# On the Role of 3's for the 1-2-3 Conjecture

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**Abstract.** The 1-2-3 Conjecture states that every connected graph different from $K_2$ admits a proper 3-labelling, i.e., can have its edges labelled with $1, 2, 3$ so that no two adjacent vertices are incident to the same sum of labels. In connection with recent optimisation variants of this conjecture, we study the role of label 3 in proper 3-labellings of graphs. Previous studies suggest that, in general, it should always be possible to produce proper 3-labellings assigning label 3 to a only few edges. We prove that, for every $p \geq 0$, there are various graphs needing exactly $p$ 3's in their proper 3-labellings. Actually, deciding whether a given graph can be labelled with $p$ 3's is NP-complete for every $p \geq 0$. We also focus on particular classes of 3-chromatic graphs (cacti, triangle-free planar graphs, etc.), for which we prove there is no $p \geq 1$ such that they all admit proper 3-labellings assigning label 3 to at most $p$ edges. In such cases, we give lower and upper bounds on the number of needed 3's.

**Keywords:** Proper labellings · 3-chromatic graphs · 1-2-3 Conjecture.

## 1 Introduction

This work is mainly motivated by the so-called **1-2-3 Conjecture**, which can be defined through the following terminology and notation. Let $G$ be a graph and consider a $k$-labelling $\ell : E(G) \to \{1, \dots, k\}$, i.e., an assignment of labels $1, \dots, k$ to the edges of $G$. To every vertex $v \in V(G)$, we associate, as its *colour* $c_\ell(v)$, the sum of labels assigned by $\ell$ to its incident edges. That is, $c_\ell(v) = \sum_{u \in N(v)} \ell(vu)$. We say that $\ell$ is *proper* if we have $c_\ell(u) \neq c_\ell(v)$ for every $uv \in E(G)$, that is, if no two adjacent vertices of $G$ get incident to the same sum of labels by $\ell$.

The complete graph on two vertices, $K_2$, is the only connected graph admitting no proper labellings. Thus, when studying the 1-2-3 Conjecture, we focus on *nice graphs*, which are those graphs with no connected component isomorphic to $K_2$, i.e., admitting proper labellings. If a graph $G$ is nice, then we can investigate the smallest $k \geq 1$ such that proper $k$-labellings of $G$ exist. This parameter is denoted by $\chi_\Sigma(G)$. A natural question to ask, is whether this parameter $\chi_\Sigma(G)$ can be large for a given graph $G$. This question is precisely at the heart of the 1-2-3 Conjecture [11], which states that if $G$ is a nice graph, then $\chi_\Sigma(G) \leq 3$.

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To date, most of the progress towards the 1-2-3 Conjecture can be found in [13]. Let us highlight that the conjecture was verified mainly for 3-colourable graphs [11] and complete graphs [4]. Regarding the tightness of the conjecture, it was proved that deciding if a given graph $G$ verifies $\chi_\Sigma(G) \leq 2$ (denoted as the 2-Labelling problem) is NP-complete in general [8], and remains so even in the case of cubic graphs [6]. Hence, there is no nice characterisation of graphs admitting proper 2-labellings (or, the other way round, of graphs needing 3's in their proper 3-labellings), unless P=NP. Lastly, to date, the best result towards the 1-2-3 Conjecture, from [10], is that $\chi_\Sigma(G) \leq 5$ holds for every nice graph $G$.

This work takes place in a recent line of research studying optimisation problems related to the 1-2-3 Conjecture which arise when considering proper labellings fulfilling additional constraints. In a way, one of the main sources of motivation here is further understanding the very mechanisms that lie behind proper labellings. In particular, towards better comprehending the connection between proper labellings and proper vertex-colourings, the authors of [1,3] studied proper labellings $\ell$ for which the resulting vertex-colouring $c_\ell$ is required to be close to an optimal proper vertex-colouring (i.e., with the number of distinct resulting vertex colours being close to the chromatic number). Due to one of the core motivations behind the 1-2-3 Conjecture, the authors of [2] also investigated proper labellings minimising the sum of labels assigned to the edges.

Each of these previous studies led to presumptions of independent interest. In particular, it is believed in [3], that every nice graph $G$ admits a proper labelling where the maximum vertex colour is at most $2\Delta(G)$ (recall that $\Delta(G)$ and $\delta(G)$ are used to denote the maximum and the minimum, resp., degree of any vertex of $G$), while, from [2], it is believed that every $G$ should admit a proper labelling where the sum of assigned labels is at most $2|E(G)|$. One of the main reasons why these presumptions are supposed to hold is that, in general, it seems that nice graphs admit 2-labellings that are almost proper, in the sense that they need only a few 3's to design proper 3-labellings. This belief on the number of 3's is long-standing, as, in a way, it lies behind the 1-2 Conjecture of Przybyło and Woźniak [12], which states that we should be able to build a proper 2-labelling of every graph if we can also locally alter each vertex colour a bit.

Our goal in this work is to study and formally establish the intuition that, in general, graphs should admit proper 3-labellings assigning only a few 3's. First, we study whether, given a (possibly infinite) class $\mathcal{F}$ of graphs, the members of $\mathcal{F}$ admit proper 3-labellings assigning only a constant number of 3's. Note that this holds, for instance, for all nice trees since they admit proper 2-labellings [4]. In case $\mathcal{F}$ admits no such constant $c_\mathcal{F}$, i.e., the number of 3's the members of $\mathcal{F}$ need in their proper 3-labellings is a function of their number of edges, the second question we consider is whether the number of 3's needed can be "large" for a given member of $\mathcal{F}$, with respect to the number of its edges.

