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Γ-SUPERCYCLICITY OF FAMILIES OF TRANSLATES IN WEIGHTED L^p-SPACES ON LOCALLY COMPACT GROUPS

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Abstract. Let ω be a weight function defined on a locally compact group G, $1 \leqslant p < +\infty$, $S \subset G$ and let us assume that for any $s \in S$, the left translation operator T_s is continuous from the weighted L^p-space $L^p(G, \omega)$ into itself. For a given set $\Gamma \subset \mathbb{C}$, a vector $f \in L^p(G, \omega)$ is said to be (Γ, S)-dense if the set $\{\lambda T_s f : \lambda \in \Gamma,\ s \in S\}$ is dense in $L^p(G, \omega)$. In this paper, we characterize the existence of (Γ, S)-dense vectors in $L^p(G, \omega)$ in terms of the weight and the set Γ.

1. Introduction

Let G be a locally compact group with a left Haar measure μ, and let $\omega : G \to \mathbb{R}_+$ be a weight on G, that is, a positive locally p-integrable function. Consider now the weighted L^p-space, $1 \leqslant p < +\infty$:

$$L^p(G, \omega) := \left\{ f : G \to \mathbb{C} :\ f \text{ measurable}, \int_G |f(t)|^p \omega(t)^p \, d\mu(t) < \infty \right\}$$

endowed with the norm

$$\|f\|_{p,\omega} := \left(\int_G |f(t)|^p \omega(t)^p \, d\mu(t) \right)^{1/p}.$$

For $s \in G$, the left translation operator T_s is defined by

$$(T_s f)(t) = f(s^{-1}t), \quad t \in G,\ f \in L^p(G, \omega).$$

It is known that T_s maps $L^p(G, \omega)$ into itself and is bounded if and only if

$$M(s) := \operatorname*{ess\,sup}_{t \in G} \frac{\omega(st)}{\omega(t)} < +\infty,$$

and $M(s)$ is equal to the norm of T_s in this case. Let S be a subset of G. We say that ω is an S-*admissible* weight if, for all $s \in S$, $M(s) < +\infty$. In this case the following notion makes sense: A function $f \in L^p(G, \omega)$ is called S-*dense* if its S-orbit

$$Orb_S(f) := \{T_s f :\ s \in S\}$$

is dense in $L^p(G, \omega)$.

Whether S-dense functions exist, depends on S and on the weight. The first criterion of the existence of S-dense functions in this context was obtained by H. Salas in [14]: In the case $G = \mathbb{Z}$ and $S = \mathbb{N}$, a necessary and sufficient condition is $\liminf\{\omega(n + q) + \omega(-n + q) : n \in \mathbb{N}\} = 0$ for all $q \in \mathbb{N}$, independently of p.

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In [1], E. Abakumov and Y. Kuznetsova gave an S-density criterion in the case of a general group G and its subset S. The (necessary and sufficient) condition involves series of local p-norms of ω, similar to that of Theorem A below, and is in general not simplifiable. In the case when the subgroup generated by S is abelian, the condition simplifies to the following: *For every compact set $F \subset G$ and any given $\delta > 0$, there exists $s \in S$ and a compact set $K \subset F$ such that $\mu(F \setminus K) < \delta$ and $\operatorname*{ess\,sup}_{sK \cup s^{-1}K} \omega < \delta$.* This is a generalization of the result of Salas, as well as that of W. Desch et al. [6] ($G = \mathbb{R}$, $S = \mathbb{R} \setminus \mathbb{R}_+$) and C-C. Chen [4] (a single translation operator T_{s_0} on $L^p(G, \omega)$).

An interesting necessary condition for the existence of an S-dense vector is that G needs to be second-countable, that is, to have a countable base of topology [1, Proposition 3], which is equivalent to saying that $L^p(G, \omega)$ is separable (Recall that S and thus the orbit of f need not be countable).

If S is the sub-semigroup generated by a single point $s_0 \in G$, then S-density is equivalent to the hypercyclicity of T_{s_0}. Recall that an operator T defined on a Banach space X into itself is **hypercyclic**, if there exists $x \in X$ such that the orbit of x, i.e., $\operatorname{Orb}(x, T) := \{T^n x : n \in \mathbb{N}\}$ is dense in X. The result of Salas [14] is thus a criterion of hypercyclicity of the backward shift operator on $L^p(\mathbb{Z}, \omega)$. We refer to [2, 10, 11] for more information about hypercyclicity and universality.

A bounded linear operator T on a Banach space X is said to be **supercyclic** if there exists a vector $x \in X$ whose projective orbit, i.e., $\operatorname{Orb}(\mathbb{C}x, T) := \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X. A characterization of supercyclicity of weighted shift operators was also obtained by Salas in [15], while M. Matsui et al. [13] characterized the supercyclicity of one-parameter translation semigroups. Moreover, if S is the sub-semigroup generated by $s_0 \in G$, then a characterization of supercyclicity of T_{s_0} was also obtained by C-C. Chen [5].

For $\Gamma \subset \mathbb{C}$, Γ-supercyclicity is a recent intermediate notion between hypercyclicity and supercyclicity introduced in [3]. An operator T defined on a Banach space X is **Γ-supercyclic** if there exists $x \in X$ such that the orbit of Γx, i.e., $\operatorname{Orb}(\Gamma x, T) := \{\lambda T^n x : \lambda \in \Gamma, n \in \mathbb{N}\}$ is dense in X. Instead of a single operator, one can consider a family of translation operators parametrized by a subset $S \subset G$. On this way we come to the following definition, which can be regarded as Γ-supercyclicity of a family of translates:

Definition 1.1. We say that $f \in L^p(G, \omega)$ is (Γ, S)-*dense* in $L^p(G, \omega)$ if its (Γ, S)-orbit, i.e.,

$$\operatorname{Orb}_S(\Gamma f) := \{\lambda T_s f : \lambda \in \Gamma, s \in S\},$$

is dense in $L^p(G, \omega)$.

We have clearly a chain of implications:

$$S\text{-density} \Rightarrow (\Gamma, S)\text{-density} \Rightarrow (\mathbb{C}, S)\text{-density}.$$

In particular, if Γ is reduced to a non-zero point, then S-density coincides with (Γ, S)-density. Furthermore, if S is the sub-semigroup generated by $s_0 \in G$, then (Γ, S)-density coincides with Γ-supercyclicity of T_{s_0}.

Besides the case of a trivial group $G = \{e\}$, the group must be non-compact second-countable in order to have (Γ, S)-dense vectors for some p, ω, S, Γ. This is proved in Section 2. Under the above assumptions on the group, we get the following density criterion extending the results of [1]. Denote, for a measurable function f on G and a subset K of G,

$$\|f\|_{p,K} := \left(\int_K |f(t)|^p \, d\mu(t) \right)^{1/p}.$$

Theorem A. *Let G be a second-countable locally compact non-compact group. Let $S \subset G$, $1 \leqslant p < +\infty$, $\Gamma \subset \mathbb{C}$ be such that $\Gamma \setminus \{0\}$ is non-empty, and let ω be a locally p-integrable S-admissible weight on G. Then the following conditions are equivalent:*

(1) *There is a (Γ, S)-dense vector in $L^p(G, \omega)$.*
(2) *For every increasing sequence of compact subsets $(F_n)_{n\geqslant 1}$ of G of positive measure and for every sequence $(\delta_n)_{n\geqslant 1}$ of positive numbers, there are sequences $(s_n)_{n\geqslant 1} \subset S$, $(\lambda_n)_{n\geqslant 1} \subset \Gamma \setminus \{0\}$ and a sequence of compact subsets $K_n \subset F_n$ such that the sets $s_n^{-1} F_n$ are pairwise disjoint, $\mu(F_n \setminus K_n) < \delta_n$ and*

$$\sum_{n,k\geqslant 0; n\neq k} \frac{|\lambda_n|^p}{|\lambda_k|^p} \|\omega\|^p_{p, s_n s_k^{-1} K_k} < +\infty,$$

with $s_0 = e$ being the identity, $\lambda_0 = 1$ and $K_0 = \emptyset$.

This theorem is stated again, with yet another equivalent condition, and proved in Section 4.

In [16], S. Shkarin proved the equivalence of $\mathbb{R}_+$-supercyclicity and supercyclicity. By Theorem A, we obtain the following generalization of this result as a corollary:

Corollary 1.2. *Let $|\Gamma|$ denote the set $\{|\lambda| : \lambda \in \Gamma\}$. $L^p(G, \omega)$ has a (Γ, S)-dense vector if and only if it has a $(|\Gamma|, S)$-dense vector. In particular, $L^p(G, \omega)$ has an $(\mathbb{R}_+, S)$-dense vector if and only if it has a $(\mathbb{C}, S)$-dense vector.*

Let T be an operator on a Banach space X. If $\Gamma \subset \mathbb{C}$ is such that $\Gamma \setminus \{0\}$ is non-empty, bounded and bounded away from zero, we know that T is Γ-supercyclic if and only if T is hypercyclic [3]. Here we have the following similar result for a family of translation operators:

Corollary 1.3. *If $\Gamma \setminus \{0\}$ is bounded and bounded away from zero, then $L^p(G, \omega)$ has an S-dense vector if and only if it has a (Γ, S)-dense vector.*

If Γ is unbounded or zero is its limit point, then $L^p(G, \omega)$ may have a (Γ, S)-dense vector, but no S-dense vectors, see Example 4.3.

If the subgroup generated by S is abelian, Theorem A simplifies to the following, under the same assumptions on p, G, S, Γ and ω as in Theorem A:

Theorem B. *If the subgroup generated by S is abelian, then the following conditions are equivalent:*

(1) *There is a (Γ, S)-dense vector in $L^p(G, \omega)$.*
(2) *For any compact subset $F \subset G$ and $\varepsilon > 0$, there are $s \in S$, $\lambda \in \Gamma \setminus \{0\}$ and a compact subset $E \subset F$ such that $\mu(F \setminus E) < \varepsilon$ and*

$$|\lambda| \operatorname*{ess\,sup}_{t\in E} \omega(st) < \varepsilon \quad \text{and} \quad \frac{1}{|\lambda|} \operatorname*{ess\,sup}_{t\in E} \omega(s^{-1}t) < \varepsilon.$$

For the proof of Theorem B, with an additional equivalent condition, see Section 6. It is worth mentioning that the condition (2) above does not depend on p, but we need the usual assumption of local p-summability of ω which does depend on p.

In the case where Γ is equal to one of these sets: the complex plane, $[0,1]$, $[1,+\infty[$, or a set which is bounded and bounded away from zero, we get a complete characterization of (Γ, S)-density only in terms of the weight, see Corollary 6.2.

And finally, in the case when all translations are bounded (and not only by $s \in S$), the criterion is as follows:

Proposition 1.4. *Let G be abelian and let ω be a continuous G-admissible weight. Then there is a (Γ, S)-dense vector in $L^p(G,\omega)$ if and only if there exist two sequences $(s_n)_{n\geqslant 1} \subset S$ and $(\lambda_n)_{n\geqslant 1} \subset \Gamma \setminus \{0\}$ such that*

$$\lim_{n\to+\infty} \max\left\{ |\lambda_n|\omega(s_n)\,;\, \frac{1}{|\lambda_n|}\omega(s_n^{-1}) \right\} = 0.$$

Proposition 1.4 is a consequence of Proposition 5.4 below when G is abelian. In the case where G is abelian, and Γ is equal to one the particular mentioned above sets, we get a complete characterization of (Γ, S)-density only in terms of the weight, see Corollary 5.5.

2. Restrictions on the group

It has been shown in [1] that S-dense vectors can exist in $L^p(G,\omega)$ only if G is non-compact second-countable. For (Γ, S)-density, the restrictions are the same, even if the proofs have to be changed.

In this section and further on we denote by χ_K the characteristic function of a set K, and by e the identity of G.

