COHOMOLOGY OF $\mathfrak{aff}(2|1)$ ACTING ON THE SPACE OF LINEAR DIFFERENTIAL OPERATORS ON THE SUPERSPACE $\mathbb{R}^{1|2}$

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ABSTRACT. We compute the first cohomology of affine Lie superalgebra $\mathfrak{aff}(2|1)$ on the $(1,2)$ -dimensional real superspace with coefficients in the superspace $\mathfrak{D}_{\lambda;\mu}^2$ of linear differential operators acting on weighted densities. We also compute the same, but $\mathfrak{aff}(1|1)$ -relative, cohomology. We explicitly give 1-cocycles spanning these cohomology.

1. Introduction

Deformation theory plays a crucial role in all branches of mathematics and physics. In physics the mathematical theory of deformations has proved to be a powerful tool in modeling physical reality. The concepts symmetry and deformations are considered to be two fundamental guiding principle for developing the physical theory further. Formal deformations of arbitrary rings and associative algebras, and the related cohomology questions, were first investigated by Gerstenhaber, in a series of articles [11, 12, 13, 14]. Deformations of Lie algebras with multi-parameters were intensively studied by Fialowski [9, 10].

The space of weighted densities of weight λ on $\mathbb{R}$ (or λ -densities for short), denoted by:

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}, \quad \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes \lambda}$. The Lie algebra $\text{Vect}(\mathbb{R})$ of vector fields $X_h = h \frac{d}{dx}$, where $h \in C^\infty(\mathbb{R})$, acts by the Lie derivative. Alternatively, this action can be written as follows:

$$(1.1) \quad X_h.(f dx^\lambda) = L_{X_h}^\lambda(f) dx^\lambda \text{ with } L_{X_h}^\lambda(f) = hf' + \lambda h'f,$$

where f', h' are $\frac{df}{dx}, \frac{dh}{dx}$. Each linear differential operator A on $\mathbb{R}$ gives thus rise to a morphism from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$, for any $\lambda, \mu \in \mathbb{R}$, by $f dx^\lambda \mapsto A(f) dx^\mu$. The Lie algebra $\text{Vect}(\mathbb{R})$ acts on the space $\mathcal{D}_{\lambda;\mu}$ of these differential operators by:

$$(1.2) \quad X_h.A = L_{X_h}^\mu \circ A - A \circ L_{X_h}^\lambda,$$

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In [1], found the first cohomology $H_{\text{diff}}^1(\mathfrak{aff}(1), \mathcal{D}_{\lambda, \mu})$ and the analogue super structures: $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^1)$, where $\mathfrak{aff}(1)$ and $\mathfrak{aff}(1|1)$ are realized as the Lie subalgebra of $\text{Vect}(\mathbb{R})$ and $\mathcal{K}(1)$ spanned by $\{X_1, X_x\}$ and $\{X_1, X_x, X_\theta\}$ and where H_{diff}^1 denotes the differential cohomology; that is, only cochains given by differential operators are considered.

In this paper we consider the Lie superalgebra $\mathfrak{aff}(2|1)$ naturally realized as a subalgebra of $\mathcal{K}(2)$. We compute here $H_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{D}_{\lambda, \mu}^2)$ and $H_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}^2)$. Moreover, we give explicit formula for all the nontrivial 1-cocycles. These spaces arise in the classification of infinitesimal deformations of the $\mathfrak{aff}(2|1)$ -module

$$(1.3) \quad \mathcal{S}_{\mu-\lambda}^2 = \bigoplus_{k \geq 0} \mathfrak{F}_{\mu-\lambda-\frac{k}{2}}^2.$$

We hope to be able to describe in the future all the deformations of this module.

2. Definitions and Notation

2.1. The Lie superalgebra of contact vector fields on $\mathbb{R}^{1|2}$. We define the superspace $\mathbb{R}^{1|2}$ in terms of its superalgebra of functions, denoted by $C^\infty(\mathbb{R})$ and consisting of elements of the form:

$$F(x, \theta_1, \theta_2) = f_0(x) + f_1(x)\theta_1 + f_2(x)\theta_2 + f_{12}(x)\theta_1\theta_2,$$

where x is the even variable, θ_1 and θ_2 are the odd indeterminates, i.e., $\theta_i\theta_j = -\theta_j\theta_i$ and $f_0, f_1, f_2, f_{12} \in C^\infty(\mathbb{R})$. Even elements in $C^\infty(\mathbb{R})$ are the functions $F(x, \theta) = f_0(x) + f_{12}(x)\theta_1\theta_2$, the functions $F(x, \theta) = f_1(x)\theta_1 + f_2(x)\theta_2$ are odd elements. The parity of homogenous elements F will be denoted $|F|$. Let $\text{Vect}(\mathbb{R}^{1|2})$ be the superspace of vector fields on $\mathbb{R}^{1|2}$:

$$\text{Vect}(\mathbb{R}^{1|2}) = \{F_0\partial_x + F_1\partial_1 + F_2\partial_2 \mid F_i \in C^\infty(\mathbb{R}^{1|2})\}$$

where $\partial_i = \frac{\partial}{\partial \theta_i}$. Let $\mathcal{K}(2)$ be the Lie superalgebra of conformal vector fields on $\mathbb{R}^{1|2}$:

$$\mathcal{K}(2) = \{X \in \text{Vect}(\mathbb{R}^{1|2}) \mid \text{there exists } H \in C^\infty(\mathbb{R}^{1|2}) \text{ such that } \mathfrak{L}_X(\alpha_2) = H\alpha_2\},$$

where $\mathfrak{L}_X$ is the Lie derivative along the vector field X and

$$\alpha_2 = dx + \theta_1 d\theta_1 + \theta_2 d\theta_2.$$

Any contact vector field on $\mathbb{R}^{1|2}$ has the following explicit form:

$$X_H = H\partial_x - \frac{1}{2}(-1)^{|H|} \sum_{i=1}^2 \bar{\eta}_i(H)\bar{\eta}_i, \text{ where } H \in C^\infty(\mathbb{R}^{1|2}).$$

