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Existence of solutions to a phase-field model for the isothermal solidification process of a binary alloy *

J. Rappaz¹, J.F. Scheid²

Abstract. We investigate the well-posedness of a phase-field model for the isothermal solidification of a binary alloy due to Warren-Boettinger [12]. Existence of weak solution as well as regularity and uniqueness results are established under Lipschitz and boundeness assumptions for the nonlinearities. A maximum principle holds that guarantees the existence of a solution under physical assumptions on the nonlinearities.

MOS subject classification : 82C26, 35K55, 35B65, 35B50.

1 Introduction.

In this paper we study the well-posedness of a solutal phase-field model for the solidification of a binary alloy at a constant temperature. Such a model was derived by Warren-Boettinger [12]. It involves the relative concentration c and an order parameter ϕ which accounts for the solidification state of the alloy by being equal to 0 if the system is in a solid phase and equal to 1 if it is in a liquid phase. The time evolution of c and ϕ is governed by the following equations :

$$(P) \quad \begin{cases} \frac{\partial \phi}{\partial t} = \varepsilon^2 \Delta \phi + F_1(\phi) + c F_2(\phi) & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial c}{\partial t} = \operatorname{div} (D_1(\phi) \nabla c + D_2(c, \phi) \nabla \phi) & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \phi}{\partial n} = \frac{\partial c}{\partial n} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ \phi(0) = \phi_0, \quad c(0) = c_0 & \text{in } \Omega, \end{cases}$$

where Ω is an open subset of $\mathbb{R}^d$ with $1 \leq d \leq 3$ and with boundary $\partial\Omega$, n is the unit normal to $\partial\Omega$ and $\varepsilon > 0$ is a given positive constant.

The functions F_1 , F_2 , D_1 and D_2 appearing in (P) have the following properties :

- F_1 and F_2 are regular functions such that $F_i(0) = F_i(1) = 0$ for $i = 1, 2$.
- (H) • D_1 is a positive and regular function bounded above and below by two positive constants.
- D_2 is a regular function such that $D_2(c, \cdot) = 0$ for $c = 0$ and $c = 1$.

Moreover, initial physical data c_0 and ϕ_0 are given and have to stay between 0 and 1. The solutions c and ϕ of Problem (P) must be found with the same property.

Phase-field models have been principally used to describe phase transitions of pure materials due to thermal effects and they lead to nonlinear parabolic systems for the phase-field and the temperature. However, the nonlinearities are then different from those of problem (P). Such thermal models have been well studied and we refer for instance to [2], [3], [1] and [8] where existence and uniqueness results are investigated.

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In the next section, we give a sketch of the modelling leading to problem (P). This derivation is based on [12] and [7]. We establish the well-posedness of problem (P) by first assuming that the nonlinear terms F_i and D_i , $i = 1, 2$ are bounded and Lipschitz functions. Then, under these assumptions, we prove the existence of a weak solution in Section 3 by the use of a Faedo-Galerkin method. Next, regularity and uniqueness results are investigated in Section 4. Finally, we obtain in Section 5 a maximum principle under the additional assumptions that the nonlinear functions F_1 , F_2 and D_2 vanish beyond the sets $\{0 < c < 1\}$ and $\{0 < \phi < 1\}$. This maximum principle ensures that c and ϕ stay between 0 and 1 so long as the initial data c_0 and ϕ_0 do. Then this justifies the assumptions made in Section 3 about the boundeness and Lipschitz conditions for the nonlinear terms.

2 Modelling

This section is devoted to a brief introduction of the derivation of the phase-field model we study. For a complete description of the modelling, we refer to [12], [7].

Let us consider an alloy of two components A and B in a spacial domain Ω of $\mathbb{R}^d$ with $d \leq 3$. The system is on the one-hand characterized by the relative concentration $c = c(x, t)$ of the component B with respect to the mixture, for $x \in \Omega$ and at time t . The function c is then always between 0 and 1. On the other-hand, the solidification state of the alloy is taken into account by an order parameter $\phi = \phi(x, t)$, the so-called *phase-field*. More precisely, the phase-field equals 0 if the system is in a solid phase and equals 1 if it is in a liquid phase, but may take continuous values between 0 and 1.

We suppose that the solidification process takes place at a constant and homogeneous temperature T which is fixed somewhere between the two melting temperatures of the components A and B . To have a thermodynamical description of the system, we introduce a Ginzburg-Landau free energy functional by using a free energy density f which depends on c and ϕ . The thermodynamical laws together with the mass conservation law imply the following equations for ϕ and c (see [12], [7]) :

$$\alpha \frac{\partial \phi}{\partial t} = \varepsilon^2 \Delta \phi - \frac{\partial f}{\partial \phi} \quad \text{in } \Omega, \quad (1)$$

$$\frac{\partial c}{\partial t} = \operatorname{div} \left(M \nabla \frac{\partial f}{\partial c} \right) \quad \text{in } \Omega, \quad (2)$$

where ε and α are positive parameters and $M = M(c, \phi)$ is a positive function.

Moreover, c and ϕ are subject to homogeneous Neumann boundary conditions on the boundary $\partial\Omega$ of Ω .

Using some basic thermodynamical principles, we can infer (see [12], [7]) that

$$-\frac{\partial f}{\partial \phi} = F_1(\phi) + cF_2(\phi), \quad (3)$$

where F_1 , F_2 are functions of ϕ that vanish for $\phi = 0$ and $\phi = 1$, and

$$\nabla \frac{\partial f}{\partial c} = -F_2(\phi) \nabla \phi + \frac{RT}{v_m} \frac{1}{c(1-c)} \nabla c, \quad (4)$$

where R is the Boltzman constant and v_m the molar volume.

We point out that F_1 and F_2 can be choosen as polynomial functions of 4th degree that guarantee some thermodynamical constraints.

Hence, equation (1) with (3) becomes

$$\alpha \frac{\partial \phi}{\partial t} = \varepsilon^2 \Delta \phi + F_1(\phi) + c F_2(\phi) \quad \text{in } \Omega. \quad (5)$$

The parameters α and ε^2 can be related to physical parameters by exhibiting a special solution for the phase-field in a one-dimensional case for a pure element ($c \equiv 0$ or $c \equiv 1$) and performing an asymptotic analysis as ε tends towards 0 (see [12], [7], [6]). In that way, ε^2 is proportional to the interface thickness.

Moreover, in order to recover a classical diffusion equation for c whenever $\phi \equiv 0$ or $\phi \equiv 1$ i.e. in a totally solid or liquid phase, we choose

$$M(c, \phi) = \frac{v_m}{RT} D_1(\phi) c(1 - c), \quad (6)$$

where D_1 is an increasing smooth function such that $D_1(0) > 0$ and $D_1(1) > 0$ are respectively the solid and liquid diffusion coefficients, and D_1 is bounded above and below by two positive constants $D_s < D_l$.

Thus, equation (2) with (4) and (6) leads to

$$\frac{\partial c}{\partial t} = \operatorname{div} (D_1(\phi) \nabla c + D_2(c, \phi) \nabla \phi) \quad \text{in } \Omega, \quad (7)$$

where $D_2(c, \phi) = -\frac{v_m}{RT} D_1(\phi) c(1 - c) F_2(\phi)$ is a smooth function vanishing for $c = 0$ and $c = 1$.

Without loss of generality for the purpose of this article, we may choose $\alpha = 1$ and then the equations (5) and (7) together with homogeneous Neumann boundary conditions and initial conditions lead to problem (P).

Notations and assumptions . Throughout the article, we assume that Ω is an open bounded domain in $\mathbb{R}^d$, $d \leq 3$ with a smooth boundary $\partial\Omega$ of class C^∞ . We note $V = H^1(\Omega)$ and V' is the dual space $(H^1(\Omega))'$ of $H^1(\Omega)$. We denote by $\langle \cdot, \cdot \rangle_{V', V}$ the duality product between V' and V and we also denote $Q_T = \Omega \times (0, T)$, for $T > 0$.

3 Existence results.

In this section, we prove the existence of a weak solution to problem (P) under Lipschitz and boundeness assumptions on the nonlinear functions F_i and D_i , $i = 1, 2$. More precisely, we assume that

(H1) · $F_1, F_2 \in C(\mathbb{R})$ are Lipschitz and bounded functions with

$$|F_i(r)| \leq M_1 \quad \text{for } i = 1, 2 \text{ and } \forall r \in \mathbb{R}.$$

(H2) · $D_1 \in C(\mathbb{R})$ is a Lipschitz positive and bounded function with

$$0 < D_s \leq D_1(r) \leq D_l, \quad \forall r \in \mathbb{R}.$$

(H3) · $D_2 \in C(\mathbb{R} \times \mathbb{R})$ is a Lipschitz and bounded function with

$$|D_2(r_1, r_2)| \leq M_2, \quad \forall (r_1, r_2) \in \mathbb{R} \times \mathbb{R}.$$

Before expressing and proving the existence result relative to the previous assumptions, we recall a well-known result from elliptic theory (see for instance [9], Chap. 2, Th. 5.1) that will be useful later on.

