Almost global attitude stabilisation of a 3-D pendulum by means of two control torques

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Abstract: We investigate the problem of stabilising the attitude of a 3-D axially symmetric pendulum. The system is assumed to be actuated by two torques acting on a plane orthogonal to the symmetry axis. We develop a smooth control law to stabilise the pendulum to the upright position with a given orientation starting from almost all initial conditions. Our approach consists in two steps: first, stabilising the kinematic subsystem by using the angular velocity as a virtual input; second, exploiting the actual inputs to force the angular velocity to follow the reference designed in the previous step.

Keywords: Attitude control, 3-D spherical pendulum, stabilisation, nonlinear control systems

1. INTRODUCTION

The 3-D spherical pendulum is a benchmark mechanical system providing a simplified model for robotic and spacecraft systems Crouch (1984); Krishnan et al. (1992); Morin et al. (1994); Tsiotras et al. (1995); Coron and Keraï (1996) as well as for the human stance Elhasairi and Pechev (2015). The space of its configurations is characterised by 3 (spatial pendulum) or 2 (planar pendulum) translational degrees of freedom (DOFs) and by 3 rotational DOFs. If the pendulum presents a symmetry axis (axially symmetric pendulum), a reduced attitude can be considered that ignores the angle around the symmetry axis. In this case, only two rotational DOFs are used and it is usually referred to as 2-D spherical pendulum.

Despite its deceiving simplicity, the 3-D pendulum is a source of many challenging control problems (see for instance Chaturvedi et al. (2008, 2009); Mayhew and Teel (2010) for a glimpse of the recent literature on the topic). A couple of inputs is sufficient to control the position of a 3-D spherical pendulum on a plane (see e.g. Bloch et al. (2000)) and even to force the pivot of a 2-D spherical pendulum to follow a circular path, while keeping its attitude confined in a cone close to the upright position (see Greco et al. (2017)).

Stabilising the attitude of a 3-D pendulum (for instance in the upright position with a given angle around the symmetry axis) is possible, albeit not always trivial, with three control inputs. We recall that, while a locally stabilising, time invariant smooth feedback can be defined in the case of three independent inputs Byrnes and Isidori (1991), topological obstructions prevent the construction of a global feedback with the same characteristics Sontag (1998); Bhat and Bernstein (2000). In Chaturvedi et al. (2009) it has been shown that three torques allow the almost global, asymptotic stabilisation of the complete attitude in the upright equilibrium with a smooth control law.

The stabilisation becomes tougher when a stronger underactuation is present, i.e. only two control inputs are available. The complete attitude cannot be locally asymptotically stabilised to an equilibrium by any time-invariant continuous state feedback control law Krishnan et al. (1992). In the case the two control inputs span a plan orthogonal to the symmetry axis, the linearised system about the upright equilibrium is not even controllable. Therefore, in Crouch (1984) a discontinuous feedback and in Morin et al. (1994); Coron and Keraï (1996) time-varying smooth feedbacks have been proposed to locally stabilise the attitude of a spacecraft, essentially a 3-D pendulum without gravity, by means of two inputs. In Chaturvedi et al. (2008) two smooth inputs (torques) are used to almost globally, asymptotically stabilise a 2-D spherical pendulum in the upright position. We stress that the reduced attitude only is stabilised here. The full attitude is globally asymptotically stabilised in Krishnan et al. (1992) via a discontinuous control law based on sequential manouvers and in Casagrande et al. (2007); Teel and Sanfelice (2008) by means of hybrid feedbacks. We remark that in Teel and Sanfelice (2008) the asymptotic stability is achieved in a practical sense. In Tsiotras et al. (1995) the dynamics of an axially symmetric spacecraft is considered. Two torques are used to stabilise the complete attitude, but the control law depends on the initial conditions, which have to belong to a compact annular set of the state space not containing the target equilibrium. The feedback is smooth except in the origin, where it is singular.

In this paper we focus on an axially symmetric 3-D pendulum actuated by two torques acting on a plane orthogonal to the symmetry axis. We address the problem of stabilising the complete attitude in the upright position, assuming zero angular velocity along the symmetry axis. Our main result provides the first example (to our knowledge) of a family of smooth feedback laws that almost globally asymptotically stabilise the system to the target configuration. The stabilisation problem is tackled in two steps. First, we define a virtual feedback for the angular velocity guaranteeing the attitude stabilisation for almost every initial condition. Second, we look for a couple of control torques ensuring the convergence of the actual angular velocity to the virtual feedback. We show that such a control law almost globally stabilises the full system. Our approach revolves around a quaternion formalism, which proves to be well suited for describing the rotation kinematics.

