# Eigenfunctions of Ultrametric Morphological Openings and Closings

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**Abstract.** This paper deals with the relationship between spectral analysis in min–max algebra and ultrametric morphological operators. Indeed, morphological semigroups in ultrametric spaces are essentially based on that algebra. Theory of eigenfunctionals in min–max analysis is revisited, including classical applications (preference analysis, percolation and hierarchical segmentation). Ultrametric distance is the fix point functional in min–max analysis and from this result, we prove that the ultrametric distance is the key ingredient to easily define the eigenfunctions of ultrametric morphological openings and closings.

**Keywords:** ultrametric space; ultrametric image processing; (min, max)-analysis; morphological semigroups

## 1 Introduction

Given a square matrix $A$, one of the fundamental linear algebra problems, known as spectral analysis of $A$, is to find a number $\lambda$, called eigenvalue, and a vector $v$, called eigenvector, such that $Av = \lambda v$. This problem is ubiquitous in both mathematics and physics. In the infinity dimensional generalization the problem is also relevant for linear operators. An example of its interest is the case of the spectral analysis of Laplace operator. The spectrum of the Laplace operator consists of all eigenvalues $\lambda_i$ for which there is a corresponding eigenfunction $\phi_i$ with: $-\Delta\phi_i = \lambda_i\phi_i$. Then, for instance, given any initial heat distribution $f(x)$ in a bounded domain $\Omega$, the solution of the heat equation at time $t$, $u(x,t) = (f * k_t)(x) = P_t f(x)$, can be written either by its convolution form or by its expectral expansion, i.e.,

$$P_t f(x) = \frac{1}{(4\pi t)^{d/2}} \int_\Omega f(y) e^{-|x-y|^2/4t} dy = \sum_{i=1}^{+\infty} e^{-t\lambda_i} \langle u_0, \phi_i \rangle \phi_i(x), \qquad (1)$$

with the heat kernel as $k_t(x,y) = \sum_{i=1}^{+\infty} e^{-t\lambda_i}\phi_i(x)\overline{\phi_i}(y)$.

From the 60s and 70s of last century, different applied mathematics fields have been studying a more general eigenproblem where the addition and multiplication in matrix and vectors operations are replaced by other pairs of operations.

If one replaces the addition and multiplication of vectors and matrices by operations of maximum and sum, the corresponding "linear algebra" is called max–plus algebra, which has been extensively studied, including the eigenproblem, see for instance the book [11] for exhaustive list of references. But that problem is out of the scope of this paper. Readers interested on max–plus matrix algebra and spectral analysis from the perspective of mathematical morphology are referred to the excellent survey by Maragos [13]. In the case where one replaces respectively by operations of maximum and minimum, we work on the so-called max–min algebra (also known as bottleneck algebra [5]). Spectral analysis in max–min algebra is also relatively classic from their first interpretation in the field of hierarchical clustering [8]. Eigenvectors of max-min matrices and their connection with paths in digraphs were widely investigated by Gondran and Minoux, see overview papers [9,12], by Cechlárova [5] and by Gavalec [7]. Spectral analysis in max–min algebras is also relatively classic in fuzzy reasoning [15]. This eigenproblem in distributive lattice was studied in [16]. Procedures and efficient algorithms to compute the maximal eigenvector of a given max-min matrix has been also considered [5]. Max-min algebra is also very relevant in several morphological frameworks, such as fuzzy logic, viscous morphology or geodesic reconstruction, see our overview in [1].

In this work, we are interested on relating the notion of spectral analysis in max–min algebra to ultrametric morphological operators [2]. Indeed, morphological semigroups in ultrametric spaces are essentially based on that algebra. The expansion provided by (1) for the diffusion process using the Laplacian eigenfunctions can be similarly formulated in ultrametric spaces [3]. The interpretation of ultrametric Laplace eigenfunctions depends obvioulsy of the hierarchical organisation of ultrametric balls according to the ultrametric distance. We show here that the ultrametric distance is also the key ingredient to define the eigenfunctions of ultrametric morphological openings and closings. The theoretical results of this paper are mainly based on Gondran and Minoux theory, where the discrete case was considered in [9] and later, in [10] the continuous (and infinite dimensional) one. The later study also considered the preliminary interest in nonlinear physics such as percolation. This paper is another step forwards in our program of revisiting classical image/data processing on ultrametric representations. The rest of its contents is organised as follows. Section 2 provides a short reminder of the main definitions and properties of ultrametric morphological operators. Gondran and Minoux theory of min–max analysis of operators and matrices is reviewed is Section 3. Section 4 discusses our contribution to the study of the eigensystem of ultrametric morphological operators. Finally, Section 5 closes the paper with some conclusions and perspectives.