Lemma 2.1. *Let X be a non-separable Banach space and let $0 < M < 1$. Then there exist an uncountable set $\{x_\alpha\}_{\alpha\in I} \subset X$ of norm 1 vectors such that for all $\alpha_1, ..., \alpha_n \in I$ and $c_1, ..., c_n \in \mathbb{C}\setminus\{0\}$,*

$$\|\sum_{i=1}^{n} c_i x_{\alpha_i}\| > M \min\{|c_i| : 1 \leqslant i \leqslant n\}.$$

Proof. This is an easy application of Zorn's lemma and Riesz's lemma: note that $\|\sum_{i=1}^n c_i x_{\alpha_i}\| \geqslant |c_k| \operatorname{dist}(x_{\alpha_k}, \operatorname{span}\{x_{\alpha_i} : k \neq i \in \{1,\dots,n\}\}$ for every $k = 1,\dots,n$. □

Lemma 2.2. *Let G be a σ-compact locally compact group. The orbit $\operatorname{Orb}_G(f)$ of any function f in $L^p(G)$ is separable (in the norm topology).*

Proof. This fact is not new, and proved for example in [17] for $p = 1$. Since it is not stated for $p > 1$ up to our knowledge, we include a proof here. It is known [12, Theorem 20.4] that the map $\tau : s \mapsto T_s f$ is continuous, from G to $L^p(G)$ with the norm topology. Since G is a countable union of compact sets, the same holds for $\tau(G) = \operatorname{Orb}_G(f)$. This is in addition a metric space, which implies that it is separable. □

Theorem 2.3. *If for some p, S, and ω there exists a $(\mathbb{C}, S)$-dense vector in $L^p(G,\omega)$, then G is second-countable which is equivalent to saying that $L^p(G,\omega)$ is separable.*

Proof. Let us prove first that G is σ-compact, which is a strictly weaker condition. Suppose that f is a $(\mathbb{C}, S)$-dense vector in $L^p(G,\omega)$ for some p, ω and S. It is known that f can be assumed to vanish outside a σ-compact set [12, Theorem 11.40], so that the support of f is contained in an open σ-compact subgroup H of G.

Suppose now that G is not σ-compact, then $H \neq G$. Pick $g \in G \setminus H$. Let $U \subset H$ be a compact symmetric neighbourhood of identity. There exist $\gamma \in \mathbb{C}$ and $s \in S$ such that $\|\chi_U + \chi_{gU} - \gamma T_s f\|_{p,\omega} < \min(\|\chi_U\|_{p,\omega}, \|\chi_{gU}\|_{p,\omega})$. But this is only possible if the support of $T_s f$ intersects both H and gH, which is false by the choice of H.

This shows that from now on, we can assume that G is σ-compact. According to [18, Theorem 2], we have G is second-countable if and only if $L^p(G)$ is separable, and the same is valid for $L^p(G,\omega)$ since this space is isometrically isomorphic to $L^p(G)$.

Suppose that $L^p(G,\omega)$ is not separable but $f \in L^p(G,\omega)$ is a $(\mathbb{C}, S)$-dense vector. According to Lemma 2.1, there exists an uncountable set $\{g_\alpha\}_{\alpha \in I} \subset L^p(G,\omega)$ such that $\|g_\alpha\|_{p,\omega} = 1$ for all α and

$$\|\sum_{i=1}^{n} c_i g_{\alpha_i}\|_{p,\omega} > \frac{9}{10} \min\{|c_i| : 1 \leqslant i \leqslant n\}$$

for all $\alpha_1, ..., \alpha_n \in I$ and $c_1, ..., c_n \in \mathbb{C} \setminus \{0\}$. For all $\alpha \in I$, we can approximate g_α so that $\|g_\alpha - \lambda_\alpha T_{s_\alpha} f\|_{p,\omega} < \dfrac{1}{10}$, with $s_\alpha \in S$ and $\lambda_\alpha \in \mathbb{C} \setminus \{0\}$. We can choose $\delta > 0$ so that the set $J = \{\alpha \in I : \delta < |\lambda_\alpha| < 2\delta\}$ is uncountable. Now for every $\alpha, \beta \in J$ we have

$$\|\lambda_\alpha^{-1} g_\alpha - T_{s_\alpha} f\|_{p,\omega} < \frac{1}{10|\lambda_\alpha|} < \frac{1}{10\delta},$$

and

$$\|T_{s_\alpha} f - T_{s_\beta} f\|_{p,\omega} \geqslant \|\frac{1}{\lambda_\alpha} g_\alpha - \frac{1}{\lambda_\beta} g_\beta\|_{p,\omega} - \frac{1}{5\delta} \geqslant \frac{9}{10} \min\{\frac{1}{|\lambda_\alpha|}, \frac{1}{|\lambda_\beta|}\} - \frac{1}{5\delta} > \frac{1}{4\delta}.$$

We will show next that the set J allows to find a function in $L^p(G)$ with a non-separable orbit, which contradicts Lemma 2.2 and will prove the theorem. For every $\alpha \in I$, we have

$$\|T_{s_\alpha} f\|_{p,\omega}^p = \lim_{N \to +\infty} \int_{\{x : \omega(x) \leqslant N\}} |T_{s_\alpha} f(x)|^p \omega(x)^p d\mu(x)$$

so that there exists $N_\alpha \in \mathbb{N}$ such that $\int_{\{x:\omega(x) > N_\alpha\}} |T_{s_\alpha} f(x)|^p \omega(x)^p dx < (16\delta)^{-1}$. There is $N \in \mathbb{N}$ and an uncountable set $J_0 \subset J$ such that $N_\alpha = N$ for every $\alpha \in J_0$. Moreover, there exists $C > 0$ and an uncountable set $J_1 \subset J_0$ such that $\|T_{s_\alpha}\| \leqslant C$ for every $\alpha \in J_1$.

Next, for every $\varepsilon > 0$ the function

$$f_\varepsilon(x) = f(x)\chi_{\{\omega \geqslant \varepsilon\}}(x)$$

is in $L^p(G)$. Pick $\varepsilon > 0$ such that $\|f_\varepsilon - f\|_{p,\omega} < (32C\delta)^{-1}$. We have now for $\alpha, \beta \in J_1$:

$$\begin{aligned}\|T_{s_\alpha} f_\varepsilon - T_{s_\beta} f_\varepsilon\|_p &\geqslant \frac{1}{N} \|\chi_{\omega \leqslant N} \cdot \big[T_{s_\alpha} f_\varepsilon - T_{s_\beta} f_\varepsilon\big] \|_{p,\omega} \\ &\geqslant \frac{1}{N} \Big[\|T_{s_\alpha} f - T_{s_\beta} f\|_{p,\omega} - \|\chi_{\omega > N} \cdot \big[T_{s_\alpha} f - T_{s_\beta} f\big] \|_{p,\omega} \\ &- \|\chi_{\omega \leqslant N} \cdot \big[T_{s_\alpha}(f_\varepsilon - f) - T_{s_\beta}(f_\varepsilon - f)\big] \|_{p,\omega} \Big] \\ &\geqslant \frac{1}{N} \Big[\frac{1}{4\delta} - \frac{2}{16\delta} - \|T_{s_\alpha}(f_\varepsilon - f)\|_{p,\omega} - \|T_{s_\beta}(f_\varepsilon - f)\|_{p,\omega} \Big] \\ &\geqslant \frac{1}{N} \Big[\frac{1}{8\delta} - 2C\|f_\varepsilon - f\|_{p,\omega} \Big] > \frac{1}{16\delta N} > 0.\end{aligned}$$

This means that the S-orbit of f_ε is nonseparable in $L^p(G)$. This is the expected contradiction, which proves the theorem. □

Proposition 2.4. *If G is a nontrivial compact group, then the space $L^p(G, \omega)$ has no (Γ, S)-dense vectors for any Γ, ω, S, p.*

Proof. Suppose that $L^p(G, \omega)$ has a (Γ, S)-dense vector f. If G is finite, then $\mathrm{Orb}_S(\Gamma f)$ is contained in the finite union $\cup\{\mathbb{C} T_g f : g \in G\}$ of one-dimensional lines, which cannot be dense in $L^p(G, \omega)$ unless $G = \{e\}$. We suppose therefore that G is infinite.

Scale the Haar measure so that $\mu(G) = 1$. Fix $\varepsilon \in (0, 1/12)$. There is $\delta > 0$ such that $\mu(E_\delta) > 1 - \varepsilon$ where $E_\delta = \{t \in G : \omega(t) > \delta\}$. There exist $s_1 \in S$ and $\gamma_1 \in \Gamma$ such that $\|\chi_G - \gamma_1 T_{s_1} f\|_{p,\omega} < \varepsilon\delta$. Then

$$\delta^p \varepsilon^p > \int_{E_\delta} \delta^p |1 - \gamma_1 f(s_1^{-1} t)|^p d\mu(t),$$

so that

$$\int_{s_1^{-1} E_\delta} |1 - \gamma_1 f(t)|^p d\mu(t) < \varepsilon^p.$$

Since G is infinite, μ has no atoms. We know that G is second-countable, thus e has a countable base of neighbourhoods (U_n) and one can assume them decreasing; there exists U_n with $\mu(U_n) < \varepsilon$. By compactness, G is covered by a finite number of translates of U_n (all of the same measure $< \varepsilon$); picking a sufficient number of them, one can form a set $E \subset G$ such that $1/2 - \varepsilon < \mu(E) < 1/2 + \varepsilon$. Let now $s_2 \in S$, $\gamma_2 \in \Gamma$ be such that $\|\chi_{G \setminus E} - \gamma_2 T_{s_2} f\|_{p,\omega} < \delta\varepsilon$. As above, this implies

$$\delta^p \varepsilon^p > \int_{E_\delta \cap E} \delta^p |\gamma_2|^p |f(s_2^{-1} t)|^p d\mu(t),$$

so that

$$\int_{s_2^{-1}(E_\delta \cap E)} |\gamma_2 f(t)|^p d\mu(t) < \varepsilon^p,$$

and

$$\int_{s_2^{-1}(E_\delta \setminus E)} |1 - \gamma_2 f(t)|^p d\mu(t) < \varepsilon^p.$$

Denote $A = s_1^{-1} E_\delta$, $B = s_2^{-1}(E_\delta \cap E)$, $C = s_2^{-1}(E_\delta \setminus E)$. Now $\mu(A \cap B) \geqslant 1 - \mu(G \setminus A) - \mu(G \setminus B) > 1 - \varepsilon - 1/2 - 2\varepsilon = 1/2 - 3\varepsilon$, and similarly $\mu(A \cap C) \geqslant 1/2 - 3\varepsilon$.

At the same time (the norms below are unweighted),

$$\mu(A\cap B)^{1/p} = \|\chi_{A\cap B}\|_p \leqslant \|\chi_{A\cap B}\gamma_1 f\|_p + \|\chi_{A\cap B}(1-\gamma_1 f)\|_p < \frac{|\gamma_1|}{|\gamma_2|}\varepsilon + \varepsilon,$$

so if we denote $z = \gamma_1/\gamma_2$ then $\varepsilon|z| > (1/2-3\varepsilon)^{1/p} - \varepsilon \geqslant 1/2 - 4\varepsilon > 1/6$ by the choice of ε and $|z| > 1/(6\varepsilon)$. In particular, $|z| > 2$. On the other hand,

$$\begin{aligned}\mu(A\cap C)^{1/p}\Big|1-\frac{\gamma_1}{\gamma_2}\Big| &= \|\chi_{A\cap C}\Big(1-\frac{\gamma_1}{\gamma_2}\Big)\|_p \\ &\leqslant \|\chi_{A\cap C}(1-\gamma_1 f)\|_p + \|\chi_{A\cap C}\frac{\gamma_1}{\gamma_2}(1-\gamma_2 f)\|_p < \varepsilon + \frac{|\gamma_1|}{|\gamma_2|}\varepsilon,\end{aligned}$$

whence similarly $|1-z|/4 < \varepsilon(1+|z|)$ and $|z| < 1 + 4\varepsilon(1+|z|)$ which implies $|z| < (1+4\varepsilon)/(1-4\varepsilon)$. This is however incompatible with $|z| > 2$. This contradiction proves the proposition. □

3. Continuity of translations

The condition for the translation operator T_s to be continuous was given in the introduction. The question of continuity of the map $S \to L^p(G,\omega)$, $s \mapsto T_s f$ for fixed f is more delicate.