In this work, we investigate these two questions in general and for restricted classes of graphs. We begin in Section 2 by formally introducing the terminology that we employ throughout this work. In Section 3, we introduce proof techniques for establishing lower and upper bounds on the number of 3's needed in proper

3-labellings for some graph classes. In Section 4, we use these tools to establish that, for several classes of graphs, the number of required 3's in their proper 3-labellings is not bounded by an absolute constant. In such cases, we exhibit bounds (functions depending on the size of said graphs) on this number.

## 2 Terminology and a conjecture

For any notation on graph theory not defined in the paper, we refer the reader to [7]. Let $G$ be a (nice) graph, and $\ell$ be a $k$-labelling of $G$. For any $i \in \{1, \dots, k\}$, we denote by $\mathrm{nb}_\ell(i)$ the number of edges assigned label $i$ by $\ell$. Focusing now on proper 3-labellings, we denote by $\mathrm{mT}(G)$ the minimum number of edges assigned label 3 by a proper 3-labelling of $G$. That is, $\mathrm{mT}(G) = \min\{\mathrm{nb}_\ell(3) : \ell \text{ is a proper 3-labelling of } G\}$. We extend this parameter mT to classes $\mathcal{F}$ of graphs by defining $\mathrm{mT}(\mathcal{F})$ as the maximum value of $\mathrm{mT}(G)$ over the members $G$ of $\mathcal{F}$. Clearly, $\mathrm{mT}(\mathcal{F}) = 0$ for every class $\mathcal{F}$ of graphs admitting proper 2-labellings (i.e., $\chi_\Sigma(G) \leq 2$ for every $G \in \mathcal{F}$). Given a graph class $\mathcal{F}$, we are interested in determining whether $\mathrm{mT}(\mathcal{F}) \leq p$ for some $p \geq 0$. From that perspective, for every $p \geq 0$, we denote by $\mathcal{G}_p$ the class of graphs $G$ with $\mathrm{mT}(G) = p$. For convenience, we also define $\mathcal{G}_{\leq p} := \mathcal{G}_0 \cup \dots \cup \mathcal{G}_p$.

Since nice trees admit proper 2-labellings [4], if $\mathcal{T}$ is the class of all nice trees, then the notation above allows us to state that $\mathcal{T} \subset \mathcal{G}_0$. More generally speaking, bipartite graphs form perhaps the most investigated class of graphs in the context of the 1-2-3 Conjecture. A notable result, due to Thomassen, Wu, and Zhan [14], is that bipartite graphs verify the 1-2-3 Conjecture. These graphs were further studied in several works, such as [2], in which it was proved that:

**Theorem 1 ([2]).** *If $G$ is a nice bipartite graph, then $G \in \mathcal{G}_{\leq 2}$.*

Theorem 1 is worrisome since, even without additional constraints, we do not know much about how proper 3-labellings behave beyond the scope of bipartite graphs. Our take in this work is to focus on the next natural case to consider, that of 3-chromatic graphs, which fulfil the 1-2-3 Conjecture [11]. Unfortunately, as will be seen later on, a result equivalent to Theorem 1 for 3-chromatic graphs does not exist, even for very restricted classes of 3-chromatic graphs.

As mentioned earlier, we will see throughout this work that, for several graph classes $\mathcal{F}$, there is no $p \geq 0$ such that $\mathcal{F} \subset \mathcal{G}_{\leq p}$. For such a class, we want to know whether the proper 3-labellings of their members require assigning label 3 many times, with respect to their number of edges. We study this aspect through the following terminology. For a nice graph $G$, we define $\rho_3(G) := \mathrm{mT}(G)/|E(G)|$. We extend this ratio to a class $\mathcal{F}$ by setting $\rho_3(\mathcal{F}) = \max\{\rho_3(G) : G \in \mathcal{F}\}$.

In this work, we are interested in determining bounds on $\rho_3(\mathcal{F})$ for graph classes $\mathcal{F}$ of 3-chromatic graphs, and, generally speaking, in how large this ratio can be. Among the sample of small connected graphs (e.g., of order at most 6), the maximum ratio $\rho_3$ is exactly $1/3$, and is attained by $C_3$ and $C_6$. These are the worst graphs we know of, which leads us to raising the following conjecture.

*Conjecture 1.* If $G$ is a nice connected graph, then $\rho_3(G) \leq \frac{1}{3}$.

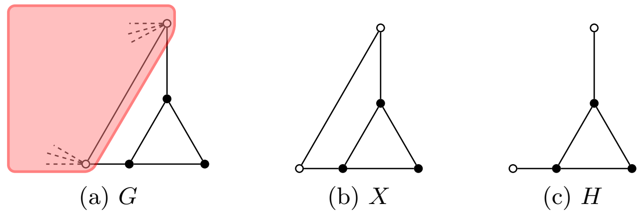



Fig. 1: A graph $G$ containing a graph $H$ as a weakly induced subgraph $X$. In $G$, the white vertices can have arbitrarily many neighbours in the red part, while the full neighbourhood of the black vertices is as displayed. In $H$, the white vertices are the border vertices, while the black vertices are the core vertices.