Notation

The unit 3-sphere, i.e. the set of unit vectors in $\mathbb{R}^4$, is denoted by S^3. We denote by $SO(3)$ the group of matrices $R \in \mathbb{R}^{3\times 3}$ satisfying $R^{-1} = R^T$ and $\det(R) = 1$ (special orthogonal group). The symbol $\wedge$ denotes the usual cross product in $\mathbb{R}^3$. With each vector $w = (w_1, w_2, w_3)^T$ we associate a skew-symmetric matrix

$$\widehat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$

Recall that $\widehat{w}x = w \wedge x$ for any $w, x \in \mathbb{R}^3$.

2. PROBLEM FORMULATION

We consider here a simplified model of a 3-D pendulum of mass m, whose pivot is constrained on the horizontal plane. An inertial frame is centred in the pivot with the first two axes lying in the horizontal plane and the third one pointing opposite to the gravity vector. A body fixed frame is centred in the pivot, with the third axis aligned with the vector from the pivot to the centre of mass (the symmetry axis). We denote with $J_{\text{piv}} = \operatorname{diag}(J_1, J_2, J_3)$ the inertia matrix with respect to the pivot in the body fixed frame. Due to symmetry, in the following we assume that $J_1 = J_2 = J$ and $J \neq J_3$. We use a matrix rotation $R \in SO(3)$ to describe the state of the 3-D pendulum: R describes the orientation of the body fixed frame with respect to the inertial frame. The angular velocity vector in the body fixed frame is represented by $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$.

Assume that the pendulum is actuated by a pair of torques τ_1, τ_2 acting on a plane orthogonal to the symmetry axis. We set $\tau = (\tau_1, \tau_2, 0)^T$. The dynamics of the pendulum is given by

$$J_{\text{piv}}\dot{\omega} = (J_{\text{piv}}\omega) \wedge \omega + mg(R^T e_3) \wedge w_{\text{cm}} + \tau, \tag{1}$$

where g is the gravity acceleration, $e_3 = (0, 0, 1)^T$, $w_{\text{cm}} = le_3$ is the centre of mass of the pendulum in the body fixed frame. The rotational kinematics equation is

$$\dot{R} = R\widehat{\omega}. \tag{2}$$

From (1) it is easy to see that $\dot{\omega}_3 = 0$, which implies that the system is not completely controllable. Therefore, we assume that $\omega_3 \equiv 0$ and we focus on the dynamics of the remaining variables. The fact that $J_1 = J_2$ and $\omega_3 \equiv 0$ implies that $(J_{\text{piv}}\omega) \wedge \omega \equiv 0$ in (1).

We put

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = mglP\widehat{e}_3 R^T e_3 + Ju \tag{3}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4}$$

and $u = (u_1, u_2)^T$ are the new control variables. By using (3), equation (1) reduces to

$$\begin{aligned} \dot{\omega}_1 &= u_1 \\ \dot{\omega}_2 &= u_2\,. \end{aligned} \tag{5}$$

In order to analyse the rotational kinematics (2) it is convenient to rewrite rotations in terms of quaternions. Recall that any rotation matrix may be identified with a rotation axis, represented by a unit vector p, and an angle α, that is $R = \exp(\alpha\,\widehat{p})$. This allows us to define the associated unit quaternion as

$$\begin{aligned} &\mathbf{q} = (q_0, q) \in S^3 \\ &q_0 = \cos\frac{\alpha}{2}, \qquad q = (q_1, q_2, q_3) = p\sin\frac{\alpha}{2}. \end{aligned}$$

Note that the quaternions $\mathbf{q}$ and $-\mathbf{q}$ identify the same rotation. The kinematics (2) in the quaternion setting takes the form Chou (1992)

$$\begin{aligned} \dot{q}_0 &= -\frac{1}{2}q^T\omega \\ \dot{q} &= \frac{1}{2}q \wedge \omega + \frac{1}{2}q_0\omega. \end{aligned} \tag{6}$$

We remark that the coupled system (5)-(6) is an equivalent formulation of the one considered in Tsiotras et al. (1995).

