Homogenization of the Stokes system in a non-periodically perforated domain

Sylvain Wolf

Homogenization of the Stokes system in a non-periodically perforated domain

S. Wolf[1]

[1]Université de Paris, Sorbonne Paris-Cité, Sorbonne Université, CNRS, Laboratoire Jacques-Louis Lions, F-75013 Paris

January 11, 2021

Abstract

In our recent work [8], we have studied the homogenization of the Poisson equation in a class of non periodically perforated domains. In this paper, we examine the case of the Stokes system. We consider a porous medium in which the characteristic distance between two holes, denoted by ε, is proportional to the characteristic size of the holes. It is well known (see [1],[17] and [19]) that, when the holes are periodically distributed in space, the velocity converges to a limit given by the Darcy's law when the size of the holes tends to zero. We generalize these results to the setting of [8]. The non-periodic domains are defined as a local perturbation of a periodic distribution of holes. We obtain classical results of the homogenization theory in perforated domains (existence of correctors and regularity estimates uniform in ε) and we prove H^2-convergence estimates for particular force fields.

Contents

1 Introduction

In this paper, we study the three dimensional Stokes system in a perforated domain for an incompressible fluid with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in} \quad \Omega_\varepsilon \\ \text{div } u_\varepsilon = 0 \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{1.1}$$

In Equation (1.1), $\Omega_\varepsilon \subset \mathbb{R}^3$ denotes the perforated domain, the vector valued function f is the force field, the unknowns u_ε and p_ε refer respectively to the velocity and the pressure of the fluid. The distance between two neighbouring holes is denoted by ε. We assume that the charasteristic size of the holes is ε. Our purpose is to understand the limit of $(u_\varepsilon, p_\varepsilon)$ when $\varepsilon \to 0$. We construct classical objects of the homogenization theory such as correctors (Theorem 2.1) and we give new rates of convergence of u_ε to its limit when f is smooth, compactly supported and $\text{div}(Af) = 0$ where A is the so-called permeability tensor (see Theorem 2.3).

To our knowledge, the first paper on the homogenization of the Stokes system in perforated domains is [19]. In this work, Equation (1.1) is studied for a periodic distribution of perforations in the macroscopic domain Ω (that is, each cell of a periodic array of size ε contains a perforation). It is in particular proved that $(u_\varepsilon/\varepsilon^2, p_\varepsilon)$ converges in some sense to a couple (u_0, p_0) given by the Darcy's law. This result can be guessed by performing a standard two scale expansion of $(u_\varepsilon, p_\varepsilon)$, see [17]. Error estimates between u_ε and its first order term in ε are proved in [14, 15] for particular situations namely the two-dimensional case in [15] and the case of a periodic macroscopic domain in [14]. Sharp error estimates under general assumptions on f have been obtained in [18]. The case of boundary layers in an infinite two-dimensional rectangular has been addressed in [13]. The results of [19] have been extended in [1] to porous medium in which both solid and fluid parts are connected. The case of holes that scale differently as ε is examined in [3]. Recently, the homogenization of the Stokes system at higher order has been adressed in [10].

In this paper, we adapt the results of [19] to the setting of [8], that is to perforated domains that are defined as a local perturbation of the periodically perforated domain considered in [19]. This framework is inspired by the papers [5, 6, 7] (see [8, Remark 1.5]). The purpose of these works is to study the homogenization of ellitptic PDEs with coefficients that are periodic and perturbed by a defect which belongs to L^r, $1 < r < +\infty$.

The paper is organized as follows. We recall in subsection 1.2 the main results of the homogenization of the Stokes system in the periodic case. We introduce in subsection 1.3 the non-periodic setting. We state in Section 2 the main results of this paper and we make some remarks. These results are proved in Section 3. Some technical Lemmas are given in Appendix A. In Appendix B, we give more specific geometric assumptions on the non-periodic perforations that allow to obtain the results of Section 2.

1.1 General notations

The canonical basis of $\mathbb{R}^3$ is denoted e_1, e_2, e_3. We denote the euclidian scalar product between two vectors u and v by $u \cdot v$. The euclidian distance to a subset $A \subset \mathbb{R}^3$ will be written $d(\cdot, A)$. The diameter of A will be denoted by $\text{diam}(A)$. If A is a Lipschitz domain, we denote by n the outward normal vector. $|\cdot|$ will be the Lebesgue measure on $\mathbb{R}^3$.

If A, B are two real matrices, we write $A : B := \sum_{i,j=1}^3 A_{i,j} B_{i,j}$. If X is a vector or a matrix, its transpose will be denoted by X^T. If $A \subset \mathbb{R}^3$, the complementary set of A will be written A^c. We define $Q :=]-\frac{1}{2}, \frac{1}{2}[^3$ and, for $k \in \mathbb{Z}^3$, $Q_k := \prod_{j=1}^3 \left]-\frac{1}{2} + k_j, \frac{1}{2} + k_j\right[^3 = Q + k$. If $x \in \mathbb{R}^3$ and $r > 0$, we denote by $B(x, r)$ the open ball centered in x of radius r.

The gradient operator of a real or vector valued function will be denoted $\nabla\cdot$ and the second order derivative of a real or vector valued function will be written $D^2\cdot$. The divergence operator will be denoted $\text{div}\,\cdot$ and the scalar or vectorial Laplacian $\Delta\cdot$.

Functional spaces. If ω is an open subset of $\mathbb{R}^3$ and $1 \leq p \leq +\infty$, we denote by $L^p(\omega)$ the standard Lebesgue spaces and $H^s(\omega), W^{m,p}(\omega)$ the standard Sobolev spaces. For $s \in \mathbb{R}$ and $m \in \mathbb{N}^*$, we denote by $[L^p(\omega)]^3$, $[H^s(\omega)]^3$ and $[W^{m,p}(\omega)]^3$ the spaces of vector valued functions whose components are respectively elements of $L^p(\omega)$, $H^s(\omega)$ and $W^{m,p}(\omega)$. The space $L^p(\omega)/\mathbb{R}$ corresponds to the equivalence classes for the relation $\sim$ defined by: for all $f, g \in L^p(\omega)$, $f \sim g$ if and only if $f - g$ is a.e constant in ω. $\mathcal{D}(\omega)$ will be the set of smooth and compactly supported functions in ω. We denote by $\mathcal{C}^\infty(\omega)$ (resp. $\mathcal{C}^\infty(\overline{\omega})$) the set of smooth functions defined on ω (resp. $\overline{\omega}$).

1.2 Review of the periodic case

In this subsection, we recall the results of the homogenization of the Stokes system in periodically perforated domains with large holes. For more details, see [19, 17, 1].

Notations. We fix a locally Lipschitz bounded domain $\Omega \subset \mathbb{R}^3$ and a subset $\mathcal{O}_0^{\text{per}}$ such that $\mathcal{O}_0^{\text{per}} \subset\subset Q$, $\mathcal{O}_0^{\text{per}}$ is of class $\mathcal{C}^{2,\alpha}$ and $Q\backslash\overline{\mathcal{O}_0^{\text{per}}}$ is connected. We define for $k \in \mathbb{Z}^3$, $\mathcal{O}_k^{\text{per}} := \mathcal{O}_0^{\text{per}} + k$. $\mathcal{O}^{\text{per}}$ will be the set of perforations, that is, $\mathcal{O}^{\text{per}} := \bigcup_{k\in\mathbb{Z}^3} \mathcal{O}_k^{\text{per}}$.

We define some periodic functional spaces that will be used in the sequel. Using the notations of our problem, we set for $1 \leq p \leq +\infty$,

$$L^{p,\text{per}}\left(Q\backslash\overline{\mathcal{O}_0^{\text{per}}}\right) := \left\{u \in L^p_{\text{loc}}(\mathbb{R}^3\backslash\overline{\mathcal{O}^{\text{per}}}) \text{ s.t. } u \text{ is } Q-\text{periodic}\right\}$$

and

$$H^{1,\text{per}}\left(Q\backslash\overline{\mathcal{O}_0^{\text{per}}}\right) := \left\{u \in H^1_{\text{loc}}(\mathbb{R}^3\backslash\overline{\mathcal{O}^{\text{per}}}) \text{ s.t. } u \text{ is } Q-\text{periodic and } \partial_i u \text{ are } Q-\text{periodic}, i = 1, 2, 3\right\}.$$

The space of H^1-periodic vector valued functions will be $\left[H^{1,\text{per}}\left(Q\backslash\overline{\mathcal{O}_0^{\text{per}}}\right)\right]^3$. The space of H^1-periodic functions that vanish on the perforations is

$$H_0^{1,\text{per}}\left(Q\backslash\overline{\mathcal{O}_0^{\text{per}}}\right) := \left\{u \in H^{1,\text{per}}\left(Q\backslash\overline{\mathcal{O}_0^{\text{per}}}\right) \text{ s.t. } u = 0 \text{ on } \partial\mathcal{O}_0^{\text{per}}\right\}.$$

Similarly, we define $\left[H_0^{1,\text{per}}\left(Q\backslash\overline{\mathcal{O}_0^{\text{per}}}\right)\right]^3$. In the sequel, we use the summation convention on repeated indices.

For $\varepsilon > 0$, we denote $Y_\varepsilon^{\text{per}} := \{k \in \mathbb{Z}^3,\ \varepsilon Q_k \subset \Omega\}$. We define the periodically perforated domain $\Omega_\varepsilon^{\text{per}}$ by (see Figure 1)

$$\Omega_\varepsilon^{\text{per}} := \Omega \setminus \bigcup_{k\in Y_\varepsilon^{\text{per}}} \varepsilon\overline{\mathcal{O}_k^{\text{per}}}.$$

It is easily seen that $\Omega_\varepsilon^{\text{per}}$ is open and connected.