The bracket on $\mathcal{K}(2)$ is given by $[X_F, X_G] = X_{\{F, G\}}$:

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|} \sum_{i=1}^2 \bar{\eta}_i(F)\bar{\eta}_i(G).$$

2.2. The superalgebra $\mathfrak{aff}(2|1)$. The Lie superalgebra $\mathfrak{aff}(1|1)$ as a superalgebra of $\mathcal{K}(1)$:

$$\mathfrak{aff}(1|1) = \text{Span}(X_1, X_x, X_\theta).$$

Similarly, we now consider the Lie superalgebra $\mathfrak{aff}(2|1)$ as a superalgebra of $\mathcal{K}(2)$:

$$\mathfrak{aff}(2|1) = \text{Span}(X_1, X_x, X_{\theta_1}, X_{\theta_2}, X_{\theta_1\theta_2}).$$

The space of even elements is isomorphic to $\mathfrak{aff}(1|1)$, while the space of odd elements is :

$$(\mathfrak{aff}(2|1))_{\bar{1}} = \text{Span}(X_{\theta_2}, X_{\theta_1\theta_2}).$$

2.3. The space of weighted densities on $\mathbb{R}^{1|2}$. We define the space of λ -densities as

$$\mathfrak{F}_\lambda^2 = \{F(x, \theta_1, \theta_2)\alpha_2^\lambda \mid F(x, \theta_1, \theta_2) \in C^\infty(\mathbb{R}^{1|2})\}.$$

As a vector space, $\mathfrak{F}_\lambda^2$ is isomorphic to $C^\infty(\mathbb{R}^{1|2})$, but the Lie derivative of the density $F\alpha_2^\lambda$ along the vector field X_H in $\mathcal{K}(2)$ is now:

$$(2.1) \quad \mathfrak{L}_{X_H}(F\alpha_2^\lambda) = \mathfrak{L}_{X_H}^\lambda(F)\alpha_2^\lambda, \text{ with } \mathfrak{L}_{X_H}^\lambda(F) = \mathfrak{L}_{X_H}(F) + \lambda H'F.$$

Here, we restrict ourselves to the subalgebra $\mathfrak{aff}(2|1)$, thus we obtain a one-parameter family of $\mathfrak{aff}(2|1)$ -modules on $C^\infty(\mathbb{R}^{1|2})$ still denoted by $\mathfrak{F}_\lambda^2$. As an $\mathfrak{aff}(2|1)$ -module, we have

$$(2.2) \quad \mathfrak{F}_\lambda^2 \simeq \mathfrak{F}_\lambda^1 \oplus \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1)$$

where Π is the change of parity operator.

Since $-\bar{\eta}_i^2 = \partial_x$ and $\partial_i = \bar{\eta}_i - \theta_i \bar{\eta}_i^2$,

Of course, for all λ , $\mathfrak{F}_\lambda^2$ is a $\mathcal{K}(2)$ -module:

$$[\mathfrak{L}_{X_F}^\lambda, \mathfrak{L}_{X_G}^\lambda] = \mathfrak{L}_{[X_F, X_G]}^\lambda.$$

2.4. Differential operators on weighted densities. A differential operator on $\mathbb{R}^{1|2}$ is an operator on $C^\infty(\mathbb{R}^{1|2})$ of the following form:

$$(2.3) \quad A(F\alpha_2^\lambda) = \sum_{l,m} a_{l,m}(x, \theta) \bar{\eta}_i^l \bar{\eta}_i^m(F) \alpha_2^\mu,$$

where the coefficients $a_{l,m}(x, \theta)$ are arbitrary functions.

Of course, any differential operator defines a linear mapping from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ for any $\lambda, \mu \in \mathbb{R}$, thus, the space of differential operators becomes a family of $\mathcal{K}(2)$ and $\mathfrak{aff}(2|1)$ modules denoted $\mathfrak{D}_{\lambda,\mu}^2$ for the natural action:

$$X_H.A = \mathfrak{L}_{X_H}^\mu \circ A - (-1)^{|A||H|} A \circ \mathfrak{L}_{X_H}^\lambda.$$

Proposition 2.1. *As a $\mathfrak{aff}(2|1)$ -module, we have*

$$(2.4) \quad \mathfrak{D}_{\lambda,\mu}^2 \simeq \mathfrak{D}_{\lambda,\mu}^1 \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^1 \oplus \Pi(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}^1 \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu}^1)$$

Proof. Any element $F \in C^\infty(\mathbb{R}^{1|2})$ can be uniquely written as follows: $F = F_1 + F_2\theta_2$, where $\partial_2 F_1 = \partial_2 F_2 = 0$. Therefore, for any $X_F \in \mathfrak{aff}(2|1)$, we easily check that

$$\mathfrak{L}_{X_H}^\lambda(F) = \mathfrak{L}_{X_H}^\lambda(F_1) + \mathfrak{L}_{X_H}^{\lambda+\frac{1}{2}}(F_2)\theta_2.$$