## 2 Remind on ultrametric morphological openings and closings

Before going further, let us recall basic facts on ultrametric morphological openings and closings. One can refer to [2] for details.

Given a separable and complete ultrametric space $(X, d)$, let us consider the family of non-negative bounded functions $f$ on $(X, d)$, $f : X \to [0, M]$. The complement (or negative) function of $f$, denoted $f^c$, is obtained by the involution $f^c(x) = M - f(x)$. The set of non-negative bounded functions on ultrametric space is a lattice with respect to the pointwise maximum $\vee$ and minimum $\wedge$.

**Definition 1.** *The canonical isotropic ultrametric structuring function is the parametric family* $\{b_t\}_{t>0}$ *of functions* $b_t : X \times X \to (-\infty, D]$ *given by*

$$b_t(x, y) = M - \frac{d(x, y)}{t}. \tag{2}$$

**Definition 2.** *Given an ultrametric structuring function* $\{b_t\}_{t>0}$ *in* $(X, d)$*, for any non-negative bounded function* $f$ *the ultrametric dilation* $D_t f$ *and the ultrametric erosion* $E_t f$ *of* $f$ *on* $(X, d)$ *according to* $b_t$ *are defined as*

$$D_t f(x) = \sup_{y \in X} \{f(y) \wedge b_t(x, y)\}, \quad \forall x \in X, \tag{3}$$

$$E_t f(x) = \inf_{y \in X} \{f(y) \vee b_t^c(x, y)\}, \quad \forall x \in X. \tag{4}$$

We can easily identify that the ultrametric dilation is a kind of product in $(\max, \min)$-algebra of function $f$ by $b_t$. Considering the classical algebraic definitions of morphological operators [17] for the case of ultrametric semigroups $\{D_t\}_{t\geq 0}$, resp. $\{E_t\}_{t\geq 0}$, they have the properties of increasingness and commutation with supremum, resp. infimum, which involves that $D_t$ is a dilation and $E_t$ is an erosion. In addition, they are extensive, resp. anti-extensive, operators and, by the supremal semigroup, both are idempotent operators, i.e., $D_t D_t = D_t$ and $E_t E_t = E_t$, which implies that $D_t$ is a closing and $E_t$ is an opening. Finaly, these semigroups are just the so-called granulometric semigroup [17] and therefore $\{D_t\}_{t\geq 0}$ is an anti-granulometry and $\{E_t\}_{t\geq 0}$ is a granulometry, which involve interesting scale-space properties useful for filtering and decomposition.

Let $(X, d)$ be a discrete ultrametric space. Choose a sequence $\{c_k\}_{k=0}^{\infty}$ of positive reals such that $c_0 = 0$ and $c_{k+1} > c_k \geq 0$, $k = 0, 1, \cdots$. Then, given $t > 0$, ones defines the sequence $\{b_{k,t}\}_{k=0}^{\infty}$, such that

$$b_{k,t} = M - t^{-1} c_k. \tag{5}$$

Let us define $\forall k$, $\forall x \in X$, the ultrametric dilation and erosion of radius $k$ on the associated partition as

$$\mathrm{Q}_k^{\vee} f(x) = \sup_{y \in B_k(x)} f(y), \tag{6}$$

$$\mathrm{Q}_k^{\wedge} f(x) = \inf_{y \in B_k(x)} f(y). \tag{7}$$

Using now (6) and (7), it is straightforward to see that the ultrametric dilation and ultrametric erosion of $f$ by $b_{k,t}$ can be written as

$$D_t f(x) = \sup_{0 \leq k \leq \infty} \{\mathrm{Q}_k^{\vee} f(x) \wedge b_{k,t}\}, \tag{8}$$

$$E_t f(x) = \inf_{0 \leq k \leq \infty} \{\mathrm{Q}_k^{\wedge} f(x) \vee (M - b_{k,t})\}. \tag{9}$$

From this formulation, one does not need to compute explicitly the ultrametric distance between all-pairs of points $x$ and $y$ since $D_t f(x)$ and $E_t f(x)$ are obtained by working on the supremum and infimum mosaics $\mathrm{Q}_k^{\vee} f(x)$ and $\mathrm{Q}_k^{\wedge} f(x)$ from the set of partitions, which is usually finite, i.e., $k = 0, 1, \cdots, K$.

