lepelley and Merlin (2001) that the abstention paradox is less likely to occur under the single-peaked domain than under the universal domain. Our analysis with three candidates lead us to notice that among the numerous (theoretical) critics against the runoff system in use in France, many of them will no be valid anymore in the real life such as the Negative Participation Paradox and the Negative Abstention Paradox.

Top and Merlin (2016) investigated the four-candidates case and they found that the task is quite arduous as under the Single-peaked domain, numerous subcases are to be analyzed. They also showed that with four candidates, the Abstention Paradox is no more the combination of only the PAP and the NAP but also of a new type of Abstention paradox : the Intermediary Abstention Paradox. This paradox occurs when, by deleting some voters, the candidate ranking in the second place (with the winner of the initial situation ranking third or fourth) or the third place (with the winner of the initial situation ranking fourth) by these voters becomes the winner. As the number of cases to investigate increases, we fear that the values of the participation and abstention paradoxes will not longer be negligible, even if we consider only the single-peaked domain.

Appendices

Appendix 1: Proof of Theorem 3

The sufficient part is just the consequence of Theorems 1 and 2. The necessary part is provided in Lemmata 2 and 3.

Lemma 2. *Consider the single peaked domain for three alternatives. For $a, b \in A \setminus \{c\}$, if candidate a wins versus candidate b, it is possible to build examples where the PPP and this possible in favor of c for any $s \in]0\ \frac{1}{2}[$.*

Proof. Let us show that the PPP can happen for $s \in]0\ \frac{1}{2}[$. Using the labels from Table 2, Table 9 describe the initial voting situation $\tilde{n}$ such as (with $\lceil x \rceil$ the smallest integer greater than x).

Table 9: Initial Profile for the PPP

acb:	cab:	cba:	bca:
$n_2 = n_5$	$n_3 = \lceil \frac{1}{s} \rceil$	$n_4 = 0$	$n_2 = n_5 = \lceil \frac{n_3}{1-2s} \rceil$

Then,the scores for $\tilde{n}$ are:

$$\begin{aligned} w_s(a) &= n_2 + sn_3 \\ w_s(b) &= n_2 \\ w_s(c) &= n_3 + 2sn_2 \end{aligned}$$

Note that $n_3 \geq \frac{1}{s} > 1$. So, $w_s(a) > w_s(c)$ and $w_s(b) - w_s(c) = (1-2s)n_2 - n_3$. By definition, $n_2 = \lceil \frac{n_3}{1-2s} \rceil \geq \frac{n_3}{1-2s}$. Thus, $w_s(b) - w_s(c) \geq 0$. Candidate c eliminated at the first run and a wins the majority contest versus b by $n_2 + n_3$ supports against $n_4 + n_5 = n_2$. Let us add $k = n_3$ voters of type 2 to create profile $\tilde{n}'$. The new scores are :

$$\begin{aligned} w_s'(a) &= n_2 + (1+s)n_3 \\ w_s'(b) &= n_2 \\ w_s'(c) &= (1+s)n_3 + 2sn_2 \end{aligned}$$

We still get $w_s'(a) > w_s'(b)$ but now:

$$w_s'(b) - w_s'(c) = (1-2s)n_2 - (1+s)n_3.$$

Since $n_2 = \lceil \frac{n_3}{1-2s} \rceil$, we have $n_2 \leq \frac{n_3}{1-2s} + 1$. Hence:

$$\begin{aligned} w_s'(b) - w_s'(c) &\leq (1-2s)\left(\frac{n_3}{1-2s} + 1\right) - (1+s)n_3 \\ \Leftrightarrow w_s'(b) - w_s'(c) &\leq 1 - 2s - sn_3 \end{aligned}$$

As $n_3 = \lceil \frac{1}{s} \rceil$, $s\lceil \frac{1}{s} \rceil \geq 1$ and the score of b is inferior to the score of c. Then, b is eliminated and c is elected if we use a tie break in his favor. Thus, the PPP can happen for any $s \in]0\ \frac{1}{2}[$. □

Lemma 3. *Consider the single peaked domain for three alternatives, as described by Table 2. If candidate c wins versus candidate a, it is possible to build at least one example where the PAP occurs in favor of a for any* $s \in]0\ \frac{1}{2}[$

Proof. The proof is just the mirror of the one we described for Lemma 2, by starting from the final profile.

□

Appendix 2: Proof of Theorem 4

The sufficient part is just the consequence of Theorems 1 and 2. The necessary part is provided in Lemmata 4 and 5.

Lemma 4. *Consider the voting situation of Table 2 and assume that candidate c wins against candidate a. The NPP can occur for any* $s \in [0\ \frac{1}{2}[$ *at some profile in* $D(U)^{SP}$.

Proof. First, notice that the NPP is not possible if we try to favor candidate b by adding some voters of type 2 or 3; in either of the cases candidate b's score remain unchanged while those of a and c increase. So, the only possibility to get the NPP is to imagine a situation in which we favor candidate a. In this case, the NPP will be possible only if after one adds some detractors of a (type 4 and/or 5), candidate c is eliminated after the first run and that candidate a wins versus candidate b. For this, consider the profile described in Table 10 with $x = \lfloor \frac{1}{1-2s} \rfloor$.