Lemma 1 *Let $k \in \mathbb{N}$ and $u \in H^2(\Omega)$ satisfying $\Delta u \in H^k(\Omega)$ and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Then $u \in H^{k+2}(\Omega)$ and there exists a constant $C > 0$ independent of u such that*

$$\|u\|_{H^{k+2}(\Omega)} \leq C \left(\|\Delta u\|_{H^k(\Omega)} + \|u\|_{H^k(\Omega)} \right). \quad (8)$$

Now, we state the main result of this section :

Theorem 1 *Let assumptions (H1)-(H3) be fulfilled.*

1) *For any $(\phi_0, c_0) \in L^2(\Omega) \times L^2(\Omega)$ and $T > 0$, there exists a couple of functions (ϕ, c) satisfying*

$$\phi, c \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V'),$$

such that $\phi(0) = \phi_0$, $c(0) = c_0$ and

$$\begin{aligned} &< \frac{\partial \phi}{\partial t}, v >_{V', V} + \varepsilon^2 \int_{\Omega} \nabla \phi \cdot \nabla v \, dx = \int_{\Omega} (F_1(\phi) + cF_2(\phi)) v \, dx, \\ &< \frac{\partial c}{\partial t}, w >_{V', V} + \int_{\Omega} (D_1(\phi) \nabla c + D_2(c, \phi) \nabla \phi) \cdot \nabla w \, dx = 0, \end{aligned} \quad (9)$$

for all $v, w \in H^1(\Omega)$ and a.e. in $(0, T)$.

2) *For any $(\phi_0, c_0) \in H^1(\Omega) \times L^2(\Omega)$ and $T > 0$, there exists a couple of functions (ϕ, c) satisfying*

$$\begin{aligned} \phi &\in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\ c &\in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V'), \end{aligned}$$

such that $\phi(0) = \phi_0$, $c(0) = c_0$ and

$$\begin{aligned} &\frac{\partial \phi}{\partial t} - \varepsilon^2 \Delta \phi = F_1(\phi) + cF_2(\phi) \quad \text{a.e. in } Q_T, \\ &\frac{\partial \phi}{\partial n} = 0 \quad \text{a.e. on } \partial\Omega \times (0, T), \\ &< \frac{\partial c}{\partial t}, v >_{V', V} + \int_{\Omega} (D_1(\phi) \nabla c + D_2(c, \phi) \nabla \phi) \cdot \nabla v \, dx = 0, \\ &\text{for all } v \in H^1(\Omega) \text{ and a.e. in } (0, T). \end{aligned} \quad (10)$$

REMARK : Since $\phi, c \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V')$ it follows that $\phi, c \in C([0, T]; L^2(\Omega))$. Moreover, as soon as $\phi \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$, we infer that $\phi \in C([0, T]; H^1(\Omega))$ (see for instance [9], Th. 3.1, p. 23).

Proof of part 1) of Theorem 1 : We employ a Faedo-Galerkin method. We consider the sequence λ_i of the eigenvalues of the operator $(-\Delta)$ with homogeneous Neumann boundary conditions such that

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots,$$

and the corresponding eigenfunctions $\{v_j\}_{j \geq 1}$ such that $(v_j, v_k)_{L^2(\Omega)} = \delta_{jk}$ for $1 \leq j, k$. Note that the eigenfunctions satisfy $(\nabla v_j, \nabla v_k)_{L^2(\Omega)} = 0$ for $j \neq k$, $1 \leq j, k$. Moreover, the eigenfunctions are smooth functions and $\{v_j\}_{j \geq 1}$ form a complete orthogonal basis in $L^2(\Omega)$ and in $H^1(\Omega)$. We denote by V_m the finite vector space spanned by $\{v_j\}_{1 \leq j \leq m}$. Then $\bigcup_{m \geq 1} V_m$

is dense in $L^2(\Omega)$ and also in $H^1(\Omega)$. In the sequel, we will make use of the L^2 -orthogonal projector p_m on the space V_m . Note that we have, for all $\varphi \in H^1(\Omega)$

$$(\nabla(p_m \varphi - \varphi), \nabla v)_{L^2(\Omega)} = 0 \quad \forall v \in V_m, \quad (11)$$

and then p_m is also the H^1 -orthogonal projector on V_m .

For each $m \geq 1$, we then consider the approximate problem of finding $\phi_m, c_m \in H^1(0, T; V_m)$ satisfying

$$(P_m) \begin{cases} \int_{\Omega} \frac{\partial \phi_m}{\partial t} v \, dx + \varepsilon^2 \int_{\Omega} \nabla \phi_m \cdot \nabla v \, dx = \int_{\Omega} (F_1(\phi_m) + c_m F_2(\phi_m)) v \, dx, & (12) \\ \int_{\Omega} \frac{\partial c_m}{\partial t} w \, dx + \int_{\Omega} (D_1(\phi_m) \nabla c_m + D_2(c_m, \phi_m) \nabla \phi_m) \cdot \nabla w \, dx = 0, & (13) \\ \text{for all } v, w \in V_m \text{ and for a.e. } t \in (0, T), \\ \phi_m(0) = \phi_{0m} \in V_m, \quad c_m(0) = c_{0m} \in V_m. & (14) \end{cases}$$

We choose $\phi_{0m} = p_m \phi_0$ and $c_{0m} = p_m c_0$. Then, the initial data have the following properties

$$\|\phi_{0m}\|_{L^2(\Omega)} \leq \|\phi_0\|_{L^2(\Omega)} \quad \text{and} \quad \phi_{0m} \rightarrow \phi_0 \text{ in } L^2(\Omega) \text{ as } m \rightarrow +\infty, \quad (15)$$

$$\|c_{0m}\|_{L^2(\Omega)} \leq \|c_0\|_{L^2(\Omega)} \quad \text{and} \quad c_{0m} \rightarrow c_0 \text{ in } L^2(\Omega) \text{ as } m \rightarrow +\infty. \quad (16)$$

We look for an approximate solution (ϕ_m, c_m) of the form

$$\phi_m(t) = \sum_{i=1}^m \varphi_{im}(t) v_i, \quad c_m(t) = \sum_{i=1}^m c_{im}(t) v_i,$$

so that Problem (P_m) is in fact an initial value problem for a system of $2m$ ordinary differential equations which can read as

$$\begin{cases} U'_m(t) + B(U_m(t)) U_m(t) = \mathcal{F}(U_m(t)), \\ U_m(0) = U_0, \end{cases}$$

where $U_m(t) = \begin{pmatrix} (\varphi_{im}(t)) \\ (c_{im}(t)) \end{pmatrix} \in \mathbb{R}^{2m}$.

Since the nonlinear terms F_1, F_2, D_1 and D_2 are Lipschitz continuous functions and since (v_j) are smooth functions, the above problem has a unique solution $U_m(t)$ on some maximal time interval $[0, T_m)$ with $T_m > 0$ and $U_m \in C^1([0, T_m))$. Hence Problem (P_m) has a unique solution (ϕ_m, c_m) on the time interval $[0, T_m)$ and $(\phi_m, c_m) \in C^1([0, T_m), V_m \times V_m)$.

Next we derive some estimates which show in particular that $T_m = +\infty$. From now on, we will denote by C a positive constant which depends on $\varepsilon^2, M_1, M_2, D_s, D_l, |\Omega|, T$, on the Lipschitz constants of nonlinear terms $F_i, D_i, i = 1, 2$ and also on $\|\phi_0\|_{L^2(\Omega)}$ and $\|c_0\|_{L^2(\Omega)}$ but which is independent of m .