Thus, the following map is an $\mathfrak{aff}(1|1)$ -isomorphism:

$$(2.5) \quad \begin{aligned} \Phi_\lambda : \quad \mathfrak{F}_\lambda^2 &\rightarrow \mathfrak{F}_\lambda^1 \oplus \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1) \\ F\alpha_2^\lambda &\mapsto (F_1\alpha_1^\lambda, \Pi(F_2\alpha_1^{\lambda+\frac{1}{2}})) \end{aligned}$$

So, we deduce an $\mathfrak{aff}(1|1)$ -isomorphism:

$$(2.6) \quad \begin{aligned} \Psi_{\lambda,\mu} : \quad \mathfrak{D}_{\lambda,\mu}^2 &\rightarrow \mathfrak{D}_{\lambda,\mu}^1 \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^1 \oplus \Pi(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}^1 \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu}^1) \mathfrak{F}_\lambda^1 \oplus \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1) \\ A &\mapsto \Phi_\mu^{-1} \circ A \circ \Phi_\lambda \end{aligned}$$

Here, we identify the $\mathfrak{aff}(1|1)$ -modules via the following isomorphisms:

$$\left\{ \begin{array}{ll} \Pi(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}^1) &\rightarrow \text{Hom}_{\text{diff}}(\mathfrak{F}_\lambda^1, \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^1)) & \Pi(A) \mapsto \Pi \circ A, \\ \Pi(\mathfrak{D}_{\lambda+\frac{1}{2},\mu}^1) &\rightarrow \text{Hom}_{\text{diff}}(\Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1), \mathfrak{F}_\mu^1) & \Pi(A) \mapsto A \circ \Pi, \\ \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^1 &\rightarrow \text{Hom}_{\text{diff}}(\Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1), \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^1)) & \Pi(A) \mapsto \Pi \circ A \circ \Pi. \end{array} \right.$$

□

3. Cohomology of $\mathfrak{aff}(2|1)$ acting on $\mathcal{D}_{\lambda,\mu}^2$

3.1. Cohomology. Let us first recall some fundamental from cohomology theory. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a superspace $V = V_0 \oplus V_1$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. (If $\mathfrak{h}$ is omitted it assumed to be $\{0\}$.) The space of $\mathfrak{h}$ -relative $\mathfrak{n}$ -cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$ -module

$$C^n(\mathfrak{g}, \mathfrak{h}; V) := \text{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h}); V).$$

The coboundary operator $\delta_n : C^n(\mathfrak{g}, \mathfrak{h}; V) \rightarrow C^{n+1}(\mathfrak{g}, \mathfrak{h}; V)$ is a $\mathfrak{g}$ -map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n denoted $Z^n(\mathfrak{g}, \mathfrak{h}; V)$, is the space $\mathfrak{h}$ -relative n -cocycle, among them, the elements in the range on δ_{n-1} are called $\mathfrak{h}$ -relative n -coboundaries. We denote $B^n(\mathfrak{g}, \mathfrak{h}; V)$ the space of n -coboundaries.

By definition, the n^{th} $\mathfrak{h}$ -relative cohomology space in the quotient space

$$H^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V) / B^n(\mathfrak{g}, \mathfrak{h}; V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0 and 1: for $v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{h}}$, $\delta v(g) := (-1)^{p(g)p(v)} g.v$, where

$$V^{\mathfrak{h}} = \{v \in V \mid h.v = 0 \text{ for all } h \in \mathfrak{h}\},$$

and for $\Upsilon \in C^0(\mathfrak{g}, \mathfrak{h}; V)$,

$$\delta(\Upsilon)(g, h) := (-1)^{p(g)p(\Upsilon)} g.\Upsilon(h) - (-1)^{p(h)(p(g)+p(\Upsilon))} h.\Upsilon(g) - \Upsilon([g, h]) \text{ for any } g, h \in \mathfrak{g}.$$

3.2. The main theorem. The main result in this paper is the following:

Theorem 3.1. *The space $H_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{D}_{\lambda, \mu}^2)$ has the following structure:*

$$(3.1) \quad H_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{D}_{\lambda, \mu}^2) \simeq \begin{cases} \mathbb{R}^2 & \text{if } \mu - \lambda = 0, \\ \mathbb{R}^4 & \text{if } \mu - \lambda = k, \quad k \in \mathbb{N}^*, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mu - \lambda = 0$, the following 1-cocycles span the corresponding cohomology spaces:

$$(3.2) \quad \begin{aligned} \Upsilon_{\lambda, \lambda}(X_G) &= \bar{\eta}_1^2(\partial_2(\theta_2 G)), \\ \Omega_{\lambda, \lambda}(X_G) &= \bar{\eta}_1 \bar{\eta}_2(G). \end{aligned}$$