Let $C_c(G)$ denote the space of continuous functions on G with compact support. On this space, no problem occurs. In a standard way (similarly to [12, Theorem (20.4)], using uniform continuity of f and local p-summability of ω), one proves the following

Proposition 3.1. *Let G be a locally compact group, $1 \leqslant p < +\infty$, and let ω be a G-admissible weight on G. Then for every $f \in C_c(G)$ the map*

$$\begin{array}{ccc} G & \longrightarrow & L^p(G,\omega) \\ s & \longmapsto & T_s f \end{array}$$

is continuous.

If ω is a G-admissible weight (all translations are bounded), the map in question is still continuous for any $f \in L^p(G,\omega)$:

Theorem 3.2. *Let G be a locally compact group, $1 \leqslant p < +\infty$, and let ω be a G-admissible weight on G. Then for every $f \in L^p(G,\omega)$ the map*

$$\begin{array}{ccc} G & \longrightarrow & L^p(G,\omega) \\ s & \longmapsto & T_s f \end{array}$$

is continuous.

Proof. It is known [8, Theorem 2.7] that every G-admissible locally summable weight ω is equivalent to a continuous weight $\tilde\omega$, in the sense that there exists $C > 0$ such that $1/C \leqslant \omega(t)/\tilde\omega(t) \leqslant C$ for all $t \in G$. For ω locally p-summable the same is true of course. We can thus assume that ω is continuous.

Set $M(s) := \sup\limits_{t\in G} \dfrac{\omega(st)}{\omega(t)}$ for every $s \in G$. According to [7, Proposition 1.16], $M(s)$ is locally bounded on G. Fix $f \in L^p(G,\omega)$. Let $s_0 \in G$ and $\varepsilon > 0$. Since G is locally compact, there exists an open neighborhood W of e whose closure is compact. Let $\delta > 0$ such that

$$\big(\|T_{s_0}\| + 1 + \sup_{s\in s_0\overline{W}} M(s)\big)\delta < \varepsilon.$$

Since ω is continuous, the space $C_c(G)$ is dense in $L^p(G,\omega)$, hence, there exists $\phi \in C_c(G)$ such that

$$\|f-\phi\|_{p,\omega} < \delta.$$

By Proposition 3.1, there exists a symmetric open neighborhood U of e whose closure is compact such that

$$\|T_{s_0}\phi - T_s\phi\|_{p,\omega} < \delta \text{ for every } s \in s_0U.$$

Let $V = U \cap W$, then V is an open neighborhood of e. Fix $s \in s_0V$. Thus $\|T_{s_0}\phi - T_s\phi\|_{p,\omega} < \delta$ and $\|T_s\phi - T_sf\|_{p,\omega} < \delta \sup_{s\in s_0\overline{W}} M(s)$. Hence

$$\begin{aligned}\|T_{s_0}f - T_sf\|_{p,\omega} &\leqslant \|T_{s_0}f - T_{s_0}\phi\|_{p,\omega} + \|T_{s_0}\phi - T_s\phi\|_{p,\omega} + \|T_s\phi - T_sf\|_{p,\omega}\\ &< (\|T_{s_0}\| + 1 + \sup_{s\in s_0\overline{W}} M(s))\delta < \varepsilon.\end{aligned}$$

□

With a weight as above, it is not difficult to show the following "non-compactness" of translations, necessary in the sequel:

Proposition 3.3. *Let G be a locally compact non-trivial group, $S \subset G$, $1 \leqslant p < +\infty$, and let $\Gamma \subset \mathbb{C}$ be such that $\Gamma \setminus \{0\}$ is non-empty. If ω is a G-**admissible** weight on G and $f \in L^p(G,\omega)$ is a (Γ, S)-dense vector, then for every compact set $K \subset G$, f is also a $(\Gamma, S \setminus K)$-dense vector in $L^p(G,\omega)$.*

Proof. As pointed out above, we can assume that ω is continuous. Since G is not a finite group, $L^p(G,\omega)$ is infinite-dimensional. By Theorem 3.2, the set $\{T_sf : s \in K\}$ is compact in $L^p(G,\omega)$, thus the set $\mathbb{C}\{T_sf : s \in K\}$ is nowhere dense. Hence

$$L^p(G,\omega) = \operatorname{int}\overline{\operatorname{Orb}_S(\Gamma f) \setminus \mathbb{C}\{T_sf : s \in K\}} = \operatorname{int}\overline{\operatorname{Orb}_{(S\setminus K)}(\Gamma f)},$$

which implies that f is a $(\Gamma, S \setminus K)$-dense vector in $L^p(G,\omega)$. □

However, if translations T_s are bounded only for s in a subset $S \subset G$, then the translation map $s \mapsto T_sf$ may be discontinuous on S. This explains the need of the long Lemma 3.5 below.

In the following example, we construct a continuous weight on $\mathbb{R}$, prove first that it is $\mathbb{R}_+$-admissible, and then exhibit a function $f \in L^p(\mathbb{R},\omega)$ such that $T_sf \not\to f = T_0f$ as $s \to 0$.

Example 3.4. Set $G = \mathbb{R}$, $S = \mathbb{R}_+$ and $a_n = n!$ for $n \geqslant 2$. Define the weight as

$$\omega(t) := \begin{cases} 1 & \text{if } t \leqslant 0 \text{ or } t \in [a_{n-1}+1, a_n - n], n \geqslant 3\\ 2^{t-(a_n-n)} & \text{if } t \in [a_n - n, a_n - 1]\\ 2^{n-1} & \text{if } t \in [a_n - 1, a_n - 1/2]\\ 1 + (2^n - 2)(a_n - t) & \text{if } t \in [a_n - 1/2, a_n]\\ 1 + (2^n - 1)2^n(t - a_n) & \text{if } t \in [a_n, a_n + 2^{-n}]\\ \dfrac{1}{t - a_n} & \text{if } t \in [a_n + 2^{-n}, a_n + 1]\end{cases}$$

This weight is thus equal to 1 at a_n and "far from" any a_n; around a_n, it has two peaks: one on the left, with the maximum value 2^{n-1} taken on a whole segment $[a_n - 1, a_n - 1/2]$, and another on the right, with the maximum 2^n taken in $a_n + 2^{-n}$. It is clear that ω is continuous.

Claim 1. ω is an $\mathbb{R}_+$-admissible weight.

Proof. Set
$$M(s)=\sup_{t\in\mathbb{R}}\frac{\omega(s+t)}{\omega(t)}.$$
Suppose that $0<s<1/2$, and let n be such that $2^{-n-1}<s\leqslant 2^{-n}$. Clearly
$$M(s)\geqslant\frac{\omega(a_n+s)}{\omega(a_n)}=1+(2^{2n}-2^n)s\geqslant 2^{2n-1}s\geqslant\frac{1}{8s}.$$
Let us show that in fact $M(s)\leqslant 2/s$. To estimate $M(s)$, let us consider all possible cases for $t\in\mathbb{R}$.

- If $t\leqslant a_{n+1}-(n+1)$, then $\omega(t)$ and $\omega(t+s)$ are between 1 and 2^n, so that their ratio is bounded by 2^n.
- If $t\in[a_m-m,a_m-1-s]$ for $m>n$, then
$$\frac{\omega(t+s)}{\omega(t)}=2^s\leqslant 2.$$
- If $t\in[a_m-1-s,a_m-1/2]$ for $m>n$, then
$$\frac{\omega(t+s)}{\omega(t)}\leqslant\frac{2^{m-1}}{2^{m-1-s}}=2^s\leqslant 2.$$
- If $t\in[a_m-1/2,a_m-s]$ for $m>n$, then $\omega(t+s)\leqslant\omega(t)$ and
$$\frac{\omega(t+s)}{\omega(t)}\leqslant 1.$$
- If $t\in[a_m-s,a_m+2^{-m}-s]$ for $m>n$, denote $u=a_m-t$. We have $u\in[s-2^{-m},s]$, and
$$\begin{aligned}\frac{\omega(t+s)}{\omega(t)}&=\frac{1+(2^m-1)2^m(t+s-a_m)}{1+(2^m-2)(a_m-t)}\\&=\frac{1+(2^m-1)2^m(s-u)}{1+(2^m-2)u}=:P(u).\end{aligned}$$
  P is a decreasing function of u on $[s-2^{-m},s]$, hence
$$\begin{aligned}\frac{\omega(t+s)}{\omega(t)}&\leqslant P(s-2^{-m})=\frac{2^m}{2^ms-2s+2^{-m+1}}\\&\leqslant\frac{2^m}{(2^m-2)s}\leqslant\frac{1}{2s}.\end{aligned}$$
- If $t\in[a_m-s+2^{-m},a_m]$ for $m>n$, denote again $u=a_m-t$. We have $u\in[0,s-2^{-m}]$, and
$$\frac{\omega(t+s)}{\omega(t)}=\frac{1}{[1+(2^m-2)(a_m-t)](t+s-a_m)}=\frac{1}{Q(u)},$$
  where $Q(u):=(1+(2^m-2)u)(s-u)$. It is easy to see that Q is a concave function, and thus its minimum on any segment is attained at one of its ends. On $[0,s-2^{-m}]$, we have $Q(0)=s$ and
$$Q(s-2^{-m})=s+2^{1-2m}-2^{1-m}s\geqslant(1-2^{1-m})s\geqslant\frac{s}{2},$$
  as we suppose $m>n\geqslant 1$. It follows that
$$\frac{\omega(t+s)}{\omega(t)}=\frac{1}{Q(u)}\leqslant\frac{2}{s}.$$

- If $t \in [a_m, a_m + 2^{-m}]$ with $m > n$, denote $u = t - a_m \in [0, 2^{-m}]$. We have
$$\frac{\omega(t+s)}{\omega(t)} = \frac{1}{(t+s-a_m)(1+(2^m-1)2^m(t-a_m))} = R(u),$$
where $R(u) := \dfrac{1}{(s+u)((2^{2m}-2^m)u+1)}$. Then R is decreasing on $[0, 2^{-m}]$, hence
$$\frac{\omega(t+s)}{\omega(t)} \leqslant R(0) = \frac{1}{s}.$$
- If $t \in [a_m + 2^{-m}, a_m + 1]$, then $\omega(t+s) \leqslant \omega(t)$.

Taking maximum of these estimates, we arrive at $M(s) \leqslant \dfrac{2}{s}$.

Being finite on $[0, 1/2)$ and submultiplicative (i.e., $M(s+s') \leqslant M(s)M(s')$), M is finite on $[0, \infty)$ so that ω is an $\mathbb{R}_+$-admissible weight. □

One can verify that $M(s) < \infty$ exactly when $s \in \mathbb{R}_+$: if $s \leqslant -1$, then
$$M(s) \geqslant \sup_{n\geqslant 2} \frac{\omega(a_n + 2^{-n})}{\omega(a_n + 2^{-n} + |s|)} \geqslant \sup_{n\geqslant 2} 2^n = +\infty,$$
and if $-1 < s < 0$ then
$$M(s) \geqslant \sup_{n:2^{-n}\leqslant 1-|s|} \frac{\omega(a_n + 2^{-n})}{\omega(a_n + 2^{-n} + |s|)} = \sup_{n:2^{-n}\leqslant 1-|s|} 2^n(2^{-n} + |s|) = +\infty.$$

Claim 2. There exists $f \in L^p(\mathbb{R}, \omega)$ such that $\|f - T_s f\|_{p,\omega} \not\to 0$, $s \to 0$.

Proof. Set $f = \sum_{k=2}^{\infty} \chi_{[a_k - 2^{-k}, a_k]}$. Check first that its p, ω-norm is finite:
$$\|f\|_{p,\omega}^p = \sum_{k=2}^{\infty} \int_{a_k - 2^{-k}}^{a_k} \left(1 + (2^k - 2)(a_k - t)\right)^p dt$$
$$\leqslant \sum_{k=2}^{\infty} 2^{-k}(1 + (2^k - 2)2^{-k})^p < \infty.$$
If $s = 2^{-n}$, $2 \leqslant n \in \mathbb{N}$, then $T_s f = \sum_{k=2}^{\infty} \chi_{[a_k - 2^{-k} + 2^{-n}, a_k + 2^{-n}]}$; it equals 1 on $[a_n, a_n + 2^{-n}]$ while f vanishes (almost everywhere) on this segment. Thus
$$\|f - T_s f\|_{p,\omega}^p \geqslant \int_{a_n}^{a_n + 2^{-n}} \omega(t)^p dt = \int_{a_n}^{a_n + 2^{-n}} \left(1 + (2^n - 1)2^n(t - a_n)\right)^p dt$$
$$\geqslant 2^{p(n-1)+pn} \int_0^{2^{-n}} t^p dt = \frac{1}{(p+1)2^p} 2^{2pn} 2^{-n(p+1)} = \frac{1}{(p+1)2^p} 2^{n(p-1)}$$
which is constant if $p = 1$ and tends to infinity as $n \to \infty$ if $p > 1$. □

The following lemma will be used in the proof of Theorem A.