A priori estimates.

i) We choose $v = \phi_m$ in (12) and $w = c_m$ in (13) to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \phi_m^2 + \varepsilon^2 \int_{\Omega} |\nabla \phi_m|^2 = \int_{\Omega} F_1(\phi_m) \phi_m + \int_{\Omega} F_2(\phi_m) c_m \phi_m, \quad (17)$$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} c_m^2 + \int_{\Omega} D_1(\phi_m) |\nabla c_m|^2 = - \int_{\Omega} D_2(c_m, \phi_m) \nabla \phi_m \cdot \nabla c_m. \quad (18)$$

Multiplying (18) by $\delta > 0$ that will be choosen later, adding it to (17) and using also (H1)-(H3), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\phi_m^2 + \delta c_m^2) + \varepsilon^2 \int_{\Omega} |\nabla \phi_m|^2 + \delta D_s \int_{\Omega} |\nabla c_m|^2 \\ & \leq M_1 \left(\int_{\Omega} |\phi_m| + \int_{\Omega} |c_m| |\phi_m| \right) + M_2 \delta \int_{\Omega} |\nabla \phi_m| |\nabla c_m| \\ & \leq C \left(1 + \int_{\Omega} \phi_m^2 + \int_{\Omega} c_m^2 \right) + \frac{M_2 \delta}{2\eta} \int_{\Omega} |\nabla \phi_m|^2 + \frac{M_2 \delta}{2} \eta \int_{\Omega} |\nabla c_m|^2, \end{aligned}$$

for all $\eta > 0$, by the use of Young's inequality. Choosing $\delta = D_s \frac{\varepsilon^2}{M_2^2}$ and $\eta = \frac{D_s}{M_2}$, we get

$$\frac{d}{dt} \int_{\Omega} (\phi_m^2 + \delta c_m^2) + \varepsilon^2 \int_{\Omega} |\nabla \phi_m|^2 + \delta D_s \int_{\Omega} |\nabla c_m|^2 \leq 2C \left(1 + \int_{\Omega} \phi_m^2 + \int_{\Omega} c_m^2 \right). \quad (19)$$

It follows by Gronwall's lemma and (14), (15), (16) that there exists a constant $C > 0$ depending on T_m such that, for all $0 \leq t < T_m$

$$\|\phi_m(t)\|_{L^2(\Omega)} + \|c_m(t)\|_{L^2(\Omega)} \leq C. \quad (20)$$

The estimate (20) shows that $T_m = +\infty$. Moreover, we can see that if t belongs to the fixed interval $[0, T]$, then the estimate (20) holds with a constant C independent of m . Integrating inequality (19) from 0 to T and using (20), (19), we conclude there exists a constant C such that

$$\|(c_m, \phi_m)\|_{(L^2(0,T;H^1(\Omega)))^2} \leq C, \quad (21)$$

$$\|(c_m, \phi_m)\|_{(L^\infty(0,T;L^2(\Omega)))^2} \leq C. \quad (22)$$

ii) Let us now give estimates of the time derivatives of ϕ_m and c_m . Due to the fact that the L^2 -orthogonal projector p_m is also the $H^1(\Omega)$ -orthogonal projector on V_m , we have for $\varphi_m \in V_m$

$$\|\varphi_m\|_{V'} \equiv \sup_{\substack{\psi \in V \\ \psi \neq 0}} \frac{(\varphi_m, \psi)_{V',V}}{\|\psi\|_V} = \max_{\substack{\psi_m \in V_m \\ \psi_m \neq 0}} \frac{(\varphi_m, \psi_m)_{L^2(\Omega)}}{\|\psi_m\|_V}.$$

Thus, from equation (12), we infer that for all $0 \leq t \leq T$

$$\left\| \frac{\partial \phi_m(t)}{\partial t} \right\|_{V'} \leq \varepsilon^2 \|\nabla \phi_m(t)\|_{L^2(\Omega)} + C (1 + \|c_m(t)\|_{L^2(\Omega)})$$

and then

$$\left\| \frac{\partial \phi_m}{\partial t} \right\|_{L^2(0,T;V')} \leq C (1 + \|c_m\|_{L^2(0,T)} + \|\phi_m\|_{L^2(0,T;H^1(\Omega))}). \quad (23)$$

In an analogous way, we obtain the following estimate for the time derivative of c_m :

$$\left\| \frac{\partial c_m}{\partial t} \right\|_{L^2(0,T;V')} \leq C (\|c_m\|_{L^2(0,T;H^1(\Omega))} + \|\phi_m\|_{L^2(0,T;H^1(\Omega))}). \quad (24)$$

Thus, using estimate (21) in (23) and (24), we get that

$$\left\| \left(\frac{\partial c_m}{\partial t}, \frac{\partial \phi_m}{\partial t} \right) \right\|_{(L^2(0,T;V'))^2} \leq C. \quad (25)$$

Taking the limit.

We deduce from (21), (22) and (25) that c_m and ϕ_m are bounded (uniformly with respect to m) in

$$W_1 = \left\{ u \in L^2(0, T; H^1(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0, T; V') \right\}$$

and in

$$W_2 = \left\{ u \in L^\infty(0, T; L^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0, T; V') \right\}.$$

Since the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, W_1 is compactly embedded into $L^2(Q_T)$ (see for instance [11], Chap. 3, Th. 2.1). Moreover, due to the compactness of the embedding of $L^2(\Omega)$ into V' , the space W_2 is compactly embedded into $C([0, T], V')$ (see [10], Cor. 4). Hence, there exist

$$c, \phi \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V')$$

and subsequences of c_m and ϕ_m (we still denote c_m and ϕ_m) such that, as $m \rightarrow +\infty$,

$$(c_m, \phi_m) \rightharpoonup (c, \phi) \quad \text{in } (L^2(0, T; H^1(\Omega)))^2 \text{ weakly,} \quad (26)$$

$$(c_m, \phi_m) \rightarrow (c, \phi) \quad \text{in } (L^2(Q_T))^2 \cap (C([0, T], V'))^2 \text{ strongly,} \quad (27)$$

$$\left(\frac{\partial c_m}{\partial t}, \frac{\partial \phi_m}{\partial t} \right) \rightharpoonup \left(\frac{\partial c}{\partial t}, \frac{\partial \phi}{\partial t} \right) \quad \text{in } (L^2(0, T, V'))^2 \text{ weakly.} \quad (28)$$

In order to take the limit in Problem (P_m) , we need the following convergence results for the nonlinear terms.

Lemma 2

$$F_i(\phi_m) \rightarrow F_i(\phi) \quad i = 1, 2 \quad \text{in } L^p(Q_T), \quad \forall p \in [1, +\infty), \quad (29)$$

$$D_1(\phi_m) \rightarrow D_1(\phi) \quad \text{in } L^p(Q_T), \quad \forall p \in [1, +\infty), \quad (30)$$

$$c_m F_2(\phi_m) \rightarrow c F_2(\phi) \quad \text{in } L^q(Q_T), \quad \forall q \in [1, 2), \quad (31)$$

$$D_1(\phi_m) \nabla c_m \rightharpoonup D_1(\phi) \nabla c \quad \text{in } L^q(Q_T) \text{ weakly, } \forall q \in [1, 2), \quad (32)$$

$$D_2(c_m, \phi_m) \nabla \phi_m \rightharpoonup D_2(c, \phi) \nabla \phi \quad \text{in } L^q(Q_T) \text{ weakly, } \forall q \in [1, 2). \quad (33)$$

Proof of Lemma 2 : The convergences in (29), (30) and (31) are obtained using the fact that $\phi_m \rightarrow \phi$ and $c_m \rightarrow c$ in $L^2(Q_T)$ strongly and the assumptions (H1), (H2) on F_i and D_1 . The convergences in (32) and (33) required the weak convergences (26) but the strong convergence (27) is necessary to obtain the convergence in (33). We only give details for the proof of (33).

Let $v \in L^{q^*}(Q_T)$ with $\frac{1}{q^*} + \frac{1}{q} = 1$, $q \in [1, 2)$ and consider

$$\mathcal{A}_m = \int_{Q_T} (D_2(c_m, \phi_m) \nabla \phi_m - D_2(c, \phi) \nabla \phi) v \, dx dt.$$

Then we write $\mathcal{A}_m$ in the following form

$$\mathcal{A}_m = \int_{Q_T} D_2(c, \phi) (\nabla \phi_m - \nabla \phi) v \, dx dt + \int_{Q_T} (D_2(c_m, \phi_m) - D_2(c, \phi)) \nabla \phi_m v \, dx dt.$$

The first term tends to 0 as $m \rightarrow 0$ thanks to (26) and to the fact that $D_2(c, \phi)v \in L^2(Q_T)$. Let $\mathcal{A}'_m$ denote the second term in the expression of $\mathcal{A}_m$. Then, using Hölder's inequality, we have

$$|\mathcal{A}'_m| \leq \|D_2(c_m, \phi_m) - D_2(c, \phi)\|_{L^s(Q_T)} \|\nabla \phi_m\|_{L^2(Q_T)} \|v\|_{L^{q^*}(Q_T)},$$

with $\frac{1}{q^*} + \frac{1}{2} + \frac{1}{s} = 1$. Moreover, since $s \geq 2$ and D_2 is a bounded Lipschitz function, it follows that

$$\|D_2(c_m, \phi_m) - D_2(c, \phi)\|_{L^s(Q_T)}^s \leq C \left(\|c_m - c\|_{L^2(Q_T)}^2 + \|\phi_m - \phi\|_{L^2(Q_T)}^2 \right).$$

We then show that $\mathcal{A}_m \rightarrow 0$ by the use of (27) and (25). □

We now take the limit in Problem (P_m) and find that c and ϕ satisfy equations (9) for all $v, w \in H^1(\Omega)$. Moreover, we conclude from (14), (15), (16) and (27) that $\phi(0) = \phi_0$ and $c(0) = c_0$.

