On Initials and the Fundamental Theorem of Tropical Partial Differential Geometry

Sebastian Falkensteiner, Cristhian Emmanuel Garay-Lopez, Mercedes Haiech, Marc Paul Noordman, François Boulier, Zeinab Toghani

On Initials and the Fundamental Theorem of Tropical Partial Differential Algebraic Geometry

Sebastian Falkensteiner Cristhian Garay-López Mercedes Haiech
Marc Paul Noordman François Boulier Zeinab Toghani

January 23, 2021

Abstract

Tropical Differential Algebraic Geometry considers difficult or even intractable problems in Differential Equations and tries to extract information on their solutions from a restricted structure of the input. The Fundamental Theorem of Tropical Differential Algebraic Geometry and its extensions state that the support of power series solutions of systems of ordinary differential equations (with formal power series coefficients over an uncountable algebraically closed field of characteristic zero) can be obtained either, by solving a so-called tropicalized differential system, or by testing monomial-freeness of the associated initial ideals. Tropicalized differential equations work on a completely different algebraic structure which may help in theoretical and computational questions.

We show here that both of these methods can be generalized to the case of systems of partial differential equations, this is, one can go either with the solution of tropicalized systems, or test monomial-freeness of the ideal generated by the initials when looking for supports of power series solutions of systems of differential equations, regardless the (finite) number of derivatives. The key are the vertex sets of Newton polytopes, upon which relies the definition of both tropical vanishing condition and the initial of a differential polynomial.

1 Introduction

Given an algebraically closed field of characteristic zero K, we consider the partial differential ring $(R_{m,n}, D)$, where

$$R_{m,n} = K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$$

and $D = (\frac{\partial}{\partial t_k} \, : \, k = 1, \ldots, m)$ for $n, m \geq 1$ (see Section 2 for definitions). Up to now, tropical differential algebra has been limited to the study of the relation between the set of solutions $\mathrm{Sol}(G) \subseteq K[[t]]^n$ of differential ideals G in $R_{1,n}$ and their corresponding *tropicalizations*, which are certain polynomials p with coefficients in a tropical semiring $\mathbb{T}_1 = (\mathbb{Z}_{\geq 0} \cup \{\infty\}, +, \min)$ with a set of solutions $\mathrm{Sol}(p) \subseteq \mathcal{P}(\mathbb{Z}_{\geq 0})^n$, or with the additional characterisation of the differential version of the fundamental theorem in terms of monomial containment of tropical initial ideals. See [9], [1] and [10].

In sum, the solutions $S \in \mathrm{Sol}(p)$ can be found by looking at the evaluations $p(S) \in \mathbb{T}_1$ where the usual tropical vanishing condition holds, or, by checking the monomial-freeness of a certain tropical initial ideal [10].

In this paper, which is an extended version of [6], we consider the case $m > 1$ of these two methods. On this account, we work with elements in $\mathbb{Z}_{\geq 0}^m$, which requires new techniques. We show that considering the Newton polytopes and their vertex sets is the appropriate method for

2010 Mathematics Subject Classification. 13P15, 13N99, 14T99, 52B20.
Keywords and phrases. Differential Algebra, Tropical Differential Algebraic Geometry, Power Series Solutions, Newton Polyhedra, Arc Spaces, Tropical Differential Equations, Initial forms of Differential Polynomials.

formulating and proving our generalization of the Fundamental Theorem of Tropical Differential Algebraic Geometry. We remark that in the case of $m = 1$ the definitions and properties presented here coincide with the corresponding ones from both [1] and [10, Theorem 1, Lemma 3], therefore this work can indeed be seen as a generalization.

The problem of finding power series solutions of systems of partial differential equations has been extensively studied in the literature, but is very limited in the general case. In fact, we know from [5, Theorem 4.11] that there is already no algorithm for deciding whether a given linear partial differential equation with polynomial coefficients has a solution or not. The Fundamental Theorem, as it is stated in here, helps to find necessary conditions for the support of possible solutions.

The last two sections of this paper contain the additional material, in which we extend the notion of initial ideals of a differential ideal to the partial case ($m > 1$), and we present the corresponding extension of the Fundamental Theorem about monomial containment.

The structure of the paper is as follows. In Section 2 we cover the necessary material from partial differential algebra. In Section 3 we introduce the semiring of supports $\mathcal{P}(\mathbb{Z}_{\geq 0}^m)$, the semiring of vertex sets $\mathbb{T}_m$ and the vertex homomorphism $\text{Vert}\colon \mathcal{P}(\mathbb{Z}_{\geq 0}^m) \longrightarrow \mathbb{T}_m$. In Section 4 we introduce the support and the tropicalization maps. In Section 5 we define the set of tropical differential polynomials $\mathbb{T}_{m,n}$, the notion of tropical solutions for them, and the tropicalization morphism $\text{trop}\colon R_{m,n} \to \mathbb{T}_{m,n}$. The main result is Theorem 6.1, which is proven in Section 6. The proof we give here differs essentially from the one in [1] for the case of $m = 1$. In Section 7 we give some examples to illustrate our results. The last part of the paper extends the previous work [6]. In Section 8 we define and prove basic properties of the initial part of a differential polynomial and of the initial ideal of a differential ideal. In Section 9 we extend the Fundamental Theorem with an extra characterization in terms of these initial ideals. The main result is Theorem 9.6, in which we present in a unified form the three descriptions of the same class of objects.

In the following we will use the conventions that for a set S we denote by $\mathcal{P}(S)$ its power set, and by K we denote an algebraically closed field of characteristic zero.

2 Partial differential algebra

Here we recall the preliminaries for partial differential algebraic geometry. The reference book for differential algebra is [11].

A **partial differential ring** is a pair (R, D) consisting of a commutative ring R with unit and a set $D = \{\delta_1, \ldots, \delta_m\}$ of $m > 1$ **derivations** which act on R and are pairwise commutative. We denote by Θ the free commutative monoid generated by D. If $J = (j_1, \ldots, j_m)$ is an element of the monoid $\mathbb{Z}_{\geq 0}^m = (\mathbb{Z}_{\geq 0}^m, +, 0)$, we denote $\Theta(J) = \delta_1^{j_1} \cdots \delta_m^{j_m}$ the **derivative operator** defined by J. If φ is any element of R, then $\Theta(J)\varphi$ is the element of R obtained by application of the derivative operator $\Theta(J)$ on φ.

Let (R, D) be a partial differential ring with $R \supseteq \mathbb{Q}$ and $x_1, \ldots, x_n$ be n **differential indeterminates**. The monoid Θ acts on the differential indeterminates, giving the infinite set of the **derivatives** which are denoted by $x_{i,J}$ with $1 \leq i \leq n$ and $J \in \mathbb{Z}_{\geq 0}^m$. Given any $1 \leq k \leq m$ and any derivative $x_{i,J}$, the action of δ_k on $x_{i,J}$ is defined by $\delta_k(x_{i,J}) = x_{i,J+e_k}$ where e_k is the m-dimensional vector whose k-th coordinate is 1 and all other coordinates are zero. One denotes $R\{x_1, \ldots, x_n\}$ the ring of the polynomials, with coefficients in R, the indeterminates of which are the derivatives. More formally, $R\{x_1, \ldots, x_n\}$ consists of all R-linear combinations of differential monomials, where a differential monomial in n independent variables of order less than or equal to r is an expression of the form

$$E_M := \prod_{\substack{1 \leq i \leq n \\ ||J||_\infty \leq r}} x_{i,J}^{M_{i,J}} \tag{1}$$

where $J = (j_1, \dots, j_m) \in \mathbb{Z}^m_{\geq 0}$, $||J||_\infty := \max_i\{j_i\} = \max(J)$ and $M = (M_{i,J}) \in (\mathbb{Z}_{\geq 0})^{n\times(r+1)^m}$.

The pair $(R\{x_1, \dots, x_n\}, D)$ then constitutes a **differential polynomial ring**. A differential polynomial $P \in R\{x_1, \dots, x_n\}$ induces an evaluation map from R^n to R given by

$$P\colon R^n \to R, \quad (\varphi_1, \dots, \varphi_n) \mapsto P|_{x_{i,J}=\Theta(J)\varphi_i},$$

where $P|_{x_{i,J}=\Theta(J)\varphi_i}$ is the element of R obtained by substituting $\Theta(J)\varphi_i$ for $x_{i,J}$.

A **zero** or **solution** of $P \in R\{x_1, \dots, x_n\}$ is an n-tuple $\varphi = (\varphi_1, \dots, \varphi_n) \in R^n$ such that $P(\varphi) = 0$. An n-tuple $\varphi \in R^n$ is a solution of a system of differential polynomials $\Sigma \subseteq R\{x_1, \dots, x_n\}$ if it is a solution of every element of Σ. We denote by $\mathrm{Sol}(\Sigma)$ the solution set of the system Σ.

A **differential ideal** of $R\{x_1, \dots, x_n\}$ is an ideal of that ring which is stable under the action of Θ. A differential ideal is said to be **perfect** if it is equal to its radical. If $\Sigma \subseteq R\{x_1, \dots, x_n\}$, one denotes by $[\Sigma]$ the **differential ideal generated by** Σ and by $\{\Sigma\}$ the **perfect differential ideal generated by** Σ, which is defined as the intersection of all perfect differential ideals containing Σ.

For $m, n \geq 1$, we will denote by R_m the partial differential ring

$$(K[[t_1, \dots, t_m]], D)$$

where $D = \{\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_m}\}$, and by $R_{m,n}$ the partial differential ring $(R_m\{x_1, \dots, x_n\}, D)$. The proof of the following proposition can be found in [3].

Proposition 2.1. *For any $\Sigma \subseteq R_{m,n}$, there exists a finite subset Φ of Σ such that* $\mathrm{Sol}(\Sigma) = \mathrm{Sol}(\Phi)$.

3 The semirings of supports and vertex sets

In this part we introduce and give some properties on our main idempotent semirings, namely the semiring of supports $\mathcal{P}(\mathbb{Z}^m_{\geq 0})$, the semiring of vertex sets $\mathbb{T}_m$ and the map $\mathrm{Vert}\colon \mathcal{P}(\mathbb{Z}^m_{\geq 0}) \to \mathbb{T}_m$ which is a homomorphism of semirings.

Recall that a commutative semiring S is a tuple $(S, +, \times, 0, 1)$ such that $(S, +, 0)$ and $(S, \times, 1)$ are commutative monoids and additionally, for all $a, b, c \in S$ it holds that

1. $a \times (b + c) = a \times b + a \times c$;
2. $0 \times a = 0$.

A semiring is called **idempotent** if $a + a = a$ for all $a \in S$. A map $f\colon S_1 \longrightarrow S_2$ between semirings is a morphism if it induces morphisms at the level of monoids.

For $m \geq 1$, we denote by $\mathcal{P}(\mathbb{Z}^m_{\geq 0})$ the idempotent semiring whose elements are the subsets of $\mathbb{Z}^m_{\geq 0}$ equipped with the union $X \cup Y$ as sum and the Minkowski sum $X + Y = \{x + y : x \in X, y \in Y\}$ as product. We call it the **semiring of supports**. For $n \in \mathbb{Z}_{\geq 1}$ and $X \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$, the notation nX will indicate $\underbrace{X + \cdots + X}_{n \text{ times}}$. By convention we set $0X = \{(0, \dots, 0)\}$.

