Quasi-regularity verification for 2D polygonal objects based on medial axis analysis
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A digital object $X \subset \mathbb{Z}^2$ is often the result of a digitization process, namely sampling or quantification, applied on a continuous object $X \subset \mathbb{R}^2$. Because of this process, the resulting object may have different properties than those of the original continuous one. This article addresses topological properties of objects in the case of the Gauss digitization $D$. In this context, some studies were proposed for providing topological guarantees by $D$. Pavlidis, in the 1980s, can be considered as the pioneer of the topology-preserving conditions of digitized objects. More precisely, he introduced in $\mathbb{R}$ the notion of $r$-regularity, and established the topological links between a continuous shape and its digitized counterpart. However, he was interested in the “smooth” objects, i.e. with boundaries having differentiable properties. Later, Stelldinger and Terzic presented in $[3]$ a generalization of $r$-regularity, named $r$-halfregularity, which includes objects with non-differentiable boundaries. It was shown that the $r$-halfregularity allows a topologically correct digitization using an additional repairing step. Recently, a morphology-based notion, called quasi-$r$-regularity, was proposed in $[4]$ and further formalized in 2D in $[5]$. A preliminary extension in 3D was presented in $[6]$. This notion provides sufficient conditions for simple-connectedness preservation by $D$. It was presented together with a rigid motion model that allows to preserve topology and some geometrical properties of digitized objects.

However, no algorithmic framework was given for quasi-regularity verification. Based on the relations between the quasi-regularity and the notion of medial axis, we propose hereafter such framework, dedicated to polygonal objects, frequently used in image analysis or computer graphics. An application for topology-preserving rigid motion of digital objects is also proposed.
Let \( X \subset \mathbb{R}^2 \) be a bounded, simply connected object. If \( X \) is quasi-regular, then \( X = X \cap \mathbb{Z}^2 \) and \( \overline{X} = \overline{X} \cap \mathbb{Z}^2 \) are both 4-connected.

The used notion of 4-connectedness usually considered in digital topology \([3]\) is the equivalence relation derived from the 4-adjacency in \( \mathbb{Z}^2 \) (\( p, q \in \mathbb{Z}^2 \) are 4-adjacent if \( |p - q|_1 = 1 \)).

Note that from Prop. \(2\) it derives that \( X \) and \( \overline{X} \) are not only simply connected, but also well-composed \([10]\).

Quasi-regularity is then a sufficient condition for guaranteeing the topological preservation of a simply connected object when it undergoes a digitization process from \( \mathbb{R}^2 \) to \( \mathbb{Z}^2 \).

2.3. Medial axis

Let \( X \subset \mathbb{R}^2 \) be a closed, bounded set such that the boundary of \( X \) is a 1-manifold. We note \( B(y, r) \) the Euclidean ball of center \( y \in \mathbb{R}^2 \) and radius \( r \in \mathbb{R}_+ \). Let \( x \in X \). We set \( r(x) \in \mathbb{R}_+ \) as the radius of the maximal Euclidean ball of center \( x \), included in \( X \):

\[
    r(x) = \max \{ r \mid B(x, r) \subseteq X \} \tag{2}
\]

Blum defined the medial axis \( M(X) \) of \( X \) as the set of points in \( X \) having more than one closest point on its boundary \( \partial X \) \([11]\):

\[
    M(X) = \{ x \in X \mid |\exists y \in X, B(x, r(x)) \subset B(y, r(y)) \} \tag{3}
\]

Alternatively, the medial axis of \( X \) can be defined as the locus of the centers of the maximal balls included in \( X \):

\[
    M(X) = \{ x \in X \mid |\exists y \in X, B(x, r(x)) \subset B(y, r(y)) \} \tag{4}
\]

The slight difference between these two definitions lies in the case of non-differentiable points of \( \partial X \). For such points, the maximal Euclidean ball has a null radius. As a consequence, these points belong to the medial axis w.r.t Eq. \(3\) but not w.r.t Eq. \(4\). However, both medial axes are equal up to closure.