For $\mu - \lambda = k$, the following 1-cocycles span the corresponding cohomology spaces :

$$(3.3) \quad \begin{aligned} \Upsilon_{\lambda, \lambda+k}^1(X_G) &= \bar{\eta}_1^2(\partial_2(\theta_2 G)) \partial_x^k, \\ \Omega_{\lambda, \lambda+k}^1(X_G) &= \bar{\eta}_1^2(\partial_2(\theta_2 G)) \bar{\eta}_1 \bar{\eta}_2 \partial_x^{k-1}, \\ \Upsilon_{\lambda, \lambda+k}^2(X_G) &= \bar{\eta}_1 \bar{\eta}_2(G) \partial_x^k, \\ \Omega_{\lambda, \lambda+k}^2(X_G) &= \bar{\eta}_1 \bar{\eta}_2(G) \bar{\eta}_1 \bar{\eta}_2 \partial_x^{k-1}. \end{aligned}$$

The proof of Theorem 3.1 will be the subject of Section 5. In fact, we need first the description of $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^2)$ and the $\mathfrak{aff}(1|1)$ -relative cohomology

$H_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}^2)$. To describe the latter one, we need also the description of some bilinear $\mathfrak{aff}(1|1)$ -invariant mappings.

4. Invariant Operators and Cohomology of $\mathfrak{aff}(1|1)$

4.1. Invariant linear differential operators. Observe that, as a $\mathfrak{aff}(1|1)$ -module, we have

$$\mathfrak{aff}(2|1) \simeq \mathfrak{aff}(1|1) \oplus \Pi(\mathcal{H}),$$

where $\mathcal{H}$ is the subspace of $\mathfrak{F}_{-\frac{1}{2}}$ spanned by $\{\alpha_1^{-\frac{1}{2}}, \theta_1 \alpha_1^{-\frac{1}{2}}\}$. To be more precise, any element X_F is decomposed into $X_F = X_{F_1} + X_{F_2 \theta_2}$ where $\partial_2 F_1 = \partial_2 F_2 = 0$, and then $X_{F_1} \in \mathfrak{aff}(1|1)$ and $X_{F_2 \theta_2}$ can be identified to $\Pi(F_2 \alpha_1^{-\frac{1}{2}}) \in \Pi(\mathcal{H})$. To compute the $\mathfrak{aff}(1|1)$ -relative cohomology of $\mathfrak{aff}(2|1)$, we need the description of $\mathfrak{aff}(1|1)$ -invariant mappings from $\mathcal{H} \otimes \mathfrak{F}_{\lambda}^1$ to $\mathfrak{F}_{\mu}^1$. To do that, we first, describe the $\mathfrak{aff}(1)$ -invariant mappings from $\mathcal{H} \otimes \mathcal{F}_{\lambda}$ to $\mathcal{F}_{\mu}$. Obviously, as a $\mathfrak{aff}(1)$ -module, we have $\mathcal{H} \simeq \mathcal{H}_0 \oplus \Pi(\mathcal{H}_1)$, where $\mathcal{H}_0$ is the subspace of $\mathcal{F}_{-\frac{1}{2}}$ spanned by $\{(dx)^{-\frac{1}{2}}\}$ and $\mathcal{H}_1$ is the subspace of $\mathcal{F}_0$ spanned by 1. Moreover, we can see easily that

$$(4.1) \quad [\mathfrak{aff}(1|1), \Pi(\mathcal{H})] \subset \Pi(\mathcal{H}) \quad \text{and} \quad [\Pi(\mathcal{H}), \Pi(\mathcal{H})] \subset \mathfrak{aff}(1|1).$$

Lemma 4.1. (see, e.g., [1]). *Let*

$$A : \mathcal{H}_0 \times \mathcal{F}_{\lambda} \rightarrow \mathcal{F}_{\mu}, (h_0 dx^{-\frac{1}{2}}, f dx^{\lambda}) \mapsto A(h_0, f) dx^{\mu}$$

be a bilinear differential operator.

If A is $\mathfrak{aff}(1)$ -invariant then the following relation holds $A_k(h_0, f) = a_k h_0 f^{(k)}$, where $a_k(\mu + \frac{1}{2} - \lambda - k) = 0$.

Lemma 4.2. *Let*

$$B : \mathcal{H}_1 \times \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu, (h_1, f dx^\lambda) \mapsto A(h_1, f) dx^\mu$$

be a bilinear differential operator.

If B is $\mathfrak{aff}(1)$ -invariant then the following relation holds $B_k(h_1, f) = b_k h_1 f^{(k)}$, where $b_k(\mu - \lambda - k) = 0$.

Proposition 4.3. *Let*

$$\mathcal{A} : \mathcal{H} \times \mathfrak{F}_\lambda \rightarrow \mathfrak{F}_\mu, (H \alpha_1^{-\frac{1}{2}}, F \alpha_1^\lambda) \mapsto \mathcal{A}(H, F) \alpha_1^\mu$$

be a non-zero $\mathfrak{aff}(1|1)$ -invariant bilinear differential operator. Then one of the following holds:

i) If $\mu = \lambda + k - \frac{1}{2}$ where k is a non-negative integer, then, up to a scalar factor, the map $\mathcal{A}$ is given by:

$$(4.2) \quad \mathcal{A}(H, F) = H F^{(k)} - (-1)^{|H|} \overline{\eta}_1(H) \overline{\eta}_1(F^{(k-1)}).$$

ii) If $\mu = \lambda + k$, where k is a non-negative integer, then, up to a scalar factor, the map $\mathcal{A}$ is given by:

$$(4.3) \quad \mathcal{A}(H, F) = (-1)^{|H|} H \overline{\eta}_1(F^{(k)}) + \overline{\eta}_1(H) F^{(k)}.$$