For $f \in \left[L^2(\Omega)\right]^3$, there exists a unique couple $(u_\varepsilon, p_\varepsilon) \in \left[H_0^1(\Omega_\varepsilon^{\text{per}})\right]^3 \times L^2(\Omega_\varepsilon^{\text{per}})/\mathbb{R}$ solution of System (1.1). The Poincaré inequality in perforated domains (see e.g. [19, Lemma 1]) and standard energy estimates yield the bound

$$\|u_\varepsilon\|_{[L^2(\Omega_\varepsilon^{\text{per}})]^3} \leq C\varepsilon^2$$

where C is a constant independent of ε. Thus, after extraction of a subsequence, $u_\varepsilon/\varepsilon^2$ converges L^2-weakly to some limit velocity u^*. Besides, it can be proved (see [19, Theorem 1]) that the pressure p_ε converges $L^2(\Omega)/\mathbb{R}-$strongly to the macroscopic pressure p_0 which is defined up to the addition of a constant. The couple (u^*, p_0) is determined by the Darcy's law which we recall here

$$\begin{cases} \text{div}(u^*) = 0 \quad \text{in} \quad \Omega \\ \qquad u^* = A(f - \nabla p_0) \\ \ u^* \cdot n = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

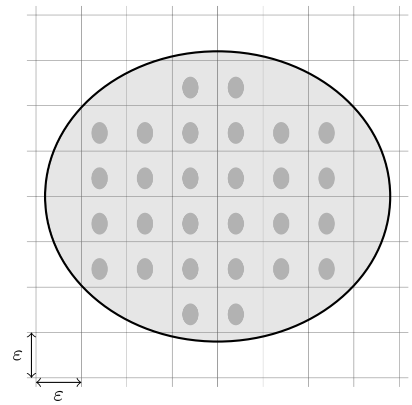


Figure 1: Periodic domain $\Omega_\varepsilon^{\mathrm{per}}$

In (1.2), the symmetric and positive definite matrix A is the so-called permeability tensor. Its coefficients are defined by

$$A_i^j = \int_{Q\setminus\overline{\mathcal{O}_0^{\mathrm{per}}}} w_j^{\mathrm{per}} \cdot e_i = \int_{Q\setminus\overline{\mathcal{O}_0^{\mathrm{per}}}} \nabla w_i^{\mathrm{per}} : \nabla w_j^{\mathrm{per}}, \quad 1 \le i,j \le 3, \tag{1.3}$$

where the functions $w_j^{\mathrm{per}}, j = 1,2,3$ are the cell periodic first correctors and solve the following Stokes problems:

$$\begin{cases} -\Delta w_j^{\mathrm{per}} + \nabla p_j^{\mathrm{per}} = e_j & \text{in} \quad Q \setminus \overline{\mathcal{O}_0^{\mathrm{per}}} \\ \mathrm{div}\ w_j^{\mathrm{per}} = 0 & \\ w_j^{\mathrm{per}} = 0 & \text{on} \quad \partial\mathcal{O}_0^{\mathrm{per}}. \end{cases} \tag{1.4}$$

We note that for fixed $j \in \{1,2,3\}$, Problem (1.4) is well-posed in the space $\left[H_0^1(Q \setminus \overline{\mathcal{O}_0^{\mathrm{per}}})\right]^3 \times L^{2,\mathrm{per}}(Q \setminus \overline{\mathcal{O}_0^{\mathrm{per}}})/\mathbb{R}$ (see [17]). A central point in the proof of the convergence of p_ε to p_0 is the construction of an extension of the pressure p_ε in the periodic holes. This extension is constructed in [19] by a duality argument.

The corrector equations (1.4) can be guessed by a standard two-scale expansion of u_ε and p_ε of the form

$$u_\varepsilon = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$
$$p_\varepsilon = p_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon p_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 p_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 p_3\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

where the functions $u_i(x,\cdot)$ and $p_i(x,\cdot)$ are $Q-$periodic for fixed $x \in \Omega$ (see [17, Section 7.2]). It can be proved that the function p_0 is independent of the microscopic variable, that is $p_0(x, \frac{x}{\varepsilon}) = p_0(x)$ for all $x \in \Omega$ (which is coherent with (1.2)). Besides, the functions u_0 and u_1 vanish and (we use, as indicated above, the summation convention over repeated indices)

$$u_2\left(x, \frac{x}{\varepsilon}\right) = w_j\left(\frac{x}{\varepsilon}\right)(f_j - \partial_j p_0)(x) \quad \text{and} \quad p_1\left(x, \frac{x}{\varepsilon}\right) = p_j\left(\frac{x}{\varepsilon}\right)(f_j - \partial_j p_0)(x).$$

We define the remainders

$$R_\varepsilon := u_\varepsilon - \varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right)(f_j - \partial_j p_0) \quad \text{and} \quad \pi_\varepsilon := p_\varepsilon - p_0 - \varepsilon p_j\left(\frac{\cdot}{\varepsilon}\right)(f_j - \partial_j p_0).$$

The strong convergence $R_\varepsilon/\varepsilon^2 \to 0$ in $L^2(\Omega_\varepsilon^{\mathrm{per}})-$norm is proved in [2, Theorem 1.3]. An H^1-quantitative estimate of this convergence is given in [18], provided that Ω is of class $\mathcal{C}^{2,\alpha}$. We will provide a new H^2-convergence estimate when $\mathrm{div}(Af) = 0$ and f is compactly supported in Ω (see Theorem 2.3 and Remark 2.5 below).

In what follows, we extend w_j^{per} by zero in the periodic perforations. The pressure p_j^{per} is extended by a constant λ_j (for example zero) in the perforations.

1.3 The non-periodic setting

We fix a periodic set of perforations as described in the previous subsection. We describe the non-periodic setting (see [8] for more details). For $k \in \mathbb{Z}^3$ and $\alpha > 0$, we define (see figure 3a)

$$\mathcal{O}_k^{\text{per},+}(\alpha) := \{x \in Q_k, \quad d(x, \mathcal{O}_k^{\text{per}}) < \alpha\},$$

and

$$\mathcal{O}_k^{\text{per},-}(\alpha) := \{x \in \mathcal{O}_k^{\text{per}}, \quad d(x, \partial\mathcal{O}_k^{\text{per}}) > \alpha\}.$$

For all $k \in \mathbb{Z}^3$, we fix an open subset $\mathcal{O}_k$ of Q_k. We suppose that the sequence $(\mathcal{O}_k)_{k\in\mathbb{Z}^3}$ satisfies Assumptions **(A1)-(A5)** below. We define the non periodic set of perforations by

$$\mathcal{O} := \bigcup_{k\in\mathbb{Z}^3} \mathcal{O}_k.$$

(A1) For all $k \in \mathbb{Z}^3$, we have $\mathcal{O}_k \subset\subset Q_k$ and $Q_k \setminus \overline{\mathcal{O}_k}$ is connected.

(A2) For all $k \in \mathbb{Z}^3$, the perforation $\mathcal{O}_k$ is Lipschitz continuous.

(A3) There exists a sequence $(\alpha_k)_{k\in\mathbb{Z}^3} \in \ell^1(\mathbb{Z}^3)$ such that for all $k \in \mathbb{Z}^3$, $\alpha_k > 0$ and we have the following chain inclusion:

$$\mathcal{O}_k^{\text{per},-}(\alpha_k) \subset \mathcal{O}_k \subset \mathcal{O}_k^{\text{per},+}(\alpha_k).$$

We refer to figure 3a for an illustration of **(A3)**.

The assumptions **(A1)-(A2)** are analogous to the one made on $\mathcal{O}^{\text{per}}$ and guarantee connectedness and some regularity on the perforated domain. Assumption **(A3)** is the geometric assumption that makes precise that $(\mathcal{O}_k)_{k\in\mathbb{Z}^3}$ is a perturbation of $(\mathcal{O}_k^{\text{per}})_{k\in\mathbb{Z}^3}$. We recall (see [8, Lemma A.1 and Lemma A.3]) that Assumptions **(A1)-(A3)** imply the following facts:

- There exists $\delta > 0$ such that for all $k \in \mathbb{Z}^3$, $d(\mathcal{O}_k, \partial Q_k) \geq \delta$. In other words, $\mathcal{O}_k$ is strictly included in Q_k, uniformly with respect to k.
- We have
$$\sum_{k\in\mathbb{Z}^3} |\mathcal{O}_k \Delta \mathcal{O}_k^{\text{per}}| < +\infty \tag{1.5}$$
where Δ stands for the sets symmetric difference operator.

Using the first point above, we can introduce two smooth open sets Q' and Q'' such that (see Figure 2) $Q' \subset\subset Q \subset\subset Q''$ and for all $k \in \mathbb{Z}^3$, $(Q' + k) \cap \mathcal{O} = (Q'' + k) \cap \mathcal{O} = \mathcal{O}_k$. We define, for $k \in \mathbb{Z}^3$,

$$Q'_k := Q' + k \quad \text{and} \quad Q''_k := Q'' + k. \tag{1.6}$$

The sets Q'_k and Q''_k, $k \in \mathbb{Z}^3$ will be used several times in the sequel.

(A4) This assumption is divided into two sub-assumptions **(A4)**$_0$ and **(A4)**$_1$.

(A4)$_0$ For all $1 < q < +\infty$, there exists a constant $C_q^0 > 0$ such that for all $k \in \mathbb{Z}^3$, the problem

$$\begin{cases} \operatorname{div} v = f & \text{in} \quad Q_k \setminus \overline{\mathcal{O}_k} \\ \quad v = 0 & \text{on} \quad \partial\left[Q_k \setminus \overline{\mathcal{O}_k}\right] \end{cases} \tag{1.7}$$

with $f \in L^q(Q_k \setminus \overline{\mathcal{O}_k})$ completed with the compatibility condition

$$\int_{Q_k\setminus\overline{\mathcal{O}}_k} f = 0 \tag{1.8}$$

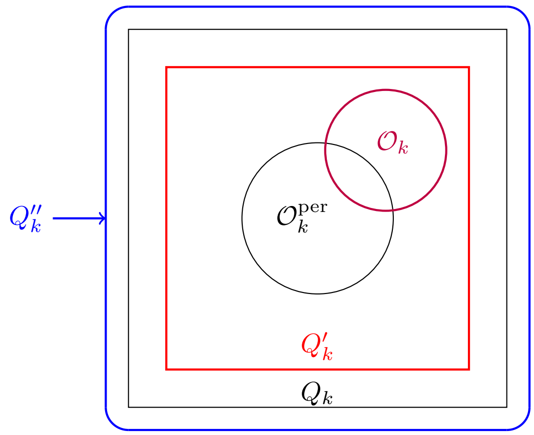


Figure 2: A cell Q_k, $k \in \mathbb{Z}^3$

admits a solution v such that $v \in \left[W^{1,q}(Q_k \backslash \overline{\mathcal{O}_k})\right]^3$ and

$$\|v\|_{\left[W^{1,q}(Q_k \backslash \overline{\mathcal{O}_k})\right]^3} \leq C_q^0 \|f\|_{L^q(Q_k \backslash \overline{\mathcal{O}_k})}. \tag{1.9}$$