Proof of part 2) of Theorem 1 : We suppose now that $\phi_0 \in H^1(\Omega)$. Since we chose $\phi_{0m} = p_m \phi_0$ as initial data for Problem (P_m) , we deduce from the property (11) about p_m , that

$$\|\nabla \phi_{0m}\|_{L^2(\Omega)} \leq \|\nabla \phi_0\|_{L^2(\Omega)}. \quad (34)$$

We derive then some new estimates for ϕ_m . From now on, we still denote by C a positive constant as in the proof of part 1), but it depends now also on $\|\nabla \phi_0\|_{L^2(\Omega)}$.

i) First, let us choose $v = \frac{\partial \phi_m}{\partial t}$ in (12). We obtain

$$\int_{\Omega} \left| \frac{\partial \phi_m}{\partial t} \right|^2 + \varepsilon^2 \int_{\Omega} \nabla \phi_m \cdot \nabla \frac{\partial \phi_m}{\partial t} = \int_{\Omega} F_1(\phi_m) \frac{\partial \phi_m}{\partial t} + \int_{\Omega} F_2(\phi_m) c_m \frac{\partial \phi_m}{\partial t}.$$

Using (H1) and Young's inequality, we have for all $\eta_1, \eta_2 > 0$:

$$\int_{\Omega} \left| \frac{\partial \phi_m}{\partial t} \right|^2 + \frac{\varepsilon^2}{2} \frac{d}{dt} \int_{\Omega} |\nabla \phi_m|^2 \leq \eta_1 M_1 |\Omega| + \left(\frac{M_1}{4\eta_1} + \frac{M_2}{4\eta_2} \right) \int_{\Omega} \left| \frac{\partial \phi_m}{\partial t} \right|^2 + M_2 \eta_2 \int_{\Omega} |c_m|^2.$$

We choose $\eta_1 = M_1$ and $\eta_2 = M_2$ so that we obtain

$$\int_{\Omega} \left| \frac{\partial \phi_m}{\partial t} \right|^2 + \varepsilon^2 \frac{d}{dt} \int_{\Omega} |\nabla \phi_m|^2 \leq C \left(1 + \int_{\Omega} |c_m|^2 \right).$$

Integrating this inequality over $(0, t)$ for $t \in (0, T)$, we get

$$\int_0^t \int_{\Omega} \left| \frac{\partial \phi_m}{\partial t} \right|^2 + \varepsilon^2 \int_{\Omega} |\nabla \phi_m(t)|^2 \leq C \left(t + \int_0^t \int_{\Omega} |c_m|^2 \right) + \varepsilon^2 \int_{\Omega} |\nabla \phi_0|^2.$$

Hence, thanks to (21) we conclude there exists C such that

$$\left\| \frac{\partial \phi_m}{\partial t} \right\|_{L^2(Q_T)} + \|\phi_m\|_{L^\infty(0, T; H^1(\Omega))} \leq C. \quad (35)$$

ii) Finally, by remarking that $-\Delta \phi_m \in V_m$, we choose $v = -\Delta \phi_m$ in (12). Integrating by parts the two terms in the resulting left hand-side and using the fact that ϕ_m satisfies the homogeneous Neumann boundary condition on $\partial\Omega$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \phi_m|^2 + \varepsilon^2 \int_{\Omega} |\Delta \phi_m|^2 = - \int_{\Omega} F_1(\phi_m) \Delta \phi_m - \int_{\Omega} F_2(\phi_m) c_m \Delta \phi_m.$$

Using (H1) and Young's inequality, we get for all $\eta_1, \eta_2 > 0$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \phi_m|^2 + \varepsilon^2 \int_{\Omega} |\Delta \phi_m|^2 \leq \eta_1 M_1 |\Omega| + \left(\frac{M_1}{4\eta_1} + \frac{M_2}{4\eta_2} \right) \int_{\Omega} |\Delta \phi_m|^2 + M_2 \eta_2 \int_{\Omega} |c_m|^2.$$

Then choosing $\eta_1 = \frac{M_1}{\varepsilon^2}$ and $\eta_2 = \frac{M_2}{\varepsilon^2}$, it follows there exists C such that

$$\frac{d}{dt} \int_{\Omega} |\nabla \phi_m|^2 + \varepsilon^2 \int_{\Omega} |\Delta \phi_m|^2 \leq C \left(1 + \int_{\Omega} |c_m|^2 \right).$$

Integrating this inequality from 0 to T and using (21), we obtain that

$$\|\Delta \phi_m\|_{L^2(Q_T)} \leq C. \quad (36)$$

From the elliptic estimate (8) of Lemma 1, combined with estimates (21) and (36), we conclude that

$$\|\phi_m\|_{L^2(0,T;H^2(\Omega))} \leq C. \quad (37)$$

We deduce from (35) and (37) that when $\phi_0 \in H^1(\Omega)$, ϕ_m is bounded (uniformly with respect to m) in

$$W_3 = \left\{ u \in L^2(0, T; H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)) \right\}. \quad (38)$$

Since the embedding of $H^2(\Omega)$ into $H^1(\Omega)$ is compact, we conclude (see for instance [11], Chap. 3, Th. 2.1) that W_3 is compactly embedded into $L^2(0, T; H^1(\Omega))$. By taking the limit ϕ of a subsequence of ϕ_m , we can conclude the proof of part 2) of Theorem 1 in the same way for part 1). More precisely, we have

$$\phi \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$$

and there exists a subsequence of ϕ_m (we still denote ϕ_m) such that, as $m \rightarrow +\infty$,

$$\phi_m \rightarrow \phi \quad \text{in } L^2(0, T; H^1(\Omega)) \text{ strongly,} \quad (39)$$

$$\frac{\partial \phi_m}{\partial t} \rightharpoonup \frac{\partial \phi}{\partial t} \quad \text{in } L^2(Q_T) \text{ weakly.} \quad (40)$$

We then pass to the limit in Problem (P_m) and find that c and ϕ satisfy (10). □

REMARK : Theorem 1 is valid for any smooth domain $\Omega \subset \mathbb{R}^d$ with $d \geq 1$. The restriction $d \leq 3$ does not occur in the proof.

4 Regularity and uniqueness

In this section, we prove a regularity and uniqueness result under the additional assumption that the initial data are smooth enough. To begin with, we recall some properties about Sobolev spaces that will be useful later on.

Lemma 3 *Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set with a smooth boundary. Then the following properties hold :*

1) $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq 6$ with continuous embedding. (41)

2) Gagliardo-Nirenberg inequalities (see [5]).

There exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$\|u\|_{L^3(\Omega)} \leq C_1 \|u\|_{H^1(\Omega)}^{1/2} \|u\|_{L^2(\Omega)}^{1/2} \quad \text{for all } u \in H^1(\Omega). \quad (42)$$

$$\|u\|_{L^6(\Omega)} \leq C_2 \|u\|_{H^2(\Omega)}^{1/2} \|u\|_{L^2(\Omega)}^{1/2} \quad \text{for all } u \in H^2(\Omega). \quad (43)$$

We state the following regularity and uniqueness result.

Theorem 2 *Let assumptions (H1)-(H3) be fulfilled.*

Let $\phi_0 \in H^2(\Omega)$ such that $\frac{\partial \phi_0}{\partial n} = 0$ on $\partial\Omega$ and $c_0 \in H^1(\Omega)$. Then for any $T > 0$, there exists a unique couple of functions (ϕ, c) satisfying

$$\begin{aligned} \phi &\in L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega)), \\ c &\in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \end{aligned} \quad (44)$$

such that $\phi(0) = \phi_0$, $c(0) = c_0$ and

$$\begin{aligned} \frac{\partial \phi}{\partial t} - \varepsilon^2 \Delta \phi &= F_1(\phi) + c F_2(\phi) \quad \text{a.e. in } Q_T, \\ \frac{\partial c}{\partial t} &= \operatorname{div}(D_1(\phi) \nabla c + D_2(c, \phi) \nabla \phi) \quad \text{a.e. in } Q_T, \\ \frac{\partial \phi}{\partial n} = \frac{\partial c}{\partial n} &= 0 \quad \text{a.e. on } \partial\Omega \times (0, T). \end{aligned} \quad (45)$$

REMARK : We infer from the regularity of the solution (see e.g. [9]) that $\phi \in C([0, T]; H^2(\Omega))$ and $c \in C([0, T]; H^1(\Omega))$

Proof: **a) Existence with regularity.** To establish the regularity of a solution, we derive further a priori estimates in the Faedo-Galerkin procedure. Subsequently we will denote by C a positive constant depending in particular on $\|\phi_0\|_{H^2(\Omega)}$ and $\|c_0\|_{H^1(\Omega)}$ but which is independent of m .