## 3 Eigen-functionals in $(\min, \max)$-analysis

In this section, the main elements of the Gondran and Minoux theory [9,10] of eigenvalues and eigen-functionals of diagonally dominant endomorphophisms in min–max analysis is reviewed. The theory is the background to the specific problem of the study of eigenfunction of ultrametric morphological operators.

### 3.1 Inf-diagonal dominant kernel (idd-kernel) in $(\min, \max)$-algebra and its powers

Let us first introduce the axiomatic definition of an *inf-diagonal dominant kernel (idd-kernel).*

**Definition 3.** *A proper l.s.c. (with closed and bounded lower-level sets) functional $\alpha : X \times X \to \overline{\mathbb{R}}$ is called an idd-kernel if the following two conditions are satisfied*

1. *Boundedness and diagonal uniformity: there exists a finite value $0_\alpha$ such that*

$$\alpha(x, x) = 0_\alpha, \ \ \forall x \in X;$$

2. *inf-diagonal dominance, i.e.,*

$$\alpha(x, y) \geq 0_\alpha, \ \ \forall x, y \in X;$$

*which is equivalent to $\forall x \in X$:*

$$\alpha(x, x) \leq \inf_{y \neq x} \{\alpha(x, y)\}. \tag{10}$$

Let us denote by $\mathcal{A}$ the set of idd-kernels in $X$. Using the $(\min, \max)$-associativity property, the succesive $(\min, \max)$-powers of an idd-kernel $\alpha \in \mathcal{A}$ in $X$ may be defined recursively as:

$$\alpha^n(x, y) = \min_{z \in X} \left\{\alpha^{n-1}(x, z) \vee \alpha(z, y)\right\}, \ \ \forall n \in \mathbb{N}, n > 2. \tag{11}$$

Let $\alpha \in \mathcal{A}$ be a idd-kernel. Considering for instance $n = 2$, one has

$$\alpha^2(x, y) = \min_{z \in X} \{\alpha(x, z) \vee \alpha(z, y)\}.$$

In particular, taking $z = y$ above and using the inf-diagonal dominance, we have

$$\alpha^2(x, y) \leq \alpha(x, y) \vee \alpha(y, y) = \alpha(x, y), \quad x, y \in X,$$

which provides a non-increasing behaviour. Indeed, there exists a stronger convergence result to a fix-point which is easily see for the fact that the sequence $\alpha(x, y)$, $\alpha^2(x, y) \cdots \ \alpha^n(x, y)$ is non-increasing, together with the fact that it is bounded from below by $\alpha(x, x) = 0_\alpha$. More formally [10]:

**Proposition 1.** *The endomorphism $\alpha^*$ defined by limit*

$$\alpha^*(x,y) = \lim_{n\to\infty} \alpha^n(x,y) \tag{12}$$

*always exists (i.e., limit is convergent) and satisfies the relationships*

$$\alpha^* = (\alpha^*)^2 = \alpha \cdot_{\min,\max} \alpha^* = \alpha^* \cdot_{\min,\max} \alpha.$$

### 3.2 $(\min,\max)$-eigenfunctions of $\alpha$ and $\alpha^*$

Let us introduce the $(\min,\max)$-product of a function $f \in \mathcal{F}(X,\overline{\mathbb{R}})$ and an idd-kernel $\alpha \in \mathcal{A}$ as follows

$$(f *_{\min,\max} \alpha)(x) = \inf_{y\in X} \{f(y) \vee \alpha(x,y)\}. \tag{13}$$

**Definition 4.** *Given an idd-kernel $\alpha \in \mathcal{A}$ and $\lambda \in \mathbb{R}$, a $\psi$ is called a $(\min,\max)$-eigenfunction of $\alpha$ for the eigenvalue $\lambda$ if and only if*

$$\psi *_{\min,\max} \alpha = \lambda \vee \psi.$$

The following two propositions provide on the one hand, the equivalence of the eigenfunctions of $\alpha$ and $\alpha^*$ and other hand an explicit way to compute the eigenfucntions from the "columns" of $\alpha^*$ [10]. We include the proof of the second proposition to justify the simplicity of the construction.

**Proposition 2.** *Let $\lambda > 0_\alpha$. If $\psi$ is a $(\min,\max)$-eigenfunction of $\alpha$ for the eigenvalue $\lambda$ then $\psi(x) \geq \lambda$, $\forall x \in X$. In addition, one has*

- $\lambda \vee \psi = 0_\alpha \vee \psi = \psi$ ;
- $\lambda \vee \psi = \psi *_{\min,\max} \alpha = \psi *_{\min,\max} \alpha^*$.