Lemma 3.5. *Let G be a locally compact non-compact group, $S \subset G$, $1 \leqslant p < +\infty$, and ω an S-admissible weight. Let f be a (Γ, S)-dense vector in $L^p(G, \omega)$. Then for every compact set $K \subset G$ of positive measure, any $\varepsilon > 0$, $\lambda \in \mathbb{C} \setminus \{0\}$ and every compact set $L \subset G$ there are $\gamma \in \Gamma$, $s \in S \setminus L$ such that*
$$\|\gamma T_s f - \lambda \chi_K\|_{p,\omega} < \varepsilon.$$

Proof. We can scale the Haar measure if necessary to have $\mu(K) = 1$. There is clearly $C > 0$ such that for $K_C := \{t \in K : \omega(t) > 1/C\}$ we have $\mu(K_C) > \dfrac{3}{4}$. Increasing L if necessary, we can suppose that $e \in L = L^{-1}$. We can also take $\varepsilon > 0$ small enough to guarantee $4\varepsilon < \|\lambda\chi_K\|_{p,\omega}$ and $2C\varepsilon < |\lambda|$.

Suppose the contrary, that is, $\|\lambda\chi_K - \gamma T_s f\|_{p,\omega} \geqslant \varepsilon$ for every $\gamma \in \Gamma$, $s \in S \setminus L$. Let us prove the following

Claim. There exists $M > 0$ such that for all $\gamma \in \Gamma$ and $s \in S$ satisfying $\|\lambda\chi_K - \gamma T_s f\|_{p,\omega} < \varepsilon$, we have $s \in L$ and $|\gamma| > \xi$ with $\xi := M^{-1}(|\lambda| - 2^{1/p}C\varepsilon) > 0$.

Proof of the claim. There exists $M > 0$ such that the set $X_M = \{t \in LK : |f(t)| > M\}$ has measure less than $\dfrac{1}{10}\mu(K_C)$. Let $\gamma \in \Gamma$ and $s \in S$ be such that $\|\lambda\chi_K - \gamma T_s f\|_{p,\omega} < \varepsilon$. By assumption we have $s \in L$ and

$$\begin{aligned}\varepsilon^p > \|\lambda\chi_K - \gamma T_s f\|^p_{p,\omega} &\geqslant \int_{K_C} |\lambda - \gamma f(s^{-1}t)|^p \omega(t)^p \, d\mu(t) \\ &\geqslant C^{-p} \int_{K_C} |\lambda - \gamma f(s^{-1}t)|^p \, d\mu(t) \geqslant C^{-p} \int_{(s^{-1}K_C)\setminus X_M} |\lambda - \gamma f(t)|^p \, d\mu(t).\end{aligned}$$

If $|\gamma|M < |\lambda|$, then

$$\begin{aligned}\varepsilon^p &\geqslant C^{-p} \int_{(s^{-1}K_C)\setminus X_M} (|\lambda| - |\gamma|M)^p \, \mathrm{d}\mu(t) \\ &> C^{-p}(|\lambda| - |\gamma|M)^p(\mu(K_C) - \mu(X_M)) \\ &> \frac{9\mu(K_C)}{10C^p}(|\lambda| - |\gamma|M)^p > \frac{1}{2C^p}(|\lambda| - |\gamma|M)^p,\end{aligned}$$

which implies

$$|\gamma| > M^{-1}(|\lambda| - 2^{1/p}C\varepsilon) =: \xi > 0$$

If $|\gamma|M \geqslant |\lambda|$ then also $|\gamma| > \xi$, since $\xi < \dfrac{|\lambda|}{M}$. □

The set LK is compact, so its measure is finite; pick $N \in \mathbb{N}$ with $\mu(LK) < N/2$. Set $r := \max\{1, \|\chi_K\|_{p,\omega}\}$ and $\lambda_j = \lambda + j\varepsilon/2Nr$, $j = 0, ..., N-1$.

For $j = 0, ..., N-1$ we choose $\delta_j > 0$, $\gamma_j \in \Gamma$, $s_j \in S$ such that

$$\|\lambda_j\chi_K - \gamma_j T_{s_j} f\|_{p,\omega} < \delta_j.$$

The choice of δ_j is specified later, but it will be less than $\varepsilon/4$. Since

$$\|\lambda\chi_K - \gamma_j T_{s_j} f\|_{p,\omega} < \delta_j + |\lambda - \lambda_j|\|\chi_K\|_{p,\omega} < \delta_j + \frac{\varepsilon}{2} < \varepsilon,$$

by the claim we have $s_j \in L$ and $|\gamma_j| > \xi$. Moreover, we have

$$\begin{aligned}\int_{K_C} |\lambda_j - \gamma_j f(s_j^{-1}t)|^p \, \mathrm{d}\mu(t) &< C^p \int_{K_C} \omega(t)^p |\lambda_j - \gamma_j f(s_j^{-1}t)|^p \, \mathrm{d}\mu(t) \\ &\leqslant C^p \|\lambda_j\chi_K - \gamma_j T_{s_j} f\|^p_{p,\omega} < C^p\delta_j^p.\end{aligned}$$

Set $K_j := \{t \in K_C : |\lambda_j - \gamma_j f(s_j^{-1}t)| \geqslant 4^{1/p}\delta_j C\}$. By trivial estimates,

$$C^p\delta_j^p > \int_{K_j} |\lambda_j - \gamma_j f(s_j^{-1}t)|^p \, \mathrm{d}\mu(t) \geqslant 4C^p\delta_j^p\mu(K_j),$$

so that

$$\mu(K_j) < \frac{1}{4}.$$

For $t \in K_j' := s_j^{-1}(K_C \setminus K_j)$ we have the estimate

$$|f(t) - \frac{\lambda_j}{\gamma_j}| < 4^{1/p} C \frac{\delta_j}{|\gamma_j|} < 4^{1/p} C \frac{\delta_j}{\xi}$$

and $\mu(K_j') \geqslant \mu(K_C) - \mu(K_j) > \frac{1}{2}$. The aim is to choose δ_j so that these sets are pairwise disjoint. For this, it is sufficient to choose δ_j so that the disks $B_j := \{z \in \mathbb{C} : |z - \lambda_j/\gamma_j| < 4^{1/p} C\delta_j/\xi\}$ are pairwise disjoint. Suppose by induction that this is done for $k < j$. Set $\delta_j = \min\{\delta_k : k < j\}$. If $B_j \cap (\cup_{k<j} B_k) = \emptyset$, then we are done. If not, divide δ_j by 2 and pick $\gamma_j \in \Gamma$, $s_j \in S$ accordingly. If we still have $B_j \cap (\cup_{k<j} B_k) \neq \emptyset$, continue in the same way. Either we arrive at an empty intersection on some step — in this case we can pass to the next j — or we continue infinitely and get a sequence $\gamma_{j,n} \in \Gamma$, $s_{j,n} \in S$ such that $\|\lambda_j \chi_K - \gamma_{j,n} T_{s_{j,n}} f\|_{p,\omega} \to 0$ as $n \to \infty$. According to the Claim, we have $s_{j,n} \in L$ and $|\gamma_{j,n}| > \xi$, so that

$$\begin{aligned}\int_{s_{j,n}^{-1}(G\setminus K)} |f(t)|^p \omega(s_{j,n} t)^p \,\mathrm{d}\mu(t) &= \int_{G\setminus K} |(T_{s_{j,n}} f)(t)|^p \omega(t)^p \,\mathrm{d}\mu(t) \\ &\leqslant \|\frac{\lambda_j}{\gamma_{j,n}} \chi_K - (T_{s_{j,n}} f)(t)\|_{p,\omega}^p \\ &\leqslant \frac{1}{\xi} \|\lambda_j \chi_K - \gamma_{j,n} T_{s_{j,n}} f\|_{p,\omega} \to 0.\end{aligned}$$

Since $G \setminus LK \subset s_{j,n}^{-1}(G \setminus K)$, we have

$$\int_{G\setminus LK} |f(t)|^p \omega(s_{j,n} t)^p \,\mathrm{d}\mu(t) \to 0 \text{ as } n \to \infty. \tag{1}$$

This is an exercise to show that (1) implies $f \equiv 0$ on $G \setminus LK$. [If not, there is a compact set $F \subset G \setminus LK$ of positive measure and $\delta > 0$ such that $|f| \geqslant \delta$ on F. Recall that $s_{j,n} \in L$ for every j, n. Pick $\delta_1 > 0$ such that the set $D = \{t \in LF : \omega(t) \leqslant \delta_1\}$ has measure $\mu(D) < \frac{1}{10}\mu(F)$. Now

$$\begin{aligned}\int_F |f(t)|^p \omega(s_{j,n} t)^p d\mu(t) &\geqslant \int_{F\setminus s_{j,n}^{-1} D} \delta^p \delta_1^p \,\mathrm{d}\mu(t) \\ &\geqslant \delta^p \delta_1^p \big(\mu(F) - \mu(D)\big) > \frac{9}{10} \delta^p \delta_1^p \mu(F)\end{aligned}$$

and cannot tend to 0.]

But now it is easy to show that f cannot be (Γ, S)-dense. Indeed, let $g \in G$ be such that $LKK^{-1}g \cap LKK^{-1} = \emptyset$. For every $s \in G$ we have then either $s^{-1}K \cap LK = \emptyset$ or $s^{-1}gK \cap LK = \emptyset$. In the first case, $T_s f$ vanishes on K, and in the second case, on gK. For all $\gamma \in \Gamma$ and $s \in S$, we have then

$$\begin{aligned}\|\chi_K + \chi_{gK} - \gamma T_s f\|_{p,\omega}^p &\geqslant \int_K |1 - \gamma (T_s f)(t)|^p \omega^p(t) \,\mathrm{d}\mu(t) + \int_{gK} |1 - \gamma (T_s f)(t)|^p \omega^p(t) \,\mathrm{d}\mu(t) \\ &\geqslant \min\Big\{ \int_K \omega^p(t) \,\mathrm{d}\mu(t), \int_{gK} \omega^p(t) \,\mathrm{d}\mu(t) \Big\}\end{aligned}$$

so that $\gamma T_s f$ cannot approximate $\chi_K + \chi_{gK}$ arbitrarily well.

This shows that the choice of required δ_j is always possible, yielding the pairwise disjoint sets K'_j. It follows that $\mu(\cup K'_j) > \frac{N}{2} > \mu(LK)$, in particular, $\cup K'_j \nsubseteq LK$. But by assumption, we have $K'_j \subset s_j^{-1}K \subset LK$ for every j. This contradiction proves the lemma. □

Remark 3.6. This lemma implies that in case of existence of (Γ, S)-dense vectors, not only G cannot be compact, but also S cannot be contained in a compact subset of a non-compact group G.

4. Proof of Theorem A

From now on, we assume the following.

Definition 4.1. We say that $(G, S, p, \omega, \Gamma)$ is an *admissible tuple* if G is a second-countable locally compact non-compact group, $S \subset G$, $1 \leqslant p < +\infty$, $\Gamma \subset \mathbb{C}$ is such that $\Gamma \setminus \{0\}$ is non-empty, and ω is a locally p-integrable S-admissible weight on G.