Additional a priori estimates for ϕ_m .

i) First, we choose $v = \Delta^2 \phi_m \in V_m$ in equation (12) of the approximate problem (P_m) . From the assumption (H1) we have that $F_1, F_2 \in W^{1,\infty}(\mathbb{R})$. Then, integrating by parts the equation resulting from the specific choice of the test function and owing to the homogeneous Neumann boundary condition satisfied by $\frac{\partial \phi_m}{\partial t}$ and $\Delta \phi_m$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\Delta \phi_m\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 = \mathcal{F}_m, \quad (46)$$

where $\mathcal{F}_m = - \int_{\Omega} (F_1'(\phi_m) \nabla \phi_m + F_2(\phi_m) \nabla c_m + F_2'(\phi_m) c_m \nabla \phi_m) \cdot \nabla \Delta \phi_m \, dx$.

We estimate $\mathcal{F}_m$ as follows. Using Young's inequality, we get the existence of a constant C such that

$$\mathcal{F}_m \leq \frac{\varepsilon^2}{4} \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 + C \left(\|\nabla \phi_m\|_{L^2(\Omega)}^2 + \|\nabla c_m\|_{L^2(\Omega)}^2 + \int_{\Omega} |c_m \nabla \phi_m \cdot \nabla \Delta \phi_m| dx \right). \quad (47)$$

The term $\mathcal{B}_m = \int_{\Omega} |c_m \nabla \phi_m \cdot \nabla \Delta \phi_m| dx$ appearing in (47) is estimated by Hölder's inequality and the Gagliardo-Nirenberg inequalities (42)-(43) :

$$\begin{aligned} \mathcal{B}_m &\leq \|c_m\|_{L^3(\Omega)} \|\nabla \phi_m\|_{L^6(\Omega)} \|\nabla \Delta \phi_m\|_{L^2(\Omega)} \\ &\leq C_1 C_2 \|c_m\|_{H^1(\Omega)}^{1/2} \|c_m\|_{L^2(\Omega)}^{1/2} \|\nabla \phi_m\|_{H^2(\Omega)}^{1/2} \|\nabla \phi_m\|_{L^2(\Omega)}^{1/2} \|\nabla \Delta \phi_m\|_{L^2(\Omega)}. \end{aligned}$$

Using (22) and (35), i.e. the fact that c_m is bounded in $L^\infty(0, T; L^2(\Omega))$ and ϕ_m is bounded in $L^\infty(0, T; H^1(\Omega))$ together uniformly with m , we have

$$\mathcal{B}_m \leq C \|c_m\|_{H^1(\Omega)}^{1/2} \|\nabla \phi_m\|_{H^2(\Omega)}^{1/2} \|\nabla \Delta \phi_m\|_{L^2(\Omega)}.$$

Using Young's inequality, we deduce that for all $\delta > 0$:

$$\mathcal{B}_m \leq C \left(\frac{\delta}{2} \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 + \frac{1}{2\delta} \left(\|c_m\|_{H^1(\Omega)} \|\nabla \phi_m\|_{H^2(\Omega)} \right) \right). \quad (48)$$

Moreover, the elliptic estimate (8) gives

$$\|\phi_m\|_{H^3(\Omega)} \leq C \left(\|\Delta \phi_m\|_{H^1(\Omega)} + \|\phi_m\|_{H^1(\Omega)} \right)$$

and consequently

$$\|\nabla \phi_m\|_{H^2(\Omega)} \leq C \left(\|\Delta \phi_m\|_{L^2(\Omega)} + \|\phi_m\|_{H^1(\Omega)} + \|\nabla \Delta \phi_m\|_{L^2(\Omega)} \right).$$

Using again Young's inequality, we have for all $\gamma > 0$:

$$\mathcal{B}_m \leq C \left(\frac{\delta}{2} \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 + \frac{1}{2\delta} \left(\frac{1}{2\gamma} \|c_m\|_{H^1(\Omega)}^2 + \frac{\gamma}{2} \left(\|\Delta \phi_m\|_{L^2(\Omega)}^2 + \|\phi_m\|_{H^1(\Omega)}^2 + \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 \right) \right) \right).$$

By choosing $\gamma = 2\delta^2$, we obtain

$$\mathcal{B}_m \leq C \left(\delta \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 + \frac{1}{8\delta^3} \|c_m\|_{H^1(\Omega)}^2 + \frac{\delta}{2} \left(\|\Delta \phi_m\|_{L^2(\Omega)}^2 + \|\phi_m\|_{H^1(\Omega)}^2 \right) \right). \quad (49)$$

By choosing $C\delta = \frac{\varepsilon^2}{4}$, we obtain with (47) :

$$\mathcal{F}_m \leq \frac{\varepsilon^2}{2} \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 + C \left(\|c_m\|_{H^1(\Omega)}^2 + \|\Delta \phi_m\|_{L^2(\Omega)}^2 + \|\phi_m\|_{H^1(\Omega)}^2 \right),$$

that gives, with the identity (46) :

$$\frac{d}{dt} \|\Delta \phi_m\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 \leq C \left(\|\Delta \phi_m\|_{L^2(\Omega)}^2 + \|\phi_m\|_{H^1(\Omega)}^2 + \|c_m\|_{H^1(\Omega)}^2 \right). \quad (50)$$

Integrating inequality (50) over $(0, t)$ for $t \in (0, T)$ and using (21) i.e. the fact that ϕ_m and c_m are bounded in $L^2(0, T; H^1(\Omega))$ uniformly with m , we get

$$\|\Delta \phi_m(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \int_0^t \|\nabla \Delta \phi_m\|_{L^2(\Omega)}^2 \leq \|\Delta \phi_{0m}\|_{L^2(\Omega)}^2 + C \left(1 + \int_0^t \|\Delta \phi_m(t)\|_{L^2(\Omega)}^2 \right). \quad (51)$$

Moreover, since $(\phi_0 - \phi_{0m}, v)_{L^2(\Omega)} = 0$ for all $v \in V_m$, we obtain through iterated integrations by parts

$$\begin{aligned}\|\Delta\phi_{0m}\|_{L^2(\Omega)}^2 &= (\phi_{0m}, \Delta^2\phi_{0m})_{L^2(\Omega)} = (\phi_0, \Delta^2\phi_{0m})_{L^2(\Omega)} \\ &= (\Delta\phi_0, \Delta\phi_{0m})_{L^2(\Omega)} - \int_{\partial\Omega} \frac{\partial\phi_0}{\partial n} \Delta\phi_{0m} d\sigma.\end{aligned}\quad (52)$$

But since we assume that $\frac{\partial\phi_0}{\partial n} = 0$ on $\partial\Omega$, the boundary term in (52) vanishes and we conclude by the Cauchy-Schwarz inequality that

$$\|\Delta\phi_{0m}\|_{L^2(\Omega)} \leq \|\Delta\phi_0\|_{L^2(\Omega)}. \quad (53)$$

Then applying Gronwall's lemma to inequality (51) and taking into account (53), we obtain that there exists a constant $C > 0$ independent of m such that

$$\|\Delta\phi_m\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad (54)$$

$$\|\nabla\Delta\phi_m\|_{L^2(Q_T)} \leq C. \quad (55)$$

We conclude from the elliptic estimate (8) and the uniform estimates (22), (35), (36), (54), (55) that

$$\|\phi_m\|_{L^\infty(0,T;H^2(\Omega))} \leq C, \quad (56)$$

$$\|\phi_m\|_{L^2(0,T;H^3(\Omega))} \leq C, \quad (57)$$

where $C > 0$ does not depend on m .

ii) Now, we choose $v = -\frac{\partial\Delta\phi_m}{\partial t} \in V_m$ in equation (12). Since $F_1, F_2 \in W^{1,\infty}(\mathbb{R})$, we integrate by parts the resulting equation and owing to the homogeneous Neumann boundary condition satisfied by $\frac{\partial\phi_m}{\partial t}$ we obtain, as in the previous a priori estimate i) :

$$\begin{aligned}\left\|\nabla\frac{\partial\phi_m}{\partial t}\right\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{2}\frac{d}{dt}\|\Delta\phi_m\|_{L^2(\Omega)}^2 &\leq C\left(\left(\|\nabla\phi_m\|_{L^2(\Omega)} + \|\nabla c_m\|_{L^2(\Omega)}\right)\left\|\nabla\frac{\partial\phi_m}{\partial t}\right\|_{L^2(\Omega)}\right. \\ &\quad \left.+ \int_{\Omega} \left|c_m \nabla\phi_m \cdot \nabla\frac{\partial\phi_m}{\partial t}\right| dx\right).\end{aligned}\quad (58)$$