We consider the following control problem: define a smooth feedback law capable of asymptotically steering the fixed body frame to the inertial frame. This is tantamount to requiring that the rotation R asymptotically converges to the identity matrix. In terms of quaternions, the problem is equivalent to the following:

Problem 1. Let $\mathbf{q}_d = (1, 0, 0, 0)$. Find a smooth feedback control (u_1, u_2) for the system (5)-(6) capable of asymptotically steering $(\omega_1, \omega_2, \mathbf{q}) \in \mathbb{R}^2 \times S^3$ to $(0, 0, \mathbf{q}_d)$.

It is worth noting that a well-known topological obstruction impedes the global stabilisation of the system to the equilibrium by means of a smooth feedback. More precisely, the manifold $\mathbb{R}^2 \times S^3$ turns out to be not contractible as a consequence of the non-contractibility of S^3. Then (Sontag, 1998, Corollary 5.9.13) implies that there is no globally stabilising feedback. A similar reasoning applies also to the original system (1)-(2). Thus, in the following we will focus on the almost global stabilisation of the system, that is we will look for smooth feedback laws solving *Problem 1* except for a zero measure set of initial conditions $(\omega_1, \omega_2, \mathbf{q}) \in \mathbb{R}^2 \times S^3$.

Also, note that the components q may be used as a local set of coordinates to describe the quaternion variables around $(0, \mathbf{q}_d)$. In these coordinates the equilibrium becomes the origin in $\mathbb{R}^5$ and the linearised system is simply given by $\dot{\omega}_1 = u_1$, $\dot{\omega}_2 = u_2$, $\dot{q} = \frac{1}{2}\omega$ and is therefore not controllable (recall that $\omega_3 = 0$). In particular this system does not admit a locally stabilising linear feedback control and system (5)-(6) cannot be exponentially stabilised with a smooth control. Indeed, for every stabilising feedback, the linearisation of the closed loop system has necessarily a singular dynamical matrix.

3. MAIN RESULTS

We divide the stabilisation problem in two steps. First, we consider the system (6) with ω as a control variable, and we look for a feedback ω_{ref} ensuring that q_0 goes to 1 for almost every initial condition. Second, we look for a feedback u such that the solution ω of the system (5) converges to ω_{ref} and we show that such a control law also almost globally stabilises the full system (5)-(6).

3.1 Stabilisation of the rotation kinematics

We look for functions $\omega(\mathbf{q}) = (\omega_1, \omega_2, 0)$ of the form

$$\omega = \gamma_1(\mathbf{q})(e_3 \wedge q) + \gamma_2(\mathbf{q})(e_3 \wedge (e_3 \wedge q)), \tag{7}$$

for some smooth γ_1, γ_2. Note that in the feedback above, $\omega = 0$ whenever q is parallel to e_3, that is whenever $q_1 = q_2 = 0$. In other words, for any choice of the functions γ_1, γ_2 the set of unit quaternions $Q_0 = \{\mathbf{q} \in S^3 \ : \ q_1 = q_2 = 0\}$ is made up of equilibria of the system [1]. We would like to find γ_1, γ_2 such that, whenever we start outside Q_0, the trajectory always converges to $\mathbf{q}_d$.

The advantage of the form (7) is that the dynamics of q_0 and q_3 are described by very simple equations in terms of γ_1, γ_2. Indeed, setting $f(\mathbf{q}) = q_1^2 + q_2^2$ we have

$$\begin{aligned}
\dot{q}_0 &= -\frac{1}{2} q^T \omega(\mathbf{q}) \\
&= -\frac{1}{2}\gamma_2(\mathbf{q}) q^T (e_3 \wedge (e_3 \wedge q)) \\
&= \frac{1}{2}\gamma_2(\mathbf{q}) |e_3 \wedge q|^2 \\
&= \frac{1}{2}\gamma_2(\mathbf{q}) f(\mathbf{q})
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\dot{q}_3 &= e_3^T \dot{q} \\
&= \frac{1}{2} e_3^T (q \wedge \omega(\mathbf{q})) \\
&= \frac{1}{2}\gamma_1(\mathbf{q}) e_3^T (q \wedge (e_3 \wedge q)) + \frac{1}{2}\gamma_2(\mathbf{q}) e_3^T (q \wedge (e_3 \wedge (e_3 \wedge q))) \\
&= \frac{1}{2}\gamma_1(\mathbf{q}) |e_3 \wedge q|^2 - \frac{1}{2}\gamma_2(\mathbf{q}) q^T \hat{e}_3{}^3 q \\
&= \frac{1}{2}\gamma_1(\mathbf{q}) f(\mathbf{q}),
\end{aligned} \tag{9}$$

which also imply

$$\begin{aligned}
\dot{f}(\mathbf{q}) &= -\frac{d}{dt}(q_0^2 + q_3^2) \\
&= -(q_0\gamma_2(\mathbf{q}) + q_3\gamma_1(\mathbf{q})) f(\mathbf{q}).
\end{aligned} \tag{10}$$

We have the following result.