**Proposition 3.** *For $\alpha \in \mathcal{A}$ and $\lambda \in \mathbb{R}$ and for an arbitrary fixed $y \in X$, let $\phi_\lambda^y$ denote the functional in $f \in \mathcal{F}(\mathcal{X},\overline{\mathbb{R}})$ defined by*

$$\phi_\lambda^y(x) = \lambda \vee \alpha^*(x,y). \tag{14}$$

*Then, $\phi_\lambda^y$ is a $(\min,\max)$-eigenfunction of $\alpha$ for the eigenvalue $\lambda$.*

*Proof.* Since $\max\{\alpha(x,z); \phi_\lambda^y(z)\} = \max\{\alpha(x,z); \alpha(z,y); \lambda\}$, we obtain

$$\begin{aligned}
\left(\phi_\lambda^y *_{\min,\max} \alpha\right)(x) &= \inf_{z\in X} \{\alpha(x,z) \vee \phi_\lambda^y(z)\} \\
&= \max\left\{\inf_{z\in X} \{\alpha(x,z) \vee \alpha^*(z,y)\}; \lambda\right\} \\
&= \max\{\alpha \cdot_{\min,\max} \alpha^*(x,y); \lambda\}.
\end{aligned}$$

From properties of $\alpha^*$, $\alpha \cdot_{\min,\max} \alpha^*(x,y) = \alpha^*(x,y)$, thus $\forall x$,

$$\begin{aligned}
\left(\phi_\lambda^y *_{\min,\max} \alpha\right)(x) &= \max\{\alpha^*(x,y); \lambda\} = \max\{\max(\alpha^*(x,y); \lambda); \lambda\} \\
&= \max\{\phi_\lambda^y(x); \lambda\} = \phi_\lambda^y(x) \vee \lambda.
\end{aligned}$$

Finally, the following representation theorem provides us the interest of the theory.

**Theorem 1 (Gondran and Minoux, 1998).** *Let $\alpha \in \mathcal{A}$, $\lambda > 0_\alpha$, and for any $x, y \in X$, one computes $\phi^y_\lambda(x) = \lambda \vee \alpha^*(x,y)$. Then, the set*

$$G_\lambda = \{\phi^y_\lambda(x) \,:\, y \in X\},$$

*is the unique minimal generator of the set of* $(\min,\max)$*-eigenfunctions of $\lambda$. Consequently, any* $(\min,\max)$*-eigenfunction $\psi$ of $\alpha$ with eigenvalue $\lambda$, there exists a functional $h \in \mathcal{F}(\mathcal{X}, \overline{\mathbb{R}})$ such that $\psi$ can be expressed in terms of the $\phi^y_\lambda(x)$ as*

$$\psi(x) = \inf_{y\in X} \{h(y) \vee \phi^y_\lambda(x)\} = \langle h, \phi^y_\lambda \rangle_{\min,\max}. \tag{15}$$

### 3.3 Discrete case of iid-kernels

Let $X$ be a finite discrete space with $|X| = n$. The functional $\alpha(x_i, x_j)$ is represented by a matrix $A = (a_{ij}) \in M_n(\mathbb{R})$, i.e., $a_{ij} = \alpha(x_i, x_j)$. The corresponding eigenproblem is written as

$$A \cdot_{\min,\max} v = \lambda \vee v$$

where the matrix operations are given as follows. Given three matrices $A, B, C \in M_n(\mathbb{R})$, a scalar $\lambda \in \mathbb{R}$ and two vector $v, w \in \mathbb{R}^n$, one has matrix multiplication $A \cdot_{\min,\max} B = C \Leftrightarrow \bigwedge_{1\leq k\leq n}(a_{ik} \vee b_{kj}) = c_{ij}$, multiplication of a matrix by a scalar $\lambda \vee A = B \Leftrightarrow \lambda \vee a_{ij} = b_{ij}$ and multiplication of a vector by a matrix $A \cdot_{\min,\max} v = w \Leftrightarrow \bigwedge_{1\leq j\leq n}(a_{ij} \vee v_j) = w_i$. Thus

$$A^{(k)} = A \cdot_{\min,\max} A^{(k-1)},$$

is just the matrix product in the matrix algebra $(\min,\max)$. The limit

$$A^* = \lim_{k\to\infty} A^{(k)} = \alpha^*(x_i, x_j)$$

is called the quasi-inverse of $A$ in $(\min,\max)$-matrix algebra [9]. Obivously, $(\min,\max)$-eigenfunctions theory is valid for the discrete case.