We use Hölder's inequality to estimate the last term $\mathcal{C}_m = \int_{\Omega} \left|c_m \nabla\phi_m \cdot \nabla\frac{\partial\phi_m}{\partial t}\right| dx$ in the right hand-side of (58) :

$$\mathcal{C}_m \leq \|c_m\|_{L^3(\Omega)} \|\nabla\phi_m\|_{L^6(\Omega)} \left\|\nabla\frac{\partial\phi_m}{\partial t}\right\|_{L^2(\Omega)}.$$

Thanks to the embedding (41) and the uniform estimate (56) i.e. the fact that ϕ_m is bounded in $L^\infty(0, T; H^2(\Omega))$ uniformly with respect to m , we have

$$\mathcal{C}_m \leq C \left(\left\|\nabla\frac{\partial\phi_m}{\partial t}\right\|_{L^2(\Omega)} \|c_m\|_{H^1(\Omega)} \right). \quad (59)$$

Combining estimates (58) with (59) and using Young's inequality, we obtain

$$\left\|\nabla\frac{\partial\phi_m}{\partial t}\right\|_{L^2(\Omega)}^2 + \varepsilon^2\frac{d}{dt}\|\Delta\phi_m\|_{L^2(\Omega)}^2 \leq C \left(\|\phi_m\|_{H^1(\Omega)}^2 + \|c_m\|_{H^1(\Omega)}^2 \right). \quad (60)$$

Integration of (60) from 0 to T with the use of the uniform estimates (21) and (53) yields

$$\left\| \nabla \frac{\partial \phi_m}{\partial t} \right\|_{L^2(Q_T)} \leq C$$

and due to (35), we conclude that

$$\left\| \frac{\partial \phi_m}{\partial t} \right\|_{L^2(0,T;H^1(\Omega))} \leq C, \quad (61)$$

where $C > 0$ does not depend on m .

Additional a priori estimates for c_m .

iii) We choose $w = -\Delta c_m \in V_m$ in equation (13). Using the fact that from the assumptions (H2)-(H3) we have $D_1 \in W^{1,\infty}(\mathbb{R})$ and $D_2 \in W^{1,\infty}(\mathbb{R} \times \mathbb{R})$ and integrating by parts the equation resulting from this specific choice of the test function, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla c_m\|_{L^2(\Omega)}^2 + \int_{\Omega} \operatorname{div} (D_1(\phi_m) \nabla c_m) \Delta c_m = - \int_{\Omega} \operatorname{div} (D_2(c_m, \phi_m) \nabla \phi_m) \Delta c_m.$$

Hence, using assumption (H2)-(H3) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla c_m\|_{L^2(\Omega)}^2 + D_s \|\Delta c_m\|_{L^2(\Omega)}^2 &\leq C \left(\int_{\Omega} |\nabla \phi_m \cdot \nabla c_m \Delta c_m| + \int_{\Omega} |\nabla \phi_m|^2 |\Delta c_m| \right. \\ &\quad \left. + \int_{\Omega} |\Delta \phi_m \Delta c_m| \right). \end{aligned} \quad (62)$$

We set $\mathcal{E}_1 = \int_{\Omega} |\nabla \phi_m \cdot \nabla c_m \Delta c_m|$, $\mathcal{E}_2 = \int_{\Omega} |\nabla \phi_m|^2 |\Delta c_m|$ and $\mathcal{E}_3 = \int_{\Omega} |\Delta \phi_m \Delta c_m|$. By Hölder's and Young's inequalities and thanks to the embedding (41) and the uniform estimate (56) i.e. the fact that ϕ_m is bounded in $L^\infty(0, T; H^2(\Omega))$ uniformly with respect to m , we deduce that

$$\begin{aligned} \mathcal{E}_2 &\leq C \|\nabla \phi_m\|_{L^4(\Omega)}^2 \|\Delta c_m\|_{L^2(\Omega)} \leq \frac{D_s}{6} \|\Delta c_m\|_{L^2(\Omega)}^2 + C \|\phi_m\|_{L^\infty(0,T;H^2(\Omega))}^4 \\ &\leq \frac{D_s}{6} \|\Delta c_m\|_{L^2(\Omega)}^2 + C \end{aligned} \quad (63)$$

and likewise

$$\mathcal{E}_3 \leq \frac{D_s}{6} \|\Delta c_m\|_{L^2(\Omega)}^2 + C. \quad (64)$$

To estimate $\mathcal{E}_1$ we proceed as follows. We use Hölder's inequality and the embedding (41) with (56) i.e. the fact that ϕ_m is bounded $L^\infty(0, T; H^2(\Omega))$ uniformly with m and then the Gagliardo-Nirenberg inequality (42) in order to get, if $C_1, C_2, \dots$ are constants independent of m :

$$\begin{aligned} \mathcal{E}_1 &\leq C_1 \|\nabla \phi_m\|_{L^6(\Omega)} \|\nabla c_m\|_{L^3(\Omega)} \|\Delta c_m\|_{L^2(\Omega)} \leq C_2 \|\phi_m\|_{H^2(\Omega)} \|\nabla c_m\|_{L^3(\Omega)} \|\Delta c_m\|_{L^2(\Omega)} \\ &\leq C_3 \|\nabla c_m\|_{H^1(\Omega)}^{1/2} \|\nabla c_m\|_{L^2(\Omega)}^{1/2} \|\Delta c_m\|_{L^2(\Omega)} \leq C_4 \|c_m\|_{H^2(\Omega)}^{1/2} \|\nabla c_m\|_{L^2(\Omega)}^{1/2} \|\Delta c_m\|_{L^2(\Omega)}. \end{aligned}$$

Now, by Young's inequality and the elliptic estimate (8) we obtain

$$\begin{aligned} \mathcal{E}_1 &\leq \frac{D_s}{12} \|\Delta c_m\|_{L^2(\Omega)}^2 + C_5 \left(\|\Delta c_m\|_{L^2(\Omega)} + \|c_m\|_{L^2(\Omega)} \right) \|\nabla c_m\|_{L^2(\Omega)} \\ &\leq \frac{D_s}{6} \|\Delta c_m\|_{L^2(\Omega)}^2 + C_6 \left(\|c_m\|_{L^2(\Omega)}^2 + \|\nabla c_m\|_{L^2(\Omega)}^2 \right) \\ &\leq \frac{D_s}{6} \|\Delta c_m\|_{L^2(\Omega)}^2 + C_7 \left(1 + \|\nabla c_m\|_{L^2(\Omega)}^2 \right), \end{aligned} \quad (65)$$

thanks to (22). Hence, the estimates (63), (64) and (65) in (62) imply there exists C such that

$$\frac{d}{dt} \|\nabla c_m\|_{L^2(\Omega)}^2 + D_s \|\Delta c_m\|_{L^2(\Omega)}^2 \leq C \left(1 + \|\nabla c_m\|_{L^2(\Omega)}^2\right). \quad (66)$$

Since we suppose $c_0 \in H^1(\Omega)$, we infer from (11) that $\|\nabla c_{0m}\|_{L^2(\Omega)} \leq \|\nabla c_0\|_{L^2(\Omega)}$. Thus applying Gronwall's lemma to (66) and then using the elliptic estimate (8) with (21) we conclude that there exists a constant $C > 0$ independent of m such that

$$\|c_m\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \quad (67)$$

$$\|c_m\|_{L^2(0,T;H^2(\Omega))} \leq C. \quad (68)$$

iv) We choose $w = \frac{\partial c_m}{\partial t} \in V_m$ in equation (13). Using the fact that $D_1 \in W^{1,\infty}(\mathbb{R})$ and $D_2 \in W^{1,\infty}(\mathbb{R} \times \mathbb{R})$ and integrating by parts we obtain

$$\left\| \frac{\partial c_m}{\partial t} \right\|_{L^2(\Omega)}^2 = \int_{\Omega} \operatorname{div} (D_1(\phi_m) \nabla c_m + D_2(c_m, \phi_m) \nabla \phi_m) \frac{\partial c_m}{\partial t}.$$

Hence, using assumption (H2)-(H3) we get

$$\left\| \frac{\partial c_m}{\partial t} \right\|_{L^2(\Omega)}^2 \leq C \int_{\Omega} (|\nabla \phi_m \cdot \nabla c_m| + |\nabla \phi_m|^2 + |\Delta \phi_m| + |\Delta c_m|) \left| \frac{\partial c_m}{\partial t} \right|. \quad (69)$$

We proceed as in (62) to estimate the right hand-side of (69). This leads to

$$\left\| \frac{\partial c_m}{\partial t} \right\|_{L^2(\Omega)}^2 \leq C \left(1 + \|c_m\|_{H^2(\Omega)}^2\right),$$

Hence, according to (68), we have that

$$\left\| \frac{\partial c_m}{\partial t} \right\|_{L^2(Q_T)} \leq C, \quad (70)$$

where $C > 0$ does not depend on m .