From the very definition of medial axis, we have \( M(X) \subseteq X \) and:

\[
    X = \bigcup_{x \in M(X)} B(x, r(x)) \tag{5}
\]

In other words, \( X \) can be reconstructed from \( M(X) \). We now define from \( M(X) \) the \( \lambda \)-level medial axis, noted \( M_\lambda(X) \), as:

\[
    M_\lambda(X) = \{ x \in M(X) \mid r(x) \geq \lambda \} \tag{6}
\]

In particular, we have \( \lambda_1 \leq \lambda_2 \Rightarrow M_{\lambda_2}(X) \subseteq M_{\lambda_1}(X) \), and \( M_0(X) = M(X) \). We can also define:

\[
    M_{\lambda_1\lambda_2}(X) = \{ x \in M(X) \mid \lambda_1 \leq r(x) \leq \lambda_2 \} \tag{7}
\]

We state two important results that will be used in the sequel.

Proposition 3 \([12]\): \( X \) and \( M(X) \) have the same homotopy type.

In \([12]\), the property is established for \( X \) open. Here, it remains valid for \( X \) closed, since we assume that \( \partial X \) is a manifold. From now on, we denote \( X \sim Y \) if \( X \) and \( Y \) have the same homotopy type.

Proposition 4 \([9]\): Let \( B_\lambda \) be the Euclidean ball of centre \( 0_{\mathbb{Z}^2} \) and radius \( \lambda \geq 0 \). We have:

\[
    X \oplus B_\lambda = \bigcup_{x \in M(X)} B(x, r(x) - \lambda) \tag{8}
\]

\[
    X \oplus B_\lambda = \bigcup_{x \in M(X)} B(x, r(x) + \lambda) \tag{9}
\]

\[
    M(X \oplus B_\lambda) = M_\lambda(X) \tag{10}
\]

3. QUASI-REGULARITY WITH MEDIAL AXIS

The above notions are valid for general objects of \( \mathbb{R}^2 \). From now on, we focus on polygonal objects. Our purpose is to assess the quasi-regularity of such objects, in order to know if they may preserve their topological properties when undergoing a Gauss digitization. Such verification is indeed difficult based on Def. \([7]\) in particular since conditions (iii) and (iv) require to compute the erosions and dilations of sets and to compare them with respect to inclusion. To tackle this issue, we propose to rely on the medial axis as a way to model these polygonal objects. Using this paradigm, we present hereafter a way of assessing the quasi-regularity of a polygonal object, thus leading to a tractable algorithm (Sec. \([4]\).

Property 5 Let \( X \subset \mathbb{R}^2 \) be a bounded, simply connected polygon. If \( M(X) \sim M_1(X) \) and \( M(\overline{X}) \sim M_1(\overline{X}) \) then conditions (i) and (ii) of Def. \([7]\) hold.

Proof Let us suppose that \( M(X) \sim M_1(X) \). We have \( X \sim M(X) \) (Prop. \([5]\), \( M(X) \sim M_1(X) \) by hypothesis, \( M_1(X) = M(X \oplus B_1) \) (Eq. \([10]\)), and \( M(X \oplus B_1) \sim X \oplus B_1 \) (Prop. \([2]\)). Since \( X \) is non-empty and simply connected by hypothesis, it follows that \( X \oplus B_1 \) is non-empty and connected; then (i) holds. If we assume \( M(\overline{X}) \sim M_1(\overline{X}) \), the same reasoning holds for \( \overline{X} \) chosen sufficiently large\(^4\) and connected.