We define the **Newton polytope** $\mathcal{N}(X) \subseteq \mathbb{R}^m_{\geq 0}$ of $X \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ as the convex hull of $X + \mathbb{Z}^m_{\geq 0}$. We call $x \in X$ a **vertex** if $x \notin \mathcal{N}(X \setminus \{x\})$, and we denote by $\mathrm{Vert}\, X$ the set of vertices of X.

The following lemma shows that the subset relation between Newton polytopes are preserved in a weaker form, which will be needed later in the extension of the Fundamental Theorem. As a consequence we will obtain that Newton polytopes are equal if and only if its vertices are equal.

Lemma 3.1. *Let $S, T \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ such that $\mathcal{N}(T) \subset \mathcal{N}(S)$. Then $\mathcal{N}(T) \cap \mathrm{Vert}(S) \subset \mathrm{Vert}(T)$. Consequently, if $\mathcal{N}(T) = \mathcal{N}(S)$, then* $\mathrm{Vert}\, T = \mathrm{Vert}\, S$.

Proof. Let $s \in \operatorname{Vert} S$ and we assume that $s \in \mathcal{N}(T \setminus \{s\})$. Then there are $t_i \in T \setminus \{s\}$, $w_i \in \mathbb{Z}^m_{\geq 0}$ and positive $\lambda_i \in \mathbb{R}$ adding up to 1 such that

$$s = \sum_i \lambda_i (t_i + w_i).$$

Since $t_i \in \mathcal{N}(S)$, we can write the t_i as

$$t_i = \sum_j \mu_{i,j} (s_{i,j} + z_{i,j}),$$

where $s_{i,j} \in S$, $z_{i,j} \in \mathbb{Z}^m_{\geq 0}$ and $\mu_{i,j} \in \mathbb{R}$ are positive and adding up to 1. Thus,

$$s = \sum_{i,j} \lambda_i \mu_{i,j} (s_{i,j} + z_{i,j} + w_i) = \sum_{i,j} \lambda_i \mu_{i,j} s_{i,j} + v,$$

where v is a vector with non-negative coordinates. By excluding in the sum those summands $s_{i,j}$ which are equal to s, we obtain

$$s = cs + \sum_{\substack{i,j \\ s_{i,j} \neq s}} \lambda_i \mu_{i,j} s_{i,j} + v$$

where $c = \sum_{i,j : s_{i,j} = s} \lambda_i \mu_{i,j} \in [0, 1]$. If $c < 1$ we can solve the equation above for s to get

$$s = \sum_{\substack{i,j \\ s_{i,j} \neq s}} \frac{\lambda_i \mu_{i,j}}{1 - c} s_{i,j} + \frac{v}{1 - c}.$$

The coefficients for the $s_{i,j}$ are positive and sum to 1, so the summation in the right hand side gives an element of $\mathcal{N}(S \setminus \{s\})$. Since $\mathcal{N}(S \setminus \{s\})$ is closed under adding elements of $\mathbb{R}^m_{\geq 0}$, and the coordinates of $v/(1 - c)$ are non-negative, we then find that $s \in \mathcal{N}(S \setminus \{s\})$ in contradicting to the assumption that s is a vertex of S. If $c = 1$, then all $s_{i,j}$ are equal to s and we get $s = s + v$. Therefore, $v = 0$ and $t_i = s$ for each i, and in particular $s \in T \setminus \{s\}$, which is a contradiction. So we conclude that $s \notin \mathcal{N}(T \setminus \{s\})$ and s is a vertex of T.

For the second part of this lemma we obtain from $\mathcal{N}(T) \cap \operatorname{Vert}(S) \subset \operatorname{Vert}(T)$ and $\operatorname{Vert}(S) \subset \mathcal{N}(S) = \mathcal{N}(T)$ that $\operatorname{Vert}(S) \subset \operatorname{Vert}(T)$. The relation $\operatorname{Vert}(T) \subset \operatorname{Vert}(S)$ follows similarly and completes the proof. □

Lemma 3.2. *Let $X \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$. Then $\mathcal{N}(\operatorname{Vert} X) = \mathcal{N}(X)$.*

Proof. By Dickson's lemma [4, chap. 2, Thm 5], there is a finite subset $S \subseteq X$ with $X \subseteq S + \mathbb{Z}^m_{\geq 0}$. For such S, it holds that $\mathcal{N}(X) = \mathcal{N}(S)$ and by Lemma 3.1, we get $\operatorname{Vert} X = \operatorname{Vert} S$. Therefore, replacing X by S, we may assume that X is finite.

We proceed by induction on $\#X$. Indeed, if $X = \emptyset$, the statement is obvious. Let X be an arbitrary finite set. If every element of X is a vertex of X, then $\mathcal{N}(X) = \mathcal{N}(\operatorname{Vert} X)$ is trivially true. Else, take $x \in X \setminus \operatorname{Vert} X$ and let $Y = X \setminus \{x\}$. Then $\mathcal{N}(X) = \mathcal{N}(Y)$ by definition, so applying Lemma 3.1 again we obtain $\operatorname{Vert} X = \operatorname{Vert} Y$. Since $\#Y < \#X$, we may apply the induction hypothesis to Y, and get that $\mathcal{N}(X) = \mathcal{N}(Y) = \mathcal{N}(\operatorname{Vert} Y) = \mathcal{N}(\operatorname{Vert} X)$. □

Corollary 3.3. *For $X, Y \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ we have $\operatorname{Vert} X = \operatorname{Vert} Y$ if and only if $\mathcal{N}(X) = \mathcal{N}(Y)$.*

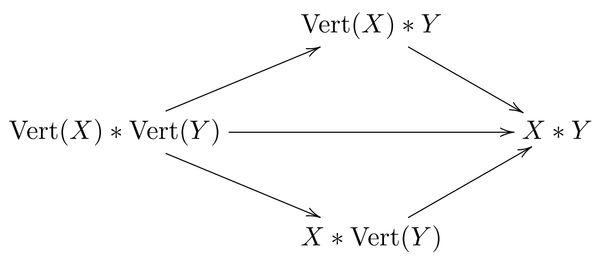
Lemma 3.4. *For $X, Y \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$, we have*

$$\operatorname{Vert}(\operatorname{Vert}(X) \cup \operatorname{Vert}(Y)) = \operatorname{Vert}(\operatorname{Vert}(X) \cup Y) = \operatorname{Vert}(X \cup \operatorname{Vert}(Y)) = \operatorname{Vert}(X \cup Y)$$

and

$$\operatorname{Vert}(\operatorname{Vert}(X) + \operatorname{Vert}(Y)) = \operatorname{Vert}(\operatorname{Vert}(X) + Y) = \operatorname{Vert}(X + \operatorname{Vert}(Y)) = \operatorname{Vert}(X + Y).$$

Proof. Let $*$ be either $\cup$ or $+$. We have the following diagram of inclusions



We show that these four sets generate the same Newton polytope. For this, it is enough to show that $X * Y \subseteq \mathcal{N}(\mathrm{Vert}(X) * \mathrm{Vert}(Y))$.

For $* = \cup$, we have $X \subseteq \mathcal{N}(\mathrm{Vert}\, X) \subseteq \mathcal{N}(\mathrm{Vert}(X) \cup \mathrm{Vert}(Y))$ and similarly $Y \subseteq \mathcal{N}(\mathrm{Vert}(X) \cup \mathrm{Vert}(Y))$. Hence, $X \cup Y \subseteq \mathcal{N}(\mathrm{Vert}(X) \cup \mathrm{Vert}(Y))$

Now suppose that $* = +$. Let $t \in X + Y$, and write $t = x + y$ with $x \in X$ and $y \in Y$. Using the inclusions $X \subseteq \mathcal{N}(\mathrm{Vert}\, X)$ and $Y \subseteq \mathcal{N}(\mathrm{Vert}\, Y)$, there are $x_i \in \mathrm{Vert}(X)$, $y_j \in \mathrm{Vert}(Y)$, $u_i, v_j \in \mathbb{Z}^m_{\geq 0}$ and $\alpha_i, \beta_j \in \mathbb{R}_{\geq 0}$ satisfying $\sum_i \alpha_i = 1$ and $\sum_j \beta_j = 1$ such that

$$t = \sum_i \alpha_i (x_i + u_i) + \sum_j \beta_j (y_j + v_j).$$

Rewriting this gives

$$t = \sum_{i,j} \alpha_i \beta_j (x_i + y_j + u_i + v_j).$$

For each pair i, j, the expression between parentheses is an element of $\mathrm{Vert}(X) + \mathrm{Vert}(Y) + \mathbb{Z}_{\geq 0}$ and the coefficients are non-negative and sum up to 1. This shows that $t \in \mathcal{N}(\mathrm{Vert}(X) + \mathrm{Vert}(Y))$, which ends the proof of the inclusions. ☐

The next result shall be needed later.

Lemma 3.5. *Let $X, Y \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$, and let $t \in \mathrm{Vert}(X + Y)$. Then there is a unique $x \in X$ and unique $y \in Y$ such that $t = x + y$. Moreover, x and y are vertices of X and Y respectively.*

Proof. By Lemma 3.4 we have $\mathrm{Vert}(X + Y) \subset \mathrm{Vert}(X) + \mathrm{Vert}(Y)$. Therefore, t is the sum of a vertex of X and a vertex of Y. Thus it only remains to prove uniqueness. Therefore, suppose that $t = x + y = x' + y'$ with $x, x' \in X$ and $y, y' \in Y$. Suppose $x \neq x'$. Then also $y \neq y'$, and $x + y' \neq t \neq x' + y$. So we have that $x' + y \in X + Y \setminus \{t\}$ and also $x + y' \in X + Y \setminus \{t\}$. But then

$$t = \frac{1}{2}(x' + y) + \frac{1}{2}(x + y') \in \mathcal{N}(X + Y \setminus \{t\})$$

contradicting that t is a vertex of $X + Y$. ☐

Example 3.6. An element $X \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ generates a monomial ideal which contains a unique minimal basis $B(X)$ (see e.g. [4]). In general, $\mathrm{Vert}(X) \subset B(X)$ and this inclusion may be strict. Consider the set $X = \{A_1 = (1,4), A_2 = (2,3), A_3 = (3,3), A_4 = (4,1)\} \subseteq \mathbb{Z}^2_{\geq 0}$. The Newton polytope $\mathcal{N}(X)$ can be visualized as in Figure 1 and $\mathrm{Vert}(X) = \{A_1, A_4\}$ which is a strict subset of $B(X) = \{A_1, A_2, A_4\}$.

We deduce from Corollary 3.3 that the map $\mathrm{Vert}\colon \mathcal{P}(\mathbb{Z}^m_{\geq 0}) \longrightarrow \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ is a projection operator in the sense that $\mathrm{Vert}^2 = \mathrm{Vert}$.

Definition 3.7. We denote by $\mathbb{T}_m$ the image of the operator Vert, and call its elements **vertex sets**. For $S, T \in \mathbb{T}_m$, we define

$$S \oplus T = \mathrm{Vert}(S \cup T) \quad \text{and} \quad S \odot T = \mathrm{Vert}(S + T).$$

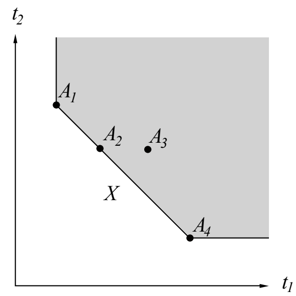


Figure 1: The Newton polytope of X. The vertex set of X is $\{A_1, A_4\}$.