Proposition 1. Assume that $\gamma_2 > 0$ outside Q_0 and that $\frac{\gamma_2}{1-q_0}$ is a smooth function. Let $\gamma_1 = -(\frac{\gamma_2}{2(1-q_0)} + \mu) q_3$ for some $\mu \geq 0$. Then for every initial condition $\mathbf{q} \notin Q_0 \setminus \{\mathbf{q}_d\}$ the corresponding trajectory of (6) with the feedback law (7) converges asymptotically to $\mathbf{q}_d$.

[1] The set $Q_0 \setminus \{\mathbf{q}_d\}$ corresponds to the configurations which differ from the target configuration only by a rotation about the symmetry axis.

Proof. Let us define

$$V(\mathbf{q}) = (1 - q_0)^2.$$

Since $\dot{V}(\mathbf{q}) = -\gamma_2(\mathbf{q})(1 - q_0) f(\mathbf{q})$, if γ_2 is chosen to be positive outside Q_0 we obtain from LaSalle invariance principle that any trajectory of the system must necessarily converge to Q_0.

It remains to show that any trajectory starting outside Q_0 converges exactly to $\mathbf{q}_d$. To this aim, let us consider the function

$$W(\mathbf{q}) = \frac{1 - q_0}{f(\mathbf{q})},$$

which is well defined outside Q_0. Since $\gamma_2 > 0$ outside Q_0, f goes to zero, independently of the choice of γ_1. If we show that $\dot{W} \leq 0$ on $S^3 \setminus Q_0$, we can conclude that the function $1 - q_0$ is dominated by a multiple of f and thus must also converge to 0. We have

$$\begin{aligned}
\dot{W} &= \frac{-\dot{q}_0 f - (1 - q_0)\dot{f}}{f^2} \\
&= \frac{-\frac{1}{2}\gamma_2 f^2 + (1 - q_0)(q_0\gamma_2 + q_3\gamma_1) f}{f^2} \\
&= \frac{-\frac{1}{2}\gamma_2(1 - q_0^2 - q_3^2) + (1 - q_0)(q_0\gamma_2 + q_3\gamma_1)}{f}
\end{aligned} \tag{11}$$

which is non-positive if and only if the numerator satisfies

$$\begin{aligned}
-\frac{1}{2}&\gamma_2(1 - q_0^2 - q_3^2) + (1 - q_0)(q_0\gamma_2 + q_3\gamma_1) \\
&= -\frac{1}{2}\gamma_2(1 - q_0)^2 + q_3(\gamma_2 q_3/2 + \gamma_1(1 - q_0)) \leq 0.
\end{aligned}$$

With the choice of γ_1 and γ_2 in the hypothesis of the proposition we have that $-\gamma_2(1 - q_0)^2/2 \leq 0$ and

$$q_3(\gamma_2 q_3/2 + \gamma_1(1 - q_0)) = -\mu q_3^2(1 - q_0) \leq 0,$$

hence the thesis.

3.2 Almost global stabilisation of the complete system

Consider now the more general problem of finding a stabilising feedback control for the complete system (5)-(6). Let us denote with $\omega_{\text{ref}}(\mathbf{q})$ a stabilising control law (7) satisfying Proposition 1. For simplicity, let us call $G(\omega, \mathbf{q})$ the right-hand side of (6). In the following, when necessary, we identify $\omega \in \mathbb{R}^2 \times \{0\}$ as an element of $\mathbb{R}^2$.

We choose a feedback control law forcing $\omega(t)$ to asymptotically approximate the function $\omega_{\text{ref}}(\mathbf{q}(t))$. For this purpose we define $\tilde{\omega} = \omega - \omega_{\text{ref}}(\mathbf{q})$ and we impose $\dot{\tilde{\omega}} = -K\tilde{\omega}$ for some $K > 0$.