Theorem A. *If $(G, S, p, \omega, \Gamma)$ is admissible, the following conditions are equivalent:*

1. *There is a (Γ, S)-dense vector in $L^p(G, \omega)$.*
2. *For every increasing sequence of compact subsets $(F_n)_{n\geqslant 1}$ of G of positive measure and for every sequence $(\delta_n)_{n\geqslant 1}$ of positive numbers, there are sequences $(s_n)_{n\geqslant 1} \subset S$, $(\lambda_n)_{n\geqslant 1} \subset \Gamma \setminus \{0\}$ and a sequence of compact subsets $K_n \subset F_n$ such that the sets $s_n^{-1}F_n$ are pairwise disjoint, $\mu(F_n \setminus K_n) < \delta_n$ and*
$$\sum_{n,k\geqslant 0; n\neq k} \frac{|\lambda_n|^p}{|\lambda_k|^p} \|\omega\|^p_{p, s_n s_k^{-1} K_k} < +\infty,$$
with $s_0 = e$, $\lambda_0 = 1$ and $K_0 = \emptyset$.
3. *For every $N \geqslant 1$, there exist N vectors $\{f_1, ..., f_N\} \subset L^p(G, \omega)$ such that the set*
$$\{\lambda(T_s f_1, ..., T_s f_N) : \lambda \in \Gamma,\ s \in S\}$$
is dense in the direct sum of N copies of $L^p(G, \omega)$.

Proof of Theorem A. Firstly, it is clear that (3) ⇒ (1). Let us show that (1) ⇒ (2). Assume that $s_0 = e$, $\lambda_0 = 1$ and $K_0 = \emptyset$. We can suppose that the sequence $(\delta_n)_{n\geqslant 1}$ is decreasing with $\sum_{k\geqslant 1} \delta_k < +\infty$ and $\delta_k < \mu(F_k)$. For each $k \geqslant 1$, there exist a compact set $F'_k \subset F_k$ and $0 < c_k < \frac{1}{4}$ such that $\mu(F_k \setminus F'_k) < \frac{\delta_k}{2}$ and
$$\operatorname*{ess\,inf}_{t\in F'_k} \omega(t) \geqslant c_k \quad , \quad \operatorname*{ess\,sup}_{t\in F'_k} \omega(t) \leqslant c_k^{-1}. \tag{2}$$

Let $f \in L^p(G, \omega)$ be a (Γ, S)-dense vector. According to Lemma 3.5, there exist two sequences $(s_k)_{k\geqslant 1} \subset S$ and $(\lambda_k)_{k\geqslant 1} \subset \Gamma \setminus \{0\}$ such that, for each $k \geqslant 1$, we have
$$\|2\chi_{F'_k} - \lambda_k T_{s_k} f\|^p_{p,\omega} < c_k^{2p} \delta_k, \tag{3}$$
and $s_k^{-1}F_k \cap s_j^{-1}F_j = \emptyset$ for every $1 \leqslant j < k$. By (2) and (3), we get
$$c_k^{2p}\delta_k > \int_{F'_k} |\lambda_k(T_{s_k}f)(t) - 2|^p \omega(t)^p \,\mathrm{d}\mu(t) \geqslant c_k^p \int_{F'_k} |\lambda_k(T_{s_k}f)(t) - 2|^p \,\mathrm{d}\mu(t),$$
that is
$$\|\lambda_k T_{s_k} f - 2\chi_{F'_k}\|^p_{p,F'_k} < c_k^p \delta_k < \frac{\delta_k}{4^p} < \delta_k.$$

Set $K'_k := \{t \in F'_k : |\lambda_k f(s_k^{-1}t)| > 1\}$. It follows that

$$\frac{\delta_k}{4} > c_k^p \delta_k > \int_{F'_k \setminus K'_k} |2 - \lambda_k f(s_k^{-1}t)|^p \, \mathrm{d}\mu(t) \geqslant \mu(F'_k \setminus K'_k),$$

There exists a compact subset $K_k \subset K'_k$ such that $\mu(K'_k \setminus K_k) < \dfrac{\delta_k}{4}$. According to the above estimates, we get $\mu(F_k \setminus K_k) < \delta_k$. Moreover,

$$\sum_{k>0} \frac{1}{|\lambda_k|^p} \|\omega\|^p_{p, s_k^{-1}K_k} \leqslant \sum_k \int_{s_k^{-1}K_k} \omega(t)^p |f(t)|^p \, d\mu(t) \leqslant \|f\|^p_{p,\omega} < +\infty. \tag{4}$$

Fix $n \geqslant 1$. Note that, for every $k \neq n$ and $t \in s_n s_k^{-1} K_k$, we have $t \in G \setminus K'_n$ and $|\lambda_k (T_{s_n} f)(t)| > 1$. Moreover, the sets $s_n s_k^{-1} K_k$ are pairwise disjoint. Hence

$$\begin{aligned}
\sum_{k, k\neq n} \frac{|\lambda_n|^p}{|\lambda_k|^p} \|\omega\|^p_{p, s_n s_k^{-1} K_k} &\leqslant \sum_{k,k\neq n} \int_{s_n s_k^{-1} K_k} \omega(t)^p |\lambda_n (T_{s_n} f)(t)|^p \, \mathrm{d}\mu(t) \\
&\leqslant \int_{G \setminus K'_n} \omega(t)^p |\lambda_n (T_{s_n} f)(t)|^p \, \mathrm{d}\mu(t) \\
&\leqslant c_n^{-p} \int_{F'_n \setminus K'_n} |\lambda_n (T_{s_n} f)(t)|^p \, \mathrm{d}\mu(t) \\
&\quad + \int_{G \setminus F'_n} \omega(t)^p |2\chi_{F'_n} - \lambda_n (T_{s_n} f)(t)|^p \, \mathrm{d}\mu(t) \\
&\leqslant c_n^{-p} \mu(F'_n \setminus K'_n) + \|2\chi_{F'_n} - \lambda_n T_{s_n} f\|^p_{p,\omega} \\
&< \delta_n + c_n^{2p} \delta_n < 2\delta_n.
\end{aligned} \tag{5}$$

According to (4) and (5) and since $\sum_{n \geqslant 1} \delta_n < +\infty$, we obtain that the series in the condition (2) of Theorem A converges.

(2) $\Rightarrow$ (3). Without loss of generality we can assume that $N = 2$. Since $L^p(G, \omega)$ is separable, there exists a countable sequence $\{(p_n, q_n) : n \geqslant 1\} \subset \mathcal{K}(G) \times \mathcal{K}(G)$ dense in $L^p(G,\omega) \times L^p(G,\omega)$ (where $\mathcal{K}(G)$ is the set of essentially bounded functions on G with compact support). Let $\{(g_n, h_n) : n \geqslant 1\}$ be a sequence composed by the terms (p_n, q_n) so that each term appears infinitely many times.

Set $F_n := \cup_{k=1}^n S_k$ where $S_k := \operatorname{supp} g_k \cup \operatorname{supp} h_k$ and $\alpha_n := \max\{\|g_n\|^p_\infty, \|h_n\|^p_\infty\}$. Let also (δ'_n) be a decreasing sequence such that $0 < \delta'_n < \alpha_n^{-1} 2^{-n}$. Since $\|\omega\|^p_{p,F_n} < +\infty$ for every $n \geqslant 1$, there exists a positive decreasing sequence $(r_n)_{n\geqslant 1}$ such that $\|\omega\|^p_{p,E} < \delta'_n$ for every $E \subset F_n$ satisfying $\mu(E) < r_n$. Set $\delta_n = \min(r_n; \delta'_n) > 0$.

By assumption, there exist $(s_n)_{n\geqslant 1} \subset S$, $(\lambda_n)_{n \geqslant 1} \subset \Gamma \setminus \{0\}$, and $K_n \subset F_n$ such that the sets $s_n^{-1} F_n$ are pairwise disjoint, $\mu(F_n \setminus K_n) < \delta_n$ and

$$\sum_{k \geqslant 0} a_k < +\infty \quad \text{where } a_k = \sum_{n; n \neq k} \frac{|\lambda_n|^p}{|\lambda_k|^p} \|\omega\|^p_{p, s_n s_k^{-1} K_k}$$

with $s_0 = e$, $\lambda_0 = 1$ and $K_0 = \emptyset$. Hence $a_k \underset{k \to +\infty}{\longrightarrow} 0$, thus there exists a subsequence $(a_{l_k})_{k \geqslant 0}$ of $(a_k)_{k \geqslant 0}$ (with $l_0 = 0$, $l_k \geqslant k$) such that $\sum_{k \geqslant 1} \alpha_k a_{l_k} < +\infty$. We can also choose l_k so that the pair (g_{l_k}, h_{l_k}) is the same as (g_k, h_k) for every $k \geqslant 1$. We then have

$$\sum_{n,k; n \neq k} \alpha_k \frac{|\mu_n|^p}{|\mu_k|^p} \|\omega\|^p_{p, t_n t_k^{-1} E_k} < +\infty \tag{6}$$

where $t_n = s_{l_n}$, $\mu_n = \lambda_{l_n}$ and $E_k = K_{l_k}$. Moreover, the sets $t_n^{-1}E_n$ are pairwise disjoint, and for every $n \geqslant 1$, $\mu(S_n \setminus E_n) \leqslant \mu(F_{l_n} \setminus K_{l_n}) < \delta_{l_n} \leqslant \delta_n$. So, the series

$$f_1 := \sum_{k\geqslant 0} \frac{1}{\mu_k} T_{t_k^{-1}}(g_k\chi_{E_k}) \quad \text{and} \quad f_2 =: \sum_{k\geqslant 0} \frac{1}{\mu_k} T_{t_k^{-1}}(h_k\chi_{E_k}),$$

are convergent. Indeed, according to (6), we get

$$\|f_1\|_{p,\omega}^p \leqslant \sum_k \frac{1}{|\mu_k|^p} \int_{t_k^{-1}E_k} \omega(t)^p |g_k(t_k t)|^p dt \leqslant \sum_k \alpha_k \frac{1}{|\mu_k|^p} \|\omega\|_{p,t_k^{-1}E_k}^p < +\infty.$$

Similarly, one can check that $\|f_2\|_{p,\omega}^p < \infty$. To finish the proof, we have to show that $\|\mu_n(T_{t_n}f_1, T_{t_n}f_2) - (g_n, h_n)\| \underset{n\to+\infty}{\longrightarrow} 0$. Note that, for any $n \geqslant 1$ and $i \neq j$ such that $i, j \neq n$, we have $t_n t_i^{-1}E_i \cap t_n t_j^{-1}E_j = \emptyset$ and $E_n \cap t_n t_i^{-1}E_i = \emptyset$. Now, for $n \geqslant 1$, we have

$$\begin{aligned}
\|\mu_n T_{t_n} f_1 - g_n\|_{p,\omega}^p &= \sum_{k,k\neq n} \frac{|\mu_n|^p}{|\mu_k|^p} \|T_{t_n t_k^{-1}}(g_k\chi_{E_k})\|_{p,\omega}^p + \|g_n(\chi_{E_n} - 1)\|_{p,\omega}^p \\
&\leqslant \sum_{k,k\neq n} \frac{|\mu_n|^p}{|\mu_k|^p} \|g_k\|_\infty^p \|\omega\|_{p,t_n t_k^{-1}E_k}^p + \|g_n\|_\infty^p \|\omega\|_{p,\mathrm{supp}(g_n)\setminus E_n}^p \\
&\leqslant \sum_{k,k\neq n} \frac{|\mu_n|^p}{|\mu_k|^p} \alpha_k \|\omega\|_{p,t_n t_k^{-1}E_k}^p + \alpha_n \delta'_{l_n} \\
&\leqslant \varepsilon_n + 2^{-n},
\end{aligned}$$

where

$$\varepsilon_n := \sum_{k;k\neq n} \frac{|\mu_n|^p}{|\mu_k|^p} \alpha_k \|\omega\|_{p,t_n t_k^{-1}E_k}^p.$$

Similarly, $\|\mu_n T_{t_n} f_2 - h_n\|_{p,\omega}^p \leqslant \varepsilon_n + 2^{-n}$. By (6), we get $\varepsilon_n \to 0$. Hence,

$$\|\mu_n T_{t_n} f_1 - g_n\|_{p,\omega} + \|\mu_n T_{t_n} f_2 - h_n\|_{p,\omega} \leqslant 2(2^{-n} + \varepsilon_n)^{1/p} \underset{n\to+\infty}{\longrightarrow} 0.$$

□

Remark 4.2. If G is a countable discrete group, then in condition (2) of Theorem A we can suppose that the set K_n is equal to F_n.

The (Γ, S)-density condition, for Γ unbounded or unbounded away from zero, is strictly weaker than just S-density, as can be seen on the following example.