(A4)$_1$ For all $1 < q < +\infty$, there exists a constant $C_q^1 > 0$ such that for all $k \in \mathbb{Z}^3$, Problem (1.7) with $f \in W_0^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})$ completed with the compatibility condition (1.8) admits a solution v such that $v \in \left[W_0^{2,q}(Q_k \setminus \overline{\mathcal{O}_k})\right]^3$ and

$$\|v\|_{\left[W^{2,q}(Q_k \backslash \overline{\mathcal{O}_k})\right]^3} \leq C_q^1 \|f\|_{W^{1,q}(Q_k \backslash \overline{\mathcal{O}_k})}. \tag{1.10}$$

(A5) For all $1 < q < +\infty$, there exists a constant $C_q > 0$ such that for all $k \in \mathbb{Z}^3$, if $(v, p) \in \left[W^{1,q}(Q_k \backslash \overline{\mathcal{O}_k})\right]^3 \times L^q(Q_k \backslash \overline{\mathcal{O}_k})$ is solution to the Stokes problem

$$\begin{cases} -\Delta v + \nabla p = f & \text{in} \quad Q_k'' \setminus \overline{\mathcal{O}_k} \\ \quad \text{div } v = 0 & \\ \qquad v = 0 & \text{on} \quad \partial \mathcal{O}_k \end{cases} \tag{1.11}$$

with $f \in L^q(Q_k'' \backslash \overline{\mathcal{O}_k})$, then $(v, p) \in \left[W^{2,q}(Q_k \backslash \overline{\mathcal{O}_k})\right]^3 \times W^{1,q}(Q_k \backslash \overline{\mathcal{O}_k})$ and

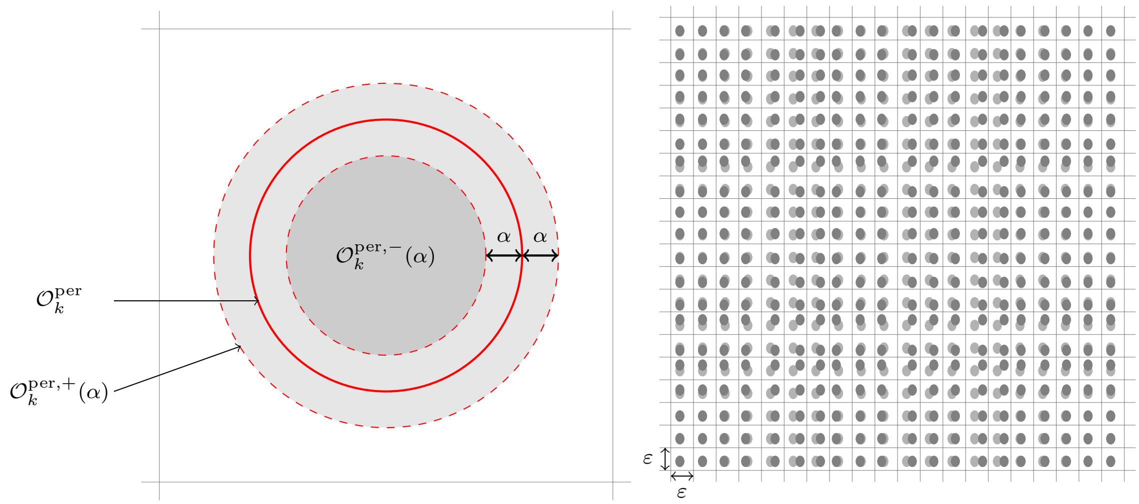
$$\|v\|_{\left[W^{2,q}(Q_k \backslash \overline{\mathcal{O}_k})\right]^3} + \|p\|_{W^{1,q}(Q_k \backslash \overline{\mathcal{O}_k})} \leq C_q \left[\|f\|_{L^q(Q_k'' \backslash \overline{\mathcal{O}_k})^3} + \|v\|_{\left[W^{1,q}(Q_k'' \backslash \overline{\mathcal{O}_k})\right]^3} + \|p\|_{L^q(Q_k'' \backslash \overline{\mathcal{O}_k})}\right]. \tag{1.12}$$

Remark 1.1. *For each fixed $k \in \mathbb{Z}^3$, the estimates* (1.9) *and* (1.10) *are satisfied with constants $C_{q,k}^i$, $i = 0, 1$, depending on k, see [11, Theorem III.3.3]. Similarly, as long as $\mathcal{O}_k$ is of class $\mathcal{C}^2$,* (1.12) *is satisfied when k is fixed (see [11, Theorem IV.5.1]). Assumptions **(A4)-(A5)** require that the constants appearing in* (1.9), (1.10) *and* (1.12) *are uniform with respect to $k \in \mathbb{Z}^3$.*

Assumptions **(A4)-(A5)** are the weakest possible given our method of proof. However, they are associated to PDEs and we would like a somewhat more geometric interpretation of these assumptions, in the spirit of **(A3)**. In fact, we may replace **(A4)-(A5)** by the likely stronger (but geometric) Assumptions **(A4)'-(A5)'** below.

We suppose that there exist $r > 0$ and $M > 0$ such that for all $k \in \mathbb{Z}^3$ and for all $x \in \partial \mathcal{O}_k$, there exists $\zeta_x : U_x \to \mathbb{R}$ where $U_x \subset \mathbb{R}^2$, $0 \in U_x$ and $r_x > r$ such that, after eventually rotating and/or translating the local coordinate system, we have that $\zeta_x(0) = 0$ and

$$\left(Q_k \setminus \overline{\mathcal{O}_k}\right) \cap B(x, r_x) = \{(y_1, y_2, y_3) \in \mathbb{R}^3, y_3 > \zeta_x(y_1, y_2) \quad \text{and} \quad (y_1, y_2) \in U_x\}. \tag{1.13}$$



(a) Illustration of **(A3)**

(b) A non periodically perforated grid

Figure 3: The non-periodic setting

We assume the following uniform regularity properties:

(A4)' The functions $\zeta_x, x \in \partial\mathcal{O}_k$ are Lipschitz functions with Lipschitz constant $\|\zeta_x\|_{\text{Lip}(U_x)}$ satisfying $\|\zeta_x\|_{\text{Lip}(U_x)} \leq M$.

(A5)' The functions $\zeta_x, x \in \partial\mathcal{O}_k$ are of class $\mathcal{C}^2$ and satisfy $\nabla\zeta_x(0) = 0$ with the estimate $\|\zeta_x\|_{W^{2,\infty}(U_x)} \leq M$.

In Assumptions **(A4)'-(A5)'** above, we emphasize that M is independent of k and x.

We prove in Appendix B that Assumptions **(A3)** and **(A4)'** imply Assumption **(A4)** and that Assumptions **(A3)** and **(A5)'** imply Assumption **(A5)**. We also note that **(A5)'** implies **(A4)'**.

Example 1.2. *We give some examples of perforations satisfying **(A1)-(A5)**:*

- *Compactly supported perturbations, that is, we change $\mathcal{O}_k^{\text{per}}$ in a finite number of cells Q_k;*
- *We remove a finite number of perforations;*
- *We make ℓ^1-translations of the periodic perforations that is we choose a sequence $(\delta_k)_{k\in\mathbb{Z}^3}$ such that $\delta_k \in \mathbb{R}^3$, $\sum_{k\in\mathbb{Z}^3} |\delta_k| < +\infty$ and for all $k \in \mathbb{Z}^3$, $\mathcal{O}_k \subset\subset Q_k$ and $\mathcal{O}_k = \mathcal{O}_k^{\text{per}} + \delta_k$.*
- *We give on Figure 6 in Appendix B some counter-examples to Assumptions **(A3)-(A4)'-(A5)'**.*

Remark 1.3. *The assumption $\mathcal{O}_k \subset\subset Q_k$ is automatically implied by **(A3)** except for a finite number of cells. Dropping it would change some technical details but not the results of the paper.*

The perforated domain. We recall that Ω is a smooth bounded domain of $\mathbb{R}^3$. We denote

$$Y_\varepsilon := \{k \in \mathbb{Z}^3, \varepsilon\mathcal{O}_k \subset Q_k\} \tag{1.14}$$

and define (see Figure 3b)

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{k\in Y_\varepsilon} \varepsilon\overline{\mathcal{O}_k}. \tag{1.15}$$

We notice that Ω_ε is a bounded, locally Lipschitz and connected open subset of $\mathbb{R}^3$.

For $f \in \left[L^2(\Omega)\right]^3$, there is a unique solution $(u_\varepsilon, p_\varepsilon) \in \left[H^1_0(\Omega_\varepsilon)\right]^3 \times L^2(\Omega_\varepsilon)/\mathbb{R}$ to the Stokes system

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in} \quad \Omega_\varepsilon \\ \text{div } u_\varepsilon = 0 & \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{1.16}$$

In the sequel, we study the homogenization of $(u_\varepsilon, p_\varepsilon)$.

2 Results

The first result concerns the existence of the first order correctors. We can perform a two scale expansion of the form

$$u_\varepsilon(x) = \varepsilon^2 \left[u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_3\left(x, \frac{x}{\varepsilon}\right) + \cdots\right], \quad p_\varepsilon(x) = p_0(x) + \varepsilon p_1\left(x, \frac{x}{\varepsilon}\right) + \cdots \tag{2.1}$$

to (1.16) and find that

$$u_2(x,y) = \sum_{j=1}^{3} w_j(y)(f_j - \partial_j p_0)(x) \quad \text{and} \quad p_1(x,y) = \sum_{j=1}^{3} p_j(y)(f_j - \partial_j p_0)(x) \tag{2.2}$$

where f_1, f_2, f_3 denote the components of the vector field f, (w_j, p_j) is solution to the following Stokes system for $j = 1, 2, 3$:

$$\begin{cases} -\Delta w_j + \nabla p_j = e_j & \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ \text{div } w_j = 0 & \\ w_j = 0 & \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{2.3}$$

and p_0 is given by the Darcy's law (1.2).