This corresponds to choosing the feedback control

$$\begin{aligned}
u(\omega, \mathbf{q}) &= P\left(\dot{\omega}_{\text{ref}}(\mathbf{q}) - K(\omega - \omega_{\text{ref}}(\mathbf{q}))\right) \\
&= P\left(\frac{d\omega_{\text{ref}}}{d\mathbf{q}}(\mathbf{q}) G(\omega, \mathbf{q}) - K\omega + K\omega_{\text{ref}}(\mathbf{q})\right)
\end{aligned} \tag{12}$$

where P is as in (4).

Theorem 2. The feedback control (12) almost globally stabilises system (5)-(6) to the equilibrium $(0, \mathbf{q}_d)$.

In order to prove Theorem 2, it is convenient to rewrite system (5)-(6) in terms of the error variable $\tilde{\omega}$:

$$\dot{\tilde{\omega}} = -K\tilde{\omega} \tag{13}$$
$$\dot{\mathbf{q}} = \tilde{G}(\tilde{\omega}, \mathbf{q}) \tag{14}$$

where $\tilde{G}(\tilde{\omega}, \mathbf{q}) = G(\tilde{\omega} + \omega_{\mathrm{ref}}(\mathbf{q}), \mathbf{q})$. Note that the map $(\omega, \mathbf{q}) \mapsto (\tilde{\omega}, \mathbf{q})$ is a diffeomorphism from $\mathbb{R}^2 \times S^3$ to itself.

We need some preliminary results. First, we show the following non-smooth extension of the classical LaSalle invariance theorem. [2]

Proposition 3. Consider the system

$$\dot{x} = F(x), \tag{15}$$

where x belongs to a manifold $\mathcal{M}$ and F is Lipschitz continuous. Let Ω be a compact invariant subset of $\mathcal{M}$, D be a compact subset of Ω such that both D and $\Omega \setminus D$ are positively invariant. Moreover assume that there exists a continuous function $V : \Omega \to \mathbb{R}$ strictly decreasing along the flow of (15) on $\Omega \setminus D$. Then for any trajectory $x(\cdot)$ in $\Omega \setminus D$ there exists $c \in \mathbb{R}$ such that $x(\cdot)$ converges to a connected component of $D \cap V^{-1}(c)$.

Proof. Let $\phi(x_0, t)$ be the trajectory of the system (15) starting from $x_0 \in \Omega \setminus D$. Then $V(\phi(x_0, t))$ is decreasing by assumption and, since the continuous function V admits a minimum in the compact set Ω, $V(\phi(x_0, t))$ must converge to a constant $c \in \mathbb{R}$. Since, by the positively invariance of Ω, the $\omega-$limit set of x_0 is connected, it remains to show that $\phi(x_0, t)$ necessarily converges to the set D. Let $\varepsilon > 0$ and define K_ε as the compact set formed by the points of Ω whose distance from D is larger or equal than ε. From the assumptions on V and by compactness of K_ε, the function $V(\phi(x, 1)) - V(x) \leq -\delta$ for some $\delta > 0$ and for $x \in K_\varepsilon$. Then $V(\phi(x_0, t+1)) - V(\phi(x_0, t)) \leq -\delta$, whenever $\phi(x_0, t) \in K_\varepsilon$. But we also know that there exists $T > 0$ such that $V(\phi(x_0, t)) < c + \delta$ for any $t > T$. Hence

$$\begin{aligned} c <& V(\phi(x_0, t+1)) < V(\phi(x_0, t)) < c + \delta \\ \Rightarrow& \quad V(\phi(x_0, t+1)) - V(\phi(x_0, t)) > -\delta \end{aligned}$$

for any $t > T$, implying that $\phi(x_0, t) \notin K_\varepsilon$. Being ε arbitrary this concludes the proof.

We apply Proposition 3 to the system (13)-(14). We define $\Omega = [-1, 1] \times [-1, 1] \times S^3$ and $D = \{0\} \times Q_0$ and

$$V(\tilde{\omega}, \mathbf{q}) = (1 - q_0)^2 + 3|\tilde{\omega}|/K.$$

We have

$$\begin{aligned} \dot{V} &= -\gamma_2(1 - q_0) f(\mathbf{q}) + (1 - q_0) q^T \tilde{\omega} - 3|\tilde{\omega}| \\ &\leq -\gamma_2(1 - q_0) f(\mathbf{q}) - |\tilde{\omega}| \end{aligned}$$

which is well-defined and strictly negative outside $\{0\} \times Q_0$. It is easy to see that for any $c \in \mathbb{R}$ there exists at most two distinct points in $D \cap V^{-1}(c)$. Then we deduce from Proposition 3 that any trajectory of the system (13)-(14) starting in $\Omega \setminus D$ asymptotically converges to a single point of $D = \{0\} \times Q_0$. Since the set Ω defined above is globally attractive in finite time the result extends to $(\mathbb{R}^2 \times S^3) \setminus D$. Summing up we get the following.