## 3 Tools for establishing bounds on mT and $\rho_3$

### 3.1 Weakly induced subgraphs – A tool for lower bounds

Our lower bounds on mT and $\rho_3$ exhibited in Section 4 are through a graph construction requiring the following terminology. For two graphs $G$ and $H$, we say that $G$ *contains* $H$ *as a weakly induced subgraph* $X$ (see Figure 1) if there exists an induced subgraph $X$ of $G$ such that $H$ is a spanning subgraph of $X$, and, for every vertex $v \in V(H)$, either $d_H(v) = 1$ or $d_H(v) = d_G(v)$ (note that $d_G(v) = |\{v \in V(G) : uv \in E(G)\}|$ denotes the degree of $v$ in a (sub)graph $G$). In other words, if we add to $H$ the edges of $G$ that connect the vertices of degree 1 in $H$, we get $X$. That is, for every edge $uv \in E(G)$, if $u \in V(X)$ and $v \in V(G) \setminus V(X)$, then $d_H(u) = 1$; we call these the *border vertices* of $H$. Also, we call the other vertices of $H$ (those that are not border vertices) its *core vertices*. By the definitions, if $G$ contains $H$ as a weakly induced subgraph and $\delta(H) \geq 2$, then $G$ is isomorphic to $H$, and thus, this notion makes more sense when $\delta(H) = 1$.

Two weakly induced subgraphs of a graph $G$, $X_1$ and $X_2$, are *disjoint* (in $G$) if they share no core vertices. It follows from the definition that, for every $v \in V(G)$, if $v \in V(X_1) \cap V(X_2)$, then $v$ is a border vertex of both $X_1$ and $X_2$.

Let $\ell$ be a labelling of $G$. For a subgraph $H$ of $G$, we denote by $\ell|_H$ the *restriction* of $\ell$ to the edges of $H$, i.e., we have $\ell|_H(e) = \ell(e)$ for every edge $e \in E(H)$. Assume now that $G$ contains $H$ as a weakly induced subgraph $X$. Abusing the notations, we will sometimes write $\ell|_H$, which refers to the labelling of $H$ inferred from $\ell|_X$, i.e., where $\ell|_H(e) = \ell|_X(e)$ for every $e \in E(H)$.

The key result is that, if a graph $G$ contains other graphs $H_1, \ldots, H_n$ as pairwise disjoint weakly induced subgraphs, then $\mathrm{mT}(G) \geq \sum_{i=1}^n \mathrm{mT}(H_i)$.

**Lemma 1.** *Let $G$ be a graph containing nice graphs $H_1, \ldots, H_n$ as pairwise disjoint weakly induced subgraphs $X_1, \ldots, X_n$. If $\ell$ is a proper 3-labelling of $G$, then $\ell|_{H_i}$ is a proper 3-labelling of $H_i$ for every $i \in \{1, \ldots, n\}$. Consequently,* $\mathrm{mT}(G) \geq \sum_{i=1}^n \mathrm{mT}(H_i)$.

*Proof.* Consider $H_j$ for some $1 \leq j \leq n$. Since, by any labelling of a nice graph, a vertex of degree 1 cannot get the same colour as its unique neighbour, then it cannot be involved in a conflict. This implies that $\ell|_{H_j}$ is proper if and only if any two adjacent core vertices of $H_j$ get distinct colours by $\ell|_{H_j}$. By the definition of a weakly induced subgraph, recall that we have $d_{H_j}(v) = d_{X_j}(v) = d_G(v)$ for every core vertex $v$ of $H_j$, which implies that $c_{\ell|_{H_j}}(v) = c_{\ell|_{X_j}}(v) = c_\ell(v)$. Thus, for every edge $uv \in E(H_j)$ joining core vertices, we have $c_\ell(u) = c_{\ell|_{H_j}}(u) = c_{\ell|_{X_j}}(u) \neq c_{\ell|_{X_j}}(v) = c_{\ell|_{H_j}}(v) = c_\ell(v)$ since $\ell$ is proper, meaning that $\ell|_{H_j}$ is also proper. Now, since $G$ contains nice graphs $H_1, \dots, H_n$ as pairwise disjoint weakly induced subgraphs $X_1, \dots, X_n$, then $\mathrm{mT}(G) \geq \sum_{i=1}^{n} \mathrm{mT}(H_i)$. □

The next lemma points out that, in some contexts, we can add some structure to a given graph without altering its value of mT. This will be useful for applying inductive arguments or simplifying the structure of a considered graph later on.

**Lemma 2.** *Let $G$ be a nice graph with minimum degree 1 and $v \in V(G)$ be such that $d(v) = 1$. If $G'$ is the graph obtained from $G$ by adding $x > 0$ vertices of degree 1 adjacent to $v$, then $\mathrm{mT}(G') = \mathrm{mT}(G)$.*

Next, we prove each graph class $\mathcal{G}_p$ $(p \geq 1)$ contains infinitely many graphs.

**Theorem 2.** *Given a graph $G$ and any (fixed) integer $p > 1$, deciding if $G \in \mathcal{G}_{\leq p}$ is NP-complete.*

*Sketch of proof.* We do a reduction from the 2-Labelling problem, which is NP-hard even when the graph has minimum degree 1 [8]. Given an instance $H$ of 2-Labelling such that $\delta(H) = 1$, we construct a graph $G$ such that $\mathrm{mT}(G) = p$ if and only if $H$ admits a proper 2-labelling. The graph $G$ is constructed by identifying (all to one vertex $w$) a vertex of degree 1 from each of $p$ copies of a nice graph $H'$ and a vertex of degree 1 of $H$, where $\delta(H') = 1$ and $\mathrm{mT}(H') = 1$ ($H'$ exists, see Section 4), and adding many leaves adjacent to $w$. The result follows from Lemma 1 since $G$ contains $p$ copies of $H'$ and one copy of $H$ as pairwise disjoint weakly induced subgraphs (and since it is easy to deduce a proper 3-labelling of $G$ using $p + mT(H)$ 3's). ◇

### 3.2 Switching closed walks – A tool for upper bounds

Due to Theorem 1, investigating the parameters mT and $\rho_3$ only makes sense for graphs with chromatic number at least 3, i.e., that are not bipartite.