Now we are able to establish the regularity result. Indeed, we deduce from (56), (57) and (61) that

$$\phi \in L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega)).$$

Moreover, thanks to (68) and (70), c_m is bounded in the space W_3 defined by (38) and which is compactly embedded into $L^2(0, T; H^1(\Omega))$. Then

$$c \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$$

and there exists a subsequence still denoted by c_m such that as $m \rightarrow +\infty$,

$$c_m \rightarrow c \quad \text{in } L^2(0, T; H^1(\Omega)) \text{ strongly}, \quad (71)$$

$$\frac{\partial c_m}{\partial t} \rightharpoonup \frac{\partial c}{\partial t} \quad \text{in } L^2(Q_T) \text{ weakly}. \quad (72)$$

We then pass to the limit in Problem (P_m) and find that c and ϕ satisfy (45). $\square$

b) Uniqueness. Suppose there exist two solutions (ϕ_1, c_1) and (ϕ_2, c_2) satisfying (44) and (45) together with the same initial data. We note $\phi = \phi_2 - \phi_1$ and $c = c_2 - c_1$ and derive some estimates for ϕ and c . In the sequel, we will denote by C a positive constant depending on ε^2 , M_1 , M_2 , D_s , D_l , T , on the Lipschitz constants of nonlinear terms F_i , D_i , $i = 1, 2$ and also on $\|\phi_2\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|c_2\|_{L^\infty(0,T;H^1(\Omega))}$.

i) First, we multiply the equation for ϕ by ϕ and integrate over Ω . We obtain after integrating by parts, for a.e. $t \in (0, T)$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\phi(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla \phi(t)\|_{L^2(\Omega)}^2 &= \int_{\Omega} (F_1(\phi_2) - F_1(\phi_1)) \phi \, dx + \int_{\Omega} F_2(\phi_1) c \phi \, dx \\ &+ \int_{\Omega} c_2 (F_2(\phi_2) - F_2(\phi_1)) \phi \, dx. \end{aligned}$$

Using assumption (H1) with the Hölder and the Cauchy-Schwarz inequalities, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\phi(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla \phi(t)\|_{L^2(\Omega)}^2 &\leq C \left(\|\phi(t)\|_{L^2(\Omega)}^2 + \|c(t)\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. + \|c_2(t)\|_{L^4(\Omega)} \|\phi(t)\|_{L^4(\Omega)} \|\phi(t)\|_{L^2(\Omega)} \right). \end{aligned}$$

Thanks to the continuous embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and since $c_2 \in L^\infty(0, T; H^1(\Omega))$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\phi(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla \phi(t)\|_{L^2(\Omega)}^2 \leq C \left(\|\phi(t)\|_{L^2(\Omega)}^2 + \|c(t)\|_{L^2(\Omega)}^2 + \|\nabla \phi(t)\|_{L^2(\Omega)} \|\phi(t)\|_{L^2(\Omega)} \right).$$

We conclude by Young's inequality that

$$\frac{d}{dt} \|\phi(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla \phi(t)\|_{L^2(\Omega)}^2 \leq C \left(\|\phi(t)\|_{L^2(\Omega)}^2 + \|c(t)\|_{L^2(\Omega)}^2 \right). \quad (73)$$

ii) We multiply the equation for ϕ by $-\Delta \phi$ and integrate over Ω . We obtain after integrating by parts, for a.e. $t \in (0, T)$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \phi(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\Delta \phi(t)\|_{L^2(\Omega)}^2 &= - \int_{\Omega} (F_1(\phi_2) - F_1(\phi_1)) \Delta \phi \, dx - \int_{\Omega} F_2(\phi_1) c \Delta \phi \, dx \\ &- \int_{\Omega} c_2 (F_2(\phi_2) - F_2(\phi_1)) \Delta \phi \, dx. \end{aligned}$$

Using assumption (H1) and Hölder's inequality, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \phi(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\Delta \phi(t)\|_{L^2(\Omega)}^2 &\leq C \left(\|\phi(t)\|_{L^2(\Omega)} + \|c(t)\|_{L^2(\Omega)} \right. \\ &\quad \left. + \|c_2(t)\|_{L^4(\Omega)} \|\phi(t)\|_{L^4(\Omega)} \right) \|\Delta \phi(t)\|_{L^2(\Omega)}. \end{aligned}$$

We conclude as in the previous part i) that

$$\frac{d}{dt} \|\nabla \phi(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\Delta \phi(t)\|_{L^2(\Omega)}^2 \leq C \left(\|\phi(t)\|_{L^2(\Omega)}^2 + \|c(t)\|_{L^2(\Omega)}^2 + \|\nabla \phi(t)\|_{L^2(\Omega)}^2 \right). \quad (74)$$

iii) We multiply the equation for c by c and integrate over Ω . We obtain after integrating by parts, for a.e. $t \in (0, T)$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|c(t)\|_{L^2(\Omega)}^2 &+ \int_{\Omega} (D_1(\phi_2) \nabla c_2 - D_1(\phi_1) \nabla c_1) \cdot \nabla c \, dx \\ &+ \int_{\Omega} (D_2(c_2, \phi_2) \nabla \phi_2 - D_2(c_1, \phi_1) \nabla \phi_1) \cdot \nabla c \, dx = 0, \end{aligned}$$

that we write in the following form

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|c(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} D_1(\phi_1) |\nabla c|^2 dx &= \int_{\Omega} (D_1(\phi_1) - D_1(\phi_2)) \nabla c_2 \cdot \nabla c dx - \int_{\Omega} D_2(c_1, \phi_1) \nabla \phi \cdot \nabla c dx \\ &+ \int_{\Omega} (D_2(c_1, \phi_1) - D_2(c_2, \phi_2)) \nabla \phi_2 \cdot \nabla c dx. \end{aligned}$$

Hence, using assumptions (H2) and (H3), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|c(t)\|_{L^2(\Omega)}^2 + D_s \|\nabla c(t)\|_{L^2(\Omega)}^2 &\leq C \left(\int_{\Omega} |\phi| |\nabla c_2 \cdot \nabla c| dx + \int_{\Omega} |\nabla \phi \cdot \nabla c| dx \right. \\ &\quad \left. + \int_{\Omega} (|\phi| + |c|) |\nabla \phi_2 \cdot \nabla c| dx \right). \end{aligned} \quad (75)$$

We have to evaluate the right hand-side of inequality (75). Using the continuous embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ which is valid for a smooth domain $\Omega \subset \mathbb{R}^d$ with $d \leq 3$ and owing to the elliptic estimate (8), there exist positive constants C_1, C_2 such that

$$\begin{aligned} \int_{\Omega} |\phi| |\nabla c_2 \cdot \nabla c| dx &\leq \|\phi\|_{L^\infty(\Omega)} \|\nabla c_2\|_{L^2(\Omega)} \|\nabla c\|_{L^2(\Omega)} \\ &\leq C_1 \|c_2\|_{L^\infty(0,T;H^1(\Omega))} \|\phi\|_{H^2(\Omega)} \|\nabla c\|_{L^2(\Omega)} \\ &\leq C_2 \left(\|\Delta \phi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)} \right) \|\nabla c\|_{L^2(\Omega)}. \end{aligned} \quad (76)$$

The same estimate holds for the term $\int_{\Omega} |\phi| |\nabla \phi_2 \cdot \nabla c| dx$. Moreover, by the use of the Hölder inequality, the continuous embedding (41) and the Gagliardo-Nirenberg inequality (42), there exist positive constants C_3, C_4 such that

$$\begin{aligned} \int_{\Omega} |c| |\nabla \phi_2 \cdot \nabla c| dx &\leq \|c\|_{L^3(\Omega)} \|\nabla \phi_2\|_{L^6(\Omega)} \|\nabla c\|_{L^2(\Omega)} \\ &\leq C_3 \|\phi_2\|_{L^\infty(0,T;H^2(\Omega))} \|c\|_{H^1(\Omega)}^{1/2} \|c\|_{L^2(\Omega)}^{1/2} \|\nabla c\|_{L^2(\Omega)} \\ &\leq C_4 \left(\|c\|_{L^2(\Omega)} \|\nabla c\|_{L^2(\Omega)} + \|c\|_{L^2(\Omega)}^{1/2} \|\nabla c\|_{L^2(\Omega)}^{3/2} \right). \end{aligned} \quad (77)$$