In the sequel, \( Y \) can be either \( X \) or \( \overline{X} \). Let us consider the object \( Y \) closed, simply connected polygon.

\[
    Y \oplus B_1 = Y \oplus B_1 \oplus B_1, \text{the opening of } Y \text{ by } B_1. \tag{11}
\]

From Prop. \([4]\), it is plain that:

\[
    Y \oplus B_1 = \bigcup_{x \in M_1(Y)} B(x, r(x)) \subseteq Y \tag{11}
\]

Let us define the top-hat of \( Y \) as \( T_{B_1}(Y) = Y \setminus (Y \oplus B_1) \). From Eqs. \([9]\) and \([11]\), we have:

\[
    T_{B_1}(Y) \subseteq \bigcup_{x \in M_1(Y)} B(x, r(x)) \tag{12}
\]

Let \( M \subseteq M_1(Y) \) be a connected component of \( M_1(Y) \). From Prop. \([5]\), there exists a unique point \( y \in M \) such that \( r(y) = 1 \). Since \( Y \) has a polygonal shape, the set \( M \) contains \( k \) terminal points \( (k \geq 1) \), namely points \( z_1, \ldots, z_k \) such that \( r(z_i) = 0 \). These points are convex vertices of the polygon \( Y \).

We set \( C(M) \subset \mathbb{R}^2 \) as the convex hull of \( B(y, 1) \cup \{z_i\}_{i=1}^k \).

Then, we have:

\[
    \bigcup_{x \in M} B(x, r(x)) \subseteq C(M) \tag{13}
\]

In particular, if we assume that \((P)\) \( \forall 1 \leq i \leq k, \|y - z_i\|_2 \leq \sqrt{2} \), it follows that \( C(M) \subseteq B(y, \sqrt{2}) \). Since \( y \in Y \oplus B_1 \), we have:

\[
    (P) \Rightarrow \bigcup_{x \in M} B(x, r(x)) \subseteq Y \oplus B_1 \oplus B_{\sqrt{2}} \tag{14}
\]

From Eqs. \([12]\) and \([14]\), we then obtain the following result.

Proposition 6 Let \( X \subset \mathbb{R}^2 \) be a bounded, simply connected polygon. Let us suppose that \( M(X) \sim M_1(X) \), \( M(\overline{X}) \sim M_1(\overline{X}) \) and that for each connected component of \( M_1(X) \) and \( M_1(\overline{X}) \), property \((P)\) holds. Then, \( X \) is quasi-regular.

\(^4\)In order to handle the medial axes for both a polygonal object \( X \) and its (infinite) complement \( \overline{X} \), we will consider only a part of \( \overline{X} \) bounded, but “sufficiently large” with respect to \( X \), so that the notions and properties related to quasi-regularity remain unaltered. For instance one can intersect \( X \) with a square centered on \( X \) with a size as large as required by the user, which will correspond in general to the finite support of a digital image.
Proposition 6 provides sufficient conditions for establishing the quasi-regularity of a polygonal object \( X \); see Fig. 2. They are not necessary conditions because \( X \) may be quasi-regular even if \((P)\) is not satisfied for some of the connected components of \( M^2_b(X) \) or \( M^2_b(\overline{X}) \). Such cases, however, correspond to tortuous shapes of \( X \), which are rarely seen in usual applications. In other words, Prop. 6 provides us with a reasonable way of determining that a polygonal object is quasi-regular.

4. QUASI-REGULARITY VERIFICATION METHOD

Based on Prop. 6, we present hereafter a method to assess the quasi-regularity of a polygonal object \( X \subset \mathbb{R}^2 \) using the medial axes \( M(Y) \) (with \( Y \in \{X, \overline{X}\} \)). From Prop. 6, it consists of verifying the following two conditions:

(a) \( M(Y) \subset M_1(Y) \)

(b) \((P)\) holds for each connected component of \( M^2_b(Y) \).