Lemma 4. Any trajectory of the system (13)-(14) asymptotically converges to a single point of $\{0\} \times Q_0$.

It remains to show that almost every trajectory of the system (13)-(14) converges exactly to $(0, \mathbf{q}_d)$.

[2] for similar results in a much more general context, see e.g. Bacciotti and Ceragioli (1999); Sanfelice et al. (2007)

The lemma below allows a characterisation of the set of initial points such that the corresponding trajectories do not converge to $(0, \mathbf{q}_d)$.

Lemma 5. Consider the system (13)-(14). In addition to the hypotheses in Proposition 1 we assume that γ_2 is strictly positive outside $\mathbf{q}_d$. Let $\mathbf{q}^* = (q_0^*, 0, 0, q_3^*) \in Q_0 \setminus \{\mathbf{q}_d\}$. Then the linearised system at the equilibrium $(0, \mathbf{q}^*)$ (on the five-dimensional tangent space $T_{(0,\mathbf{q}^*)}(\mathbb{R}^2 \times S^3)$) is associated with a Jacobian matrix having two eigenvalues equal to $-K$ and one eigenvalue equal to 0. The remaining two eigenvalues have strictly positive real part.

Proof. To simplify the computations we embed $\mathbb{R}^2 \times S^3$ in $\mathbb{R}^6$, using the coordinates $(\tilde{\omega}_1, \tilde{\omega}_2, q_0, q_1, q_2, q_3)$.

Equation (13) immediately implies the existence of two eigenvalues of the linearised system equal to $-K$. The remaining ones are eigenvalues of the four dimensional square matrix

$$\frac{\partial \tilde{G}}{\partial \mathbf{q}}(\tilde{\omega}, \mathbf{q})|_{(0,\mathbf{q}^*)} = \frac{\partial G}{\partial \omega}(0, \mathbf{q}^*) \frac{d\omega_{\mathrm{ref}}}{d\mathbf{q}}(\mathbf{q}^*) + \frac{\partial G}{\partial \mathbf{q}}(0, \mathbf{q}^*),$$

where we have used the fact that $\omega_{\mathrm{ref}}(\mathbf{q}^*) = 0$. A direct computation shows that the kernel of this matrix is generated by e_1, e_4 and therefore contains both $\mathbf{q}^*$ and the vector tangent to Q_0 at $\mathbf{q}^*$. The zero eigenvalue corresponding to the radial direction $\mathbf{q}^*$ must be neglected, being $\mathbf{q}^*$ orthogonal to the tangent space TS^3.

The two remaining eigenvalues of the matrix may be easily computed as

$$\frac{1}{2}(-\gamma_1 q_3^* - \gamma_2 q_0^*) \pm \frac{1}{2} i(-\gamma_2 q_3^* + \gamma_1 q_0^*),$$

and, by replacing the expression of γ_1 in Proposition 1, we have

$$-\gamma_1 q_3^* - \gamma_2 q_0^* = \frac{1}{2}\gamma_2(1 - q_0^*) + \mu(q_3^*)^2 > 0 \text{ if } q_0^* \neq 1.$$

This concludes the proof of the lemma.

The previous result allows us to cast our dynamical model in the well established framework of normally hyperbolic invariant manifolds, first developed in Fenichel (1971, 1974, 1977); Hirsch et al. (1970, 1977), which generalises classical results on hyperbolic equilibrium points. A normally hyperbolic manifold V is an invariant compact submanifold of the state space such that the linearised dynamics around V may be decoupled into three parts: a stable dynamics and an unstable one, both of which are transverse to V, and a dynamics tangent to the manifold V. In addition, it is assumed that, roughly speaking, the rates of contraction and expansion of the flow respectively in the direction of the stable and unstable subspaces are larger than those along V. The latter condition is automatically satisfied if V is made of equilibrium points.

We are now ready to prove our main result.