**Theorem 3.** *If $G$ is a connected 3-chromatic graph, then $\mathrm{mT}(G) \leq |V(G)|$, and thus $\rho_3(G) \leq \frac{|V(G)|}{|E(G)|}$.*

*Proof.* Since $G$ is not bipartite, there exists an odd-length cycle $C$ in $G$. Let $H$ be a subgraph of $G$ constructed as follows. Start from $C = H$. Then, until $V(H) = V(G)$, repeatedly choose a vertex $v \in V(G) \setminus V(H)$ such that there exists a vertex $u \in V(H)$ with $uv \in E(G)$, and add the edge $uv$ to $H$. In the end, $H$ is a connected spanning subgraph of $G$ containing only one cycle, $C$, which is of odd length. Then, we have $|E(H)| = |V(G)|$.

Let $\phi : V(G) \to \{0, 1, 2\}$ be a proper 3-vertex-colouring of $G$. In what follows, we construct a 3-labelling $\ell$ of $G$ such that $c_\ell(v) \equiv \phi(v) \bmod 3$ for every vertex $v \in V(G)$, thus making $\ell$ proper. To prove the full statement, we also want $\ell$ to satisfy $\mathrm{nb}_\ell(3) \leq |V(G)|/|E(G)|$. Aiming at vertex colours modulo 3, we can instead assume that $\ell$ assigns labels $0, 1, 2$, and require $\mathrm{nb}_\ell(0) \leq |V(G)|/|E(G)|$. To obtain such a labelling, we start from $\ell$ assigning label 2 to all edges of $G$. We then modify $\ell$ iteratively until all vertex colours are as desired modulo 3.

As long as $G$ has a vertex $v$ with $c_\ell(v) \not\equiv \phi(v) \bmod 3$, we apply the following procedure. Choose $W = (v, v_1, \dots, v_n, v)$, a closed walk[3] of odd length in $G$ starting and ending at $v$, and going through edges of $H$ only. This walk exists. Indeed, consider, in $H$, a (possibly empty) path $P$ from $v$ to the closest vertex $u$ of $C$ (if $v$ lies on $C$, then note that $u = v$ and $P$ has no edge). Then, the closed walk $vPuCuPv$ is a possible $W$. We then follow the consecutive edges of $W$, starting from $v$ and ending at $v$, and, going along, we apply $+2, -2, +2, -2, \dots, +2$ (modulo 3) to the labels assigned by $\ell$ to the traversed edges. As a result, $c_\ell(x)$ is not altered modulo 3 for every vertex $x \neq v$, while $c_\ell(v)$ is incremented by 1 modulo 3. If $c_\ell(v) \equiv \phi(v) \bmod 3$, then we are done with $v$. Otherwise, we repeat this switching procedure once again, so that $v$ fulfils that property.

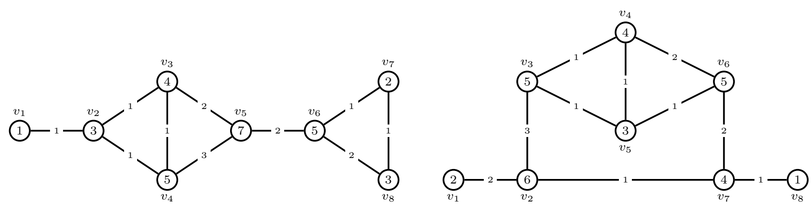
Eventually, $c_\ell(v) \equiv \phi(v) \bmod 3$ for every $v \in V(G)$, meaning that $\ell$ is proper. Recall that we have $\ell(e) = 2$ for every $e \in E(G) \setminus E(H)$. Thus, only the edges of $H$ can be assigned label 0 by $\ell$. Since there are exactly $|V(G)|$ such edges, and we can replace all assigned 0's with 3's without breaking the modulo 3 property, we have $\mathrm{mT}(G) \leq |V(G)|$, which implies that $\rho_3(G) \leq |V(G)|/|E(G)|$. $\square$

In the next lemma, we show a way to play with $\phi$ in order to reduce the number of 3's assigned by $\ell$ to certain sets of edges.

**Lemma 3.** *Let $G$ be a graph and $\ell$ be a proper $\{0, 1, 2\}$-labelling of $G$ such that $c_\ell(u) \not\equiv c_\ell(v) \bmod 3$ for every edge $uv \in E(G)$. If $H$ is a (not necessarily connected) spanning $d$-regular subgraph of $G$ for some $d \geq 1$, then there exists a proper $\{0, 1, 2\}$-labelling $\ell'$ of $G$ such that $c_{\ell'}(u) \not\equiv c_{\ell'}(v) \bmod 3$ for every edge $uv \in E(G)$ and that assigns label 0 to at most a third of the edges of $E(H)$. Moreover, for every edge $e \in E(G) \setminus E(H)$, $\ell'(e) = \ell(e)$.*

*Proof.* We construct the following labelling: starting from $\ell$, add 1 (modulo 3) to all the labels assigned by $\ell$ to the edges of $H$. The resulting labelling $\ell_1$ is a proper $\{0, 1, 2\}$-labelling of $G$ such that $c_{\ell_1}(u) \not\equiv c_{\ell_1}(v) \bmod 3$ for every edge $uv \in E(G)$. Indeed, for every $v \in V(G)$, we have $c_{\ell_1}(v) \equiv c_\ell(v) + d \bmod 3$. Thus, if there exist two vertices $u, v \in V(G)$ such that $c_{\ell_1}(u) \equiv c_{\ell_1}(v) \bmod 3$, then $c_\ell(u) \equiv c_\ell(v) \bmod 3$, a contradiction. We define $\ell_2$ in a similar way, by adding 1 (modulo 3) to all the labels assigned by $\ell_1$ to the edges of $H$. Similarly, $\ell_2$ is proper. Since, for every edge $e \in E(H)$, we have $\{\ell(e), \ell_1(e), \ell_2(e)\} = \{0, 1, 2\}$, then at least one of $\ell, \ell_1, \ell_2$ assigns label 0 to at most a third of the edges of

[3] Recall that a *walk* in a graph is a path in which vertices and edges can be repeated.