Corollary 3.8. *The set* $(\mathbb{T}_m, \oplus, \odot)$ *is a commutative idempotent semiring, with the zero element* $\emptyset$ *and the unit element* $\{(0, \dots, 0)\}$.

Proof. The only things to check are associativity of $\oplus$, associativity of $\odot$ and the distributive property. The associativity of $\oplus$ and $\odot$ follows from the equalities

$$S \oplus (T \oplus U) = \mathrm{Vert}(S \cup T \cup U) = (S \oplus T) \oplus U$$

and

$$S \odot (T \odot U) = \mathrm{Vert}(S + T + U) = (S \odot T) \odot U$$

which are consequences of Lemma 3.4. The distributivity follows from

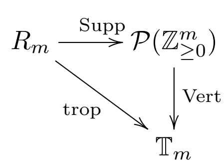
$$S \odot (T \oplus U) = \mathrm{Vert}((S + T) \cup U) = \mathrm{Vert}((S + T) \cup (S + U)) = (S \odot T) \oplus (S \odot U). \qquad \square$$

Corollary 3.9. *The map* Vert *is a homomorphism of semirings.*

Proof. Follows directly from Lemma 3.4 and Corollary 3.8. $\square$

4 The support map and the tropicalization map

We consider the differential ring R_m from Section 2, and the semirings $\mathcal{P}(\mathbb{Z}^m_{\geq 0})$, $\mathbb{T}_m$ from Section 3. In this part we introduce the support and the tropicalization maps, which are related by the following commutative diagram



If $J = (j_1, \dots, j_m)$ is an element of $\mathbb{Z}^m_{\geq 0}$, we will denote by t^J the monomial $t_1^{j_1} \cdots t_m^{j_m}$. An element of R_m is of the form $\varphi = \sum_{J \in \mathbb{Z}^m_{\geq 0}} a_J t^J$ with $a_J \in K$.

Definition 4.1. The **support** of $\varphi = \sum a_J t^J \in R_m$ is defined as

$$\mathrm{Supp}(\varphi) = \{J \in \mathbb{Z}^m_{\geq 0} \mid a_J \neq 0\}.$$

For a fixed integer n, the map which sends $(\varphi_1, \dots, \varphi_n) \in R^n_m$ to $(\mathrm{Supp}(\varphi_1), \dots, \mathrm{Supp}(\varphi_n)) \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$ will also be denoted by Supp. The **set of supports** of a subset $T \subseteq R^n_m$ is its image under the map Supp:

$$\mathrm{Supp}(T) = \{\mathrm{Supp}(\varphi) \mid \varphi \in T\} \subseteq \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$$

Definition 4.2. The map that sends each series in R_m to the vertex set of its support is called the **tropicalization** map

$$\begin{array}{rccc} \mathrm{trop}\colon & R_m & \to & \mathbb{T}_m \\ & \varphi & \mapsto & \mathrm{Vert}(\mathrm{Supp}(\varphi)) \end{array}$$

Lemma 4.3. *The tropicalization map is a non-degenerate valuation in the sense of [8, Definition 2.5.1]. This is, it satisfies*

1. $\mathrm{trop}(0) = \emptyset$, $\mathrm{trop}(\pm 1) = \{(0, \dots, 0)\}$,
2. $\mathrm{trop}(\varphi \cdot \psi) = \mathrm{trop}(\varphi) \odot \mathrm{trop}(\psi)$,
3. $\mathrm{trop}(\varphi + \psi) \oplus \mathrm{trop}(\varphi) \oplus \mathrm{trop}(\psi) = \mathrm{trop}(\varphi) \oplus \mathrm{trop}(\psi)$,
4. $\mathrm{trop}(\varphi) = \emptyset$ *implies that* $\varphi = 0$.

Proof. The first point is clear. For the second point, note that the Newton polytope has the well-known homomorphism-type property (see [12, Lemma 2.2])

$$\mathcal{N}(\mathrm{Supp}(\varphi \cdot \psi)) = \mathcal{N}(\mathrm{Supp}(\varphi)) + \mathcal{N}(\mathrm{Supp}(\psi)) = \mathcal{N}(\mathrm{Supp}(\varphi) + \mathrm{Supp}(\psi)).$$

Hence, the vertices of the left hand side coincide with the vertices of the right hand side. This gives $\mathrm{trop}(\varphi \cdot \psi) = \mathrm{Vert}(\mathcal{N}(\mathrm{Supp}(\varphi) + \mathrm{Supp}(\psi)))$. That this is equal to $\mathrm{trop}(\varphi) \odot \mathrm{trop}(\psi)$ follows from Lemma 3.4. The third point follows from the observation that $\mathrm{Supp}(\varphi + \psi) \subseteq \mathrm{Supp}(\varphi) \cup \mathrm{Supp}(\psi)$ and Corollary 3.9. The last point follows from the fact that the empty set is the only set with empty Newton polytope. $\square$

Remark 4.4. It is important to remark that for $m > 1$, the valuation trop is not a classical (Krull) valuation, since the idempotent sum operation of the semiring $\mathbb{T}_m$ is not induced by a total order. Thus we need to develop new methods to work with this formalism.

Definition 4.5. For $J = (j_1, \dots, j_m) \in \mathbb{Z}^m_{\geq 0}$, we define the **tropical derivative operator** $\Theta_{\mathrm{trop}}(J)\colon \mathcal{P}(\mathbb{Z}^m_{\geq 0}) \to \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ as

$$\Theta_{\mathrm{trop}}(J)T := \left\{ (t_1 - j_1, \dots, t_m - j_m) \;\middle|\; \begin{array}{c} (t_1, \dots, t_m) \in T, \\ t_i - j_i \geq 0 \text{ for all } i \end{array} \right\}.$$

For example, if T is the grey part in Figure 2 left and $J = (1, 2)$, then informally $\Theta_{\mathrm{trop}}(J)T$ is a translation of T by the vector $-J$ and then keeping only the non-negative part. It is represented by the grey part in Figure 2 right.

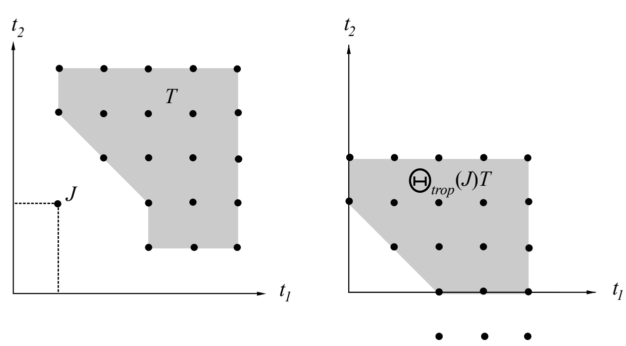


Figure 2: The operator $\Theta_{\mathrm{trop}}(J)$ for $J = (1, 2)$ applied to T.

Since K is of characteristic zero, for all $\varphi \in R_m$ and $J \in \mathbb{Z}^m_{\geq 0}$, we have

$$\mathrm{Supp}(\Theta(J)\varphi) = \Theta_{\mathrm{trop}}(J)\mathrm{Supp}(\varphi). \tag{2}$$

Consider a differential monomial E_M as in (1) and $S = (S_1, \dots, S_n) \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$. We can now define the evaluation of E_M at S as

$$E_M(S) = \sum_{\substack{1 \leq i \leq n \\ ||J||_\infty \leq r}} M_{i,J} \Theta_{\text{trop}}(J) S_i \in \mathcal{P}(\mathbb{Z}^m_{\geq 0}). \tag{3}$$

Lemma 4.6. *Given $\varphi = (\varphi_1, \dots, \varphi_n) \in R^n_m$ and a differential monomial E_M, we have*

$$\text{trop}(E_M(\varphi)) = \text{Vert}(E_M(\text{Supp}(\varphi))).$$

Proof. By applying Vert to equation (2), we have

$$\text{trop}(\Theta(J)\varphi_i) = \text{Vert}(\Theta_{\text{trop}}(J)\text{Supp}(\varphi_i)). \tag{4}$$

Using the multiplicativity of trop, equation (4) and Corollary 3.9, we obtain

$$\begin{aligned} \text{trop}(E_M(\varphi)) &= \bigodot_{i,J} \text{trop}(\Theta(J)\varphi_i)^{\odot M_{i,J}} \\ &= \bigodot_{i,J} \text{Vert}(\Theta_{\text{trop}}(J)\text{Supp}(\varphi_i))^{\odot M_{i,J}} \\ &= \text{Vert}(E_M(\text{Supp}(\varphi))). \end{aligned}$$

□

Remark 4.7. If $P = \sum_M \alpha_M E_M \in R_{m,n}$ and $\varphi = (\varphi_1, \dots, \varphi_n) \in R^n_m$, then we can consider the upper support $US(P, \varphi)$ of P at φ as

$$US(P, \varphi) = \bigcup_M (\text{Supp}(\alpha_M) + \text{Supp}(E_M(\varphi))) \in \mathcal{P}(\mathbb{Z}^m_{\geq 0}).$$

We now compute the vertex set of $US(P, \varphi)$ by applying the operation Vert and Corollary 3.9 to the above expression to find

$$\begin{aligned} \text{Vert}\Big(US(P, \varphi)\Big) &= \bigoplus_M \text{trop}(\alpha_M) \odot \text{trop}(E_M(\varphi)) \\ &= \bigoplus_M \text{trop}(\alpha_M) \odot \text{Vert}(E_M(\text{Supp}(\varphi))), \end{aligned}$$

since $\text{trop}(E_M(\varphi)) = \text{Vert}(E_M(\text{Supp}(\varphi)))$ by Lemma 4.6. This motivates the definition of tropical differential polynomials in the next section.

5 Tropical differential polynomials

In this section we define the set of tropical differential polynomials $\mathbb{T}_{m,n}$ and the corresponding tropicalization morphism $\text{trop}\colon R_{m,n} \to \mathbb{T}_{m,n}$.

Let us remark that in the case of $m = 1$ the definitions and properties presented here coincide with the corresponding ones in [1]. Moreover, later in Section 7 we illustrate in Example 7.2 the reason for the particular definitions given here.

Definition 5.1. For a set $S \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ and a multi-index $J \in \mathbb{Z}^m_{\geq 0}$ we define

$$\text{Val}_J(S) = \text{Vert}(\Theta_{\text{trop}}(J)S).$$

Note that for $\varphi \in R_m$ and any multi-index J this means that

$$\text{Val}_J(\text{Supp}(\varphi)) = \text{trop}(\Theta(J)\varphi).$$

In particular, $\text{Val}_J(\text{Supp}(\varphi)) = \emptyset$ if and only if $\Theta(J)\varphi = 0$. It follows from Corollary 3.9 that

$$\text{Vert}(E_M(S)) = \bigodot_{\substack{1 \leq i \leq n \\ ||J||_\infty \leq r}} \text{Val}_J(S_i)^{\odot M_{i,J}}.$$

Definition 5.2. A **tropical differential monomial** in the variables $x_1, \dots, x_n$ of order less or equal to r is an expression of the form

$$\epsilon_M = \bigodot_{\substack{1 \le i \le n \\ ||J||_\infty \le r}} x_{i,J}^{\odot M_{i,J}}$$

where $M = (M_{i,J}) \in (\mathbb{Z}_{\ge 0})^{n \times (r+1)^m}$.