Theorem 2.1 (Existence of correctors)**.** *Suppose that Assumptions **(A1)-(A3)** and **(A4)**$_0$ are satisfied. For all $j \in \{1,2,3\}$, System (2.3) admits a solution (w_j, p_j) of the form*

$$w_j = w_j^{\text{per}} + \widetilde{w_j} \quad \text{and} \quad p_j = p_j^{\text{per}} + \widetilde{p_j}$$

where $(\widetilde{w_j}, \widetilde{p_j}) \in \left[H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3 \times L^2_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$. Moreover, we have the following estimate

$$\|\widetilde{p_j} - \langle \widetilde{p_j} \rangle\|_{L^2(\frac{1}{\varepsilon}\Omega_\varepsilon)} \leq C\varepsilon^{-1},$$

where C is a constant independent of ε and $\langle \widetilde{p_j} \rangle$ denotes the mean value of $\widetilde{p_j}$ on $\frac{1}{\varepsilon}\Omega_\varepsilon$.

We define

$$R_\varepsilon := u_\varepsilon - \varepsilon^2 \sum_{j=1}^{3} w_j\left(\frac{\cdot}{\varepsilon}\right)(f_j - \partial_j p_0) \quad \text{and} \quad \pi_\varepsilon := p_\varepsilon - p_0 - \varepsilon \sum_{j=1}^{3} p_j\left(\frac{\cdot}{\varepsilon}\right)(f_j - \partial_j p_0).$$

Following the ideas of the proof of [2, Theorem 1.3], we can prove under the assumption $f \in \left[W^{3,\infty}(\Omega)\right]^3$ that $R_\varepsilon/\varepsilon^2 \underset{\varepsilon \to 0}{\longrightarrow} 0$ in the non-periodic setting for the $\left[L^2(\Omega)\right]^3$ $-$norm (where it is understood that u_ε and $w_j, j = 1, 2, 3$ are extended by zero in the perforations). This fact, though relevant because it makes (2.1) rigorous, is not strong enough to justify the construction of the non-periodic correctors $(w_j, p_j), j = 1, 2, 3$. Indeed, if we set

$$R_\varepsilon^{\text{per}} := u_\varepsilon - \varepsilon^2 \sum_{j=1}^{3} w_j^{\text{per}}\left(\frac{\cdot}{\varepsilon}\right)(f_j - \partial_j p_0),$$

we notice that

$$R_\varepsilon = R_\varepsilon^{\mathrm{per}} - \varepsilon^2 \sum_{j=1}^{3} \widetilde{w_j}\left(\frac{\cdot}{\varepsilon}\right)(f_j - \partial_j p_0).$$

Since $\widetilde{w_j} \in \left[L^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$, one has for $j = 1, 2, 3$:

$$\begin{aligned}\left\|\widetilde{w_j}\left(\frac{\cdot}{\varepsilon}\right)(f_j - \partial_j p_0)\right\|_{[L^2(\Omega_\varepsilon)]^3} &= \varepsilon^{\frac{3}{2}} \left\|\widetilde{w_j}(f_j - \partial_j p_0)(\varepsilon \cdot)\right\|_{\left[L^2(\frac{1}{\varepsilon}\Omega_\varepsilon)\right]^3} \\ &\leq \varepsilon^{\frac{3}{2}} \|\widetilde{w_j}\|_{[L^2(\mathbb{R}^3)]^3} \|f_j - \partial_j p_0\|_{L^\infty(\Omega)} = C\varepsilon^{\frac{3}{2}}.\end{aligned}$$

Thus $R_\varepsilon^{\mathrm{per}}/\varepsilon^2 = R_\varepsilon/\varepsilon^2 + O(\varepsilon^{3/2})$. This proves that $R_\varepsilon^{\mathrm{per}}/\varepsilon^2 \underset{\varepsilon\to 0}{\longrightarrow} 0$ for the $\left[L^2(\Omega)\right]^3$ −norm. So, using w_j^{per} instead of w_j does not change the convergence of u_ε to its first order asymptotic expansion.

Yet, since w_j and $p_j, j = 1, 2, 3$ are the *ad hoc* correctors for the non-periodic setting, there must be situations highlighting that the approximation of u_ε (resp. p_ε) by $\varepsilon^2 w_j(\cdot/\varepsilon)(f_j - \partial_j p_0)$ (resp. $p_0 + \varepsilon p_j(\cdot/\varepsilon)(f_j - \partial_j p_0)$) is improved in some sense when we use w_j instead of w_j^{per}. We exhibit in Theorem 2.3 such a situation (see Remark 2.6).

Before stating Theorem 2.3, we obtain in Theorem 2.2 H^2−estimates for the solution of a Stokes system posed in Ω_ε (see [16, Theorem 4.1] for the periodic case).

Theorem 2.2 (Estimates for a Stokes problem)**.** *Suppose that Assumptions* ***(A4)***$_0$ *and* ***(A5)*** *are satisfied. Let* $f \in \left[L^2(\Omega_\varepsilon)\right]^3$ *and* $(u, p) \in \left[H_0^1(\Omega_\varepsilon)\right]^3 \times L^2(\Omega_\varepsilon)/\mathbb{R}$ *be solution of*

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in} \quad \Omega_\varepsilon \\ \operatorname{div}(u_\varepsilon) = 0 & \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{2.4}$$

Then $(u_\varepsilon, p_\varepsilon) \in \left[H^2(\Omega_\varepsilon)\right]^3 \times H^1(\Omega_\varepsilon)/\mathbb{R}$ *and there exists a constant* $C > 0$ *such that for any domain* $\Omega'' \subset\subset \Omega$ *and all* $\varepsilon < \varepsilon_0(\Omega'')$,

$$\begin{aligned}\|D^2 u_\varepsilon\|_{[L^2(\Omega''\cap\Omega_\varepsilon)]^3} + \varepsilon^{-1}\|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} + \varepsilon^{-2}\|u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3}& \\ + \|\nabla p_\varepsilon\|_{L^2(\Omega''\cap\Omega_\varepsilon)^3} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} &\leq C\|f\|_{[L^2(\Omega_\varepsilon)]^3}.\end{aligned}$$

Furthermore, the couple $(u_\varepsilon, p_\varepsilon)$ *is unique in* $\left[H^1(\Omega_\varepsilon)\right]^3 \times L^2(\Omega_\varepsilon)/\mathbb{R}$.

Theorem 2.3 (Convergence Theorem)**.** *Suppose that assumptions* ***(A1)-(A5)*** *are satisfied. Let* $f \in \left[W^{3,\infty}(\Omega)\right]^3$ *be such that* $\operatorname{div}(Af) = 0$ *and* f *is compactly supported in* Ω. *There exists a constant* $C > 0$ *such that for all* $\varepsilon > 0$ *small enough and all domain* $\Omega'' \subset\subset \Omega$,

$$\begin{aligned}\left\|D^2\left[u_\varepsilon - \varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right) f_j\right]\right\|_{[L^2(\Omega''\cap\Omega_\varepsilon)]^3} &+ \varepsilon^{-1}\left\|\nabla\left[u_\varepsilon - \varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right) f_j\right]\right\|_{[L^2(\Omega_\varepsilon)]^3} \\ &+ \varepsilon^{-2}\left\|u_\varepsilon - \varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right) f_j\right\|_{[L^2(\Omega_\varepsilon)]^3} \leq C\varepsilon\end{aligned} \tag{2.5}$$

and

$$\left\|\nabla\left[p_\varepsilon - \varepsilon\left\{p_j\left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j\right\} f_j\right]\right\|_{L^2(\Omega''\cap\Omega_\varepsilon)} + \left\|p_\varepsilon - \varepsilon\left\{p_j\left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j\right\} f_j\right\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon, \tag{2.6}$$

where

$$\lambda_\varepsilon^j = \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} p_j\left(\frac{\cdot}{\varepsilon}\right).$$

Remark 2.4. *We note that Theorem 2.2 and Theorem 2.3 are valid in the periodic case (that is in the framework of subsection 1.2). This provides a new situation in which quantitative error estimates can be obtained, besides th ones of [14, 18].*

Remark 2.5. *The assumptions* $\operatorname{div}(Af) = 0$ *and* f *compactly supported in* Ω *make boundary effects disappear. Indeed, it is straightforward to see that in this case* $\nabla p_0 = 0$ *in* Ω *(see* (1.2)*). Since* f *is compactly supported, we have* $\varepsilon^2 w_j(\cdot/\varepsilon) f_j = 0$ *on* $\partial\Omega$*, so* u_ε *and its first order expansion coincide on* $\partial\Omega$*. This explains why the* $O(\varepsilon^2)$ H^1-*convergence rate of* R_ε *obtained in Theorem 2.3 is sharper than the* $O(\varepsilon^{3/2})$ H^1-*convergence rate obtained in [18, Theorem 1.1].*

Remark 2.6. *By applying Theorem 2.3, we get that* $R_\varepsilon \in H^2(\Omega_\varepsilon)$*. We now note that, in general, one has* $R_\varepsilon^{\mathrm{per}} \notin H^2(\Omega_\varepsilon)$*. This follows from the fact that* $w_j^{\mathrm{per}}\left(\frac{\cdot}{\varepsilon}\right) \notin H^2(\Omega_\varepsilon)$ *(unless of course* $\Omega_\varepsilon = \Omega_\varepsilon^{\mathrm{per}}$*) for* $j = 1, 2, 3$*. This is due to the normal derivative jumps of* $w_j^{\mathrm{per}}(\cdot/\varepsilon)$ *along the parts of* $\varepsilon\partial\mathcal{O}^{\mathrm{per}}$ *that are included in* Ω_ε*. This shows that, in the non-periodic case, using the periodic corrector in* (2.2) *does not give the expected convergence rate, contrary to the non-periodic corrector.*

Remark 2.7. *Theorem 2.2 and Theorem 2.3 can be proved up to the boundary of* Ω *with the same convergence rates when* Ω *is of class* $\mathcal{C}^2$*. The proof is rather technical and will be omitted here.*

Remark 2.8. *Theorem 2.2 can be proved for the* H^m-*norm,* $m > 0$ *in the periodic domain* $\Omega_\varepsilon^{\mathrm{per}}$ *(see [16, Theorem 4.2]) and in the non-periodic domain* Ω_ε*, provided that we require higher regularity of* $\mathcal{O}_k$ *in* ***(A5)'*** *(typically that* $\mathcal{O}_k$ *is uniformly with respect to* k *of class* $\mathcal{C}^{m+2}$*, see [11, Theorem IV.5.1]): if* $f \in [H^m(\Omega_\varepsilon)]^3$*, then* $(u_\varepsilon, p_\varepsilon) \in \left[H^{m+2}(\Omega_\varepsilon)\right]^3 \times H^{m+1}(\Omega_\varepsilon)/\mathbb{R}$ *and there exists a constant* C *independent of* ε *such that*

$$\|D^{m+2}u_\varepsilon\|_{[L^2(\Omega''\cap\Omega_\varepsilon)]^3} + \|D^{m+1}p_\varepsilon\|_{L^2(\Omega''\cap\Omega_\varepsilon)} \leq C\sum_{i=0}^{m}\frac{1}{\varepsilon^i}\|D^{m-i}f\|_{[L^2(\Omega_\varepsilon)]^3}.$$

Remark 2.9. *This paper presents only the three dimensional case. All that follows is true in dimension greater than 3. As for the two dimensional case, Theorem 2.1 and Theorem 2.2 are valid.*

The rest of the paper is devoted to proofs. In Section 3.1, we give the proof of Theorem 2.2 in both periodic and non periodic perforated domains. We next prove in Section 3.2 the existence of the non-periodic correctors. Finally, Section 3.3 is devoted to the proof of the convergence Theorem 2.3. Some technical Lemmas, especially concerning divergence problems, are postponed to Appendix A.