(a) $A_1$ (b) $A_2$

Fig. 2: Proper 3-labellings $\ell$ of $A_1$ and $A_2$ with $\mathrm{nb}_\ell(3) = 1$. The colours by $c_\ell$ are indicated by integers within the vertices.

$E(H)$. Since none of the labels of the edges of $E(G) \setminus E(H)$ were changed to obtain $\ell_1$ from $\ell$ and to get $\ell_2$ from $\ell_1$, the last statement of the lemma holds. □

In Lemma 3, if $d = 2$, then $H$ forms a cycle cover of $G$. Thus, when $H$ is also a unicyclic spanning connected subgraph of $G$, an application of Lemma 3 in conjunction with the proof of Theorem 3 gives the following:

**Corollary 1.** *If $G$ is Hamiltonian, of odd order, and $\chi(G) = 3$, then $\rho_3(G) \leq \frac{1}{3}$.*

# 4 Results for mT and $\rho_3$ for some graph classes

We now use the tools from Section 3 to exhibit results on the parameters mT and $\rho_3$ for some classes of 3-chromatic graphs. In particular, we prove that, for many classes $\mathcal{F}$ of 3-chromatic graphs, there is no $p \geq 1$ such that $\mathcal{F} \subset \mathcal{G}_{\leq p}$. In most cases, we provide upper bounds for $\rho_3(\mathcal{F})$.

## 4.1 Connected graphs needing lots of 3's

As mentioned before, we are aware of only two connected graphs for which $\rho_3$ is exactly 1/3, and these are $C_3$ and $C_6$[4]. One question to ask, is if the bound in Conjecture 1 is accurate in general, i.e., whether it can be attained by arbitrarily large graphs. In light of these thoughts, our goal in this section is to provide a class of arbitrarily large connected graphs achieving the largest possible ratio $\rho_3$.

We ran computer programs to find graphs $H$ with $\delta(H) = 1$, $\mathrm{mT}(H) \geq 1$, and with the fewest edges possible. It turns out that the smallest such graphs have 10 edges. Two such graphs, which we call $A_1$ and $A_2$, are depicted in Figure 2. The following observation, proven by simple case analysis, allows us to use these two graphs to build arbitrarily large connected graphs with large $\rho_3$.

**Observation 4.** $\mathrm{mT}(A_1) = \mathrm{mT}(A_2) = 1$.

**Theorem 5.** *There are arbitrarily large connected graphs $G$ with $\rho_3(G) \geq \frac{1}{10}$.*

[4] Conjecture 1 focuses on connected graphs since any disjoint union of $C_3$'s and $C_6$'s reaches that value.

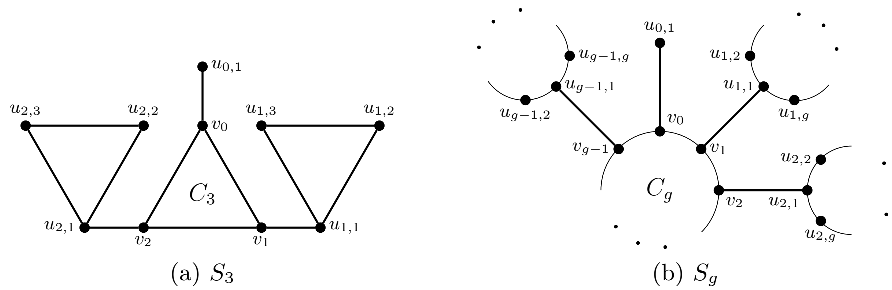



Fig. 3: The planar graphs $S_3$ (left) and $S_g$ (right) of girths 3 and $g$, respectively.

*Proof.* Let $p \geq 1$ be fixed. We construct a connected graph $G$ with $10p$ edges, such that $\mathrm{nb}_\ell(3) \geq p$ for any proper 3-labelling $\ell$ of $G$, which implies that $\rho_3(G) \geq 1/10$. Start, as $G$, with $p$ disjoint copies of $A_1$ (or $A_2$), and identify a vertex of degree 1 from each of these $p$ copies to a single vertex. The labelling property follows from Lemma 1 and Observation 4, since $G$ contains $p$ copies of $A_1$ or $A_2$ as pairwise disjoint weakly induced subgraphs. □

### 4.2 Bounds for connected planar graphs with large girth

The *girth* $g(G)$ of a graph $G$ is the length of its shortest cycle. For any $g \geq 3$, we denote by $\mathcal{P}_g$ the class of planar graphs with girth at least $g$. Note that $\mathcal{P}_3$ is the class of all planar graphs, and that $\mathcal{P}_4$ is the class of all triangle-free planar graphs. Recall that the girth of a tree is set to $\infty$, since it has no cycle.