A tropical differential monomial ϵ_M induces an evaluation map from $\mathcal{P}(\mathbb{Z}^m_{\ge 0})^n$ to $\mathbb{T}_m$ by

$$\epsilon_M(S_1, \dots, S_n) = \text{Vert}(E_M(S)) = \bigodot_{i,J} \text{Val}_J(S_i)^{\odot M_{i,J}}$$

where $\text{Val}_J(S_i)$ is given in Definition 5.1 and $E_M(S)$ as in (3). Let us recall that, by Corollary 3.8, we can also write

$$\epsilon_M(S_1, \dots, S_n) = \text{Vert}\left(\sum_{i,J} \text{Val}_J(S_i)^{\odot M_{i,J}}\right).$$

Definition 5.3. A **tropical differential polynomial** in the variables $x_1, \dots, x_n$ of order less or equal to r is an expression of the form

$$p = p(x_1, \dots, x_n) = \bigoplus_{M \in \Delta} a_M \odot \epsilon_M$$

where $a_M \in \mathbb{T}_m, a_M \neq \emptyset$ and Δ is a finite subset of $(\mathbb{Z}_{\ge 0})^{n \times (r+1)^m}$. We denote by $\mathbb{T}_{m,n} = \mathbb{T}_m\{x_1, \dots, x_n\}$ the set of tropical differential polynomials.

A tropical differential polynomial p as in Definition 5.3 induces a map from $\mathcal{P}(\mathbb{Z}^m_{\ge 0})^n$ to $\mathbb{T}_m$ by

$$p(S) = \bigoplus_{M \in \Delta} a_M \odot \epsilon_M(S) = \text{Vert}\Big(\bigcup_{M \in \Delta} (a_M + \epsilon_M(S))\Big)$$

The second equality follows again from Corollary 3.9. A differential polynomial $P \in R_{m,n}$ of order at most r is of the form

$$P = \sum_{M \in \Delta} \alpha_M E_M$$

where Δ is a finite subset of $(\mathbb{Z}_{\ge 0})^{n \times (r+1)^m}$, $\alpha_M \in K[[t_1, \dots, t_m]]$ and E_M is a differential monomial as in (1). Then the **tropicalization** of P is defined as

$$\text{trop}(P) = \bigoplus_{M \in \Delta} \text{trop}(\alpha_M) \odot \epsilon_M \in \mathbb{T}_{m,n}$$

where ϵ_M is the tropical differential monomial corresponding to E_M.

Definition 5.4. Let $G \subseteq R_{m,n}$ be a differential ideal. Its **tropicalization** $\text{trop}(G)$ is the set of tropical differential polynomials $\{\text{trop}(P) \mid P \in G\} \subseteq \mathbb{T}_{m,n}$.

Lemma 5.5. *Given a differential monomial E_M and $\varphi = (\varphi_1, \dots, \varphi_n) \in K[[t_1, \dots, t_m]]^n$, we have that*

$$\text{trop}(E_M(\varphi)) = \epsilon_M(\text{Supp}(\varphi)).$$

Proof. Follows from notations and Lemma 4.6. □

The following tropical vanishing condition is a natural generalization of the case $m = 1$, but now the evaluation $p(S)$ consists of a vertex set instead of a single minimum.

Definition 5.6. Let $p = \bigoplus_{M\in\Delta} a_M \odot \epsilon_M$ be a tropical differential polynomial. An n-tuple $S \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$ is said to be a **solution** of p if for every $J \in p(S)$ there exists $M_1, M_2 \in \Delta$ with $M_1 \neq M_2$ such that $J \in a_{M_1} \odot \epsilon_{M_1}(S)$ and $J \in a_{M_2} \odot \epsilon_{M_2}(S)$. Note that in the particular case of $p(S) = \emptyset$, S is a solution of p.

For a family of differential polynomials $H \subseteq \mathbb{T}_{m,n}$, S is called a **solution** of H if and only if S is a solution of every tropical polynomial in H. The set of solutions of H will be denoted by $\mathrm{Sol}(H)$.

Proposition 5.7. *Let G be a differential ideal in the ring of differential polynomials $R_{m,n}$. If $\varphi \in \mathrm{Sol}(G)$, then $\mathrm{Supp}(\varphi) \in \mathrm{Sol}(\mathrm{trop}(G))$.*

Proof. Let φ be a solution of G and $S = \mathrm{Supp}(\varphi)$. Let $P = \sum_{M\in\Delta} \alpha_M E_M \in G$ and $p = \mathrm{trop}(P) = \bigoplus_{M\in\Delta} a_M \odot \epsilon_M$, where $a_M = \mathrm{trop}(\alpha_M)$. We need to show that S is a solution of p. Let $J \in p(S)$ be arbitrary. By the definition of $\oplus$, there is an index M_1 such that

$$J \in a_{M_1} \odot \epsilon_{M_1}(S).$$

Hence, by Lemma 5.5, and multiplicative property of trop Lemma 4.3

$$J \in \mathrm{Vert}(\mathrm{Supp}(\alpha_{M_1} E_{M_1}(\varphi))).$$

Since $P(\varphi) = 0$, there is another index $M_2 \neq M_1$ such that

$$J \in \mathrm{Supp}(\alpha_{M_2} E_{M_2}(\varphi)),$$

because otherwise there would not be cancellation. Since J is a vertex of $p(S)$, it follows that J is a vertex of every subset of $\mathcal{N}(p(S))$ containing J and in particular of $\mathcal{N}(\mathrm{Supp}(\alpha_{M_2} E_{M_2}(\varphi)))$. Therefore,

$$J \in a_{M_2} \odot \epsilon_{M_2}(S)$$

and because J and P were chosen arbitrary, S is a solution of G. □

6 The Fundamental Theorem

Let $G \subset R_{m,n}$ be a differential ideal. Then Proposition 5.7 implies that $\mathrm{Supp}(\mathrm{Sol}(G)) \subseteq \mathrm{Sol}(\mathrm{trop}(G))$. The main result of this paper is to show that the reverse inclusion holds as well if the base field K is uncountable.

Theorem 6.1 (Fundamental Theorem)**.** *Let K be an uncountable, algebraically closed field of characteristic zero. Let G be a differential ideal in the ring $R_{m,n}$. Then*

$$\mathrm{Supp}(\mathrm{Sol}(G)) = \mathrm{Sol}(\mathrm{trop}(G)).$$

The proof of the Fundamental Theorem will take the rest of the section and is split into several parts. First let us introduce some notations. If $J = (j_1, \ldots, j_m)$ is an element of $\mathbb{Z}^m_{\geq 0}$, we define by $J!$ the component-wise product $j_1! \cdots j_m!$. The bijection between $K^{\mathbb{Z}^m_{\geq 0}}$ and R_m given by

$$\begin{array}{cccc} \psi: & K^{\mathbb{Z}^m_{\geq 0}} & \to & R_m \\ & \underline{a} = (a_J)_{J\in\mathbb{Z}^m_{\geq 0}} & \mapsto & \displaystyle\sum_{J\in\mathbb{Z}^m_{\geq 0}} \frac{1}{J!} a_J t^J \end{array}$$

allows us to identify points of R_m with points of $K^{\mathbb{Z}^m_{\geq 0}}$. Moreover, if $I \in \mathbb{Z}^m_{\geq 0}$, the mapping ψ has the following property:

$$\Theta(I)\psi(\underline{a}) = \sum_{J\in\mathbb{Z}^m_{\geq 0}} \frac{1}{J!} a_{I+J} t^J$$

which implies

$$\underline{a} = (\Theta(I)\psi(\underline{a})|_{t=0})_{I\in\mathbb{Z}^m_{\geq 0}}.$$

Fix for the rest of the section a finite set of differential polynomials $\Sigma = \{P_1, \ldots, P_s\} \subseteq G$ such that Σ has the same solution set as G (this is possible by Proposition 2.1). For all $\ell \in \{1, \ldots, s\}$ and $I \in \mathbb{Z}^m_{\geq 0}$ we define

$$F_{\ell,I} = (\Theta(I)P_\ell)|_{t_1=\cdots=t_m=0} \in K\big[x_{i,J} : 1 \leq i \leq n, J \in \mathbb{Z}^m_{\geq 0}\big]$$

and

$$A_\infty = \{(a_{i,J}) \in K^{n\times(\mathbb{Z}^m_{\geq 0})} : F_{\ell,I}(a_{i,J}) = 0 \text{ for all } 1 \leq \ell \leq s, I \in \mathbb{Z}^m_{\geq 0}\}.$$

The set A_∞ corresponds to the formal power series solutions of the differential system $\Sigma = 0$ as the following lemma shows.

Lemma 6.2. *Let $\varphi \in K[[t_1, \ldots, t_m]]^n$ with $\varphi = (\varphi_1, \ldots, \varphi_n)$, where*

$$\varphi_i = \sum_{J\in\mathbb{Z}^m_{\geq 0}} \frac{a_{i,J}}{J!} t^J.$$

Then φ is a solution of $\Sigma = 0$ if and only if $(a_{i,J}) \in A_\infty$.

Proof. This statement follows from formula

$$P_\ell(\varphi_1, \ldots, \varphi_n) = \sum_{I\in\mathbb{Z}^m_{\geq 0}} \frac{F_{\ell,I}((a_{i,J})_{i,J})}{I!} t^I,$$

which is commonly known as Taylor formula for multivariate formal power series. To prove this formula, first notice that for arbitrary $P \in R_{m,n}$ we have $P(\varphi)|_{t=0} = (P|_{t=0})((a_{i,J})_{i,J})$. Applying this to $P = \Theta(I)(P_\ell)$ for fixed I and ℓ, we find that

$$\Theta(I)(P_\ell(\varphi))\big|_{t=0} = (\Theta(I)(P_\ell)\big|_{t=0})((a_{i,J})_{i,J}) = F_{\ell,I}((a_{i,J})_{i,J}).$$

Therefore the coefficient of t^I in $P_\ell(\varphi)$ is $F_{\ell,I}((a_{i,J})_{i,J})/I!$, and this gives the formula above. □

For any $S = (S_1, \ldots, S_n) \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$ we define

$$A_{\infty,S} = \{(a_{i,J}) \in A_\infty : a_{i,J} = 0 \text{ if and only if } J \notin S_i\}.$$

This set corresponds to power series solutions of the system $\Sigma = 0$ which have support exactly S. In particular, $S \in \text{Supp}(\text{Sol}(G))$ if and only if $A_{\infty,S} \neq \emptyset$.