Proof. Let $\mathcal{A} = \mathcal{A}_{\overline{0}} + \mathcal{A}_{\overline{1}}$ be the decomposition of $\mathcal{A}$ into even and odd parts. As $\mathfrak{aff}(1)$ -module, we have

$$(4.4) \quad \mathcal{H} \times \mathfrak{F}_\lambda^1 \simeq \mathcal{H}_0 \otimes \mathcal{F}_\lambda \oplus \mathcal{H}_0 \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \oplus \Pi(\mathcal{H}_1) \otimes \mathcal{F}_\lambda \oplus \Pi(\mathcal{H}_1) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}).$$

So, the map $\mathcal{A}_{\overline{0}}$ is decomposed into four maps:

$$(4.5) \quad \begin{aligned} \mathcal{H}_0 \otimes \mathcal{F}_\lambda &\rightarrow \mathcal{F}_\mu, & \mathcal{H}_0 \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) &\rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), \\ \Pi(\mathcal{H}_1) \otimes \mathcal{F}_\lambda &\rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), & \Pi(\mathcal{H}_1) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) &\rightarrow \mathcal{F}_\mu \end{aligned}$$

and $\mathcal{A}_{\overline{1}}$ is also decomposed into four maps:

$$(4.6) \quad \begin{aligned} \mathcal{H}_0 \otimes \mathcal{F}_\lambda &\rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), & \mathcal{H}_0 \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) &\rightarrow \mathcal{F}_\mu, \\ \Pi(\mathcal{H}_1) \otimes \mathcal{F}_\lambda &\rightarrow \mathcal{F}_\mu, & \Pi(\mathcal{H}_1) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) &\rightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}) \end{aligned}$$

□

4.2. The space $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^2)$. Let $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{p}$ be a Lie superalgebra, where $\mathfrak{q}$ is a subalgebra and $\mathfrak{p}$ is a $\mathfrak{q}$ -module such that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{q}$. Consider a 1-cocycle $\Upsilon \in Z^1(\mathfrak{g}, V)$, where V is a $\mathfrak{g}$ -module. The cocycle relation reads

$$\Upsilon([g, h]) - (-1)^{|g||\Upsilon|} g \cdot \Upsilon(h) - (-1)^{|h|(|g|+|\Upsilon|)} h \cdot \Upsilon(g) = 0, \quad g, h \in \mathfrak{g}.$$

Denote $\Upsilon_q = \Upsilon|_q$ and $\Upsilon_p = \Upsilon|_p$. Obviously, Υ_q is a 1-coycle over q and if $\Upsilon_q = 0$ then Υ_p is q -invariant. Thus, the space $H^1(\mathfrak{g}, V)$ is closely related to the space $H^1(q, V)$. Furthermore, q and p subject to the following equations:

$$(4.7) \quad \Upsilon_p([h, p]) - (-1)^{|h||\Upsilon|} h \cdot \Upsilon_p(p) - (-1)^{|p|(|h|+|\Upsilon|)} p \cdot \Upsilon_q(h) = 0, \quad h \in \mathfrak{g}, p \in p.$$

$$(4.8) \quad \Upsilon_q([p, p']) - (-1)^{|p||\Upsilon|} p \cdot \Upsilon_p(p') - (-1)^{|p'|(|p|+|\Upsilon|)} p' \cdot \Upsilon_p(p) = 0, \quad p, p' \in p.$$

In our situation, $\mathfrak{g} = \mathfrak{aff}(2|1)$, $q = \mathfrak{aff}(1|1)$, $p = \Pi(\mathfrak{h})$ and $V = \mathfrak{D}_{\lambda, \mu}^2$. Thus, as a first step towards the proof of Theorem 3.1, we shall need to compute $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^2)$. According to isomorphism (2.4), we can see that the first cohomology space $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^2)$ is closely related to the spaces $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^1)$:

$$(4.9) \quad \begin{aligned} H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^2) &\simeq H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^1) \oplus H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}^1) \\ &\oplus H_{\text{diff}}^1(\mathfrak{aff}(1|1), \Pi(\mathfrak{D}_{\lambda, \mu+\frac{1}{2}}^1)) \oplus H_{\text{diff}}^1(\mathfrak{aff}(1|1), \Pi(\mathfrak{D}_{\lambda+\frac{1}{2}, \mu}^1)). \end{aligned}$$

Thus, we recall the space $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^1)$ which was computed in [1]:

$$H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^1) \simeq \begin{cases} \mathbb{R} & \text{if } \mu - \lambda = \begin{cases} k, & k \in \mathbb{N}, \\ k - \frac{1}{2}, & k \in \mathbb{N}^*. \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

A base for the nontrivial space $H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^1)$ is given by the cohomology classes of the 1-cocycles

$$(4.10) \quad \begin{aligned} \Theta_{\lambda, \lambda+k}(X_F) &= \bar{\eta}_1^2(F) \bar{\eta}_1^{2k}, \quad k \in \mathbb{N}, \\ \Theta_{\lambda, \lambda+k-\frac{1}{2}}(X_F) &= \bar{\eta}_1^2(F) \bar{\eta}_1^{2k-1}, \quad k \in \mathbb{N}^*. \end{aligned}$$

Thus, we can easily deduce the following result:

Proposition 4.4. *We have*

$$H_{\text{diff}}^1(\mathfrak{aff}(1|1), \mathfrak{D}_{\lambda, \mu}^2) \simeq \begin{cases} \mathbb{R}^4 & \text{if } \mu - \lambda = k, \quad k \in \mathbb{N}^*, \\ \mathbb{R}^2 & \text{if } \mu - \lambda = 0, \\ \mathbb{R} & \text{if } \mu - \lambda = \pm \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mu - \lambda = 0$, the following 1-cocycles span the corresponding cohomology spaces:

$$\begin{aligned} \Upsilon_{\lambda, \lambda}^1(X_G) &= \bar{\eta}_1^2(\partial_2(\theta_2 G)), \\ \Upsilon_{\lambda, \lambda}^2(X_G) &= \bar{\eta}_1 \bar{\eta}_2(G). \end{aligned}$$

For $\mu - \lambda = \pm \frac{1}{2}$, the following 1-cocycles span the corresponding cohomology spaces:

$$\Upsilon_{\lambda, \lambda \pm \frac{1}{2}}^3(X_G) = G'.$$

For $\mu - \lambda = k$, the following 1-cocycles span the corresponding cohomology spaces :

$$\begin{aligned}\Upsilon_{\lambda, \lambda+k}^4(X_G) &= \bar{\eta}_1^2(\partial_2(\theta_2 G))\partial_x^k, \\ \Upsilon_{\lambda, \lambda+k}^5(X_G) &= \bar{\eta}_1^2(\partial_2(\theta_2 G))\bar{\eta}_1\bar{\eta}_2\partial_x^k, \\ \Upsilon_{\lambda, \lambda+k}^6(X_G) &= \bar{\eta}_1^2(\partial_2(\theta_2 G))\partial_x^{k-\frac{1}{2}}, \\ \Upsilon_{\lambda, \lambda+k}^7(X_G) &= \bar{\eta}_1^2(\partial_2(\theta_2 G))\bar{\eta}_1\bar{\eta}_2\partial_x^{k-\frac{1}{2}}.\end{aligned}$$

4.3. The space $H_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}^2)$. In this subsection we compute the space $H_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}^2)$. and we prove that it is nontrivial which is not the case for $n = 1$ see, [1].

Theorem 4.5.

$$\dim H_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{aff}(1|1); \mathfrak{D}_{\lambda, \mu}^2) \simeq \begin{cases} 1 & \text{if } \mu - \lambda = 0, \\ 2 & \text{if } \mu - \lambda = k, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mu - \lambda = 0$, the cohomology classes of the following 1-cocycles generate the corresponding spaces:

$$\tilde{\Upsilon}_{\lambda, \lambda}(X_G) = \bar{\eta}_1\bar{\eta}_2(G).$$

For $\mu - \lambda = k$, the cohomology classes of the following 1-cocycles generate the corresponding spaces:

$$\begin{aligned}\tilde{\Upsilon}_{\lambda, \lambda+k}^1(X_G) &= \bar{\eta}_1\bar{\eta}_2(G)\partial_x^k, \\ \tilde{\Upsilon}_{\lambda, \lambda+k}^2(X_G) &= \bar{\eta}_1\bar{\eta}_2(G)\bar{\eta}_1\bar{\eta}_2\partial_x^{k-1}.\end{aligned}$$

To prove Theorem 4.5, we need the following classical fact:

Lemma 4.6. Let $\mathfrak{g}$ be a Lie superalgebra and $A : U \otimes V \rightarrow W$ a bilinear map, where U, V and W are $\mathfrak{g}$ -modules. We consider the following associated maps

$$\begin{aligned}A_1 : \Pi(U) \otimes V &\rightarrow W, & A_2 : \Pi(U) \otimes \Pi(V) &\rightarrow \Pi(W), \\ A_3 : \Pi(U) \otimes V &\rightarrow \Pi(W), & A_4 : \Pi(U) \otimes \Pi(V) &\rightarrow W\end{aligned}$$

defined by

$$\begin{aligned}A_1(\Pi(u) \otimes v) &= (-1)^{|u|}A(u \otimes v), & A_2(\Pi(u) \otimes \Pi(v)) &= (-1)^{|u|}\Pi(A(u \otimes v)), \\ A_3(\Pi(u) \otimes v) &= (-1)^{|v|}\Pi(A(u \otimes v)), & A_4(\Pi(u) \otimes \Pi(v)) &= (-1)^{|v|}A(u \otimes v).\end{aligned}$$

The maps A_1, A_2, A_3 and A_4 are $\mathfrak{g}$ -invariant if and only if A is $\mathfrak{g}$ -invariant.

Lemma 4.7. Any 1-cocycle $\Upsilon \in Z_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{D}_{\lambda, \mu}^2)$ is a coboundary if and only if its restriction to $\mathfrak{aff}(1|1)$ is a coboundary.