Table 10: Initial Profile for the NPP

acb:	*cab*:	*cba*:	*bca*:
$n_2 = x + 1$	$n_3 = 1$	$n_4 = 0$	$n_5 = x$

Note that x is positive and well defined for any $s \in [0, \frac{1}{2}[$. Then,the scores for $\tilde{n}$ are:

$$\begin{aligned} w_s(a) &= x + 1 + s \\ w_s(b) &= x \\ w_s(c) &= (2x+1)s + 1 \end{aligned}$$

We have immediately $w_s(a) \geq w_s(b)$. Also, we get:

$$\begin{aligned} w_s(c) - w_s(b) &= (2x+1)s + 1 - x \\ &= (2s-1)x + 1 + s \\ &= (2s-1)\lfloor \frac{x}{1-2s} \rfloor + 1 + s \geq 0 \end{aligned}$$

As $(1-2s)\lfloor \frac{x}{1-2s} \rfloor \leq 1$, we are sure that the score of c is at least superior or equal to the score of b for any $s \in [0, \frac{1}{2}[$. We can assume that a and c reach the second stage, and that c beats a by one extra vote.

Let us add 1 type 5 voter with preference bca. The new scores are :

$$\begin{aligned} w_s^{'}(a) &= x + 1 + s \\ w_s^{'}(b) &= x + 1 \\ w_s^{'}(c) &= (2x+1)s + 1 + s \end{aligned}$$

Again, a's score is superior to b's. Now:

$$\begin{aligned} w_s^{'}(b) - w_s^{'}(a) &= x + 1 - (2x+1)s - 1 - x \\ &= \lfloor \frac{1}{1-2s} \rfloor (1-2s) - 2s \end{aligned}$$

On can check that with Maple or another software that this value is always positive for any $s \in [0, \frac{1}{2}[$. Hence, a and b make it to the runoff. Both alternatives obtain the same number of votes, and provided that we break the tie in favor of a, we obtain a NPP.

□

Lemma 5. *Consider the voting situation of Table 2 and assume that candidate a wins against candidate b. The NAP can occur for any* $s \in [0\ \frac{1}{2}[$ *at some profile in* $D(U)^{SP}$.

Proof. The proof is similar to the one we proposed for Lemma 4, except that the roles of the initial and final profiles are switched. □

Appendix 3: Additional Lemmata for the Proof of Theorem 5

Lemma 6. *Consider the voting situation of Table 2 and assume that candidate b wins against candidate a. A PPP can occur in favor of c for any* $s \in [0\ \frac{1}{2}[$ *at some profile in* $D(U)^{SP}$.

Proof. Case 2 of Table 4 envisages that $w_s(a) \geq w_s(b) \geq w_s(c)$, b beat a, and adding bca voters lead to the victory of c! We will display an profile that supports this assertion for any $s \in]0, \frac{1}{2}[$ in Table 11, where x is a real number such as the values of n_2,n_3, n_4 and n_5 are integer numbers. Notice that for $s \in [0, \frac{1}{2}[$, these numbers are well defined.

Table 11: Initial Profile for the PPP

acb:	*cab*:	*cba*:	*bca*:
$n_2 = \frac{x}{6}\frac{(2+s-s^2)}{(1-s^2)}$	$n_3 = \frac{x}{6}\frac{(1-s-2*s^2)}{(1-s^2)}$	$n_4 = \frac{x}{6}\frac{(1-s-s^2)}{(1-s^2)}$	$n_5 = \frac{x}{6}\frac{(2+s-s^2)}{(1-s^2)}$

One can check that the values of the n_i's have been precisely designed such as $w_s(a) = w_s(b) = w_s(c)$. The pairwise comparison between a and b is a tie. Hence, with the appropriate tie breaking rule, a and b could go to the runoff, and b could be elected.

Now, let us add a bca voter. Hence, a is eliminated first, and b is now confronted to c. As $n_3 = n_4 > 1$, c now wins against b. □

Lemma 7. *Consider the voting situation of Table 2 and assume that candidate b wins against candidate a. A NAP can occur in favor of c for any* $s \in [0\ \frac{1}{2}[$ *at some profile in* $D(U)^{SP}$.

Proof. Again, we will start from the profile displayed in Table 11 which lead to equals scores among a, b and c. Hence, with the appropriate tie breaking rule, a and b could go to the runoff, and b could be elected. Now, let us remove a acb voter. A priori, this should help b. However, a is eliminated first, and b is now confronted to c. As $n_3 = n_4 > 1$, c now wins against b. □

Appendix 4: Probability representations for large electorates

- Limit probability of the PPP

$$\hat{P}^{\text{IAC}}_{\text{PPP}}(s,3,\infty) = \begin{cases} \frac{s(10s^3+20s^2+25s-21)(-1+2s)^3}{36(-2+s)(s-1)^2(3-5s+4s^2)(5s-3)} & \text{for } s \in [0\ \frac{1}{3}] \\ \\ \frac{(s^2+17s-2)(-1+2s)^3}{72(-2+s)(s-1)^2(1+s)} & \text{for } s \in [\frac{1}{3}\ \frac{1}{2}] \end{cases}$$

- Limit probability of the PAP

$$\hat{P}^{\text{IAC}}_{\text{PAP}}(s,3,\infty) = \frac{s(5s^3+12s^2+3s-13)(2s-1)^3}{432(s-2)(4-5s)(s-1)^2(s^2+2s-2)} \quad \text{for } s \in [0\ \frac{1}{2}]$$

- Limit probability of the NAP

$$\hat{P}^{\text{IAC}}_{\text{NAP}}(s,3,\infty) = \frac{(s+4)(2s-1)^3}{72(s-2)(s-1)^2} \quad \text{for } s \in [0\ \frac{1}{2}]$$

- Limit probability of the NPP

$$\hat{P}^{\text{IAC}}_{\text{NPP}}(s,3,\infty) = \frac{(s^2+2s+4)(-1+2s)^3}{432(s-1)(1-s+s^2)(2-s)} \quad \text{for } s \in [0\ \frac{1}{2}]$$

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