The sets A_∞ and $A_{\infty,S}$ refer to infinitely many coefficients. We want to work with a finite approximation of these sets. For this purpose, we make the following definitions. For each integer $k \geq 0$, choose $N_k \geq 0$ minimal such that for every $\ell \in \{1, \ldots, s\}$ and $||I||_\infty \leq k$ it holds that

$$F_{\ell,I} \in K[x_{i,J} : 1 \leq i \leq n, ||J||_\infty \leq N_k].$$

Note that for $k_1 \leq k_2$ it follows that $N_{k_1} \leq N_{k_2}$. Then we define

$$A_k = \{(a_{i,J}) \in K^{n\times\{1,\ldots,N_k\}^m} : F_{\ell,I}(a_{i,J}) = 0 \text{ for all } 1 \leq \ell \leq s, ||I||_\infty \leq k\}$$

and

$$A_{k,S} = \{(a_{i,J}) \in A_k : a_{i,J} = 0 \text{ if and only if } J \notin S_i\}.$$

Proposition 6.3. *Let $S \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$ and K be an uncountable algebraically closed field of characteristic zero. If $A_{\infty,S} = \emptyset$, then there exists $k \geq 0$ such that $A_{k,S} = \emptyset$.*

Proof. Assume that $A_{k,S} \neq \emptyset$ for every $k \geq 0$; we show that this implies $A_{\infty,S} \neq \emptyset$. We follow the strategy of the proof of [5, Theorem 2.10]: first we use the ultrapower construction to construct a larger field $\mathbb{K}$ over which a power series solution with support S exists, and then we show that this implies the existence of a solution with the same support and with coefficients in K. For more information on ultrafilters and ultraproducts, the reader may consult [2].

For each integer $k \geq 0$, choose an element $(a^{(k)}_{i,J})_{1\leq i\leq n, ||J||_\infty \leq N_k} \in A_{k,S}$. Fix a non-principal ultrafilter $\mathcal{U}$ on the natural numbers $\mathbb{N}$ and consider the ultrapower $\mathbb{K}$ of K along $\mathcal{U}$. In other words, $\mathbb{K} = (\prod_{r\in\mathbb{N}} K)/\sim$ where $x \sim y$ for $x = (x_r)_{r\in\mathbb{N}}$ and $y = (y_r)_{r\in\mathbb{N}}$ if and only if the set $\{r \in \mathbb{N} : x_r = y_r\}$ is in $\mathcal{U}$. We will denote the equivalence class of a sequence (x_r) by $[(x_r)]$. We consider $\mathbb{K}$ as a K-algebra via the diagonal map $K \to \mathbb{K}$. Now for each i and J, we may define $a_{i,J} \in \mathbb{K}$ as

$$a_{i,J} = [(a^{(k)}_{i,J} : k \in \mathbb{N})]$$

where we set $a^{(k)}_{i,J} = 0$ for the finitely many values of k with $||J||_\infty > N_k$. For all ℓ and I, we have that $F_{\ell,I}((a^{(k)}_{i,J})_{i,J}) = 0$ for k large enough, and so $F_{\ell,I}((a_{i,J})_{i,J}) = 0$ in $\mathbb{K}$, because the set of k such that $F_{\ell,I}((a^{(k)}_{i,J})_{i,J}) \neq 0$ is finite. Moreover, for $J \in S_i$ we have, by hypothesis, $a^{(k)}_{i,J} \neq 0$ for all sufficiently large k, so $a_{i,J} \neq 0$ in $\mathbb{K}$. On the other hand, for $J \notin S_i$ we have $a^{(k)}_{i,J} = 0$ for all k, so also $a_{i,J} = 0$.

Now we will use that K is uncountable. Consider the ring

$$R = K\left[\begin{array}{l} x_{i,J} : 1 \leq i \leq n, J \in \mathbb{Z}^m_{\geq 0} \\ x^{-1}_{i,J} : 1 \leq i \leq n, J \in S_i \end{array}\right] \Big/ \left(\begin{array}{l} F_{\ell,I} : 1 \leq \ell \leq s, I \in \mathbb{Z}^m_{\geq 0} \\ x_{i,J} : 1 \leq i \leq n, J \notin S_i \end{array}\right)$$

The paragraph above shows that the map $R \to \mathbb{K}$ defined by sending $x_{i,J}$ to $a_{i,J}$ is a well-defined ring map. In particular, R is not the zero ring. Let $\mathfrak{m}$ be a maximal ideal of R. We claim that $K = R/\mathfrak{m}$ in the sense that the map $K \to R/\mathfrak{m}$ induced by the composition of the inclusion and the projection $K \to R \to R/\mathfrak{m}$ is an isomorphism. Indeed, $R/\mathfrak{m}$ is a field, and as a K-algebra it is countably generated, since R is. Therefore, it is of countable dimension as K-vector space (it is generated as K-vector space by the products of some set of generators as a K-algebra). If $t \in R/\mathfrak{m}$ were transcendental over K, then by the theory of partial fraction decomposition, the elements $1/(t - \alpha)$ for $\alpha \in K$ would form an uncountable, K-linearly independent subset of $R/\mathfrak{m}$. This is not possible, so $R/\mathfrak{m}$ is algebraic over K. Since K is algebraically closed, we conclude that $K = R/\mathfrak{m}$.

Now let $b_{i,J} \in K$ be the image of $x_{i,J}$ in $R/\mathfrak{m} = K$. Then by construction, the set $(b_{i,J})$ satisfies the conditions $F_{\ell,I}((b_{i,J})) = 0$ for all ℓ and I, and $b_{i,J} = 0$ if and only if $J \notin S_i$. So $(b_{i,J})$ is an element of $A_{\infty,S}$, and in particular $A_{\infty,S} \neq \emptyset$. □

Proof of Theorem 6.1. We now prove the remaining direction of the Fundamental Theorem by contraposition. Let $S = (S_1, \dots, S_n)$ in $\mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$ be such that $A_{\infty,S} = \emptyset$, i.e. there is no power series solution of $\Sigma = 0$ in $K[[t_1, \dots, t_m]]^n$ with S as the support. Then by Proposition 6.3 there exists $k \geq 0$ such that $A_{k,S} = \emptyset$. Equivalently, the relation

$$V\left(\begin{array}{l} F_{\ell,I} : 1 \leq \ell \leq s, ||I||_\infty \leq k \\ x_{i,J} : 1 \leq i \leq n, J \notin S_i, ||J||_\infty \leq N_k \end{array}\right) \subseteq V\Bigg(\prod_{\substack{1\leq i\leq n \\ J\in S_i \\ ||J||_\infty \leq N_k}} x_{i,J}\Bigg)$$

holds, where V denotes the implicitly defined algebraic set. By Hilbert's Nullstellensatz, there is an integer $M \geq 1$ such that

$$E := \Bigg(\prod_{\substack{1\leq i\leq n \\ J\in S_i \\ ||J||_\infty \leq N_k}} x_{i,J}\Bigg)^M \in \left\langle \begin{array}{l} F_{\ell,I} : 1 \leq \ell \leq s, ||I||_\infty \leq k \\ x_{i,J} : 1 \leq i \leq n, J \notin S_i, ||J||_\infty \leq N_k \end{array}\right\rangle.$$

Therefore, there exist $G_{\ell,I}$ and $H_{i,J}$ in $K[x_{i,J} : 1 \leq i \leq n, ||J||_\infty \leq N_k]$ such that

$$E = \sum_{\substack{1\leq \ell \leq s \\ ||I||_\infty \leq k}} G_{\ell,I} F_{\ell,I} + \sum_{\substack{1\leq i\leq n \\ J\notin S_i \\ ||J||_\infty \leq N_k}} H_{i,J} x_{i,J}.$$

Define the differential polynomial P by

$$P = \sum_{\substack{1\leq \ell \leq s \\ ||I||_\infty \leq k}} G_{\ell,I} \Theta(I)(P_\ell).$$

Then P is an element of the differential ideal generated by $P_1, \ldots, P_s$, so in particular $P \in G$. Since $F_{\ell,I} = \Theta(I)(P_\ell)|_{t=0}$, there exist $h_i \in R_{m,n}$ such that

$$P = E - \sum_{\substack{1\leq i\leq n \\ J\notin S_i \\ ||J||_\infty \leq N_k}} H_{i,J} x_{i,J} \; + t_1 h_1 + \ldots + t_m h_m.$$

Notice that the monomial E occurs effectively in P, since it cannot cancel with other terms in the sum above. By construction we have $\text{trop}(E)(S) = \{(0, \ldots, 0)\}$. However, we have $(0, \ldots, 0) \notin \text{trop}(H_{i,J} x_{i,J})(S)$ because $J \notin S_i$, and we have $(0, \ldots, 0) \notin \text{trop}(t_i h_i)(S)$ because the factor t_i forces the ith coefficient of each element of $\text{trop}(t_i h_i)(S)$ to be at least 1. Hence, the vertex $(0, \ldots, 0)$ in $\text{trop}(P)(S)$ is attained exactly once, in the monomial E, and therefore, S is not a solution of $\text{trop}(P)$. Since $P \in G$, it follows that $S \notin \text{Sol}(\text{trop}(G))$, which proves the statement. □

7 Examples and remarks on the Fundamental Theorem

In this section we give an example to illustrate the results obtained in the previous sections. Moreover, we show that some straight-forward generalizations of the Fundamental Theorem from [1] and our version, Theorem 6.1, do not hold. Also we give more directions for further developments.

Example 7.1. Let us consider in $R_{2,2}$ the system

$$\begin{aligned}\Sigma = \{&P_1 = x_{1,(1,0)}^2 - 4\, x_{1,(0,0)}\,,\; P_2 = x_{1,(1,1)}\, x_{2,(0,1)} - x_{1,(0,0)} + 1\,, \\ &P_3 = x_{2,(2,0)} - x_{1,(1,0)}\}.\end{aligned}$$

By means of elimination methods in differential algebra such as the ones implemented in the MAPLE `DifferentialAlgebra` package, it can be proven that

$$\begin{aligned}\text{Sol}(\Sigma) = \{&\varphi_1(t_1,t_2) = 2\, c_0\, t_1 + c_0^2 + \sqrt{2}\, c_0\, t_2 + t_1^2 + \sqrt{2}\, t_1\, t_2 + \frac{1}{2}\, t_2^2, \\ &\varphi_2(t_1,t_2) = c_2\, t_1 + c_1 + \frac{1}{2}\sqrt{2}\,(c_0^2 - 1)\, t_2 + c_0\, t_1^2 \\ &\quad + \sqrt{2}\, c_0\, t_1\, t_2 + \frac{1}{2}\, c_0\, t_2^2 \\ &\quad + \frac{1}{3}\, t_1^3 + \frac{1}{2}\sqrt{2}\, t_1^2\, t_2 + \frac{1}{2}\, t_1\, t_2^2 + \frac{1}{12}\sqrt{2}\, t_2^3\},\end{aligned}$$

where $c_0, c_1, c_2 \in K$ are arbitrary constants. By setting $c_0 = c_2 = 0, c_1 \neq 0$, we obtain for example that

$$(\{(2,0),(1,1),(0,2)\}, \{(0,0),(0,1),(3,0),(2,1),(1,1),(0,3)\})$$

is in Supp(Sol(Σ)).