Proof. Of course, if Υ is a coboundary over $\mathfrak{aff}(2|1)$, then its restriction to $\mathfrak{aff}(1|1)$ is a coboundary over $\mathfrak{aff}(1|1)$. So, assume that $\Upsilon|_{\mathfrak{aff}(1|1)}$ is a coboundary, that is, there exist $\Psi \in \mathfrak{D}_{\lambda, \mu}^2$ such that

$$\Upsilon(X_{F_1}) = \delta(\Psi) \quad \text{for all } X_{F_1} \in \mathfrak{aff}(1|1).$$

By replacing Υ by $\Upsilon - \delta(\Psi)$, we can suppose that Υ vanishes on $\mathfrak{aff}(1|1)$. But, in this case, according to (4.1), the 1-cocycle relations read

$$(4.11) \quad \begin{cases} \mathfrak{L}_{X_{F_1}}^{\lambda, \mu} \Upsilon(X_{H\theta_2}) - \Upsilon([X_{F_1}, X_{H\theta_2}]) = 0, \\ \mathfrak{L}_{X_{H_1\theta_2}}^{\lambda, \mu} \Upsilon(X_{H_2\theta_2}) - (-1)^{p(H_1\theta_2)p(H_2\theta_2)} \mathfrak{L}_{X_{H_2\theta_2}}^{\lambda, \mu} \Upsilon(X_{H_1\theta_2}) = 0, \end{cases}$$

where $X_{F_1}, X_{F_2} \in \mathfrak{aff}(1|1)$ and $H\alpha_1^{-\frac{1}{2}}, H_1\alpha_1^{-\frac{1}{2}}, H_2\alpha_1^{-\frac{1}{2}} \in \mathcal{H}$. If Υ is an odd 1-cocycle then, according to the first equation of (4.11), we get

$$\Upsilon(X_{H\theta_2, F}) = \begin{cases} a_1(-1)^{|H|} (HF_1^{(k)} - (-1)^{|H|} \bar{\eta}_1(H) \bar{\eta}_1(F_1^{(k-1)})) \\ + a_2(-1)^{|F_2|} ((-1)^{|H|} H \bar{\eta}_1(F_2^{(k-1)}) + \bar{\eta}_1(H) F_2^{(k-1)}) \\ + a_3(-1)^{|H|} (HF_2^{(k)} - (-1)^{|H|} \bar{\eta}_1(H) \bar{\eta}_1(F_2^{(k-1)})) \theta_2 \\ + a_4(-1)^{|F_1|} ((-1)^{|H|} H \bar{\eta}_1(F_1^{(k)}) + \bar{\eta}_1(H) F_1^{(k)}) \theta_2 & \text{if } \mu = \lambda - \frac{1}{2} + k, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_i \in \mathbb{R}$, $F \in \mathbb{R}[x, \theta]$ and $H \in \mathbb{R}[x, \theta]$. But, the map Υ must satisfy the equation (4.11), so we obtain for the nonzero a_i the following relations: $a_3 = -a_1$ and $a_4 = -a_2$.

$$\begin{aligned} \Upsilon(X_{H\theta_2, F}) &= a_1((-1)^{|H|} H(-1)^{|F|} F^{(k)} - \bar{\eta}_1(H) \bar{\eta}_1((-1)^{|F|} F^{(k-1)})) \\ &+ a_2((-1)^{|H|} H \bar{\eta}_1 \bar{\eta}_2((-1)^{|F|} F^{(k-1)}) + \bar{\eta}_1(H) \bar{\eta}_2((-1)^{|F|} F^{(k-1)})). \end{aligned}$$

$$\Upsilon = \begin{cases} a_1 \delta(\bar{\eta}_2 \partial_x^{k-1}) + a_2 \delta(\bar{\eta}_1 \partial_x^{k-1}) & \text{if } \mu = \lambda - \frac{1}{2} + k, \\ 0 & \text{otherwise,} \end{cases}$$

Similarly, if Υ is even 1-cocycle, then, up to a scalar factor, Υ is given by

$$\Upsilon(X_{H\theta_2, F}) = \begin{cases} b_1(-1)^{|H|} ((-1)^{|H|} H \bar{\eta}_1(F_1^{(k)}) + \bar{\eta}_1(H) F_1^{(k)}) \\ + b_2(-1)^{|F_2|} (HF_2^{(k)} - (-1)^{|H|} \bar{\eta}_1(H) \bar{\eta}_1(F_2^{(k-1)})) \\ + b_3(-1)^{|H|} ((-1)^{|H|} H \bar{\eta}_1(F_2^{(k)}) + \bar{\eta}_1(H) F_2^{(k)}) \theta_2 \\ + b_4(-1)^{|F_1|} (HF_1^{(k+1)} - (-1)^{|H|} \bar{\eta}_1(H) \bar{\eta}_1(F_1^{(k)})) \theta_2 & \text{if } \mu = \lambda + k, \\ 0 & \text{otherwise,} \end{cases}$$

the map Υ must satisfy the equation (4.11), so we obtain for the nonzero b_i the following relations:

$b_3 = b_1$ and $b_4 = b_2$.

$$\begin{aligned} \Upsilon(X_{H\theta_2, F}) &= b_1(H \bar{\eta}_1(F^{(k)}) + (-1)^{|H|} \bar{\eta}_1(H) F^{(k)}) \\ &+ b_2(H \bar{\eta}_2((-1)^{|F|} F^{(k)}) - (-1)^{|H|} \bar{\eta}_1(H) \bar{\eta}_1 \bar{\eta}_2((-1)^{|F|} F^{(k-1)})). \end{aligned}$$

$$\Upsilon = \begin{cases} b_1 \tilde{\Upsilon}_{\lambda, \lambda+k} + b_2 \delta(\partial_x^k) & \text{if } \mu = \lambda + k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\Upsilon}_{\lambda, \lambda+k}$ is those given in Theorem 4.5.