Example 4.3. Let $G = \mathbb{Z}$, let $S = (n_k)_{n\geqslant 1}$ be a strictly increasing sequence of positive integers, let $1 \leqslant p < +\infty$ and let $\Gamma \subset \mathbb{C}$ be such that $\Gamma \setminus \{0\}$ is non-empty. Suppose that one of the following conditions holds:

(1) Γ is unbounded and

$$\omega(n) = \begin{cases} 2^{-n} & \text{if } n \geqslant 0 \\ 1 & \text{if } n \leqslant 0 \end{cases}.$$

(2) 0 is a limit point of Γ and

$$\omega(n) = \begin{cases} 1 & \text{if } n \geqslant 0 \\ 2^n & \text{if } n \leqslant 0 \end{cases}.$$

Then ω is a $\mathbb{Z}$-admissible weight such that $L^p(\mathbb{Z},\omega)$ has a (Γ,S)-dense vector, but has no S-dense vectors.

Proof. We prove the case (1) only, and the case (2) is proved similarly. It is clear that ω is $\mathbb{Z}$-admissible. Since $\inf_{n\in\mathbb{N}} \max(\omega_n,\omega_{-n}) = 1 \neq 0$, it follows from [1, Corollary 13] that $L^p(\mathbb{Z},\omega)$ has no S-dense vector. Let us show that the condition (2) of Theorem A holds. Let $(F_n)_{n\geqslant 1}$ be an increasing sequence of finite sets of $\mathbb{Z}$. For every $n\in\mathbb{N}$, set $D_n := \max\{t : t\in F_n\}$, $d_n := \min\{t : t\in F_n\}$, $s_0 = 0$, $\lambda_0 = 1$ and $F_0 = \emptyset$. We denote by $\operatorname{card}(F)$ the cardinality of a set F. By induction, we construct a subsequence $(s_n)_{n\geqslant 1}\subset S$ and a sequence $(\lambda_n)_{n\geqslant 1}\subset \Gamma\setminus\{0\}$ such that for every $k = 0,1,...,n-1$,

(1) $\dfrac{|\lambda_k|^p}{|\lambda_n|^p}\operatorname{card}(F_n) < \dfrac{1}{2^n}$ (this choice of λ_n is possible since Γ is unbounded);
(2) $D_n + s_k - d_k < s_n$;
(3) $\dfrac{|\lambda_n|^p}{|\lambda_k|^p}\operatorname{card}(F_n)\,\omega(d_n+s_n-s_k)^p < \dfrac{1}{2^n}$ (when choosing s_n, both (2) and (3) can be satisfied since $\omega(n)\to 0$ as $n\to+\infty$).

It is clear that condition (2) implies that $(F_k - s_k)\cap(F_n - s_n) = \emptyset$ for every $k<n$. Since ω is decreasing and bounded by 1, $F_k\subset F_n$ for every $k<n$, we have

$$\begin{aligned}
\sum_{n,k\geqslant 0; n\neq k}\frac{|\lambda_n|^p}{|\lambda_k|^p}\|\omega\|^p_{p,F_k+s_n-s_k} &= \sum_{n\geqslant 0}\sum_{k\neq n}\frac{|\lambda_n|^p}{|\lambda_k|^p}\sum_{t\in F_k}\omega(t+s_n-s_k)^p\\
&\leqslant \sum_{n\geqslant 0}\left(\sum_{k<n}\frac{|\lambda_n|^p}{|\lambda_k|^p}\sum_{t\in F_n}\omega(t+s_n-s_k)^p + \sum_{k>n}\frac{|\lambda_n|^p}{|\lambda_k|^p}\operatorname{card}(F_k)\right)\\
&\leqslant \sum_{n\geqslant 0}\left(\sum_{k<n}\frac{|\lambda_n|^p}{|\lambda_k|^p}\operatorname{card}(F_n)\,\omega(d_n+s_n-s_k)^p + \sum_{k>n}\frac{|\lambda_n|^p}{|\lambda_k|^p}\operatorname{card}(F_k)\right)\\
&< \sum_{n\geqslant 0}\Big(\sum_{0\leqslant k<n}\frac{1}{2^n} + \sum_{k>n}\frac{1}{2^k}\Big) \quad \text{(by (1) and (3))}\\
&= \sum_{n\geqslant 0}\Big(\frac{n}{2^n}+\frac{1}{2^n}\Big) < +\infty.
\end{aligned}$$

This proves that $L^p(\mathbb{Z},\omega)$ has a (Γ,S)-dense vector. □

5. Sufficient conditions

The condition of Theorem A is universal, but sometimes difficult to check. It is therefore convenient to have simpler sufficient conditions, even if they are more restrictive. We make on G,S,p,Γ,ω the same assumptions as in Section 4, see Definition 4.1.

Theorem 5.1. *Suppose that (G,S,p,ω,Γ) is an admissible tuple. If for every compact subset K of G and for all $\varepsilon,\delta>0$, there exist $s\in S$, $\lambda\in\Gamma\setminus\{0\}$ and a compact subset $E\subset K$ such that $\mu(K\setminus E)<\delta$ and*

$$|\lambda|\operatorname*{ess\,sup}_{sE}\omega<\varepsilon \qquad \text{and} \qquad \frac{1}{|\lambda|}\operatorname*{ess\,sup}_{s^{-1}E}\omega<\varepsilon,$$

then $L^p(G,\omega)$ has a (Γ,S)-dense vector.

We could of course state the theorem just with one parameter ε, implying $\delta = \varepsilon$, and get an equivalent statement. For the proof, we need the following lemma.

Lemma 5.2. *Fix a compact set $L \subset G$. Under the assumptions of Theorem 5.1, s can in addition be chosen outside of L.*

Proof. Let K, L be two compact subsets of G. Assume, towards a contradiction, that for some $\varepsilon, \delta > 0$ the assumptions of Theorem 5.1 are only satisfied if $s \in S \cap L$. This implies in particular that $\mu(K) > 0$, otherwise $E = \emptyset$ and any $s \in S \setminus L$ will do (by Lemma 3.5, S cannot be contained in L). We can clearly decrease δ as to have $0 < \delta < \mu(K)$.

Let $E_0 \subset K$, $s_0 \in S \cap L$ and $\lambda_0 \in \Gamma \setminus \{0\}$ satisfy the assumptions of the theorem with $\varepsilon, \delta > 0$. Set

$$t_0 := \begin{cases} s_0 & \text{if } |\lambda_0| \geqslant 1 \\ s_0^{-1} & \text{if } |\lambda_0| < 1 \end{cases}.$$

Then

$$\operatorname*{ess\,sup}_{t_0 E_0} \omega < \varepsilon$$

and $\mu(E_0) > \mu(K) - \delta = \delta_1 > 0$. There is next a compact subset $K_1 \subset E_0$ such that $\mu(E_0 \setminus K_1) < \dfrac{\delta_1}{9}$ and $\varepsilon_1 := \operatorname*{ess\,inf}_{t_0 K_1} \omega > 0$.

By induction, we can choose for $n \geqslant 0$ sequences E_n, K_n of compact sets, $s_n \in S \cap L$, $\lambda_n \in \Gamma \setminus \{0\}$, $\varepsilon_n > 0$ (with $K_0 := K$ and $\varepsilon_0 := \varepsilon$) as follows. For a given $n \geqslant 1$, one chooses first E_n, λ_n, s_n such that

$$E_n \subset K_n, \qquad \mu(K_n \setminus E_n) < \frac{\delta_1}{9^n}, \qquad |\lambda_n| \operatorname*{ess\,sup}_{s_n E_n} \omega < \varepsilon_n, \qquad \frac{1}{|\lambda_n|} \operatorname*{ess\,sup}_{s_n^{-1} E_n} \omega < \varepsilon_n.$$

Setting

$$t_n := \begin{cases} s_n & \text{if } |\lambda_n| \geqslant 1 \\ s_n^{-1} & \text{if } |\lambda_n| < 1 \end{cases},$$

we have then

$$\operatorname*{ess\,sup}_{t_n E_n} \omega < \varepsilon_n. \tag{7}$$

Now we choose K_{n+1} so that

$$K_{n+1} \subset E_n, \qquad \mu(E_n \setminus K_{n+1}) < \frac{\delta_1}{9^{n+1}}, \qquad \varepsilon_{n+1} := \operatorname*{ess\,inf}_{t_n K_{n+1}} \omega > 0. \tag{8}$$

Setting $D_n = t_n E_n$, as in the proof of [1, Theorem 8], we first note that

$$\mu(D_n) = \mu(E_n) > \mu(K_n) - \frac{\delta_1}{9^n} > \mu(E_{n-1}) - \frac{2\delta_1}{9^n},$$

so that by induction

$$\mu(D_n) > \mu(E_0) - \sum_{k=1}^{n} \frac{2\delta_1}{9^k} > \frac{3}{4}\,\delta_1.$$

Next, the estimates (7) and (8) imply that $\mu(t_n K_{n+1} \cap t_k E_k) = 0$ for any $k > n$, so that up to a null set,

$$D_n \cap D_k = t_n E_n \cap t_k E_k \subset t_n E_n \setminus (t_n K_{n+1}),$$

and

$$\mu\Big(\bigcup_{k>n}(D_n \cap D_k)\Big) \leqslant \mu\big(t_nE_n \setminus (t_nK_{n+1})\big) < \frac{\delta_1}{9^{n+1}}.$$

It follows that

$$\mu\Big(\bigcup_{k<n}(D_n \cap D_k)\Big) \leqslant \sum_{k<n} \frac{\delta_1}{9^{k+1}} < \frac{\delta_1}{8},$$

and as a consequence

$$\mu\Big(\bigcup_{n\leqslant N} D_n\Big) \geqslant \sum_{n=1}^{N} \mu\Big(D_n \setminus \bigcup_{k<n} D_k\Big) \geqslant \sum_{n=1}^{N}\Big[\mu(D_n) - \mu\Big(\bigcup_{k<n}(D_n \cap D_k)\Big)\Big]$$
$$> \sum_{n=1}^{N}[\frac{3}{4}\,\delta_1 - \frac{1}{8}\,\delta_1] = \frac{5N}{8}\,\delta_1.$$

But now $\mu(LE_0 \cup L^{-1}E_0) \geqslant \mu(\cup_{k=1}^{\infty} D_k) = +\infty$, which is impossible since $LE_0 \cup L^{-1}E_0$ is compact. □

Proof of Theorem 5.1. For $K \subset G$, denote $\operatorname{ess\,sup}_K \omega$ by $\|\omega\|_{\infty,K}$. Let us show that condition (2) of Theorem A holds. Let $(F_n)_{n\geqslant 1}$ be an increasing sequence of compact sets of positive measure and let $(\delta_n)_{n\geqslant 1}$ be a sequence of positive numbers. We can assume that $\delta_1 < \frac{1}{2}$, $\delta_n < \mu(F_n)$ and $\delta_{n+1} < \frac{\delta_n}{2}$ for every $n \geqslant 1$. Set $F_0 = \emptyset$, $s_0 = e$ and $\lambda_0 = 1$. By induction, we choose sequences $(s_n)_{n\geqslant 1} \subset S$ and $(\lambda_n)_{n\geqslant 1} \subset \Gamma \setminus \{0\}$ as follows. If s_k, λ_k are chosen for $k = 0, \dots, n-1$, set

$$E_n := (\bigcup_{k<n} s_kF_n) \cup (\bigcup_{k<n} F_k \cup s_k^{-1}F_k) \quad \text{and} \quad C_n := \max_{k<n} \|T_{s_k}\| \geqslant 1.$$

Choose $\varepsilon_n > 0$ so that $\varepsilon_n^p\, \mu(E_n) \max(n, C_n^p) < 2^{-n}$. There exist a compact set $E_n' \subset E_n$ and $s_n \in S$, $\lambda_n \in \Gamma$ such that $\mu(E_n \setminus E_n') < \frac{\delta_n}{2}$,

$$|\lambda_n|\|\omega\|_{\infty, s_nE_n'} < \varepsilon_n \min_{k<n} |\lambda_k| \quad \text{and} \quad \frac{1}{|\lambda_n|}\|\omega\|_{\infty, s_n^{-1}E_n'} < \frac{\varepsilon_n}{\max_{k<n}|\lambda_k|}.$$