Now we illustrate that by our results necessary conditions and relations on the support can be found. Let $(S_1, S_2) \in \mathcal{P}(\mathbb{Z}^2_{\geq 0})^2$ be a solution of trop([Σ]). Let us first consider

$$\mathrm{trop}(P_1)(S_1, S_2) = \mathrm{Vert}(2 \cdot \Theta_{\mathrm{trop}}(1,0)S_1 \cup S_1).$$

If we assume that $(0,0) \in S_1$, then $(0,0)$ is a vertex of S_1. By the definition of a solution of a tropical differential polynomial, $(0,0)$ must be a vertex of the term $2 \cdot \Theta_{\mathrm{trop}}(1,0)S_1$ as well, so we then know that $(1,0) \in S_1$. Conversely, if $(1,0) \in S_1$, then $(0,0) \in S_1$ follows. This is what we expect since the corresponding monomials in φ_1 vanish if and only if $c_0 = 0$.
Now consider

$$\begin{aligned}&\mathrm{trop}(\Theta(1,0)P_1)(S_1, S_2) = \\ &\mathrm{Vert}(\Theta_{\mathrm{trop}}(1,0)S_1 + \Theta_{\mathrm{trop}}(2,0)S_1 \cup \Theta_{\mathrm{trop}}(1,0)S_1).\end{aligned}$$

If we assume that $(0,0)$ is not a vertex of this expression, which implies that $(1,0) \notin S_1$, and $(k,0)$ is a vertex in $\Theta_{\mathrm{trop}}(1,0)S_1$ for some $k \geq 1$, then we obtain from the two tropical differential monomials that necessarily $(k,0) = (2k-1,0)$. This is fulfilled only for $k = 1$ and hence, $(2,0) \in S_1$.

Another natural way for defining $\odot$ and $\oplus$ in Section 3 would be to simply take the minimal basis of the monomial ideal generated by the support of the series rather than the (possibly smaller) vertex set, as we do. If we do this, then some intermediate results (and in particular Proposition 5.7) do not hold anymore as the following example shows.

Example 7.2. Let $\{e_1, \ldots, e_4\}$ be the standard basis for $\mathbb{Z}^4_{\geq 0}$. We consider the differential ideal in $R_{4,1} = K[[t_1, \ldots, t_4]]\{x\}$ generated by

$$P = x_{e_3}x_{e_4} + (-t_1^2 + t_2^2)x_{e_1+e_3} = \frac{\partial x}{\partial t_3} \cdot \frac{\partial x}{\partial t_4} + (-t_1^2 + t_2^2)\frac{\partial^2 x}{\partial t_1 \partial t_3}$$

and the solution $\varphi = (t_1 + t_2)t_3 + (t_1 - t_2)t_4$. Then

$$\mathrm{Supp}(\varphi) = \{e_1 + e_3, e_2 + e_3, e_1 + e_4, e_2 + e_4\}.$$

On the other hand, for $S \in \mathcal{P}(\mathbb{Z}^4_{\geq 0})$ we obtain

$$\begin{aligned}\mathrm{trop}(P)(S) = \mathrm{Vert}(\, &\mathrm{Vert}(\Theta_{\mathrm{trop}}(e_3)S + \Theta_{\mathrm{trop}}(e_4)S) \\ &\cup \mathrm{Vert}(2e_1 + \Theta_{\mathrm{trop}}(e_1+e_3)S) \\ &\cup \mathrm{Vert}(2e_2 + \Theta_{\mathrm{trop}}(e_1+e_3)S).\end{aligned}$$

If we set $S = \mathrm{Supp}(\varphi)$, we obtain

$$\mathrm{trop}(P)(S) = \mathrm{Vert}(\,\mathrm{Vert}(\{2e_1, e_1+e_2, 2e_2\}) \cup \{2e_1\} \cup \{2e_2\}).$$

Since

$$\mathrm{Vert}(\{2e_1, e_1+e_2, 2e_2\}) = \{2e_1, 2e_2\},$$

every $J \in \mathrm{trop}(P)(S)$, namely $2e_1$ and $2e_2$, occurs in three monomials in $\mathrm{trop}(P)(S)$ and S is indeed in $\mathrm{Sol}(\mathrm{trop}(P))$. Note that in the Newton polytope the point $e_1 + e_2$, which is not a vertex, comes from only one monomial in $\mathrm{trop}(P)(S)$. Therefore, it is necessary to consider the vertices instead of the whole Newton polytope such that for instance Proposition 5.7 holds.

Remark 7.3. The Fundamental Theorem for systems of partial differential equations over a countable field such as $\overline{\mathbb{Q}}$ does in general not hold anymore by the following reasoning. According to [5, Corollary 4.7], there is a system of partial differential equations G over $\mathbb{Q}$ having a solution in $\mathbb{C}[[t_1, \ldots, t_m]]$ but no solution in $\overline{\mathbb{Q}}[[t_1, \ldots, t_m]]$. Taking $K = \overline{\mathbb{Q}}$ as base field, we have $\mathrm{Sol}(\mathrm{trop}(G)) \neq \emptyset$ because $\mathrm{Sol}(\mathrm{trop}(G)) = \mathrm{Supp}(\mathrm{Sol}(G))$ is non-empty in $\mathbb{C}$, but $\mathrm{Supp}(\mathrm{Sol}(G)) = \emptyset$.

In this paper we focus on formal power series solutions. A natural extension would be to consider formal Puiseux series instead. The following example shows that with the natural extension of our definitions to Puiseux series, the fundamental theorem does not hold, even for $m = n = 1$.

Example 7.4. Let us consider $R_{1,1} = K[t]\{x\}$ and the differential ideal generated by the differential polynomial

$$P = 2tx_{(1)} - x_{(0)} = 2t \cdot \frac{\partial x}{\partial t} - x.$$

There is no non-zero formal power series solution φ of $P = 0$, but $\varphi = ct^{1/2}$ is for any $c \in K$ a solution. In fact, $\{\varphi\}$ is the set of all formal Puiseux series solutions.

On the other hand, let $S \in \mathcal{P}(\mathbb{Z}_{\geq 0})$. Then every point J in

$$\mathrm{trop}(P)(S) = \mathrm{Vert}(\mathrm{Vert}(\{1\} + (\Theta_{\mathrm{trop}}(1)S) \cup \mathrm{Vert}(S))$$

occurs in both monomials except if $0 \in S$. Hence, for every $S \in \mathrm{Sol}(\mathrm{trop}(P))$ we know that $0 \notin S$. For every $I \geq 0$ we have that

$$\Theta(I)P = 2tx_{(I+1)} + (2I - 1)x_{(I)} \in [P]$$

and

$$\mathrm{trop}(\Theta(I)P)(S) = \mathrm{Vert}(\mathrm{Vert}(\{(1)\} + (\Theta_{\mathrm{trop}}((I + 1)S) \cup \mathrm{Vert}(\Theta_{\mathrm{trop}}(I)S)).$$

Similarly to above, every $J \in \mathrm{trop}(\Theta(I)P)(S)$ occurs in both monomials except if $I \in S$. Therefore, $I \notin S$ and so the only $S \in \mathcal{P}(\mathbb{Z}_{\geq 0})$ with $S \in \mathrm{Sol}(\mathrm{trop}([P]))$ is $S = \emptyset$. Hence, $\mathrm{Sol}(\mathrm{trop}([P])) = \{\emptyset\} = \mathrm{Supp}(\mathrm{Sol}([P]))$.

Now we want to consider formal Puiseux series solutions instead of formal power series solutions. Now let us set for $S \in \mathbb{Q}^m$ and $J = (j_1, \ldots, j_m) \in \mathbb{Z}_{\geq 0}^m$, the set $\Theta_{\mathrm{trop}}(J)S$ defined as

$$\left\{ (s_1 - j_1, \ldots, s_m - j_m) \;\middle|\; \begin{array}{c} (s_1, \ldots, s_m) \in S, \\ \forall 1 \leq i \leq m,\ s_i < 0 \text{ or } s_i - j_i \notin \mathbb{Z}_{<0} \end{array} \right\}$$

This is the natural definition, since only in the case when the exponent of a monomial is a non-negative integer, the derivative can be equal to zero. We have that $\Theta_{\mathrm{trop}}(J)(\mathrm{Supp}(\psi)) = \mathrm{Supp}(\Theta(J)\psi)$ for all Puiseux series ψ. For Val_J and the operations $\odot$ and $\oplus$ the definitions remain unchanged.

Let $Q \in [P]$. Then

$$Q = \sum_{k \in \mathcal{I}} Q_k \cdot \Theta(I_k)P$$

for some index-set $\mathcal{I}$ and $Q_k \in R_{m,n}$. For every I_k we know that $\mathrm{Supp}(\varphi) = \{(1/2)\} \in \mathrm{Sol}(\mathrm{trop}(\Theta(I_k)P))$. Let $\alpha \in \mathbb{Q} \cap (0, 1)$. Then for every $J \in \mathrm{trop}(\Theta(I_k)P) \in \mathbb{Z}_{\geq 0}$ we have that $\Theta_{\mathrm{trop}}(J)\{(1/2)\} = \Theta_{\mathrm{trop}}(J)\{(\alpha)\} + \{(1/2 - \alpha)\}$. Thus, $\{\alpha\} \in \mathrm{Sol}(\mathrm{trop}(\Theta(I_k)P))$. Since

$$\mathrm{trop}(Q_k \cdot \Theta(I_k)P) = \mathrm{trop}(Q_k) \odot \mathrm{trop}(\Theta(I_k)P),$$

the solvability remains by multiplication with Q_k. Therefore, $\{\alpha\} \in \mathrm{Sol}(\mathrm{trop}(Q_k \cdot \Theta(I_k)P))$ and consequently, $\{\alpha\} \in \mathrm{Sol}(\mathrm{trop}([P]))$. However, $\{\alpha\} \notin \mathrm{Supp}(\mathrm{Sol}([P])) = \{\emptyset, \{1/2\}\}$ for $\alpha \neq 1/2$.

We remark that P is an ordinary differential polynomial and by similar computations as here, the straight-forward generalization from formal power series to formal Puiseux series fails for the Fundamental Theorem in [1] as well.

We conclude this section by emphasizing that the Fundamental Theorem may help to find necessary conditions on the support of solutions of systems of partial differential equations, but in general it cannot be completely algorithmic. In fact, according to [5, Theorem 4.11], already determining the existence of a formal power series solution of a linear system with formal power series coefficients is in general undecidable.

8 Initial parts and initial ideals

In this section we introduce the notion of initial of a differential polynomial and discuss some of its properties. We also define and discuss initial ideals. These definitions generalize those presented in [10] from the ordinary to the partial case.

We use the following notations through the remainder of the section: We write $p = \text{trop}(P)$ if $P \in R_{m,n}$, $S = \text{Supp}(\varphi)$ if $\varphi \in R_m^n$, and $\epsilon = \text{trop}(E)$ if E is a differential monomial. Additionally, let $a = \sum_{I\in\Omega} \alpha_I t^I \in R_m \setminus \{0\}$ with $\alpha_I \in K \setminus \{0\}$ and hence, $\Omega = \text{Supp}(a) \neq \emptyset$. Given $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)$, we denote by $a|_S = \sum_{I\in\Omega\cap S} \alpha_I t^I$ the restriction of a to S.

Definition 8.1. Let $a = \sum_{I\in\Omega} \alpha_I t^I \in R_m \setminus \{0\}$ with $\alpha_I \in K \setminus \{0\}$. We denote by $\overline{a} = a|_{\text{Vert}(\Omega)}$ the restriction of a to the vertices of its support.

It is worth noting that $\overline{a}$ is a polynomial, and that $\text{trop}(a) = \text{trop}(\overline{a})$.