### 3.4 Two applications

We consider now two first applications of the $(\min,\max)$-spectral theory.

**Preference analysis in $(\max,\min)$-algebra [12].** Given $n$ objects, find a total ordering between them using the pairwise comparison preferences (or votes) given by $K$ judges. The results of this kind of ranking can be represented by a preference $n \times n$-matrix $A = (a_{ij})$, where $a_{ij}$ denotes the number of judges who prefer $i$ to $j$. Note that by construction of the matrix $A$, $a_{ij} + a_{ji} = K$, $\forall i, j$, $i \neq j$. In the case of ties, it is assumed a $1/2$ contribution.

Starting from $A$, the method of partial orders is based on determining a hierarchy of preference relations on the objects with nested equivalence classes. More precisely, for any $\lambda$, the classes at level $\lambda$ are defined as the strong connected components of the graph $G_\lambda = (X, E_\lambda)$, with node set $X$ are the objects and the set of edges is $E_\lambda = \{(i,j) : a_{ij} \geq \lambda\}$.

Let us consider example with $n = 4$ and $K = 6$, given by the following matrix $A$ and its quasi-inverse in the $(\max, \min)$-algebra $A^*$:

$$A = \begin{pmatrix} 0 & 3 & 4 & 3.5 \\ 3 & 0 & 4 & 1 \\ 2 & 2 & 0 & 5 \\ 2.5 & 5 & 1 & 0 \end{pmatrix}, \quad A^* = A^3 = \begin{pmatrix} 0 & 4 & 4 & 4 \\ 3 & 0 & 4 & 4 \\ 3 & 5 & 0 & 5 \\ 3 & 5 & 4 & 0 \end{pmatrix}.$$

There is thus three eigenvalues $\lambda_1 = 5$, $\lambda_2 = 4$ and $\lambda_3 = 3$. At level $\lambda_3 = 3$, $G_3$ is just a single connected component and the four objects are therefore not ordered. Level $\lambda_2 = 4$ leads to a quotien graph $G_4$, refined into two classes, where object 1 is prefered over the three others, but the objects 2, 3 and 4 are undistinguishable. The level $\lambda_1 = 5$ provides a differentiation order between them: 3 is preferred over 4, which, in turn, is preferred over 2. The $(\max, \min)$-approach can be compared with a $(+, \times)$-based spectral analysis of $A$, associated to the method proposed by Berge [4]. It consists in a "best mean ordering" of the objects according to the non-increasing values of the components of the real eigenvector $v$ corresponding to the largest eigenvalue $\lambda$, $Av = \lambda v$. In the current example [12], the largest eigenvalue is $\lambda = 8.92$ and corresponding eigenvector is $v = (0.56\, 0.46\, 0.50\, 0.47)^T$. Thus $A^*$ provides at the end the same order, but in addition various quotient graphs corresponding to the hierarchical partial orders.

Note that replacing objects and judges by drugs and effects on patients, the problem is relevant is medical analysis [15].

**Percolation on distribution of particles in $(\max, \min)$-algebra [10].** Let us consider a continuous (or discrete) distribution of particles in the space $X$ and $\alpha(x, y)$ can be interpreted as a potential of interaction between particles located at $x$ and $y$ (instead of a "distance", it should be seen as an "affinity"); e.g., for a random function, we can for instance uses the difference of intensities to define the affinity. The $\lambda$-percolation, or connectivity problem up to threshold $\lambda$, consists in finding for any pair of distinct points $x$ and $y$ in $X$ a path with respect to the threshold $\lambda$.

Consider the dual $(\max, \min)$-eigensystem, i.e., sup-diagonal dominant kernel $\alpha(x, x) \geq \sup_{y \neq x}\{\alpha(x, y)\}$.

First, compute the limit $\alpha^*(x, y)$: as we show just below, that can can using for instance the MST on the dual of potential of interaction. The $(\max, \min)$-eigenfunctions for any eigenvalue $\lambda$ are

$$\phi_\lambda^y(x) = \lambda \wedge \alpha^*(x, y).$$

Then, for any $y \in X$, there is a percolation path between $x$ and $y$ if $\phi_\lambda^y(x) = \lambda$.

# 4 Eigensystem on ultrametric morphological operators

Eigenfunctional analysis in (min, max)-algebra is the natural framework in the case of ultrametric spaces.