3 Proofs

3.1 Proof of Theorem 2.2

We first state the following Poincaré-Friedrichs inequality:

Lemma 3.1. *Suppose that Assumptions* ***(A1)*** *and* ***(A3)*** *are satisfied. There exists a constant* $C > 0$ *independent of* ε *such that for all* $u \in \left[H_0^1(\Omega_\varepsilon)\right]^3$*, one has*

$$\int_{\Omega_\varepsilon}|u|^2 \leq C\varepsilon^2\int_{\Omega_\varepsilon}|\nabla u|^2.$$

Proof. We recall that Y_ε is defined by (1.14) and we define $Z_\varepsilon := \{k \in \mathbb{Z}^d, \varepsilon Q_k \cap \partial\Omega \neq \emptyset\}$. We have the decomposition

$$\Omega_\varepsilon = \left(\bigcup_{k\in Y_\varepsilon}\varepsilon(\overline{Q_k}\setminus\overline{\mathcal{O}_k})\right)\cup\left(\bigcup_{k\in Z_\varepsilon}\left[(\varepsilon\overline{Q_k})\cap\Omega\right]\right). \tag{3.1}$$

Thanks to Assumption **(A3)** and the proof of [8, Lemma 3.2], we know that there exists a constant $C > 0$ independent of k and ε such that for all $k \in Y_\varepsilon$, we have the inequality

$$\int_{\varepsilon(Q_k\setminus\overline{\mathcal{O}_k})} u^2 \leq C\varepsilon^2 \int_{\varepsilon(Q_k\setminus\overline{\mathcal{O}_k})} |\nabla u|^2. \tag{3.2}$$

We now fix $k \in Z_\varepsilon$. Thanks to the proof of [19, Lemma 1], there exists a constant $C > 0$ which is independent of k and ε such that

$$\int_{(\varepsilon Q_k)\cap\Omega} u^2 \leq C\varepsilon^2 \int_{(\varepsilon Q_k)\cap\Omega} |\nabla u|^2. \tag{3.3}$$

Summing the estimate (3.2) over $k \in Y_\varepsilon$, the estimate (3.3) over $k \in Z_\varepsilon$ and using (3.1) concludes the proof of Lemma 3.1. □

Let $(u_\varepsilon, p_\varepsilon)$ be the solution of (1.16). We have by classical energy estimates the following inequalities:

$$\left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\right)^{\frac{1}{2}} \leq C\varepsilon\|f\|_{[L^2(\Omega_\varepsilon)]^3} \quad \text{and} \quad \left(\int_{\Omega_\varepsilon} |u_\varepsilon|^2\right)^{\frac{1}{2}} \leq C\varepsilon^2\|f\|_{[L^2(\Omega_\varepsilon)]^3} \tag{3.4}$$

which will be useful in the proof of Theorem 2.2.

Proof of Theorem 2.2. In this proof, C will denote various constants independent of ε that can change from one line to another. We fix $\Omega'' \subset\subset \Omega$. We first show the following estimate:

$$\|D^2u_\varepsilon\|_{[L^2(\Omega''\cap\Omega_\varepsilon)^3]^{3\times3}}+\|\nabla p_\varepsilon\|_{[L^2(\Omega''\cap\Omega_\varepsilon)]^3} \leq C\left[\varepsilon^{-1}\|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3\times3}} + \varepsilon^{-2}\|u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} + \|f\|_{(L^2(\Omega_\varepsilon)]^3}\right]. \tag{3.5}$$

Proof of (3.5): we study Problem (2.4) on each periodic cell $Q_k \setminus \overline{\mathcal{O}_k}$. Let $k \in Y_\varepsilon$, where Y_ε is defined in (1.14). We recall that Q''_k is introduced in (1.6) and we define in $Q''_k \setminus \overline{\mathcal{O}_k}$ the functions

$$\begin{cases} U^k_\varepsilon := \varepsilon^{-2}u_\varepsilon(\varepsilon\cdot) \\ P^k_\varepsilon := \varepsilon^{-1}p_\varepsilon(\varepsilon\cdot) - \lambda_k \\ F^k_\varepsilon := f(\varepsilon\cdot) \end{cases}$$

where $\lambda_k \in \mathbb{R}$ is chosen such that

$$\int_{Q''_k\setminus\overline{\mathcal{O}_k}} P^k_\varepsilon = 0.$$

Then $(U^k_\varepsilon, P^k_\varepsilon) \in \left[H^1\left(Q''_k \setminus \overline{\mathcal{O}_k}\right)\right]^3 \times L^2\left(Q''_k \setminus \overline{\mathcal{O}_k}\right)$ and $(U^k_\varepsilon, P^k_\varepsilon)$ is solution to the following Stokes system

$$\begin{cases} -\Delta U^k_\varepsilon + \nabla P^k_\varepsilon = F^k_\varepsilon \quad \text{in} \quad Q''_k \setminus \overline{\mathcal{O}_k} \\ \operatorname{div}(U^k_\varepsilon) = 0 \\ U^k_\varepsilon = 0 \quad \text{on} \quad \partial\mathcal{O}_k. \end{cases} \tag{3.6}$$

By applying Assumption **(A5)** to System (3.6), we get the estimate

$$\|U^k_\varepsilon\|_{\left[H^2(Q_k\setminus\overline{\mathcal{O}_k})\right]^3} + \|P^k_\varepsilon\|_{H^1(Q_k\setminus\overline{\mathcal{O}_k})} \leq C\left[\|U^k_\varepsilon\|_{\left[H^1(Q''_k\setminus\overline{\mathcal{O}_k})\right]^3} + \|P^k_\varepsilon\|_{L^2(Q''_k\setminus\overline{\mathcal{O}_k})} + \|F^k_\varepsilon\|_{\left[L^2(Q''_k\setminus\overline{\mathcal{O}_k})\right]^3}\right]. \tag{3.7}$$

Assumption $\mathbf{(A4)}_0$ and [11, Lemma III.3.2] applied with $\Omega_1 := Q_k \setminus \overline{\mathcal{O}_k}$ and $\Omega_2 := Q''_k \setminus Q_k$ give a function $v \in \left[H^1_0(Q''_k \setminus \mathcal{O}_k)\right]^3$ such that $\operatorname{div}(v) = P^k_\varepsilon$ and

$$\|v\|_{\left[H^1(Q''_k\setminus\mathcal{O}_k)\right]^3} \leq C\|P^k_\varepsilon\|_{L^2(Q''_k\setminus\mathcal{O}_k)}, \tag{3.8}$$

where C is independent of k. Thus,

$$\|P^k_\varepsilon\|^2_{L^2(Q''_k\setminus\overline{\mathcal{O}_k})} = \langle\nabla P^k_\varepsilon, v\rangle_{H^{-1}\times H^1_0(Q''_k\setminus\overline{\mathcal{O}_k})} \leq \|\nabla P^k_\varepsilon\|_{\left[H^{-1}(Q''_k\setminus\overline{\mathcal{O}_k})\right]^3}\|v\|_{\left[H^1(Q''_k\setminus\overline{\mathcal{O}_k})\right]^3}. \tag{3.9}$$

Gathering together (3.8) and (3.9) yields

$$\|P_\varepsilon^k\|_{L^2(Q_k''\setminus\overline{\mathcal{O}_k})} \leq C\|\nabla P_\varepsilon^k\|_{\left[H^{-1}(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3}. \tag{3.10}$$

The triangle inequality applied to the first equation of (3.6) then provides the inequality

$$\begin{aligned}\|\nabla P_\varepsilon^k\|_{\left[H^{-1}(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3} &\leq \|\Delta U_\varepsilon^k\|_{\left[H^{-1}(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3} + \|F_\varepsilon^k\|_{\left[H^{-1}(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3} \\ &\leq \|\nabla U_\varepsilon^k\|_{\left[L^2(Q_k''\setminus\overline{\mathcal{O}_k})\right]^{3\times 3}} + \|F_\varepsilon^k\|_{\left[L^2(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3}.\end{aligned} \tag{3.11}$$

Collecting (3.7), (3.10) and (3.11), we get

$$\left\|U_\varepsilon^k\right\|_{\left[H^2(Q_k\setminus\overline{\mathcal{O}_k})\right]^3} + \|P_\varepsilon^k\|_{H^1(Q_k\setminus\overline{\mathcal{O}_k})} \leq C\left[\left\|U_\varepsilon^k\right\|_{\left[H^1(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3} + \left\|F_\varepsilon^k\right\|_{\left[L^2(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3}\right].$$