Moreover, by Lemma 5.2 we can choose s_n such that $s_n^{-1}F_n \cap s_k^{-1}F_k = \emptyset$ for every $k < n$. Set now

$$K_0 = \emptyset \quad \text{and} \quad K_n = F_n \cap E_n' \cap s_n(\cap_{k>n} E_k') \quad \text{for every } n \geqslant 1.$$

For fixed $n \geqslant 1$, we have $F_n \subset E_n$ and for every $k > n$, $F_n \subset s_nE_k$, thus

$$\mu(F_n \setminus K_n) \leqslant \mu(E_n \setminus E_n') + \mu(\cup_{k>n} s_nE_k \setminus s_nE_k')$$
$$\leqslant \sum_{k\geqslant n} \mu(E_k \setminus E_k') < \sum_{k\geqslant n} \frac{\delta_k}{2} < \sum_{k\geqslant n} \frac{\delta_n}{2^{k-n+1}} = \delta_n.$$

Since the sets $s_n^{-1}F_n$ are pairwise disjoint and $s_k^{-1}K_k \subset E'_n$ for every $k < n$, we have

$$\begin{aligned}
\sum_{n,k\geqslant 0; n\neq k} \frac{|\lambda_n|^p}{|\lambda_k|^p}\|\omega\|^p_{p,s_ns_k^{-1}K_k} &\leqslant \sum_n \Big(\sum_{k<n} \frac{|\lambda_n|^p}{|\lambda_k|^p}\|\omega\|^p_{p,s_ns_k^{-1}K_k} + \sum_{k>n} \|T_{s_n}\|^p \frac{|\lambda_n|^p}{|\lambda_k|^p}\|\omega\|^p_{p,s_k^{-1}K_k}\Big) \\
&\leqslant \sum_n \Big(\frac{n\,|\lambda_n|^p}{\min_{k<n}|\lambda_k|^p}\|\omega\|^p_{p,s_nE'_n} + \sum_{k>n} \|T_{s_n}\|^p \frac{|\lambda_n|^p}{|\lambda_k|^p}\|\omega\|^p_{p,s_k^{-1}E'_k}\Big) \\
&< \sum_n \Big(\frac{n\,|\lambda_n|^p}{\min_{k<n}|\lambda_k|^p}\|\omega\|^p_{\infty,s_nE'_n}\mu(E'_n) + \sum_{k>n} C_k^p \frac{|\lambda_n|^p}{|\lambda_k|^p}\|\omega\|^p_{\infty,s_k^{-1}E'_k}\mu(E'_k)\Big) \\
&\leqslant \sum_n \Big(n\,\varepsilon_n^p\,\mu(E'_n) + \sum_{k>n} C_k^p\,\varepsilon_k^p\,\mu(E'_k)\Big) \\
&< \sum_n \Big(\frac{1}{2^n} + \sum_{k>n}\frac{1}{2^k}\Big) < +\infty.
\end{aligned}$$

□

Corollary 5.3. *Let $(G, S, p, \omega, \Gamma)$ be admissible. If for every compact subset K of G there exists $\lambda \in \Gamma \setminus \{0\}$ such that*

$$\inf_{s\in S} \max\left\{ |\lambda| \operatorname*{ess\,sup}_{sK} \omega\,;\ \frac{1}{|\lambda|}\operatorname*{ess\,sup}_{s^{-1}K}\omega \right\} = 0,$$

then $L^p(G, \omega)$ has a (Γ, S)-dense vector.

If we know that all translations are bounded, on the left and on the right, then we get the following proposition. It implies Proposition 1.4 when G is abelian.

Proposition 5.4. *Suppose that $(G, S, p, \omega, \Gamma)$ is an admissible tuple. Let ω be a continuous weight such that for every $s \in G$, we have*

$$M(s) := \sup_{t\in G}\frac{\omega(st)}{\omega(t)} < +\infty \quad \text{and} \quad R(s) := \sup_{t\in G}\frac{\omega(ts)}{\omega(t)} < +\infty.$$

Then the following conditions are equivalent:

1. *There exists a (Γ, S)-dense vector in $L^p(G, \omega)$.*
2. *For every compact set $K \subset G$, there exist sequences $(s_n)_{n\geqslant 1} \subset S$ and $(\lambda_n)_{n\geqslant 1} \subset \Gamma \setminus \{0\}$ such that*
$$\lim_{n\to+\infty} |\lambda_n| \sup_{t\in K}\omega(s_nt) = 0 \quad \text{and} \quad \lim_{n\to+\infty}\frac{1}{|\lambda_n|}\sup_{t\in K}\omega(s_n^{-1}t) = 0.$$
3. *There exist sequences $(s_n)_{n\geqslant 1} \subset S$ and $(\lambda_n)_{n\geqslant 1} \subset \Gamma \setminus \{0\}$ such that*
$$\lim_{n\to+\infty} |\lambda_n|\,\omega(s_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty}\frac{1}{|\lambda_n|}\,\omega(s_n^{-1}) = 0.$$

Proof. Note that M or R being finite implies already that ω can be chosen continuous [8, Theorem 2.7], so that we can assume it from the beginning. It is clear that (2) ⇒ (3). By Theorem 5.1, we have (2) ⇒ (1). Let us show first that (3) ⇒ (2). Fix a compact set $K \subset G$. By [7, Proposition 1.16], we know that M and R are locally bounded. It is easy to see that for $s \in S$ and $\lambda \in \Gamma \setminus \{0\}$,

$$\max\{|\lambda| \sup_{t\in K}\,\omega(st)\,;\ \frac{1}{|\lambda|}\sup_{t\in K}\,\omega(s^{-1}t)\} \leqslant \sup_{t\in K} R(t)\,\max\{|\lambda|\omega(s)\,;\ \frac{1}{|\lambda|}\omega(s^{-1})\},$$

thus (2) and (3) are equivalent. Let us show now that (1) $\Rightarrow$ (3). Let $F \subset G$ be a compact set of nonzero measure. By applying condition (2) of Theorem A with $F_n = F$ and $\delta_n = \dfrac{\mu(F)}{4}$, we get sequences $(s_n)_{n\geqslant 1} \subset S$, $(\lambda_n)_{n\geqslant 1} \subset \Gamma \setminus \{0\}$ and a sequence of compact sets $K_n \subset F$ such that $\mu(F \setminus K_n) < \delta_n$ and

$$\lim_{n\to+\infty} |\lambda_n| \|\omega\|_{p,s_n s_1^{-1}K_1} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \frac{1}{|\lambda_n|} \|\omega\|_{p,s_1 s_n^{-1}K_n} = 0. \tag{9}$$

In particular, for every $n \geqslant 1$ we have $\mu(K_n \cap K_1) > \dfrac{\mu(F)}{2}$. Let $C > 0$ be such that $M(s_1^{-1}) \leqslant C$ and $R|_{K_1^{-1}\cup K_1^{-1}s_1} \leqslant C$. Then

$$\omega(s_n) = \inf_{t\in s_1^{-1}K_1} \omega(s_n t t^{-1}) \leqslant C \inf_{t\in s_1^{-1}K_1} \omega(s_n t) \leqslant C\|\omega\|_{p,s_n s_1^{-1}K_1} \mu(K_1)^{-1/p}$$

and

$$\begin{aligned}\omega(s_n^{-1}) &= \inf_{t\in K_1\cap K_n} \omega(s_1^{-1} s_1 s_n^{-1} t t^{-1}) \leqslant C^2 \inf_{t\in K_1\cap K_n} \omega(s_1 s_n^{-1} t) \\ &\leqslant C^2 \|\omega\|_{p,s_1 s_n^{-1}K_n} \mu(K_1\cap K_n)^{-1/p} < 2C^2 \mu(F)^{-1/p} \|\omega\|_{p,s_1 s_n^{-1}K_n}.\end{aligned}$$

Combining the last inequalities with (9), we obtain

$$\max\{|\lambda_n|\omega(s_n)\,;\, \frac{1}{|\lambda_n|}\omega(s_n^{-1})\} \underset{n\to+\infty}{\longrightarrow} 0.$$

□

We can now easily obtain the following corollary, which provides for several types of Γ a complete characterization of (Γ, S)-density only in terms of the weight.

Corollary 5.5. *Suppose that $(G, S, p, \omega, \Gamma)$ is an admissible tuple. If ω is a continuous weight such that for every $s \in G$,*

$$M(s) := \sup_{t\in G} \frac{\omega(st)}{\omega(t)} < +\infty \quad \text{and} \quad R(s) := \sup_{t\in G} \frac{\omega(ts)}{\omega(t)} < +\infty,$$

then the following conditions hold:

1. *If $\Gamma \setminus \{0\}$ is bounded and bounded away from zero, there is a (Γ, S)-dense vector in $L^p(G, \omega)$ if and only if*
$$\inf_{s\in S} \max\{\omega(s); \omega(s^{-1})\} = 0.$$
2. *There is a $([0, 1], S)$-dense vector in $L^p(G, \omega)$ if and only if*
$$\inf_{s\in S} \max\{\omega(s)\omega(s^{-1}); \omega(s^{-1})\} = 0.$$
3. *There is a $([1, +\infty[, S)$-dense vector in $L^p(G, \omega)$ if and only if*
$$\inf_{s\in S} \max\{\omega(s)\omega(s^{-1}); \omega(s)\} = 0.$$
4. *There is a $(\mathbb{C}, S)$-dense vector in $L^p(G, \omega)$ if and only if $\inf_{s\in S} \omega(s)\,\omega(s^{-1}) = 0$.*

Proof. In view of Proposition 5.4, the case (1) is easy to check. Suppose that the condition on ω in (2) holds. For every $k \geqslant 1$ there exists $s_k \in S$ such that

$$\omega(s_k)\omega(s_k^{-1}) \leqslant \frac{1}{k^2} \quad \text{and} \quad \omega(s_k^{-1}) \leqslant \frac{1}{k^2}.$$

If we set $\lambda_k = k\,\omega(s_k^{-1})$ then $0 \leqslant \lambda_k \leqslant \frac{1}{k} \leqslant 1$ and

$$\lambda_k\omega(s_k) = k\,\omega(s_k)\omega(s_k^{-1}) \leqslant \frac{1}{k}, \qquad \frac{1}{\lambda_k}\omega(s_k^{-1}) = \frac{1}{k}.$$

Thanks to Proposition 5.4, $L^p(G,\omega)$ has a $([0,1],S)$-dense vector. The converse follows easily from Proposition 5.4. The proof of (3) is similar. For (4), by Proposition 5.4, the condition on ω is necessary. Assume now that $\inf_{s\in S}\omega(s)\,\omega(s^{-1}) = 0$. For every $k \geqslant 1$ there exists $s_k \in S$ such that

$$\omega(s_k)\omega(s_k^{-1}) \leqslant \frac{1}{k^2}.$$

Set $\lambda_k = \omega(s_k^{-1})^{1/2}\omega(s_k)^{-1/2}$, then

$$\lambda_k\omega(s_k) = \omega(s_k^{-1})^{1/2}\omega(s_k)^{1/2} \leqslant \frac{1}{k} \underset{k\to\infty}{\longrightarrow} 0$$

and

$$\frac{1}{\lambda_k}\omega(s_k^{-1}) = \omega(s_k^{-1})^{1/2}\omega(s_k)^{1/2} \leqslant \frac{1}{k} \underset{k\to\infty}{\longrightarrow} 0,$$

so that by Proposition 5.4, $L^p(G,\omega)$ has a $(\mathbb{C},S)$-dense vector. □

The following example shows that we do need all translations bounded to have the equivalence (4) in Corollary 5.5. The following weight is S-admissible, but not G-admissible.

Example 5.6. Let $G = \mathbb{Z}$, $S = \mathbb{Z}\setminus\mathbb{N}$ and let ω be the weight defined on G by

$$\omega_n = \begin{cases} 2^{2^n} & \text{if } n \geqslant 0 \\ 2^{-2^{-n+1}} & \text{if } n < 0 \end{cases}.$$

Then, we have

$$\inf_{n\in S}\omega_n\omega_{-n} = 0 \quad \text{and} \quad \sup_{k\in\mathbb{Z}}\frac{\omega_{n+k}}{\omega_k} < +\infty \quad \text{for all } n \in S,$$

so that the condition (4) of Corollary 5.5 holds. But $L^p(G,\omega)$ doesn't have a $(\mathbb{C},S)$-dense vector, since $\liminf_{n\to+\infty}\omega_{n+1}\omega_{-n+1} \neq 0$ (see [2, Theorem 1.38]).