## 4.1 (min, max)-eigensystem in ultrametric space

Let $(X, d)$ be a length space. It is easy to see that $d(x, y)$ is just an example of an idd-kernel. The corresponding limit of (min, max)-powers (11) which can be written as [11]

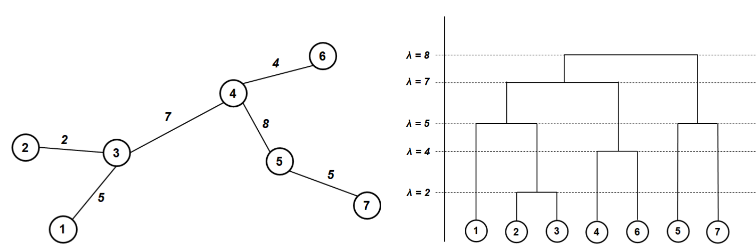
$$d^*(x, y) = \min_{\pi \in \mathrm{path}(x,y)} \max_{\pi \,:\, k=0,\cdots,p-1} d(z_k, z_{k+1}) \tag{16}$$

with $\pi = \{z_0 = x, z_1, \cdots, z_p = y\}$ is a path in $X$. We easily see that $d^*(x, y) \leq \max_z (d^*(x, z), d^*(z, y))$. Therefore $d^*(x, y)$ is the *sub-dominant ultrametric on* $(X, d)$, defined as the largest ultrametric below the given dissimilarity $d(x, y)$.

Let us note $\Lambda = \{d^*(x, y) : x, y \in X,\ x \neq y\}$. Any $\lambda \in \Lambda$ is a (min, max)-eigenvalue of $d(x, y)$ and $d^*(x, y)$ and the corresponding *minimal generator of* (min, max)-*eigenfunctions* are given by

$$\phi^y_\lambda(x) = \lambda \vee d^*(x, y). \tag{17}$$

**Application to hierarchical classification.** Let us recall the pioneering result on the connection between hierarchical classification and spectral analysis in (min, max)-algebra.



(a) (b)

**Fig. 1.** (a) Example of minimum spanning tree and (b) associated dendrogram.

**Theorem 2 (Gondran, 1976 [8]).** *At each level $\lambda$ of a hierarchical classification (dendogram) associated to the sub-dominant ultrametric of a distance matrix $D_{i,j} = d(i, j)$, where $D_{i,:}$ denote the $i$-th column of $D$. Two objects (leafs of the tree) $i$ and $j$ belong to the same class at level $\lambda$ (ultrametric ball of radius* *λ) if and only if the two* $(\min, \max)$*-eigenfunctions associated to their columns are equal, i.e.,*

$$\lambda \vee D^*_{i,:} = \lambda \vee D^*_{j,:}.$$

*The set of distinct vectors of the form* $\lambda \vee D^*_{i,:}$*, forms the unique minimal generators of* $(\min, \max)$*-eigenvectors of* $D$ *for eigenvalue* $\lambda$*. With* $\lambda = 0$*, the set of columns of* $D^*$ *minimal generator of the eigenvectors associated to the neutral element of the "product":* $0 \vee D^*_{i,:} = D^*_{i,:}$*.*

The following example, borrowed from [11] illustrates this theorem. Consider the distance matrix $D$ and its pseudo-inverse $D^*$:

$$D = \begin{pmatrix} 0 & 7 & 5 & 8 & 10 & 8 & 10 \\ 7 & 0 & 2 & 10 & 9 & 9 & 10 \\ 5 & 2 & 0 & 7 & 11 & 10 & 9 \\ 8 & 10 & 7 & 0 & 8 & 4 & 11 \\ 10 & 9 & 11 & 8 & 0 & 9 & 5 \\ 8 & 9 & 10 & 4 & 9 & 0 & 10 \\ 10 & 10 & 9 & 11 & 5 & 10 & 0 \end{pmatrix}, \quad D^* = \begin{pmatrix} 0 & 5 & 5 & 7 & 8 & 7 & 8 \\ 5 & 0 & 2 & 7 & 8 & 7 & 8 \\ 5 & 2 & 0 & 7 & 8 & 7 & 8 \\ 7 & 7 & 7 & 0 & 8 & 4 & 8 \\ 8 & 8 & 8 & 8 & 0 & 8 & 5 \\ 7 & 7 & 7 & 4 & 8 & 0 & 8 \\ 8 & 8 & 8 & 8 & 5 & 8 & 0 \end{pmatrix}.$$