According to Theorem 6.1, given a differential ideal $G \subset R_{m,n}$ and $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n$, for checking the existence of a solution $\varphi \in R_m^n$ of G with support S the following input is required:

1. the vertex set $p(S)$ for every $P \in G$, and
2. the monomials $a_M E_M$ such that $\text{trop}(a_M E_M)(S)$ contributes to $p(S)$, where $P = \sum_M a_M E_M$.

Given $P \in R_{m,n}$, we construct some sort of localization of P at S, denoted $\text{in}_S(P)$, which records these local properties of p evaluated at S. We will call this object the initial of P (with respect to S), and we will show later in Lemma 8.6 that this is the case.

Definition 8.2. Let $P = \sum_{M\in\Lambda} a_M E_M \in R_{m,n}$ and $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n$. Then we define the *initial of* P (with respect to S) as

$$\text{in}_S(P) = \sum_{\substack{M\in\Lambda \\ \text{trop}(a_M E_M)(S)\cap p(S)\neq\emptyset}} \overline{a_M} E_M. \tag{5}$$

In other words, the initial of a differential polynomial P at S is a simpler differential polynomial, formed with the restricted monomials $\overline{a_M} E_M$ which contribute, after tropicalization and evaluation at S, to the vertex set $p(S)$.

Remark 8.3. We remark that if $m = 1$, then $p(S)$ consists of at most one point, and $\text{in}_S(P)$ coincides with the definition presented in [10].

Note that $\text{in}_S(P) = 0$ if and only if $p(S) = \emptyset$. Moreover, it is immediate that if $\text{in}_S(P)$ is a single monomial, then P has no solution with support S. The converse is in general not true as the following example shows.

Example 8.4. Let $P = x_{(1,0)} + x_{(0,1)} \in R_{2,1}$ and $\varphi = \alpha\, t_1^2 + \beta\, t_2^2$. Then φ is not a solution of $P = 0$ for any $\alpha, \beta \neq 0$, but we obtain $\text{in}_S(P) = P$ for $S = \text{Supp}(\varphi) = \{(2,0),(0,2)\}$. For $J = (1,0)$ and $\Theta(J)P \in [P]$, however, we obtain the single monomial $\text{in}_S(\Theta(J)P) = x_{(2,0)}$.

In the following we will need some basic properties of the initial of a differential polynomial, which we record in the remaining part of this section.

Lemma 8.5. *Let $aE \in R_{m,n}$ be a differential monomial, and $\varphi \in R_m^n$ with $S = \text{Supp}(\varphi)$. Then*

$$\text{trop}(aE(\varphi)) = \text{trop}(aE)(S) = \text{trop}(\overline{a}E)(S) = \text{trop}(a) \odot \text{trop}(E(S))$$

Proof. This follows from Lemma 4.6 plus the multiplicativity of trop. □

In particular, the vertex set $\text{trop}(aE(\varphi))$ does not depend on the choice of $\varphi \in \text{Supp}^{-1}(S)$. We deduce that $\text{in}_S(aE) = 0$ if and only if $E(S) = \emptyset$ (or $a = 0$).

Lemma 8.6. *Let $P \in R_{m,n}$ and $\varphi \in R_m^n$ with $S = \mathrm{Supp}(\varphi)$. Then*

1. *$\mathcal{N}(\mathrm{Supp}(P(\varphi))) \subset \mathcal{N}(p(S))$.*
2. *$P(\varphi)|_{p(S)} = \mathrm{in}_S(P)(\varphi)|_{p(S)}$.*
3. *$p(S) = \mathrm{trop}(\mathrm{in}_S(P))(S)$.*

Proof. We write $P = \sum_M a_M E_M$.

1. We have
$$\mathrm{Supp}(P(\varphi)) \subset \bigcup_M \mathrm{Supp}(a_M E_M(\varphi)) \subset \bigcup_M (\mathrm{Supp}(a_M) + E_M(S))$$
and hence,
$$\mathcal{N}(\mathrm{Supp}(P(\varphi)) \subset \mathcal{N}\left(\bigcup_M (\mathrm{Supp}(a_M) + E_M(S))\right) = \mathcal{N}(p(S)).$$

2. We express $P(\varphi) = \mathrm{in}_S(P)(\varphi) + R(\varphi)$, and if $J \in \mathrm{trop}(P)(S)$ then $J \notin \mathrm{Supp}(R(\varphi))$, since $\mathrm{in}_S(P)$ contains all the terms of P that contribute to the coefficient of t^J.

3. We write $\Lambda' = \{M \in \Lambda \mid \mathrm{trop}(a_M E_M)(S) \cap p(S) \neq \emptyset\}$, so $\mathrm{in}_S(P) = \sum_{M \in \Lambda'} \overline{a_M} E_M$. Let $Y = \bigcup_{M \in \Lambda} (\mathrm{Supp}(a_M) + E_M(S))$ and $X = \bigcup_{M \in \Lambda'} (\mathrm{Supp}(\overline{a_M}) + E_M(S))$, so it is clear that $X \subset Y$. We want to show that $\mathrm{Vert}(Y) \subset X$: if $J \in \mathrm{Vert}(Y)$, then $J \in \mathrm{trop}(a_M E_M)(S)$ for some $M \in \Lambda$. Thus, by definition there exists $M \in \Lambda'$ such that $J \in \mathrm{trop}(\overline{a_M} E_M)(S) = \mathrm{trop}(a_M E_M)(S)$, so $J \in X$. The chain $\mathrm{Vert}(Y) \subset X \subset Y$ yields $\mathcal{N}(X) = \mathcal{N}(Y)$, so $p(S) = \mathrm{Vert}(Y) = \mathrm{Vert}(X) = \mathrm{trop}(\mathrm{in}_S(P))(S)$. □

Definition 8.7. Let $G \subset R_{m,n}$ be a differential ideal and $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n$. Then we define the *initial ideal* $\mathrm{in}_S(G)$ of G (with respect to S) as the algebraic ideal generated by $\{\mathrm{in}_S(P) : P \in G\}$ in $R_{m,n}$.

Note that for $P \in G \subset R_{m,n}$ it holds that $\mathrm{in}_S(P) \in \mathrm{in}_S(G)$. However, not every element belonging to $\mathrm{in}_S(G)$ arises as $\mathrm{in}_S(P)$ for some $P \in G$, as it can be seen in [10, Example 2.11].

Remark 8.8. One could consider defining the initial ideal of G as the *differential* ideal generated by $\{\mathrm{in}_S(P) : P \in G\}$, rather than the algebraic ideal. For our purposes, however, this would not be an appropriate choice. For example, consider the differential ideal G generated by $P = x_{(0)} - 1 \in R_{1,1}$. The initial of P with respect to $S = \{(0)\}$ is P itself. All the initials of derivatives of P are zero, and so $\mathrm{in}_S(G)$ is the (monomial-free) algebraic ideal defined by P. But $\Theta(1)P = x_{(1)}$ is a monomial, so the differential ideal generated by $\mathrm{in}_S(G)$ contains a monomial. Since G has solutions with support S, we expect that $\mathrm{in}_S(G)$ is monomial-free. The algebraic ideal has this property, while the differential ideal does not.

9 Extended Fundamental Theorem

In this section we extend the Fundamental Theorem 6.1 from Section 6 to Theorem 9.6 and show that the initial ideal $\mathrm{in}_S(G)$ with respect to S of a differential ideal G is monomial-free if and only if G admits a solution with support S. This generalizes Theorem 1 in [10] from the ordinary to the partial case. A related construction can also be found in [7, Theorem 3.9].

First, we note that one direction is already implicitly proved in Section 6.

Proposition 9.1. *Let K be an uncountable, algebraically closed field of characteristic zero. Let $G \subset R_{m,n}$ be a differential ideal. Let $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n$ be such that G has no solution with support S. Then $\mathrm{in}_S(G)$ contains a monomial.*

Proof. Near the end of the proof of Theorem 6.1, under the assumption that $S \notin \mathrm{Supp}(\mathrm{Sol}(G))$, a differential polynomial $P \in G$ is constructed with $\mathrm{trop}(P)(S) = \{(0,\dots,0)\}$ where the origin corresponds to a single monomial E. For this P, we have $\mathrm{in}_S(P) = E \in \mathrm{in}_S(G)$ and the statement follows. □

It remains to prove the converse: if G admits a solution with support S, then $\mathrm{in}_S(G)$ is differential monomial-free. We start by showing that if $P(\varphi) = 0$, then cancellation occurs in the evaluation $\mathrm{in}_S(P)(\varphi)$.

Lemma 9.2. *Let $P \in R_{m,n}$ and $\varphi \in R_m^n$ with $S = \mathrm{Supp}(\varphi)$. If $P(\varphi) = 0$ then $p(S) \cap \mathcal{N}(\mathrm{Supp}(\mathrm{in}_S(P)(\varphi))) = \emptyset$. In particular, if $p(S) \neq \emptyset$ then*

$$\mathrm{trop}(\mathrm{in}_S(P)(\varphi)) \neq \mathrm{trop}(\mathrm{in}_S(P))(S).$$

Proof. By Lemma 8.6.2 we have $\mathrm{in}_S(P)(\varphi)|_{p(S)} = P(\varphi)|_{p(S)} = 0$, so $p(S) \cap \mathrm{Supp}(\mathrm{in}_S(P)(\varphi)) = \emptyset$. On the other hand, from Lemma 8.6.1 we have the chain of subsets

$$\mathcal{N}(\mathrm{Supp}(\mathrm{in}_S(P)(\varphi))) \subset \mathcal{N}(\mathrm{trop}(\mathrm{in}_S(P))(S)) = \mathcal{N}(p(S)),$$

and so Lemma 3.1 gives $p(S) \cap \mathcal{N}(\mathrm{Supp}(\mathrm{in}_S(P)(\varphi))) \subset \mathrm{trop}(\mathrm{in}_S(P)(\varphi)) \subset \mathrm{Supp}(\mathrm{in}_S(P)(\varphi))$. Combined, this give $p(S) \cap \mathcal{N}(\mathrm{Supp}(\mathrm{in}_S(P)(\varphi))) = \emptyset$ as we wanted. □

The following proposition generalizes Lemma 9.2: if φ is a solution of G, then for any $R \in \mathrm{in}_S(G)$ there is cancellation in $R(\varphi)$. Since such cancellation is usually impossible for monomials, this almost implies the result that we are looking for. However, one has to be careful to account for monomials aE for which $\mathrm{trop}(E)(S) = \emptyset$, and doing this leads to Corollary 9.4.