To date, it is still unknown whether planar graphs verify the 1-2-3 Conjecture, which makes the study of the parameters mT and $\rho_3$ adventurous for this class of graphs. However, there is no $p \geq 1$ such that planar graphs lie in $\mathcal{G}_{\leq p}$ by the construction in the proof of Theorem 5, since the graphs $A_1$ and $A_2$ are planar. Thus, there exist arbitrarily large connected planar graphs $G$ with $\rho_3(G) \geq 1/10$.

To go further, we consider planar graphs with large girth. By Grötzsch's Theorem [9], triangle-free planar graphs are 3-colourable, which means they verify the 1-2-3 Conjecture (see [11]). In what follows, first, we prove that, for every $g \geq 3$, there is no $p \geq 1$ such that $\mathcal{P}_g \subset \mathcal{G}_{\leq p}$. Second, we prove that, as the girth $g(G)$ of a planar graph $G$ grows, the ratio $\rho_3(G)$ decreases. As a side result, we prove Conjecture 1 for planar graphs with girth at least 36. To prove the first result above, we cannot use the graphs $A_1$ and $A_2$ introduced previously, as they contain triangles. Instead, we use the graph $S_g$ illustrated in Figure 3.

**Lemma 4.** *For every $g \geq 3$ with $g \equiv 3 \bmod 4$, we have $\mathrm{mT}(S_g) = 1$.*

*Sketch of proof.* Let $\ell$ be a proper 2-labelling of $S_g$. Due to the length of each outer cycle, it follows that $c_\ell(u_{i,1}) = \ell(u_{i,1}v_i) + 3$. Moreover, due to $C_g$ being of odd length, we can deduce that there must exist a vertex $v_x \in V(C_g)$ such that $v_x \neq v_0$ and $c_\ell(v_x) = \ell(v_x u_{x,1}) + 3 = c_\ell(u_{x,1})$, a contradiction. The rest of the result follows from identifying a proper 3-labelling $\ell'$ of $S_g$ with $\mathrm{nb}_{\ell'}(3) = 1$. ⋄

By taking arbitrarily many copies of the $S_g$ graph and identifying their respective roots (the vertex $u_{0,1}$ in Figure 3), we can prove that:

**Theorem 6.** *For every $g' \geq 3$, there exist arbitrarily large connected planar graphs $G$ with $g(G) \geq g'$ and $\rho_3(G) \geq \frac{1}{g^2+g}$, where $g$ is the smallest natural number such that $g \geq g'$ and $g \equiv 3 \bmod 4$.*

We now proceed to prove that $\rho_3(G) \leq \frac{2}{k-1}$ for any planar graph $G$ of girth $g \geq 5k+1$, when $k \geq 7$. The next theorem from [5] is one of the tools we use to prove this result. Note that, for any $k \geq 1$, a *$k$-thread* in a graph $G$ is a path $(u_1, \ldots, u_{k+2})$, where the $k$ inner vertices $u_2, \ldots, u_{k+1}$ all have degree 2 in $G$.

**Theorem 7 ([5]).** *For any integer $k \geq 1$, every planar graph with minimum degree at least 2 and girth at least $5k+1$ contains a $k$-thread.*

We can now proceed with the main theorem.

**Theorem 8.** *Let $k \geq 7$. If $G$ is a nice planar graph with $g(G) \geq 5k+1$, then $\rho_3(G) \leq \frac{2}{k-1}$.*

*Proof.* Throughout this proof, we set $g = g(G)$. The proof is by induction on the order of $G$. The base case is when $|V(G)| = 3$. In that case, $G$ must be a path of length 2 (due to the girth assumption), and the claim is clearly true. So let us focus on proving the general case.

If $G$ is a tree, then $\chi_\Sigma(G) \leq 2$ and we have $\rho_3(G) = 0$. So, from now on, we may assume that $G$ is not a tree. We first deal with the case of planar graphs $G$ of girth $g \geq 5k+1$ for which there exists at least one cut vertex $v \in V(G)$ (meaning that $G - \{v\}$ has more connected components than $G$) such that $G - \{v\}$ contains a connected component $T'$ that is a tree such that $|E(T')| \geq 1$ and the induced subgraph of $G$ formed by the vertices of $T'$ and $v$ is also a tree. Let $u \in V(T')$ be the neighbour of $v$. By the inductive hypothesis, $\rho_3(G - V(T')) \leq \frac{2}{k-1}$ since removing a pendant tree from $G$ can neither decrease its girth nor result in a tree, and thus, by recursively applying Lemma 2, there is a proper 3-labelling of $G$ such that $\rho_3(G) \leq \frac{2}{k-1}$. The same arguments can be applied for all such connected components of $G - \{v\}$. Hence, we can assume that $G$ does not contain any such cut vertex $v$. Another way to state this, is that if $G$ contains a vertex $v$ to which a pending tree $T'$ is attached, then $T'$ is a star with center $v$.

Let $G^-$ be the graph obtained from $G$ by removing all vertices of degree 1. Note that removing vertices of degree 1 from $G$ can neither decrease its girth nor result in a tree. Since $G$ has girth $g \geq 5k+1$ and does not contain any cut vertex $v \in V(G)$ as described above, the graph $G^-$ has minimum degree 2. By Theorem 7, $G^-$ contains a $k$-thread $P$. Let $u_1, \ldots, u_{k+2}$ be the vertices of $P$, where $d_H(u_i) = 2$ for all $2 \leq i \leq k+1$. Thus, the vertices of $P$ exist in $G$ except that each of the vertices $u_i$ (for $2 \leq i \leq k+1$) may be adjacent to some vertices of degree 1 in addition to their adjacencies in $G^-$. Let $G'$ be the graph obtained from $G$ by removing the vertices $u_3, \ldots, u_k$ and all of their neighbours that have degree 1 in $G$. Note that $G'$ might contain up to two connected components. In case $G'$ has exactly two connected components, then, due to a previous assumption, none of these can be a tree, which implies that $G'$ is nice. If $G'$ is connected, then, because it has at least two edges ($u_1u_2$ and $u_{k+1}u_{k+2}$), it must be nice. Furthermore, in both cases, the girth of $G'$ is at least that of $G$. Then, by combining the inductive hypothesis and the fact that every nice tree $T$ verifies $\rho_3(T) = 0$, we deduce that $\rho_3(G') \leq \frac{2}{k-1}$.