In this discrete setting, the matrix of ultrametric distances $D^*$ can be computed efficiently using a minimum spanning tree (MST) algorithm. Fig. 1(a) depicts the MST corresponding to the graph of $D$ as adjacency matrix, matrix $D^*$ is straightforward derived from it. For instance, $d*(2,7) = \max\,(d(2,3), d(3,4), d(4,5), d(5,8)) = 8$. The associated hierarchical classification represented by a dendrogram is given in Fig. 1(b). Taking for instance $\lambda = 5$, one has the three eigenvectors

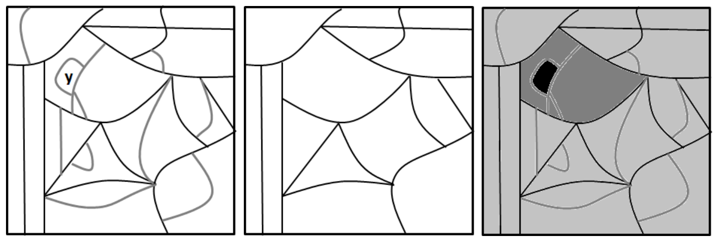
$$\begin{aligned} v_1 = \lambda \vee D^*_{1,:} = \lambda \vee D^*_{2,:} = \lambda \vee D^*_{3,:} &= (5\ 5\ 5\ 7\ 8\ 7\ 8)^T \\ v_2 = \lambda \vee D^*_{4,:} = \lambda \vee D^*_{6,:} &= (7\ 7\ 7\ 5\ 8\ 5\ 8)^T \\ v_3 = \lambda \vee D^*_{5,:} = \lambda \vee D^*_{7,:} &= (8\ 8\ 8\ 8\ 5\ 8\ 5)^T \end{aligned}$$

which are the minimal generator $G_5 = \{v_1, v_2, v_3\}$.

**Interpretation of** $(\min, \max)$**-eigenfunction in an ultrametric space.** Using the fact that in an ultrametric space if $d^*(x,y) < r$ then $B_r(x) = B_r(y)$, one has that two points $x$ and $y$ that belonging to the ultrametric ball of radius $\lambda$ have the same $(\min, \max)$-eigenfunction, i.e., if $y \in B_\lambda(x)$ (which implies that $x \in B_\lambda(y)$) then $\lambda \vee d^*(x,z) = \lambda \vee d^*(y,z),\ z \in X \quad \Leftrightarrow \quad \phi^x_\lambda = \phi^y_\lambda$ . In addition, we can easily see that

$$\phi^y_\lambda(x) = \lambda\, \mathbf{1}_{B_\lambda(y)}(x) + \underbrace{d^*(x,y)}_{\geq \lambda}\ \mathbf{1}_{X \setminus B_\lambda(y)}(x), \tag{18}$$

with the following "normalization" $\inf_{y \in X} \phi^y_\lambda(x) = \lambda \mathbf{1}_X(x),\ \forall x \in X$. Fig. 2 depicts two partitions of a discrete ultrametric space at levels $\lambda$ and $\lambda + 1$ and the corresponding $(\min, \max)$-eigenfunction at point $y \in X$ and eigenvalue $\lambda$.




(a) (b) (c)

**Fig. 2.** $(\min, \max)$-eigenfunction at point $y \in X$ and eigenvalue $\lambda$: (a) and (b) depict two partitions of a discrete ultrametric space at levels $\lambda$ and $\lambda + 1$; (c) eigenfunction $\phi_\lambda^y(x)$. Black corresponds to $\lambda$.

### 4.2 Eigenfunctions of ultrametric erosion-opening and dilation-closing

The property $d^*(x, y) \cdot_{\min,\max} d^*(x, y) = d^*(x, y)$ is the basic ingredient in [2] to prove the existency of the supremal ultrametric morphological semigroups and the fact that ultrametric erosion and dilation are idempotent operators. Let us also notice that using expression (13), one has

$$\begin{aligned} E_t f(x) &= \inf_{y \in X} \left\{ f(y) \vee t^{-1} d^*(x, y) \right\} = \left( f *_{\min,\max} t^{-1} d^* \right)(x) \\ &= t^{-1} \left( tf *_{\min,\max} d^* \right)(x), \ \ \forall x \in X, t > 0. \end{aligned} \tag{19}$$

**Proposition 4.** *Given an ultrametric space $(X, d^*)$, the corresponding $\phi_\lambda^y(x)$, $\forall y \in X$ and $\lambda \in \Lambda$, is an eigenfunction of the ultrametric erosion-opening with $t = 1$, i.e.,*

$$E_1 \phi_\lambda^y(x) = \lambda \vee \phi_\lambda^y(x). \tag{20}$$

*For $t \neq 1$, one has the following scaling*

$$E_t t^{-1} \phi_\lambda^y(x) = t^{-1} \left( \lambda \vee \phi_\lambda^y(x) \right).$$