Therefore, in order to complete the proof of Theorem 4.5, we have to study the cohomology classes of the 1-cocycle $\tilde{\Upsilon}_{\lambda, \lambda+k}$. $\square$

□

Lemma 4.8. *The maps $\tilde{\Upsilon}_{\lambda, \lambda+k}$ is nontrivial $\mathfrak{aff}(1|1)$ -relative 1-cocycle.*

Proof. First, we can easily see that, for any even element $F \in C^\infty(\mathbb{R}^{1|2})$,

$$(4.12) \quad \tilde{\Upsilon}_{\lambda, \lambda+k}(X_{\theta_1 \theta_2})(F \alpha_2^{\lambda+k}) = \theta_1 \bar{\eta}_1(F^{(k)}) + F^{(k)}.$$

Next, assume that there exists an even operator $A \in \mathfrak{D}_{\lambda, \lambda+k}^2$ such that $\tilde{\Upsilon}_{\lambda, \lambda+k}$ is equal to δA , that is

$$(4.13) \quad \tilde{\Upsilon}_{\lambda, \lambda+k} = \mathfrak{L}_{X_G}^{\lambda+k} \circ A - A \circ \mathfrak{L}_{X_G}^\lambda.$$

The operator A is of the form (2.3); the condition (4.13) implies that its coefficients are constants (which is equivalent to the fact that $X_1.A = 0$). Then, it is now easy to check that the condition (4.13) has no solution: using formula (2.1), we can see that the expression (4.12) never appear in the right hand side of (4.13). This is a contradiction with our assumption. Lemma 4.7 is proved. Thus we have completed the proof of Theorem 4.5. □

□

Corollary 4.9. *Up to a coboundary, any 1-cocycle $\Upsilon \in Z_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{D}_{\lambda, \mu}^2)$ is invariant with respect to the vector field $X_1 = \partial_x$. That is, the map Υ can be expressed with constant coefficients.*

Proof. The 1-cocycle condition reads:

$$(4.14) \quad X_1.\Upsilon(X_F) - (-1)^{|F||\Upsilon|} X_F.\Upsilon(X_1) - \Upsilon([X_1, X_F]) = 0$$

But, from (4.10) and Theorem 4.5, it follows that, up to a coboundary, we have $\Upsilon(X_1) = 0$, and therefore the equation (4.14) becomes

$$X_1.\Upsilon(X_F) - \Upsilon([X_1, X_F]) = 0$$

which is nothing but the invariance property of Υ with respect to X_1 . □

5. Proof of Theorem 3.1

Consider a 1-cocycle $\Upsilon \in Z_{\text{diff}}^1(\mathfrak{aff}(2|1), \mathfrak{D}_{\lambda, \mu}^2)$. If $\Upsilon|_{\mathfrak{aff}(1|1)}$ is trivial then the 1-cocycle Υ is completely described by Theorem 4.5. Thus, assume that $\Upsilon|_{\mathfrak{aff}(1|1)}$ is nontrivial. Of course, up to coboundary, the general form of $\Upsilon|_{\mathfrak{aff}(1|1)}$ is given by (4.10) together with the isomorphism (4.9) while $\Upsilon|_{\Pi(h)}$ can be essentially described by equation (4.7) and Corollary 4.9. More precisely, according to (4.10) and the isomorphism (4.9), the 1-cocycle Υ can be nontrivial only, a priori, if $\mu = \lambda + k$ where $k \in \mathbb{N}$. Thus, we have to distinguish all these cases. Hereafter all ξ 's are constants.

The case where $\mu = \lambda + k$

Considering (4.10) and the isomorphism (4.9), we see that there are two subcases: In this case, the map $\Upsilon|_{\mathfrak{aff}(1|1)}$ is, a priori, given by

$$\Upsilon|_{\mathfrak{aff}(1|1)}(X_{G_1})(F\alpha_2^\lambda) = (\xi_1\bar{\eta}_1^2(G_1)\bar{\eta}_1^{2k}(F_1) + \xi_2\bar{\eta}_1^2(G_1)\bar{\eta}_1^{2k}(F_2)\theta_2 + \xi_3\bar{\eta}_1^2(G_1)\bar{\eta}_1^{2k}(F_2))\alpha_2^{\lambda+k},$$

where $F = F_1 + F_2\theta_2$, with $\partial_2 F_1 = \partial_2 F_2 = 0$. By direct computation, using equations (4.7) and (4.8) and Corollary 4.9, we deduce that $\xi_1 = \xi_2, \xi_3 = 0$ and

$$\Upsilon|_{\Pi(h)} = 0.$$

The case where $\mu = \lambda + k - \frac{1}{2}$

In this case, the map $\Upsilon|_{\mathfrak{aff}(1|1)}$ is, a priori, given by

$$\Upsilon|_{\mathfrak{aff}(1|1)}(X_{G_1})(F\alpha_2^\lambda) = (\xi_1\bar{\eta}_1^2(G_1)\bar{\eta}_1^{2k-1}(F_1) + (\xi_2\bar{\eta}_1^2(G_1)\bar{\eta}_1^{2k-1}(F_1) + \xi_3\bar{\eta}_1^2(G_1)\bar{\eta}_1^{2k-1}(F_2))\theta_2)\alpha^{\lambda+k-\frac{1}{2}}.$$

The same arguments, as above, show that $\xi_1 = \xi_3, \xi_2 = 0$ and

$$\Upsilon|_{\Pi(h)} = 0.$$

The case where $\mu = \lambda + k - 1$ or $\mu = \lambda + k + \frac{1}{2}$

Using the same arguments, we prove that, up to a coboundary, in each case, we have $\Upsilon = 0$.

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