To obtain a proper 3-labelling $\ell$ of $G$ such that $\rho_3(G) \leq \frac{2}{k-1}$, we extend a proper 3-labelling $\ell'$ of $G'$ corresponding to $\rho_3(G') \leq \frac{2}{k-1}$, as follows. First, label all of the edges incident to the vertices of degree 1 and the vertices $u_3, \dots, u_k$ with 1's. Note that none of these vertices of degree 1 can, later on, be in conflict with their neighbour since they have degree 1. Now, for each $2 \leq j \leq k-2$, in increasing order of $j$, label the edge $u_ju_{j+1}$ with 1 or 2, so that the resulting colour of $u_j$ does not conflict with the colour of $u_{j-1}$. Finally, label the edges $u_{k-1}u_k$ and $u_ku_{k+1}$ with 1, 2 or 3, so that the resulting colour of $u_{k-1}$ does not conflict with that of $u_{k-2}$, the resulting colour of $u_k$ does not conflict with that of $u_{k-1}$ nor with that of $u_{k+1}$, and the resulting colour of $u_{k+1}$ does not conflict with that of $u_{k+2}$. Indeed, this is possible since there exist at least two distinct labels $\{\alpha, \beta\}$ ($\{\alpha', \beta'\}$, respectively) in $\{1, 2, 3\}$ for $u_{k-1}u_k$ ($u_ku_{k+1}$, respectively) such that the colour of $u_{k-1}$ ($u_{k+1}$, respectively) is not in conflict with that of $u_{k-2}$ ($u_{k+2}$, respectively). Thus, w.l.o.g., choose $\alpha$ and $\alpha'$ for the labels of $u_{k-1}u_k$ and $u_ku_{k+1}$, respectively. If the colour of $u_k$ does not conflict with that of $u_{k-1}$ nor with that of $u_{k+1}$, then we are done. If the colour of $u_k$ conflicts with both that of $u_{k-1}$ and that of $u_{k+1}$, then it suffices to change both the labels of $u_{k-1}u_k$ and $u_ku_{k+1}$ to $\beta$ and $\beta'$, respectively. Lastly, w.l.o.g., if the colour of $u_k$ only conflicts with that of $u_{k-1}$, then it suffices to change the label of $u_ku_{k+1}$ to $\beta'$. The resulting labelling $\ell$ of $G$ is thus proper. Moreover, $|E(G) \setminus E(G')| \geq k-1$ and $\ell$ uses label 3 at most twice more than $\ell'$, and so, the result follows. □

### 4.3 Bounds for connected cacti

A *cactus* is a graph in which any two cycles have at most one vertex in common. The graphs $S_g$ introduced in Section 4.2, and those constructed in order to prove Theorem 6, are all cacti. Since the smallest graph $S_g$ is $S_3$, which has 12 edges, that theorem directly implies that there exist arbitrarily large connected cacti $G$ with $\rho_3(G) \geq 1/12$. We now prove Conjecture 1 for cacti.

**Theorem 9.** *If $G$ is a nice cactus, then $\rho_3(G) \leq \frac{1}{3}$.*

*Sketch of proof.* The proof is by induction on $|V(G)|$. The general case is proven by focusing on *end-cycles* $C_1, \dots, C_q$, to which pending trees might be attached, and sharing a root vertex $r$ that separates these cycles from the rest of the graph. By analysing the $C_i$'s, it can be proved that the induction hypothesis can be invoked to get a desired labelling of $G$, as soon as one of their inner vertices has a pending tree attached, or the pending tree attached to $r$ is not a star with center $r$. So the $C_i$'s can be assumed to be mostly cycles, in which case we can remove their inner vertices, invoke the induction hypothesis, and extend a labelling of the remaining graph to a desired one of $G$, by labelling the edges of the $C_i$'s so that only a few 3's are assigned. $\diamond$

### 4.4 An upper bound for Halin graphs

A *Halin graph* is a planar graph with minimum degree 3 obtained as follows. Start from a tree $T$ with no vertex of degree 2, and consider a planar embedding of $T$. Then, add edges to form a cycle going through all the leaves of $T$ in the clockwise ordering in this embedding. A Halin graph is called a *wheel* if it is constructed from a tree $T$ with diameter at most 2. Halin graphs have triangles and Hamiltonian cycles going through any given edge [15]. Also, Halin graphs are 3-degenerate, so, due to the presence of triangles, each of them has chromatic number 3 or 4. The dichotomy is well understood, as a Halin graph has chromatic number 4 if and only if it is a wheel of even order [16]. It is easy to see that these wheels admit proper 2-labellings, and so, we focus on 3-chromatic Halin graphs in the proof of the next theorem. In particular, we use our tools from Section 3 to establish an upper bound on $\rho_3$ for the 3-chromatic Halin graphs.

**Theorem 10.** *If $G$ is a Halin graph, then $\rho_3(G) \leq \frac{1}{3}$.*

*Proof.* As said above, we can assume that $G$ is not a wheel of even order. Then $\chi(G) = 3$. If $|V(G)|$ is odd, then the result follows from Corollary 1. Thus, we can assume that $|V(G)|$ is even.