Proposition 9.3. *Let $G \subset R_{m,n}$ be a differential ideal and $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n$. Let $R \in \mathrm{in}_S(G)$ and $\varphi \in \mathrm{Sol}(G)$ with $\mathrm{Supp}(\varphi) = S$. Then $r(S) \cap \mathrm{Supp}(R(\varphi)) = \emptyset$. In particular, if $r(S) \neq \emptyset$ then*

$$\mathrm{trop}(R(\varphi)) \neq r(S).$$

Proof. We will proceed by contradiction, so suppose that $J \in r(S) \cap \mathrm{Supp}(R(\varphi))$. If $R = \sum_k Q_k \, \mathrm{in}_S(P_k)$, then at least one of the terms, say $Q_1(\varphi)\, \mathrm{in}_S(P_1)(\varphi)$, has t^J in its support. Hence, there exist U and W with $J = U + W$ such that

$$U \in \mathrm{Supp}(Q_1(\varphi)),\ W \in \mathrm{Supp}(\mathrm{in}_S(P_1)(\varphi)).$$

Since we have the inclusions

$$\begin{array}{rcl}\mathcal{N}(\mathrm{Supp}(Q_1(\varphi)) + \mathrm{Supp}(\mathrm{in}_S(P_1)(\varphi))) & \subset & \mathcal{N}(\mathrm{trop}(Q_1)(S) \odot \mathrm{trop}(\mathrm{in}_S(P_1))(S)) \\ & \subset & \mathcal{N}(r(S))\end{array},$$

by Lemma 3.1, we deduce that

$$r(S) \cap \mathcal{N}(\mathrm{trop}(Q_1)(S) \odot \mathrm{trop}(\mathrm{in}_S(P_1))(S)) \subset \mathrm{trop}(Q_1)(S) \odot \mathrm{trop}(\mathrm{in}_S(P_1))(S).$$

In particular, since $J \in r(S)$ and $J \in \mathrm{Supp}(Q_1(\varphi)) + \mathrm{Supp}(\mathrm{in}_S(P_1)(\varphi))$, we have that $J \in \mathrm{trop}(Q_1)(S) \odot \mathrm{trop}(\mathrm{in}_S(P_1))(S) = \mathrm{trop}(Q_1)(S) \odot \mathrm{trop}(P_1)(S)$. Hence, by Lemma 3.5,

$$U \in \mathrm{trop}(Q_1)(S),\ W \in \mathrm{trop}(P_1)(S).$$

However, since φ is a solution of G, also $P_1(\varphi) = 0$ and we deduce by Lemma 9.2 that $W \notin \mathcal{N}(\mathrm{Supp}(\mathrm{in}_S(P_1)(\varphi)))$, a contradiction. □

Corollary 9.4. *Let $G \subset R_{m,n}$ be a differential ideal and $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n$. If $\mathrm{in}_S(G)$ contains a monomial $aE \neq 0$ such that $\mathrm{trop}(E)(S) \neq \emptyset$, then G has no solution with support S.*

Proof. Let us assume that $\varphi \in \mathrm{Sol}(G)$ with $\mathrm{Supp}(\varphi) = S$. We have

$$\mathrm{trop}(aE)(S) = \mathrm{trop}(a) \odot \mathrm{trop}(E)(S) \neq \emptyset,$$

because both factors are non-empty. By applying Proposition 9.3 to $R = aE$, we obtain

$$\mathrm{trop}(aE(\varphi)) \neq \mathrm{trop}(aE)(S),$$

in contradiction to Lemma 8.5 which implies the opposite. □

It remains to check that if $\mathrm{in}_S(G)$ contains a monomial, then it contains a monomial which tropicalization is not vanishing on S in order to apply Lemma 8.5.

Proposition 9.5. *Let $G \subset R_{m,n}$ be a differential ideal and $S \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$. Suppose $\mathrm{in}_S(G)$ contains a monomial $aE \neq 0$. Then it contains a monomial bF such that $\mathrm{trop}(bF)(S) \neq \emptyset$.*

Proof. If $\mathrm{trop}(E)(S) \neq \emptyset$ there is nothing to prove. So we assume that $\mathrm{trop}(E)(S) = \emptyset$ and write

$$E = \prod_{1 \leq i \leq n, J \in \mathbb{N}^m} x_{i,J}^{M_{i,J}}.$$

Since $\mathrm{trop}(E)(S) = \emptyset$, there are indexes i_0, J_0 with $M_{i_0,J_0} > 0$ and $\Theta_{\mathrm{trop}}(J_0)S_{i_0} = \emptyset$ and consequently, $\mathrm{trop}(x_{i_0,J_0})(S) = \emptyset$. On the other hand, aE can be written as

$$aE = \sum_{k=1}^{r} Q_k \ \mathrm{in}_S(P_k)$$

for some non-zero $P_k \in G$ and $Q_k \in R_{m,n}$. By definition of the initial, for every $1 \leq k \leq r$, no monomial of $\mathrm{in}_S(P_k)$ contains x_{i_0,J_0} as factor.

We now consider the factors $Q_k = \sum_{M \in \Omega_k} b_M V_M$ (with $0 \neq b_M \in R_m$ and V_M a differential monomial) modulo the algebraic ideal (x_{i_0,J_0}) :

$$\widetilde{Q_k} := \sum_{\substack{M \in \Omega_k \\ V_M \neq 0 \mod (x_{i_0,J_0})}} b_M V_M.$$

By the choice of $\widetilde{Q_k}$, it follows that $Q_k - \widetilde{Q_k} = 0 \mod (x_{i_0,J_0})$ and therefore, $\sum_{k=1}^r (Q_k - \widetilde{Q_k}) \ \mathrm{in}_S(P_k) = 0 \mod (x_{i_0,J_0})$. Since

$$aE = \sum_{k=1}^{r} \widetilde{Q_k} \ \mathrm{in}_S(P_k) + \sum_{k=1}^{r} (Q_k - \widetilde{Q_k}) \ \mathrm{in}_S(P_k) = 0 \mod (x_{i_0,J_0}),$$

also $\sum_{k=1}^r \widetilde{Q_k} \ \mathrm{in}_S(P_k) = 0 \mod (x_{i_0,J_0})$. Because x_{i_0,J_0} is neither a factor of any monomial in $\widetilde{Q_k}$ nor in $\mathrm{in}_S(P_k)$, it follows that $\sum_{k=1}^r \widetilde{Q_k} \, \mathrm{in}_S(P_k) = 0$ and

$$aE = \sum_{k=1}^{r} (Q_k - \widetilde{Q_k}) \ \mathrm{in}_S(P_k).$$

The monomial x_{i_0,J_0} divides all $Q_k - \widetilde{Q_k}$ and we can set $E_1 = E/x_{i_0,J_0} \in R_{m,n}$ such that

$$aE_1 = \sum_{k=1}^{r} \frac{Q_k - \widetilde{Q_k}}{x_{i_0,J_0}} \ \mathrm{in}_S(P_k) \in \mathrm{in}_S(G).$$

If $\mathrm{trop}(aE_1)(S) \neq \emptyset$, we are done. Otherwise, by considering a monomial factor of E_1 whose tropicalization vanishes at S, we can iterate the previous reduction, and since the degree strictly decreases in this process, we construct in this way a finite sequence of monomials $(aE_j)_{1 \leq j \leq s} \in \mathrm{in}_S(G)^s$ until $\mathrm{trop}(E_s)(S) \neq \emptyset$. □

Theorem 9.6 (Extended Fundamental Theorem)**.** *Let K be an uncountable, algebraically closed field of characteristic zero. Let $G \subset R_{m,n}$ be a differential ideal. Then the following three subsets of $\left(\mathcal{P}(\mathbb{Z}_{\geq 0}^m)\right)^n$ coincide:*

1. $\mathrm{Supp}(\mathrm{Sol}(G))$,
2. $\mathrm{Sol}(\mathrm{trop}(G))$,
3. $\{S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n \ : \ \mathrm{in}_S(G)$ *contains no monomial*$\}$.

Proof. The equivalence of 1 and 2 is Theorem 6.1. If $S \in \mathrm{Supp}(\mathrm{Sol}(G))$, then Corollary 9.4 implies that $\mathrm{in}_S(G)$ contains no monomial aE with $\mathrm{trop}(E)(S) \neq \emptyset$, and by Proposition 9.5 this implies that $\mathrm{in}_S(G)$ contains no monomial at all. The other direction is Proposition 9.1. □

Acknowledgements

This research project and the sixth author were supported by the European Commission, having received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement number 792432. The first author was supported by the Austrian Science Fund (FWF): P 31327-N32. The second author was supported by CONACYT Project 299261. Partially supported by PAPIIT IN108320. The fifth author would like to acknowledge the support of the bilateral project ANR-17-CE40-0036 and DFG-391322026 SYMBIONT.

This work was started during the Tropical Differential Algebra workshop, which took place on December 2019 at Queen Mary University of London. We thank the organizers and participants for valuable discussions and initiating this collaboration. In particular, we want to thank Fuensanta Aroca, Alex Fink, Jeffrey Giansiracusa and Dima Grigoriev for their helpful comments during this week.

References

[1] Fuensanta Aroca, Cristhian Garay, and Zeinab Toghani. The Fundamental Theorem of Tropical Differential Algebraic Geometry. *Pacific J. Math.*, 283(2):257–270, 2016. `arXiv:1510.01000v3`.

[2] Joseph Becker, Jan Denef, Leonard Lipshitz, and Lou van den Dries. Ultraproducts and Approximation in Local Rings I. *Inventiones mathematicae*, 51:189–203, 1979.

[3] François Boulier and Mercedes Haiech. The Ritt-Raudenbush Theorem and Tropical Differential Geometry. Available at `https://hal.archives-ouvertes.fr/hal-02403365`, 2019.

[4] David Cox, John Little, and Donal O'Shea. *Ideals, Varieties and Algorithms. An introduction to computational algebraic geometry and commutative algebra.* Undergraduate Texts in Mathematics. Springer Verlag, New York, 3rd edition, 2007.

[5] J. Denef and L. Lipshitz. Power series solutions of algebraic differential equations. *Math. Ann.*, 267(2):213–238, 1984.

[6] Sebastian Falkensteiner, Cristhian Garay-López, Mercedes Haiech, Marc Paul Noordman, Zeinab Toghani, and François Boulier. The fundamental theorem of tropical partial differential algebraic geometry. In *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*, ISSAC '20, pages 178–185, New York, NY, USA, 2020. Association for Computing Machinery.

[7] Alex Fink and Zeinab Toghani. Initial forms and a notion of basis for tropical differential equations, 2020.

[8] Jeffrey Giansiracusa and Noah Giansiracusa. Equations of tropical varieties. *Duke Math. J.*, 165(18):3379–3433, 2016.

[9] Dima Grigoriev. Tropical differential equations. *Advances in Applied Mathematics*, 82:120–128, 2017.

[10] Youren Hu and Xiao-Shan Gao. Tropical Differential Gröbner Bases. *Mathematics in Computer Science*, 2020.

[11] Ellis Robert Kolchin. *Differential Algebra and Algebraic Groups*. Academic Press, New York, 1973.

[12] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz, Austria.
Web: `risc.jku.at/m/sebastian-falkensteiner`

Centro de Investigación en Matemáticas, A.C. (CIMAT)
Jalisco S/N, Col. Valenciana CP. 36023 Guanajuato, Gto, México.
e-mail: `cristhian.garay@cimat.mx`

Institut de recherche mathématique de Rennes, UMR 6625 du CNRS
Université de Rennes 1, Campus de Beaulieu
35042 Rennes cedex (France)
e-mail: `mercedes.haiech@univ-rennes1.fr`

Bernoulli Institute, University of Groningen, The Netherlands
Web: `https://www.rug.nl/staff/m.p.noordman/`

Univ. Lille, CNRS, Centrale Lille, Inria, UMR 9189 - CRIStAL - Centre de Recherche en Informatique Signalet Automatique de Lille, F-59000 Lille, France
Web: `pro.univ-lille.fr/francois-boulier`

School of Mathematical Sciences, Queen Mary University of London
e-mail: `zeinab.toghani@im.unam.mx`