The process is summarized in Alg. 1

\[
\begin{array}{l}
\text{Restore: A simply connected polygonal object } X \subset \mathbb{R}^2 \\
\text{Output: A Boolean indicating whether } X \text{ is quasi-regular}
\end{array}
\]

1. \( Y \in \{X, \overline{X}\} \) do
2. \( \text{if} \ (M(Y) \subset M_1(Y)) \text{ then return false} \)
3. foreach connected component \( M \in M^2_b(Y) \) do
4. \( y \in M \text{ s.t. } r(y) = 1 \)
5. foreach \( z_i \in M \text{ s.t. } r(z_i) = 0 \) do
6. \( \text{if } ||y - z_i||^2 > 2 \) then return false
7. return true

along the half-arcs of parabolas is monotonic. In this work, we use the package Segment Delaunay Graphs, proposed in CGAL [14] for exact computation of the medial axis of polygons. It allows us to extract the analytic expression for each graph edge as well as the associated radius function \( r(\cdot) \). This is useful for computing \( M_1(Y) \) from \( M(Y) \) and finding the unique point \( y \) of each connected component \( M \in M^2_b(Y) \). From the graph structure of \( M(Y) \), we can also extract the \( k \) vertices \( z_i \), \( 1 \leq i \leq k \), for each \( M \).

In order to verify \((a)\) the homotopy-type preservation between \( M(Y) \) and \( M_1(Y) \) (Line 2 of Alg. 1), we rely on the Euler characteristics \( \chi = b_0 - b_1 \) where \( b_0 \) (resp. \( b_1 \)) is the number of connected components (resp. cycles) of the graph. On the one hand, \( M(X) \) is simply connected, i.e. \( \chi(M(X)) = 1 - 0 = 1 \). On the other hand, \( M(\overline{X}) \) is connected with one cycle, i.e. \( \chi(M(\overline{X})) = 1 - 1 = 0 \).

By construction, any part \( Z \in M(Y) \) satisfies \( b_0(Z) \geq b_0(M(Y)) \) and \( b_1(Z) \leq b_1(M(Y)) \), thus \( \chi(Z) \geq \chi(M(Y)) \) and we have \( \chi(Z) = \chi(M(Y)) \) iff \( b_0 \) and \( b_1 \) are the same for both, i.e. \( Z \) and \( M(Y) \) have the same homotopy type. It is then sufficient to check \( \chi \), which can be computed as \( \chi = |V| - |E| \), where \( V \) and \( E \) are the sets of vertices and edges of the graph structure of the medial axis, respectively.

In order to verify \((b)\), i.e. that the property \((P)\) holds for every connected component \( M \subset M^2_b(Y) \), it is sufficient to evaluate the square of the Euclidean distance between the point \( y \in M \cap M_1(Y) \) and any of the points \( z_i \in M \cap M_1(Y) \), as stated in Line 6 of Alg. 1.

Computing the medial axes has a complexity of \( O(n \log^2 n) \), with \( n \) is the number of vertices of the polygon \( \Sigma \). Checking \((a)\) has a complexity \( O(m) \), with \( m \) the number of vertices of the medial axes, which is in the same order as \( n \). Thus it can also be written as \( O(n) \). As checking \((b)\) has a complexity \( O(n) \), the overall complexity is then \( O(n \log^2 n) \).

5. APPLICATION CASES

Quasi-regularity characterized in Def. 1 provides a theoretical way of guaranteeing the preservation of homotopy type between a continuous object \( X \subset \mathbb{R}^2 \) and its digital analogue \( X = D(X) \subset \mathbb{Z}^2 \) obtained by Gauss digitization \( D \). From a practical point of view, Prop. 6 and the induced Alg. 1 provide an algorithmic way of assessing the quasi-regularity of \( X \subset \mathbb{R}^2 \) whenever \( X \) is a simply connected polygonal object.

We can then consider quasi-regularity as a topological tool which is useful when a polygonal object has to be digitized. This is especially the case during rasterization procedures [16], which are frequent and crucial tasks, e.g. in computer graphics applications or digital image construction.