Then, from the estimates (76) and (77) in (75) with the use of Young's inequality, we deduce there exists a positive constant C such that for a.e. $t \in (0, T)$:

$$\frac{d}{dt} \|c(t)\|_{L^2(\Omega)}^2 + D_s \|\nabla c(t)\|_{L^2(\Omega)}^2 \leq C \left(\|\Delta \phi(t)\|_{L^2(\Omega)}^2 + \|\phi(t)\|_{H^1(\Omega)}^2 + \|c(t)\|_{L^2(\Omega)}^2 \right). \quad (78)$$

□

Now, we multiply inequality (78) by $\mu > 0$ to be choosen later and we add the resulting inequality with (73) and (74). This leads to

$$\begin{aligned} \frac{d}{dt} \left(\|\phi(t)\|_{H^1(\Omega)}^2 + \mu \|c(t)\|_{L^2(\Omega)}^2 \right) + \varepsilon^2 \|\Delta \phi(t)\|_{L^2(\Omega)}^2 &\leq (C_5 + \mu C_6) \left(\|\phi(t)\|_{H^1(\Omega)}^2 + \|c(t)\|_{L^2(\Omega)}^2 \right) \\ &+ \mu C_6 \|\Delta \phi(t)\|_{L^2(\Omega)}^2, \end{aligned}$$

where C_5 and C_6 are positive constants. Then we choose $\mu = \frac{\varepsilon^2}{C_6} > 0$ in order to obtain

$$\frac{d}{dt} \left(\|\phi(t)\|_{H^1(\Omega)}^2 + \|c(t)\|_{L^2(\Omega)}^2 \right) \leq C \left(\|\phi(t)\|_{H^1(\Omega)}^2 + \|c(t)\|_{L^2(\Omega)}^2 \right), \quad (79)$$

where $C > 0$ and for a.e. $t \in (0, T)$. Applying Gronwall's lemma and since $\phi(0) = c(0) = 0$, we conclude that $\phi = c = 0$ a.e. in Q_T . The uniqueness is then proved.

□

5 A maximum principle

We establish a maximum principle under extra assumptions on the nonlinear terms. This result will allow us to conclude to the existence of solution to problem (P) with the physical assumptions (H). So, in addition to (H1), (H2) and (H3), we suppose that the nonlinear terms F_1 , F_2 , and D_2 satisfy the following extra assumptions :

$$(H4) \quad \cdot \quad F_1 \equiv F_2 \equiv 0 \text{ in }]-\infty, 0] \cup [1, +\infty[.$$

$$(H5) \quad \cdot \quad D_2(\cdot, r_2) \equiv 0 \text{ in }]-\infty, 0] \cup [1, +\infty[\text{ and for all } r_2 \in \mathbb{R}.$$

Then we have the following result

Theorem 3 *Let assumptions (H1)-(H5) be fulfilled.*

Suppose that the initial data $(\phi_0, c_0) \in L^2(\Omega) \times L^2(\Omega)$ is such that

$$0 \leq \phi_0(x), c_0(x) \leq 1 \text{ for a.e. } x \in \Omega.$$

Then for any $T > 0$, every weak solution $(\phi, c) \in (L^2(0, T; H^1(\Omega)))^2 \cap (H^1(0, T; V'))^2$ satisfies for all $t \in [0, T]$

$$0 \leq \phi(x, t), c(x, t) \leq 1 \text{ for a.e. } x \in Q_T.$$

Proof: We prove that if $\phi_0, c_0 \geq 0$ a.e. in Ω then $\phi(t), c(t) \geq 0$ for all $t \in [0, T]$ and a.e. in Q_T . Let us consider the negative parts of ϕ and c namely $\phi^- = \max(-\phi, 0)$ and $c^- = \max(-c, 0)$. According to [4], we have that $\phi^- \in L^2(0, T; H^1(\Omega))$ with $\nabla \phi^- = -\nabla \phi$ if $\phi < 0$ and $\nabla \phi^- = 0$ otherwise, a.e. in Q_T and the same property holds for c^- . Then, we choose $v = -\phi^-$ and $w = -c^-$ in equations (9) to obtain for a.e. $t \in (0, T)$:

$$- < \frac{\partial \phi}{\partial t}, \phi^- >_{V', V} + \varepsilon^2 \int_{\Omega} |\nabla \phi^-(t)|^2 dx = - \int_{\Omega} (F_1(\phi) + cF_2(\phi)) \phi^- dx$$

and

$$- < \frac{\partial c}{\partial t}, c^- >_{V', V} + \int_{\Omega} D_1(\phi(t)) |\nabla c^-(t)|^2 dx = \int_{c < 0} D_2(c, \phi) \nabla \phi \cdot \nabla c dx.$$

When a function ψ belongs to $H^1(0, T; H^1(\Omega))$, according to [4] it is easy to prove that $< \frac{\partial \psi}{\partial t}, \psi^- >_{V', V} = -\frac{1}{2} \frac{d}{dt} \|\psi^-\|_{L^2(\Omega)}^2$ in $L^1(0, T)$. By using a density argument, it is possible to show that this equality holds for all functions ψ in $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; V')$. It follows that

$$\frac{1}{2} \frac{d}{dt} \|\phi^-(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \int_{\Omega} |\nabla \phi^-(t)|^2 dx = - \int_{\Omega} (F_1(\phi) + cF_2(\phi)) \phi^- dx \quad (80)$$

and

$$\frac{1}{2} \frac{d}{dt} \|c^-(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} D_1(\phi(t)) |\nabla c^-(t)|^2 dx = \int_{c < 0} D_2(c, \phi) \nabla \phi \cdot \nabla c dx. \quad (81)$$

Now, using assumption (H4) on the nonlinear terms F_1 and F_2 , we see that

$$(F_1(\phi) + cF_2(\phi)) \phi^- = 0 \text{ a.e. in } Q_T.$$

On the other hand, assumption (H5) on D_2 leads to

$$D_2(c, \phi) = 0 \text{ if } c < 0, \text{ a.e. in } Q_T.$$

Thus, equations (80) and (81) are reduced to

$$\frac{1}{2} \frac{d}{dt} \|\phi^-(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \int_{\Omega} |\nabla \phi^-(t)|^2 dx = 0 \quad (82)$$

and

$$\frac{1}{2} \frac{d}{dt} \|c^-(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} D_1(\phi(t)) |\nabla c^-(t)|^2 dx = 0, \quad (83)$$

for a.e. $t \in (0, T)$. Since $\phi_0^- = c_0^- = 0$ and due to the fact that the function D_1 is positive, we conclude that $\phi^-(t) = c^-(t) = 0$ for all $t \in [0, T]$ and a.e. in Ω . If we want to prove that $\phi_0, c_0 \leq 1$ a.e. in Ω imply $\phi(t), c(t) \leq 1$ for all $t \in [0, T]$ and a.e. in Q_T , it is sufficient to take $v = (\phi - 1)^+$ and $w = (c - 1)^+$ in equations (9). $\square$

6 Conclusion

We are now able to conclude to the well-posedness of Problem (P) with the physical assumptions (H) for the nonlinear functions $F_i, D_i, i = 1, 2$. Recall that as we point out in Section 2, the physical modelling leads to nonlinear terms F_1, F_2, D_1 and D_2 which are smooth functions and F_1, F_2 can be choosen as polynomial functions of 4th degree in the interval $[0, 1]$ and any functions outside the interval $[0, 1]$. A natural idea is to take polynomial functions of degree 4 on all the real line. Hence we cannot directly apply the existence and regularity results obtained in Section 3 and Section 4. However, due to the vanishing properties in (H) of the physical nonlinear terms, that is $F_1(\phi) = F_2(\phi) = 0$ for $\phi = 0$ and $\phi = 1$ and $D_2(c, \cdot) = 0$ for $c = 0$ and $c = 1$, we may truncate F_1, F_2 to 0 beyond the set $\{0 < \phi < 1\}$ and truncate D_2 to 0 beyond the set $\{0 < c < 1\}$. In that way, we obtain nonlinear functions $\tilde{F}_1, \tilde{F}_2$ and $\tilde{D}_2$ satisfying the Lipschitz and boundeness assumptions (H1), (H2), (H3) and then we can apply existence and regularity results of Section 3 and Section 4. Moreover, since $\tilde{F}_1, \tilde{F}_2$ and $\tilde{D}_2$ also satisfy the vanishing assumptions (H4) and (H5), the maximum principle of Section 5 holds and since the initial physical data c_0 and ϕ_0 are values belonging to the interval $[0, 1]$, we are sure that the solution (c, ϕ) remains included between 0 and 1 when the time t goes up. It follows that c and ϕ don't depend on the choice of the extension of functions F_1, F_2, D_2 outside the interval $[0, 1]$ and consequently the physical problem (P) is well-posed.

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