In particular, we deduce

$$\left\|D^2U_\varepsilon^k\right\|_{\left[L^2(Q_k\setminus\overline{\mathcal{O}_k})^3\right]^{3\times 3}} + \|\nabla P_\varepsilon^k\|_{\left[L^2(Q_k\setminus\overline{\mathcal{O}_k})\right]^3} \leq C\left[\left\|U_\varepsilon^k\right\|_{\left[H^1(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3} + \left\|F_\varepsilon^k\right\|_{\left[L^2(Q_k''\setminus\overline{\mathcal{O}_k})\right]^3}\right]. \tag{3.12}$$

Scaling back (3.12) gives

$$\begin{aligned}\|D^2u_\varepsilon\|_{\left[L^2(\varepsilon Q_k\setminus\overline{\mathcal{O}_k})^3\right]^{3\times 3}} + \|\nabla p_\varepsilon\|_{\left[L^2(\varepsilon Q_k\setminus\overline{\mathcal{O}_k})\right]^3} &\leq C\big[\varepsilon^{-1}\|\nabla u_\varepsilon\|_{\left[L^2(\varepsilon Q_k''\setminus\overline{\mathcal{O}_k})\right]^{3\times 3}} \\ &\quad + \varepsilon^{-2}\|u_\varepsilon\|_{\left[L^2(\varepsilon Q_k''\setminus\overline{\mathcal{O}_k})\right]^3} + \|f\|_{\left[L^2(\varepsilon Q_k''\setminus\overline{\mathcal{O}_k})\right]^3}\big].\end{aligned} \tag{3.13}$$

Thus,

$$\begin{aligned}\|D^2u_\varepsilon\|^2_{\left[L^2(\varepsilon Q_k\setminus\overline{\mathcal{O}_k})^3\right]^{3\times 3}} + \|\nabla p_\varepsilon\|^2_{\left[L^2(\varepsilon Q_k\setminus\overline{\mathcal{O}_k})\right]^3} &\leq C\big[\varepsilon^{-2}\|\nabla u_\varepsilon\|^2_{\left[L^2(\varepsilon Q_k''\setminus\overline{\mathcal{O}_k})\right]^{3\times 3}} \\ &\quad + \varepsilon^{-4}\|u_\varepsilon\|^2_{\left[L^2(\varepsilon Q_k''\setminus\overline{\mathcal{O}_k})\right]^3} + \|f\|^2_{\left[L^2(\varepsilon Q_k''\setminus\overline{\mathcal{O}_k})\right]^3}\big].\end{aligned} \tag{3.14}$$

We next sum (3.14) over $k \in \widetilde{Y_\varepsilon}$ where

$$\widetilde{Y_\varepsilon} := Y_\varepsilon \setminus \left\{k \in \mathbb{Z}^3,\ \mathrm{d}(\varepsilon Q_k, \Omega^c) > \varepsilon\right\}.$$

We note that for $\varepsilon < \varepsilon_0(\Omega'')$, we have the inclusion

$$\Omega'' \cap \Omega_\varepsilon \subset \bigcup_{k\in\widetilde{Y_\varepsilon}} \varepsilon Q_k''\setminus\overline{\mathcal{O}_k} \subset \Omega_\varepsilon.$$

We get

$$\begin{aligned}\|D^2u_\varepsilon\|^2_{[L^2(\Omega''\cap\Omega_\varepsilon)^3]^{3\times 3}} + \|\nabla p_\varepsilon\|^2_{[L^2(\Omega''\cap\Omega_\varepsilon)]^3} &\leq C\big[\varepsilon^{-2}\|\nabla u_\varepsilon\|^2_{[L^2(\Omega_\varepsilon)]^{3\times 3}} \\ &\quad + \varepsilon^{-4}\|u_\varepsilon\|^2_{[L^2(\Omega_\varepsilon)]^3} + \|f\|^2_{L^2(\Omega_\varepsilon)}\big].\end{aligned} \tag{3.15}$$

Estimate (3.5) is proved. We now conclude the proof of Theorem 2.2. We have, inserting (3.4) in the right hand side of (3.5),

$$\|D^2u_\varepsilon\|_{[L^2(\Omega''\cap\Omega_\varepsilon)^3]^{3\times 3}} + \varepsilon^{-1}\|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3\times 3}} + \varepsilon^{-2}\|u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} + \|\nabla p_\varepsilon\|_{[L^2(\Omega''\cap\Omega_\varepsilon)]^3} \leq C\|f\|_{[L^2(\Omega_\varepsilon)]^3}.$$

It remains to show that

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\|f\|_{[L^2(\Omega_\varepsilon)]^3}. \tag{3.16}$$

By Lemma A.3 stated in the appendix and the first line of (3.6), we get

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon^{-1}\left[\|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3\times 3}} + C\|f\|_{[H^{-1}(\Omega_\varepsilon)]^3}\right].$$

We now show that

$$\|f\|_{[H^{-1}(\Omega_\varepsilon)]^3} \leq C\varepsilon \|f\|_{[L^2(\Omega_\varepsilon)]^3}. \tag{3.17}$$

Indeed, for any $\phi \in \left[H^1_0(\Omega_\varepsilon)\right]^3$, we write that, using successively Cauchy-Schwarz inequality and Poincaré inequality (see Lemma 3.1),

$$\begin{aligned}\langle f, \phi\rangle = \int_{\Omega_\varepsilon} f\cdot\phi \leq \|f\|_{[L^2(\Omega_\varepsilon)]^3}\|\phi\|_{[L^2(\Omega_\varepsilon)]^3} &\leq C\varepsilon\|f\|_{[L^2(\Omega_\varepsilon)]^3}\|\nabla\phi\|_{[L^2(\Omega_\varepsilon)]^{3\times 3}} \\ &\leq C\varepsilon\|f\|_{[L^2(\Omega_\varepsilon)]^3}\|\phi\|_{\left[H^1_0(\Omega_\varepsilon)\right]^3}\end{aligned}$$

Thus (3.17). Finally, we conclude with the use of (3.4) that

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon^{-1}\|\nabla u_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^{3\times 3}} + C\|f\|_{[L^2(\Omega_\varepsilon)]^3} \leq C\|f\|_{[L^2(\Omega_\varepsilon)]^3}.$$

This proves (3.16) and concludes the proof of Theorem 2.2. □

3.2 Proof of Theorem 2.1

We use the periodic correctors $(w_j^{\text{per}}, p_j^{\text{per}})$ defined in (1.4) and we search w_j and p_j in the form $w_j = w_j^{\text{per}} + \widetilde{w_j}$ and $p_j = p_j^{\text{per}} + \widetilde{p_j}$. We recall (see the last paragraph of Subsection 1.3) that w_j^{per} is extended by zero in $\mathcal{O}^{\text{per}}$ and that p_j^{per} is extended by a constant λ_j. The Stokes system defining $(\widetilde{w_j}, \widetilde{p_j})$ is

$$\begin{cases} -\Delta\widetilde{w_j} + \nabla\widetilde{p_j} = e_j + \Delta w_j^{\text{per}} - \nabla p_j^{\text{per}} \quad \text{in} \quad \mathbb{R}^3\setminus\overline{\mathcal{O}} \\ \text{div}\ \widetilde{w_j} = 0 \\ \widetilde{w_j} = -w_j^{\text{per}} \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{3.18}$$

The proof consists in applying Lax-Milgram's Lemma to (3.18). We first need to prove some preparatory Lemmas. In the sequel, we will use the notation

$$T_j := e_j + \Delta w_j^{\text{per}} - \nabla p_j^{\text{per}}$$

for $j \in \{1,2,3\}$.

Lemma 3.2. *Suppose that Assumption **(A3)** is satisfied. For all $1 < q < +\infty$, we have that $T_j \in \left[W^{-1,q'}(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^3$, where $q' = q/(q-1)$.*

Proof. Let $\phi \in \left[\mathcal{D}(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^3$. We extend ϕ by 0 in the perforations. We estimate $\langle T_j, \phi\rangle$ by an integration by parts:

$$\begin{aligned}\langle T_j, \phi\rangle &= \langle e_j + \Delta w_j^{\text{per}} - \nabla p_j^{\text{per}}, \phi\rangle \\ &= \int_{\mathbb{R}^3\setminus\overline{\mathcal{O}}} e_j\cdot\phi - \int_{\mathbb{R}^3\setminus\overline{\mathcal{O}}} \nabla w_j^{\text{per}} : \nabla\phi + \int_{\mathbb{R}^3\setminus\overline{\mathcal{O}}} (p_j^{\text{per}} - \lambda_j)\text{div}(\phi) \\ &= \int_{\mathbb{R}^3} e_j\cdot\phi - \int_{\mathbb{R}^3} \nabla w_j^{\text{per}} : \nabla\phi + \int_{\mathbb{R}^3} \left(p_j^{\text{per}} - \lambda_j\right)\text{div}(\phi) \\ &= \int_{\mathbb{R}^3} e_j\cdot\phi - \int_{\mathbb{R}^3\setminus\overline{\mathcal{O}^{\text{per}}}} \nabla w_j^{\text{per}} : \nabla\phi + \int_{\mathbb{R}^3\setminus\overline{\mathcal{O}^{\text{per}}}} (p_j^{\text{per}} - \lambda_j)\text{div}(\phi).\end{aligned}$$

Since w_j^{per} (resp. $p_j^{\text{per}} - \lambda_j$) is of class $\mathcal{C}^{2,\alpha}$ (resp. of class $\mathcal{C}^{1,\alpha}$) in $\mathbb{R}^3\setminus\mathcal{O}^{\text{per}}$ (see [11, Theorem IV.7.1]), we may integrate by parts and find that