Proof of Theorem 2. From Lemma 4, in order to characterise the trajectories that do not converge to $(0, \mathbf{q}_d)$ it is enough to study the family of all trajectories converging to the equilibria $(0, \mathbf{q}^*)$ with $\mathbf{q}^* \in Q_0 \setminus \mathbf{q}_d$. For this purpose, let us fix an arbitrary small $\epsilon > 0$ and consider the compact manifold (with boundary)

$$P_0^\epsilon = \{(\tilde{\omega}, \mathbf{q}) \in \{0\} \times Q_0 \mid q_0 \leq 1 - \epsilon\}.$$

According to Lemma 5, for the linearised dynamics, the tangent space at any equilibrium point $\mathbf{p} \in P_0^\epsilon$ splits into the sum of a two-dimensional stable subspace $E_{\mathbf{p}}^s$, a two-dimensional unstable subspace $E_{\mathbf{p}}^u$, and the one-dimensional space $T_{\mathbf{p}} P_0^\epsilon$ (which coincide with the kernel of the linearised system). Thus, in the setting of e.g. Hirsch et al. (1977), P_0^ϵ is a normally hyperbolic invariant manifold. Hence, by classical results, there exists a local invariant manifold $\mathcal{W}_\epsilon^s$ tangent to $E_{\mathbf{p}}^s \oplus T_{\mathbf{p}} P_0^\epsilon$ at any $\mathbf{p} \in P_0^\epsilon$, and which is therefore of dimension 3.

An interesting and helpful characterisation of $\mathcal{W}_\epsilon^s$ is given, in a very general setting, in Bates et al. (1998). In that paper the authors show that for a small enough smooth tubular neighbourhood $\mathcal{N}$ of P_0^ϵ one can write

$$\mathcal{W}_\epsilon^s = \{\mathbf{p} \in \mathcal{N} \mid \phi^t(\mathbf{p}) \in \mathcal{N},\ \forall t \geq 0 \text{ and } \lim_{t\to\infty} \phi^t(\mathbf{p}) \in P_0^\epsilon\},$$

where $\phi^t(\mathbf{p})$ is the flow of the system at time t applied to $\mathbf{p}$. Let us further define the set

$$\overline{\mathcal{W}}_\epsilon^s = \{\mathbf{p} \in \mathbb{R}^2 \times S^3 \mid \lim_{t\to\infty} \phi^t(\mathbf{p}) \in P_0^\epsilon\}.$$

Since $\mathcal{W}_\epsilon^s$ is a three-dimensional manifold, it has zero Lebesgue measure. Recall that the flow at a (positive or negative) time t is a diffeomorphism, hence we deduce that the set $\phi^{-t}(\mathcal{W}_\epsilon^s)$ has zero measure as well. Then $\overline{\mathcal{W}}_\epsilon^s = \cup_{n\geq 0} \phi^{-n}(\mathcal{W}_\epsilon^s)$ is a countable union of zero measure sets and thus it has zero measure.

Finally, the set of initial points in $\mathbb{R}^2 \times S^3$ such that the corresponding trajectories converge to a point of $Q_0 \setminus \mathbf{q}_d$ coincides with $\cup_{m\geq 1} \overline{\mathcal{W}}_{1/m}^s$ and thus it has zero measure.

4. SIMULATIONS

We show numerical simulations of the system with the feedback control (12). According to Proposition 1 and Theorem 2 we choose $\gamma_1 = -5q_3$, $\gamma_2 = 2(1-q_0)$ and $K = 1$. In order to illustrate the effectiveness of the feedback, we choose an initial condition close to an equilibrium of $\{0\} \times Q_0$ different from $(0, \mathbf{q}_d)$. More precisely we set $\omega(0) = (0, 0)$ and $\mathbf{q}(0) = (0.8, 0, 0.06, 0.597)$. Notice that the use of two control torques acting on a plane orthogonal to the symmetry axis, impedes a direct rotation about that axis. Hence, the attitude stabilisation would require the third axis to first move sensibly away from the initial configuration before coming back. In Figure 1 the black and grey frames represent the initial and final body fixed frames, respectively. The evolution of the axes are represented in different colours. As expected, simulations show that the trajectory quickly move away from the unstable equilibrium and slowly approaches the equilibrium $(0, \mathbf{q}_d)$. Figures 2 and 3 show how after some oscillations ω and u quickly converge to 0. Figure 4 shows the evolution of q_0 which is representative of the slow convergence of the body fixed frame to the inertial frame.