6. Commutative subgroups

We keep the same assumptions on G, S, p, Γ, ω as before, see Definition 4.1. If the subgroup generated by S is abelian, Theorem B below shows that the converse implication of Theorem 5.1 is also true. For convenience, we will recall Theorem B with an additional equivalent condition.

Theorem B. *Suppose that (G,S,p,ω,Γ) is admissible. If the **subgroup generated by S is abelian**, then the following conditions are equivalent:*

(1) *There is a (Γ,S)-dense vector in $L^p(G,\omega)$.*

(2) *For any compact subset $F \subset G$ and $\varepsilon > 0$, there are $s \in S$, $\lambda \in \Gamma \setminus \{0\}$ and a compact subset $E \subset F$ such that $\mu(F \setminus E) < \varepsilon$ and*

$$|\lambda| \operatorname*{ess\,sup}_{t\in E} \omega(st) < \varepsilon \quad \text{and} \quad \frac{1}{|\lambda|} \operatorname*{ess\,sup}_{t\in E} \omega(s^{-1}t) < \varepsilon.$$

(3) *For any compact subset $F \subset G$ and $\varepsilon > 0$, there are $s \in S$, $\lambda \in \Gamma \setminus \{0\}$ and a compact subset $E \subset F$ such that $\mu(F \setminus E) < \varepsilon$ and*

$$|\lambda| \, \|\omega\|_{p,sE} < \varepsilon \quad \text{and} \quad \frac{1}{|\lambda|} \|\omega\|_{p,s^{-1}E} < \varepsilon. \tag{10}$$

Proof of Theorem B. According to Theorem 5.1, (2) $\Rightarrow$ (1) holds for any set S and S-admissible weight.

Let us show that (1) $\Rightarrow$ (3). Let F be a compact subset of G and $\varepsilon > 0$. It is clear that we can assume that F has a positive measure. For every $n \geqslant 1$, set $F_n = F$ and $\delta_n = 2^{-n}$. By Theorem A, there are $(s_n)_{n\geqslant 1} \subset S$, $(\lambda_n)_{n\geqslant 1} \subset \Gamma \setminus \{0\}$ and $E_n \subset F$ compact such that $\mu(F \setminus E_n) < \delta_n$ and

$$\sum_{n,k\geqslant 0; n\neq k} \frac{|\lambda_n|^p}{|\lambda_k|^p} \|\omega\|^p_{p,s_n s_k^{-1} E_k} < +\infty$$

with $s_0 = e$, $\lambda_0 = 1$ and $E_0 = \emptyset$. In particular,

$$\lim_{k\to+\infty} \frac{1}{|\lambda_k|^p} \|\omega\|^p_{p,s_k^{-1}E_k} = 0. \tag{11}$$

Let us fix $k \geqslant 1$ such that $\delta_k < \dfrac{\varepsilon}{2}$ and set $C := \|T_{s_k}\|$. We have

$$\|\omega\|^p_{p,s_n E_k} = \int_{s_k s_n s_k^{-1} E_k} \omega(t)^p \mathrm{d}\mu(t) \leqslant C^p \|\omega\|^p_{p,s_n s_k^{-1} E_k}$$

Since $|\lambda_n| \|\omega\|_{p,s_n s_k^{-1} E_k} \underset{n\to+\infty}{\longrightarrow} 0$, it follows

$$|\lambda_n| \|\omega\|_{p,s_n E_k} \underset{n\to+\infty}{\longrightarrow} 0. \tag{12}$$

By (11) and (12) there exists $n > k$ such that

$$\frac{1}{|\lambda_n|} \|\omega\|_{p,s_n^{-1}E_n} < \varepsilon \quad \text{and} \quad |\lambda_n| \|\omega\|_{p,s_n E_k} < \varepsilon.$$

If we set $E = E_k \cap E_n$, $s = s_n$ and $\lambda = \lambda_n$, we have $\mu(F\setminus E) \leqslant \mu(F\setminus E_k) + \mu(F\setminus E_n) < \delta_k + \delta_n \leqslant 2\delta_k < \varepsilon$, and

$$|\lambda| \, \|\omega\|_{p,sE} < \varepsilon \quad \text{and} \quad \frac{1}{|\lambda|} \|\omega\|_{p,s^{-1}E} < \varepsilon.$$

Let us show now that (3) $\Rightarrow$ (2). Let $F \subset G$ be a compact subset of G and $\varepsilon > 0$. Let $\eta > 0$ be such that $4\eta < \varepsilon$. There exist $s \in S$, $\lambda \in \Gamma \setminus \{0\}$ and a compact subset $K \subset F$ such that $\mu(F \setminus K) < \eta$ and

$$|\lambda| \|\omega\|_{p,sK} < \eta^{1+\frac{1}{p}}, \qquad \frac{1}{|\lambda|} \|\omega\|_{p,s^{-1}K} < \eta^{1+\frac{1}{p}}.$$

Set

$$E_1 = \{t \in K : \omega(st) \leqslant \frac{\eta}{|\lambda|}\}, \qquad E_2 = \{t \in K : \omega(s^{-1}t) \leqslant \eta|\lambda|\},$$

then

$$\|\omega\|_{p,sK}^p \geqslant \frac{\eta^p}{|\lambda|^p}\mu(K\setminus E_1) \quad \text{and} \quad \|\omega\|_{p,s^{-1}K}^p \geqslant \eta^p|\lambda|^p\mu(K\setminus E_2).$$

Hence $\mu(K\setminus E_1)<\eta$ and $\mu(K\setminus E_2)<\eta$, so that

$$\mu(K\setminus(E_1\cap E_2))<2\eta.$$

Let now $E\subset E_1\cap E_2$ be a compact subset such that $\mu((E_1\cap E_2)\setminus E)<\eta$. Then $\mu(F\setminus E)<4\eta<\varepsilon$, and

$$|\lambda|\operatorname*{ess\,sup}_{t\in E}\omega(st)\leqslant\eta<\varepsilon, \qquad \frac{1}{|\lambda|}\operatorname*{ess\,sup}_{t\in E}\omega(s^{-1}t)\leqslant\eta<\varepsilon.$$

□

Remark 6.1. The condition (2) is independent of p, which means that it applies to any p. We have however to make sure that the usual condition $\omega\in L^p_{loc}(G)$ is satisfied, and this one does depend on p.

For special types of Γ, one can characterize (Γ,S)-density only in terms of the weight, similarly to Corollary 5.5.

Corollary 6.2. *Suppose that (G,S,p,ω,Γ) is an admissible tuple. If the* ***subgroup generated by*** S ***is abelian****, then:*

(1) *There is an S-dense vector in $L^p(G,\omega)$ if and only if for any compact subset $F\subset G$ and $\varepsilon>0$, there are $s\in S$ and a compact subset $E\subset F$ such that $\mu(F\setminus E)<\varepsilon$ and*

$$\operatorname*{ess\,sup}_{t\in E}\omega(st)<\varepsilon, \qquad \operatorname*{ess\,sup}_{t\in E}\omega(s^{-1}t)<\varepsilon.$$

(2) *There is a $([0,1],S)$-dense vector in $L^p(G,\omega)$ if and only if for any compact subset $F\subset G$ and $\varepsilon>0$, there are $s\in S$ and a compact subset $E\subset F$ such that $\mu(F\setminus E)<\varepsilon$ and*

$$\operatorname*{ess\,sup}_{t\in E}\omega(s^{-1}t)<\varepsilon, \qquad \operatorname*{ess\,sup}_{t\in E}\omega(st)\operatorname*{ess\,sup}_{t\in E}\omega(s^{-1}t)<\varepsilon.$$

(3) *There is a $([1,+\infty[,S)$-dense vector in $L^p(G,\omega)$ if and only if for any compact subset $F\subset G$ and $\varepsilon>0$, there are $s\in S$ and a compact subset $E\subset F$ such that $\mu(F\setminus E)<\varepsilon$ and*

$$\operatorname*{ess\,sup}_{t\in E}\omega(st)<\varepsilon, \qquad \operatorname*{ess\,sup}_{t\in E}\omega(st)\operatorname*{ess\,sup}_{t\in E}\omega(s^{-1}t)<\varepsilon.$$

(4) *There is a $(\mathbb{C},S)$-dense vector in $L^p(G,\omega)$ if and only if for any compact subset $F\subset G$ and $\varepsilon>0$, there are $s\in S$ and a compact subset $E\subset F$ such that $\mu(F\setminus E)<\varepsilon$ and*

$$\operatorname*{ess\,sup}_{t\in E}\omega(st)\operatorname*{ess\,sup}_{t\in E}\omega(s^{-1}t)<\varepsilon.$$

Proof. The assertion (1) is a part of [1, Theorem 10]. In (2), if there is a $([0,1],S)$-dense vector, then, as $1/|\lambda|\geqslant 1$, the condition on ω follows from (2) of Theorem B. Let us prove the converse. Suppose that the condition on ω in (2) holds; we will check the condition (2) from Theorem B. Let $F\subset G$ be a compact set. If $\mu(F)=0$, the inequalities in (2) of Theorem B hold with $E=\emptyset$ and any s and λ. We can assume therefore that $\mu(F) > 0$. For every $k \geqslant 1$ there exist $s_k \in S$ and a compact subset $E_k \subset F$ such that $\mu(F \setminus E_k) \leqslant \frac{1}{k}$

$$\operatorname*{ess\,sup}_{t\in E_k} \omega(s_k^{-1}t) < \frac{1}{k^2} \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in E_k} \omega(s_k t) \operatorname*{ess\,sup}_{t\in E_k} \omega(s_k^{-1}t) < \frac{1}{k^2}.$$

Set $c_k = \operatorname*{ess\,sup}_{t\in E_k} \omega(s_k t)$, $d_k = \operatorname*{ess\,sup}_{t\in E_k} \omega(s_k^{-1}t)$. For k big enough, the measure of E_k is positive so that $c_k > 0$ and $d_k > 0$. Set $\lambda_k = k d_k$ (If $d_k = 0$, which may happen for a finite number of k, we set $\lambda_k = 1$). Thus for each k, we have $\lambda_k \in (0, 1]$,

$$\lambda_k c_k \underset{k\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \frac{1}{\lambda_k} d_k \underset{k\to\infty}{\longrightarrow} 0.$$

Thanks to the assertion (2) of Theorem B, $L^p(G, \omega)$ has a $([0, 1], S)$-dense vector. The proof of (3) is similar. For (4), the direct implication follows immediately from (2) of Theorem B. Let us prove that the condition on ω in (4) implies that $L^p(G, \omega)$ has a $(\mathbb{C}, S)$-dense vector. We will show that the assertion (2) of Theorem B holds. Let $F \subset G$ be a compact set, and as in (2), we can assume that $\mu(F) > 0$. For every $k \geqslant 1$ there exist $s_k \in S$ and a compact subset $E_k \subset F$ such that $\mu(F \setminus E_k) \leqslant \dfrac{1}{k}$ and

$$\operatorname*{ess\,sup}_{t\in E_k} \omega(s_k t) \operatorname*{ess\,sup}_{t\in E_k} \omega(s_k^{-1}t) < \frac{1}{k}.$$

Set $c_k = \operatorname*{ess\,sup}_{t\in E_k} \omega(s_k t)$, $d_k = \operatorname*{ess\,sup}_{t\in E_k} \omega(s_k^{-1}t)$ and $\lambda_k = d_k^{1/2} c_k^{-1/2}$ (or $\lambda_k = 1$ if $c_k = 0$; as in (2), this may happen only for a finite number of k). Now

$$\lambda_k c_k \underset{k\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \frac{1}{\lambda_k} d_k \underset{k\to\infty}{\longrightarrow} 0.$$

Thanks to the assertion (2) of Theorem B, $L^p(G, \omega)$ has a $(\mathbb{C}, S)$-dense vector. □

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