Beyond such polygon-to-pixel applications, quasi-regularity can also be useful for pixel-to-pixel applications, in particular when con-
sidering digital geometric transformations. Indeed, defining “correct” geometric transformations from $\mathbb{Z}^2$ to $\mathbb{Z}^2$ is not a trivial task. Even for simple transformations, such as rigid motions (i.e. rotations composed with translations), this correctness is hard to reach, especially with regard to topology preservation [17,18].

The notion of quasi-regularity can allow us to tackle this issue by handling such polygon-to-pixel procedures via pixel-to-polygon-to-pixel strategies. Given a digital object $X \subset \mathbb{Z}^2$ and a (continuous) rigid motion $\mathcal{R} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, a common problem is to determine a digital object $X_\mathcal{R} \subset \mathbb{Z}^2$ that fits at best the object $\mathcal{R}(X) = \{ \mathcal{R}(x) \mid x \in X \}$ which is a subset of $\mathbb{R}^2$ but most of the time not of $\mathbb{Z}^2$.

A possible solution relies on two steps: (1) defining a polygon $P(X)$ which is a relevant representation of $X$ and (2) digitizing the image of this of polygon $P(X)$ after the rigid motion. More formally, we may compute:

$$X_\mathcal{R} = D(\mathcal{R}(P(X))) \quad (15)$$

Step (1) is a pixel-to-polygon process, whereas Step (2) is a polygon-to-pixel one (actually a rasterization process).

The choice of $P(X)$ from $X$ is of course, not unique. In our experiments, we considered that a “good” polygonalization should be reversible (i.e. $D(P(X)) = X$), preserve the topology (a simply connected digital object should lead to a simple polygon) and should lead to a polygon with rational-coordinate vertices.

Since we have $X \sim P(X)$ by construction and $P(X) \sim \mathcal{R}(P(X))$ by definition, it is sufficient to guarantee that $\mathcal{R}(P(X))$ (or equivalently, $P(X)$) is quasi-regular to ensure that Eq. (15) defines a homotopy-type preserving digital transformation between the digital objects $X$ and $X_\mathcal{R}$.

6. EXPERIMENTS

In our experiments, we mainly dealt with the digital rigid motion problem of Eq. (15). Without loss of generality we considered rigid motions $\mathcal{R}$ with rational values for their translation part and rotation angles built from Pythagorean triples [19]. This choice was not penalizing, due to the high density of such rigid motions. In addition, it allowed us to maximize the parts of the experiments carried out with exact calculation (and thus without numerical errors). Regarding the polygonalization, we considered the approach based on [5,20]. For the digitization step, we used a ray-tracing based approach [21], namely the crossing number algorithm [22], to compute the Gauss digitization of the transformed polygon.

Some experimental results are illustrated in Fig. 3. Three complex digital objects $X$ are represented by polygons $P(X)$ satisfying the required properties (reversibility, topology). These polygons are then proved quasi-regular thanks to Alg. 1 via their medial axis analysis. Their images by rigid motion can then be computed from Eq. (15) with guarantees of homotopy-type preservation.

Reversely, Fig. 4 illustrates the possible topological consequences of non-quasi-regularity. One can observe two polygonal objects that are not quasi-regular. The digital objects induced by their Gauss digitization present topological properties that vary depending on the rigid motion applied beforehand, emphasizing the non-necessary preservation of the homotopy type.

7. CONCLUSION AND PERSPECTIVES

In this article, we proposed an algorithmically tractable way of verifying the quasi-regularity of continuous objects consisting of simple polygons, based on the analysis of their medial axes. This allows, in particular, to guarantee topological preservation properties when carrying out non-trivial operations such as rasterization or digital geometric transformations. Perspective works will consist of evaluating quasi-regularity for more complex objects, e.g. hierarchies of nested polygons. Longer term research will also deal with potential extensions of this approach to higher dimensions.
8. REFERENCES