5. CONCLUSION

In this paper we considered the problem of stabilising an axially symmetric 3-D pendulum to the upright vertical position with a fixed orientation by means of two torques. The stabilisation was achieved in two steps. We first designed a smooth control feedback of the kinematic subsystem by controlling directly the angular velocity.

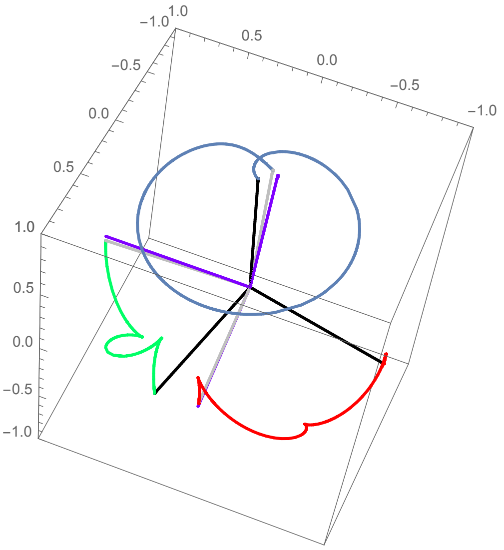
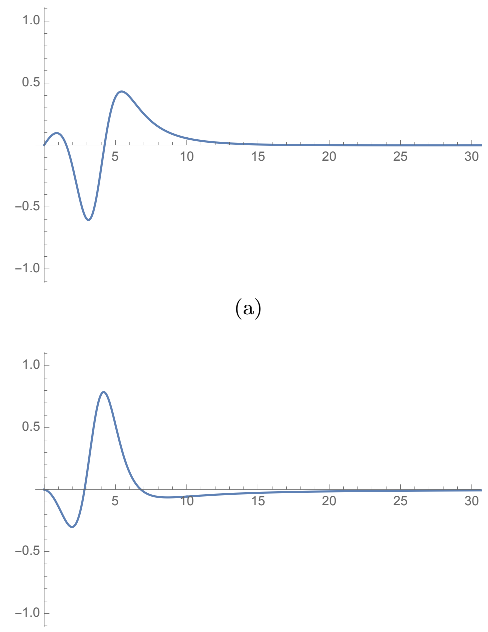


Fig. 1. Evolution of the body fixed frames from the initial position (black) to the final one (grey). The blue trace represents the evolution of the vertical axis. The purple frame represents the target.



(a)

(b)

Fig. 2. (a) evolution of ω_1 and (b) evolution of ω_2

Then, we designed a control feedback for the original inputs to force the angular velocity to follow the reference computed in the previous step. We proved that this control is capable of steering the complete system to the desired equilibrium. Future works will address the problem of the stabilisation of the 3-D pendulum by means of two planar inputs. Second, we aim at analysing the almost global exponential stabilisation problem with non-smooth feedback laws.

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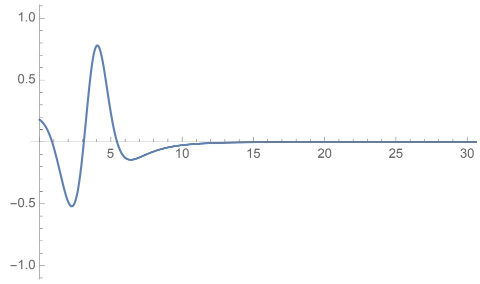
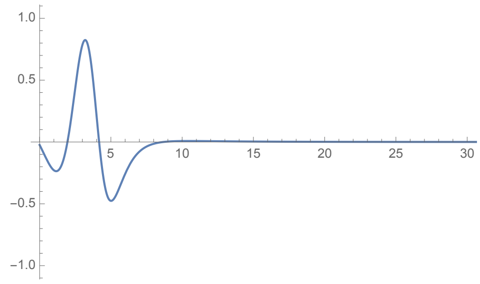


(a)



(b)

Fig. 3. (a) evolution of u_1 and (b) evolution u_2

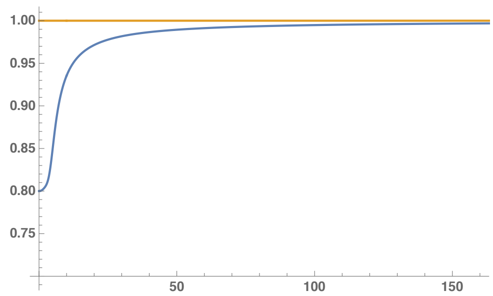


Fig. 4. Evolution of q_0

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