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\overline{\mathcal{O}^{\text{per}}}} \nabla w_j^{\text{per}} : \nabla\phi &= \int_{\partial\mathcal{O}^{\text{per}}} \frac{\partial w_j^{\text{per}}}{\partial n}\cdot\phi - \int_{\mathbb{R}^3\setminus\overline{\mathcal{O}^{\text{per}}}} \Delta w_j^{\text{per}}\cdot\phi, \\ \int_{\mathbb{R}^3\setminus\overline{\mathcal{O}^{\text{per}}}} (p_j^{\text{per}} - \lambda_j)\text{div}(\phi) &= \int_{\partial\mathcal{O}^{\text{per}}} (p_j^{\text{per}} - \lambda_j)\phi\cdot n - \int_{\mathbb{R}^3\setminus\overline{\mathcal{O}^{\text{per}}}} \nabla p_j^{\text{per}}\cdot\phi,\end{aligned}$$

where we use the notations

$$\frac{\partial w_j^{\mathrm{per}}}{\partial n} := \left(\frac{\partial w_j^{1,\mathrm{per}}}{\partial n}, \frac{\partial w_j^{2,\mathrm{per}}}{\partial n}, \frac{\partial w_j^{3,\mathrm{per}}}{\partial n}\right)^T \quad \text{and} \quad w_j^{i,\mathrm{per}} = w_j^{\mathrm{per}} \cdot e_i,$$

for $i, j \in \{1, 2, 3\}$. Thus,

$$\begin{aligned}
\langle T_j, \phi\rangle &= \int_{\mathbb{R}^3} e_j \cdot \phi + \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\mathrm{per}}}} \left[\Delta w_j^{\mathrm{per}} - \nabla p_j^{\mathrm{per}}\right] \cdot \phi + \int_{\partial \mathcal{O}^{\mathrm{per}}} (p_j^{\mathrm{per}} - \lambda_j)\phi \cdot n - \int_{\partial \mathcal{O}^{\mathrm{per}}} \frac{\partial w_j^{\mathrm{per}}}{\partial n} \cdot \phi \\
&= \int_{\mathbb{R}^3} e_j \cdot \phi - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}^{\mathrm{per}}}} e_j \cdot \phi + \int_{\partial \mathcal{O}^{\mathrm{per}}} (p_j^{\mathrm{per}} - \lambda_j)\phi \cdot n - \int_{\partial \mathcal{O}^{\mathrm{per}}} \frac{\partial w_j^{\mathrm{per}}}{\partial n} \cdot \phi \\
&= \int_{\mathcal{O}^{\mathrm{per}} \setminus \overline{\mathcal{O}}} e_j \cdot \phi + \int_{\partial \mathcal{O}^{\mathrm{per}}} (p_j^{\mathrm{per}} - \lambda_j)\phi \cdot n - \int_{\partial \mathcal{O}^{\mathrm{per}}} \frac{\partial w_j^{\mathrm{per}}}{\partial n} \cdot \phi. \\
&= (A) + (B) + (C)
\end{aligned}$$

We treat each term separetely.

Term (A). By Hölder inequality and Assumption **(A3)** (more precisely (1.5)), we obtain that

$$\left|\int_{\mathcal{O}^{\mathrm{per}} \setminus \overline{\mathcal{O}}} e_j \cdot \phi\right| \leq \left|\mathcal{O}^{\mathrm{per}} \setminus \overline{\mathcal{O}}\right|^{\frac{1}{q'}} \|\phi\|_{\left[L^q(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3} \leq C \|\phi\|_{\left[W^{1,q}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3}.$$

Term (B). We have by standard regularity results (see [11, Theorem IV.7.1]) that $p_j^{\mathrm{per}} \in L^\infty(\partial \mathcal{O}_0^{\mathrm{per}})$. We apply a Trace Theorem $W^{1,1}(\mathcal{O}_0^{\mathrm{per}}) \to L^1(\partial \mathcal{O}_0^{\mathrm{per}})$ (see e.g. [9, Theorem 1, p. 258]) that yields a constant C, which is by translation invariance independent of k, such that for all $k \in \mathbb{Z}^3$,

$$\|\phi\|_{\left[L^1(\partial \mathcal{O}_k^{\mathrm{per}})\right]^3} \leq C \|\phi\|_{\left[W^{1,1}(\mathcal{O}_k^{\mathrm{per}})\right]^3}. \tag{3.19}$$

By applying (3.19) in the second inequality, we get

$$\begin{aligned}
\left|\int_{\partial \mathcal{O}^{\mathrm{per}}} (p_j^{\mathrm{per}} - \lambda_j)\phi \cdot n\right| &\leq \left\|p_j^{\mathrm{per}} - \lambda_j\right\|_{L^\infty(\partial \mathcal{O}^{\mathrm{per}})} \int_{\partial \mathcal{O}^{\mathrm{per}}} |\phi| \\
&= C \sum_{k \in \mathbb{Z}^3} \int_{\partial \mathcal{O}_k^{\mathrm{per}}} |\phi| \leq C \sum_{k \in \mathbb{Z}^3} \int_{\mathcal{O}_k^{\mathrm{per}}} |\phi| + |\nabla \phi| = C \int_{\mathcal{O}^{\mathrm{per}} \setminus \overline{\mathcal{O}}} |\phi| + |\nabla \phi|,
\end{aligned}$$

where we used in the last equality that $\phi = 0$ in $\mathcal{O}$. Using (1.5), we conclude thanks to Hölder inequality that

$$\left|\int_{\partial \mathcal{O}^{\mathrm{per}}} (p_j^{\mathrm{per}} - \lambda_j)\phi \cdot n\right| \leq C \left|\mathcal{O}^{\mathrm{per}} \setminus \overline{\mathcal{O}}\right|^{\frac{1}{q'}} \left[\|\phi\|_{\left[L^q(\mathcal{O}^{\mathrm{per}} \setminus \overline{\mathcal{O}})\right]^3} + \|\nabla \phi\|_{\left[L^q(\mathcal{O}^{\mathrm{per}} \setminus \overline{\mathcal{O}})\right]^{3\times 3}}\right] \leq C \|\phi\|_{\left[W^{1,q}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3}.$$

Term (C). The argument is similar to **Term (B)**. This gives the existence of a constant $C > 0$ such that:

$$\left|\int_{\partial \mathcal{O}^{\mathrm{per}}} \frac{\partial w_j^{\mathrm{per}}}{\partial n} \cdot \phi\right| \leq C \|\phi\|_{\left[W^{1,q}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3},$$

where C is independent of ϕ. We conclude that there exists a constant $C = C(q) > 0$ such that

$$\forall \phi \in \left[\mathcal{D}\left(\mathbb{R}^3 \setminus \overline{\mathcal{O}}\right)\right]^3, \quad |\langle T_j, \phi\rangle| \leq C \|\phi\|_{\left[W^{1,q}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3}.$$

This proves the Lemma. □

Lemma 3.3. *Suppose that Assumptions **(A1)** and **(A3)** are satisfied. For all* $1 < q < +\infty$*, there exists a function* $\phi_j \in \left[W^{1,q}(\mathbb{R}^3)\right]^3$ *such that* $\phi_j = w_j^{\text{per}}$ *on* $\partial\mathcal{O}$*.*

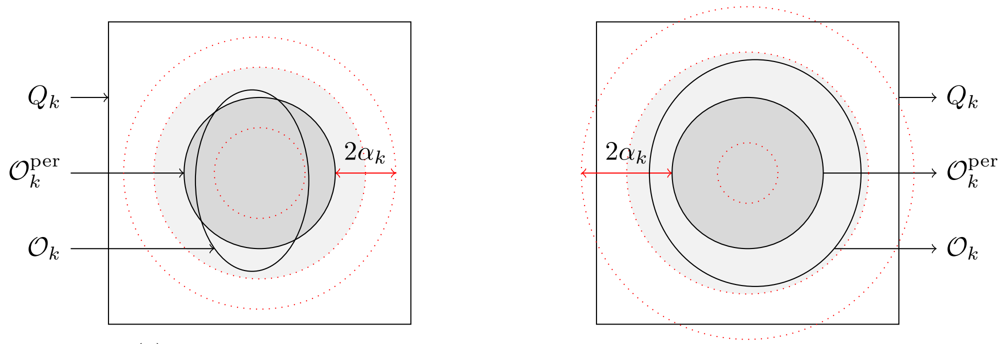
Proof. By Assumption **(A3)**, there exists a sequence $(\alpha_k)_{k\in\mathbb{Z}^3} \in \ell^1(\mathbb{Z}^3)$ such that for all $k \in \mathbb{Z}^3$, $\alpha_k > 0$ and

$$\{x \in \mathcal{O}_k^{\text{per}},\ \mathrm{d}(x, \partial\mathcal{O}_k^{\text{per}}) > \alpha_k\} \subset \mathcal{O}_k \subset \{x \in Q_k,\ \mathrm{d}(x, \mathcal{O}_k^{\text{per}}) < \alpha_k\}.$$

Let $k \in \mathbb{Z}^3$.

If $\mathcal{O}_k = \mathcal{O}_k^{\text{per}}$, then we define the function χ_k by $\chi_k(x) = 0$ for all $x \in Q_k$.

If $\mathcal{O}_k \neq \mathcal{O}_k^{\text{per}}$, there are two cases (see Figure 4).



(a) First case

(b) Second case

Figure 4: Illustration of the proof of Lemma 3.3

First case. We have $\{x \in \mathbb{R}^3,\ \mathrm{d}(x, \mathcal{O}_k^{\text{per}}) < 2\alpha_k\} \subset Q_k$. We consider a function χ_k which is smooth and compactly supported such that

$$\begin{cases} \chi_k = 1 & \text{in} \quad \{x \in Q_k,\ \mathrm{d}(x, \mathcal{O}_k^{\text{per}}) < \alpha_k\} \\ \chi_k = 0 & \text{in} \quad \{x \in Q_k,\ \mathrm{d}(x, \mathcal{O}_k^{\text{per}}) < 2\alpha_k\}^c. \end{cases}$$

We can choose χ_k such that the following estimates are satisfied:

$$|\chi_k| \leq 1 \quad ; \quad |\nabla\chi_k| \leq \frac{C}{\alpha_k} \quad \text{and} \quad \left|\mathrm{supp}(\chi_k) \cap (Q_k \backslash \overline{\mathcal{O}_k^{\text{per}}})\right| \leq C\alpha_k, \tag{3.20}$$

where the constants C are independent of k.

Second case. We have $\{x \in \mathbb{R}^3,\ \mathrm{d}(x, \mathcal{O}_k^{\text{per}}) < 2\alpha_k\} \not\subset Q_k$. We consider a smooth and compactly supported function χ_k such that

$$\begin{cases} \chi_k = 1 & \text{in} \quad \mathcal{O}_k \\ \chi_k = 0 & \text{outside} \quad \text{of } Q_k. \end{cases}$$

Because $\alpha_k \underset{|k|\to+\infty}{\longrightarrow} 0$ and because there exists $\delta > 0$ such that

$$\forall k \in \mathbb{Z}^3, \quad d(\mathcal{O}_k, \partial Q_k) \geq \delta,$$

there are only a finite number of such configurations. After possible changes of the constant C, we can suppose that (3.20) is valid for all $k \in \mathbb{Z}^3$.