*Proof.* We have $E_1 \phi_\lambda^y(x) = \left( \phi_\lambda^y *_{\min,\max} d^* \right)(x)$, then

$$\begin{aligned} (\phi_\lambda^y *_{\min,\max} d^*)(x) &= \inf_{z \in X} \{ \phi_\lambda^y(z) \vee d^*(x, z) \} = \inf_{z \in X} \{ \lambda \vee d^*(y, z) \vee d^*(z, x) \} \\ &= \lambda \vee (d^*(x, y) *_{\min,\max} d^*(x, y)) = \lambda \vee d^*(x, y) = \lambda \vee \phi_\lambda^y(x). \end{aligned}$$

When $t \neq 1$, from (19), we obtain

$$E_t t^{-1} \phi_\lambda^y(x) = t^{-1} \left( \phi_\lambda^y *_{\min,\max} d^* \right)(x) = t^{-1} E_1 \phi_\lambda^y(x),$$

so finaly, using (20), one has $E_t t^{-1} \phi_\lambda^y(x) = t^{-1} \left( \lambda \vee \phi_\lambda^y(x) \right)$.

A similar result is obtained for the eigenfunctions of the ultrametric dilation-closing

$$D_1 \bar{\phi}^y_{\bar{\lambda}}(x) = \bar{\lambda} \wedge \bar{\phi}^y_{\bar{\lambda}}(x). \tag{21}$$

where $\bar{\phi}^y_{\bar{\lambda}}(x)$ are the corresponding $(\max, \min)$-eigenfunctions, obtained from the dual ultrametric distance and the corresponding dual eigenvalues $\bar{\lambda}$.

Using the alternative representation of the discrete erosion (9) with $c_k = \lambda_k$ $E_t f(x) = \inf_{\lambda_k \in \Lambda}\{Q^{\wedge}_{\lambda_k} f(x) \vee \lambda_k\}$, we have the following result.

**Proposition 5.** *The ultrametric erosion-opening of a function $f$ on $(X, d^*)$ at $t = 1$ can be written as the expansion on the base of $(\min, \max)$-eigenfunctions $\{\phi^y_{\lambda_k}\}_{\lambda_k \in \Lambda}$ as follows:*

$$E_1 f(x) = \inf_{\lambda_k \in \Lambda} \left\{ \langle f, \phi^y_{\lambda_k} \rangle_{\min,\max} \vee \lambda \right\}. \tag{22}$$

*Scaling from (19) provides the corresponding expansion for $E_t f(x)$.*

*Proof.* The $(\min, \max)$-scalar product of a function $f$ on $(X, d^*)$ and the $(\min, \max)$-eigenfunction $\phi^y_\lambda$:

$$\langle f, \phi^y_\lambda \rangle_{\min,\max} = \inf_{y \in X} \left\{ f(y) \vee \phi^y_\lambda(x) \right\}.$$

Now, using the expression (18), one has

$$\begin{aligned} \langle f, \phi^y_\lambda \rangle_{\min,\max} &= \left[ \inf_{x \in B_\lambda(y)} \{f(x)\} \vee \lambda \right] \wedge \inf_{x \notin B_\lambda(y)} \{f(x) \vee d^*(y, x)\} \\ &= Q^{\wedge}_\lambda f(x) \vee \lambda. \end{aligned}$$

Obviously, a similar expansion can be obtained from the dilation-closing using the $(\max, \min)$-eigenfunctions of ultrametric space $X$.

## 5 Conclusions and Perspectives

Morphological semigroups in ultrametric spaces can be seen as the min–max product (and its dual) of a function and a scaled version of the ultrametric distance. Ultrametric distance is a fixed point functional in min–max analysis and its eigenfunctions are defined in a direct way. From this viewpoint, min–max spectral analysis on ultrametric spaces describes the nested organization of ultrametric balls. Eigenfunctions of ultrametric distance are just the eigenfunctions of ultrametric erosion-opening and this spectral base provide an expansion of morphological operators.

This theory is related to Meyer's theory of watershed on node- or-edge-weighed graphs [14], since min–max algebra is also connected to analysis of paths on digraphs. A better understanding of the connection between the present spectral theory and Meyer's theory could help into the integrative use of ultrametric operators in segmentation and filtering.

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