By considering any non-leaf vertex $r$ of $T$ in $G$, and defining a usual root-to-leaf (virtual) orientation, since no vertex has degree 2 in $T$, then $G$ has a triangle $(u, v, w, u)$, where $v, w$ are leaves in $T$ with parent $u$. Furthermore, $d_G(v) = d_G(w) = 3$, while $d_G(u) \geq 3$. Due to these degree properties, if we consider $C$ a Hamiltonian cycle traversing $uv$, then $C$ must also include either $wu$ or $vw$. Precisely, if we orient the edges of $C$, resulting in a spanning oriented cycle $\vec{C}$, then, at some point, $\vec{C}$ enters $(u, v, w, u)$ through one of its vertices, goes through another vertex of the triangle and then through the third of its vertices, before leaving the triangle. In other words, $C$ traverses all vertices of $(u, v, w, u)$ at once.

Up to relabelling the vertices of $(u, v, w, u)$, we can assume that $\vec{C}$ enters the triangle through $u$, then goes to $v$, before going to $w$ and leaving the triangle. Let us consider $H$, the subgraph of $G$ containing the three edges of $(u, v, w, u)$, and all successive edges traversed by $C$ after leaving the triangle except for the edge going back to $u$. Note that $H$ is a unicyclic spanning connected subgraph of $G$, in which the only cycle is the triangle $(u, v, w, u)$ to which is attached a hanging path $(w, x_1, \dots, x_{n-3})$ containing all other vertices of $G$ (i.e., $n = |V(G)|$). Furthermore, in $E(G) \setminus E(H)$, if we set $x = x_{n-3}$, then the edge $xu$ exists. Since $H$ is spanning, connected, and unicyclic, $|E(H)| = |V(G)|$, which is at most $2|E(G)|/3$, since $\delta(G) \geq 3$.

All conditions are now met to invoke the arguments in the proof of Theorem 3, from which we can deduce a proper $\{0, 1, 2\}$-labelling of $G$ where adjacent vertices get distinct colours modulo 3 and in which only the edges of the chosen $H$ are possibly assigned label 0. Let us consider the subgraph $H'$ of $G$ obtained from

$H$ by adding the edge $xu$, which is present in $G$. Recall that $\ell(xu) = 2$ by default. Note that $H'$ contains at least two disjoint perfect matchings $M_1, M_2$. Indeed, since $|V(G)|$ is even, then, in $H$, the hanging path attached at $w$ has odd length. A first perfect matching $M_1$ of $H'$ contains $x_{n-3}x_{n-4}, x_{n-5}x_{n-6}, \dots, wx_1$ and $uv$. A second perfect matching $M_2$ of $H'$ contains $x_{n-4}x_{n-5}, x_{n-6}x_{n-7}, \dots, x_2x_1$, and $wv$ and $xu$. By Lemma 3, we can assume that at most a third of the edges in $M_1 \cup M_2$ are assigned label 0 by $\ell$. Since $|M_1|+|M_2| = |E(H')|-1 = |E(H)|$, this gives $\mathrm{nb}_\ell(0) \leq \frac{E(H)}{3} + 1$, which is less than $|E(G)|/3$ since $|E(G)| \geq 3|V(G)|/2$. By turning 0's by $\ell$ into 3's, we get a proper 3-labelling of $G$ with the same upper bound on the number of assigned 3's. □

### 4.5 Bounds for outerplanar graphs

First off, we note that the construction described in the proof of Theorem 5, when performed with copies of $A_1$ only, provides graphs that are outerplanar[5], since $A_1$ is itself outerplanar. Recall as well that outerplanar graphs form a subclass of series-parallel graphs. Thus, there exist arbitrarily large connected outerplanar (series-parallel, resp.) graphs $G$ ($H$, resp.) with $\rho_3(G) \geq 1/10$ ($\rho_3(H) \geq 1/10$, resp.). Note however that the outerplanar graphs constructed above have cut vertices. So the question remains, whether or not this lower bound still holds when considering 2-connected outerplanar graphs (recall that outerplanar graphs are 2-degenerate, and thus, each of them is 3-chromatic and either separable or 2-connected). As for an upper bound, we can provide the following:

**Theorem 11.** *If $G$ is a 2-connected outerplanar graph such that $|E(G)| \geq |V(G)| + 3$, then $\rho_3(G) \leq \frac{1}{3}$.*

*Sketch of proof.* If $|V(G)|$ is odd, the result follows from Corollary 1. If $|V(G)|$ is even, then there is an odd-length cycle of $G$ that consists of consecutive vertices of the outer face of $G$, and thus, since $G$ is Hamiltonian, there is a unicyclic spanning connected subgraph $H$ containing an odd cycle. Theorem 3 can be applied using this $H$, and Lemma 3 can be applied to two disjoint perfect matchings containing all the edges of $H$ but one, as in the proof of Theorem 10. ◇

Theorem 11 covers all 2-connected outerplanar graphs with at least three chords but it can also be shown to hold when there are at most two chords.

## 5 Further Work

A first direction for further research is to prove Conjecture 1 for more classes of graphs such as for other classes of 3-chromatic graphs like separable outerplanar graphs and, more generally, series-parallel graphs. Another one is to investigate whether the bound of 1/3 in that conjecture is close to being tight or not, in particular for large graphs. Also, we were not able to come up with examples of arbitrarily large Halin graphs needing many 3's in their proper 3-labellings.

[5] An *outerplanar graph* admits a planar embedding with all vertices on the outer face.

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