Conclusion. We define

$$\phi_j := \left(\sum_{k\in\mathbb{Z}^3} \chi_k\right) w_j^{\text{per}} \in \left[W^{1,q}_{\text{loc}}(\mathbb{R}^3)\right]^3.$$

We study the $W^{1,q}-$local norm of ϕ_j. We fix $k \in \mathbb{Z}^3$; one has in Q_k:

$$|\nabla \phi_j| = \left|\nabla\left(\chi_k w_j^{\mathrm{per}}\right)\right| \le |\nabla \chi_k| \left|w_j^{\mathrm{per}}\right| + \left|\nabla w_j^{\mathrm{per}}\right| |\chi_k| .$$

We now use that $\nabla w_j^{\mathrm{per}}$ is bounded and the inequalities (3.20):

$$|\nabla \phi_j| \le C\alpha_k^{-1}|w_j^{\mathrm{per}}| + C.$$

To obtain that $|\nabla \phi_j|$ is bounded on its support, it suffices to show a bound of the type

$$|w_j^{\mathrm{per}}| \le C\alpha_k \quad \text{in} \quad \{x \in Q_k,\ \mathrm{d}(x, \mathcal{O}_k^{\mathrm{per}}) < 2\alpha_k\}.$$

Since $w_j^{\mathrm{per}} = 0$ on $\mathcal{O}_k^{\mathrm{per}}$ and $\nabla w_j^{\mathrm{per}} \in L^\infty(Q)$, this estimate follows from a classical Taylor inequality. We conclude that

$$\exists C > 0,\ \forall k \in \mathbb{Z}^3,\ \forall x \in Q_k,\ |\nabla \phi_j(x)| \le C.$$

Because

$$\forall k \in \mathbb{Z}^3,\ |\mathrm{supp}(\phi_j) \cap Q_k| = \left|\mathrm{supp}(\chi_k) \cap \left(Q_k \setminus \overline{\mathcal{O}_k^{\mathrm{per}}}\right)\right| = O(\alpha_k),$$

and because of Assumption **(A3)**, we conclude that $|\mathrm{supp}(\phi_j)| < +\infty$ and so $\nabla \phi_j \in \left[L^q(\mathbb{R}^3)\right]^{3\times 3}$. Similarly, $\phi_j \in \left[L^q(\mathbb{R}^3)\right]^3$. This concludes the Lemma. □

We define, when $R > 0$,

$$\Omega^R := R\Omega \setminus \bigcup_{k \text{ s.t. } Q_k \subset R\Omega} \overline{\mathcal{O}_k}.$$

If $R = 1/\varepsilon$, one has $\Omega^R = \frac{1}{\varepsilon}\Omega_\varepsilon$.

Lemma 3.4. *Let $T \in \left[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$. The Stokes problem*

$$\begin{cases} -\Delta w + \nabla p = T & \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ \mathrm{div}(w) = 0 & \\ w = 0 & \text{on} \quad \partial\mathcal{O} \end{cases} \tag{3.21}$$

admits a solution (w, p) such that $(w, p) \in \left[H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3 \times L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ and $\nabla p \in \left[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$. Moreover, for all $R > 0$, we have the estimate

$$\left\|p - \lambda^R\right\|_{L^2(\Omega^R)} \le CR\left[\|\nabla w\|_{\left[L^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^{3\times 3}} + \|T\|_{\left[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3}\right], \quad \lambda^R = \frac{1}{|\Omega^R|}\int_{\Omega^R} p, \tag{3.22}$$

where C is a constant independent of T and R.

Proof. We consider the space $H := \{v \in \left[H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3,\ \mathrm{div}(v) = 0\}$. This a Hilbert space as a closed subspace of $\left[H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$. We formulate the following variational problem: find $w \in H$ such that

$$\forall v \in H, \quad \int_{\mathbb{R}^3 \setminus \overline{\mathcal{O}}} \nabla w : \nabla v = \langle T, v \rangle. \tag{3.23}$$

We recall (see [8, Proof of Lemma 3.2]) that we dispose of a Poincaré inequality on $\left[H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$ and thus of a Poincaré inequality on H. We can apply Lax Milgram's Lemma and find a solution $w \in H$ of (3.23). In particular, for each vector valued function $v \in \left[\mathcal{D}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$ such that $\mathrm{div}(v) = 0$, we have

$$\langle \Delta w + T, v \rangle = 0.$$

Using [4, Theorem 2.1], this implies that there exists a distribution $p \in \mathcal{D}'(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ such that $\Delta w + T = \nabla p$. In particular, $\nabla p \in \left[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$.

Now, we fix $R > 0$. Since $\nabla p \in \left[H^{-1}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$, we have $\nabla p \in \left[H^{-1}(\Omega^R)\right]^3$ and

$$\|\nabla p\|_{[H^{-1}(\Omega^R)]^3} \leq \|\nabla p\|_{\left[H^{-1}(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^3} \leq \|\nabla w\|_{\left[L^2(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^{3\times 3}} + \|T\|_{\left[H^{-1}(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^3}$$

thanks to the triangle inequality. Lemma A.2 for $q = 2$ furnishes the estimate (3.22). □

Proof of Theorem 2.1. We fix $j \in \{1, 2, 3\}$. Lemma 3.3 gives a function $\phi_j \in \left[H^1(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^3$ such that $\phi_j = w_j^{\text{per}}$ on $\partial\mathcal{O}$. The problem

$$\begin{cases} \operatorname{div}(\widetilde{v}_j) = \operatorname{div}(\phi_j) \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ \qquad \widetilde{v}_j = 0 \quad \text{on} \quad \partial\mathcal{O} \end{cases}$$

admits a solution $\widetilde{v}_j \in \left[H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$ thanks to Lemma A.4. Indeed, we just have to check that

$$\forall k \in \mathbb{Z}^3, \int_{\partial\mathcal{O}_k} \phi_j \cdot n = \int_{\partial\mathcal{O}_k} w_j^{\text{per}} \cdot n = \int_{\mathcal{O}_k} \operatorname{div}(w_j^{\text{per}}) = 0.$$

Defining $v_j := \widetilde{v_j} - \phi_j$ yields a solution to the problem

$$\begin{cases} \operatorname{div}(v_j) = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ \qquad v_j = -w_j^{\text{per}} \quad \text{on} \quad \partial\mathcal{O}. \end{cases}$$

By Lemma 3.4, since $\Delta v_j \in \left[H^{-1}(\mathbb{R}^d \setminus \mathcal{O})\right]^3$, there exists a pair $(\widehat{v_j}, \widehat{p_j}) \in \left[H_0^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3 \times L^2_{\text{loc}}(\mathbb{R}^3\setminus\overline{\mathcal{O}})$ solution of the Problem

$$\begin{cases} -\Delta\widehat{v}_j + \nabla\widehat{p}_j = T_j + \Delta v_j \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ \qquad \operatorname{div}(\widehat{v}_j) = 0 \\ \qquad \widehat{v}_j = 0 \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{3.24}$$

We set $\widetilde{w_j} := \widehat{v_j} + v_j$ and $\widetilde{p_j} = \widehat{p_j}$ and we finish the proof of Theorem 2.1. □

3.3 Proof of Theorem 2.3

3.3.1 Strategy of the proof

We introduce

$$\mathcal{R}_\varepsilon := u_\varepsilon - \varepsilon^2 \sum_{j=1}^{3} w_j\left(\frac{\cdot}{\varepsilon}\right) f_j \quad \text{and} \quad \mathcal{P}_\varepsilon := p_\varepsilon - \varepsilon \sum_{j=1}^{3} p_j\left(\frac{\cdot}{\varepsilon}\right) f_j.$$

The strategy of the proof is to find a Stokes system satisfied by $(R_\varepsilon, \pi_\varepsilon)$ and then to apply Theorem 2.2. We need to compute the quantities

$$-\Delta\mathcal{R}_\varepsilon + \nabla\mathcal{P}_\varepsilon \quad \text{and} \quad \operatorname{div}(\mathcal{R}_\varepsilon). \tag{3.25}$$

The construction of auxiliary functions is necessary to correct the divergence equation satisfied by $\mathcal{R}_\varepsilon$, which doesn't have a suitable order in ε. This is done in subsection 3.3.2 below (Lemma 3.5). The proof of Theorem 2.3 is completed in subsection 3.3.3, in particular the computations (3.25).

3.3.2 Some auxiliary functions

We recall that the correctors w_j, $j \in \{1, 2, 3\}$ constructed in Theorem 2.1 are extended by zero in the non-periodic perforations. If $i \in \{1, 2, 3\}$, we denote $w_j^i := w_j \cdot e_i$ the $i^{\text{th}}-$component of w_j. Similarly, $w_j^{i,\text{per}}$ (resp. $\widetilde{w_j^i}$) will be the $i^{\text{th}}-$component of w_j^{per} (resp. $\widetilde{w_j^i}$). We recall that the definition of the matrix A is given in Equation (1.3).

Lemma 3.5. *Suppose that Assumption (A4)$_1$ is satisfied. Let $i,j \in \{1,2,3\}$ and χ be a function of class $\mathcal{C}^\infty$ with support in $Q \setminus \overline{Q}'$ such that $\int_Q \chi = 1$ where Q' is defined in (1.6) (see also Figure 2). We extend χ by periodicity to $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. The problem*

$$\begin{cases} -\mathrm{div} z_j^i = w_j^i - \chi A_j^i \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ \qquad z_j^i = 0 \quad \text{on} \quad \partial\mathcal{O} \end{cases} \tag{3.26}$$

admits a solution $z_j^i \in \left[H^2_{0,\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$. If we still denote z_j^i the extension of z_j^i by 0 in the perforations, we have the estimate

$$\|z_j^i\|_{[H^2(\Omega/\varepsilon)]^3} \leq C\varepsilon^{-\frac{3}{2}}\|w_j^{i,\mathrm{per}}\|_{[H^1(Q)]^3} + C\varepsilon^{-1}\|\widetilde{w_j^i}\|_{[H^1(\mathbb{R}^3)]^3} \tag{3.27}$$

for all $\varepsilon > 0$ where C is a constant independent of ε.

Proof. We fix $i,j \in \{1,2,3\}$. We search z_j^i under the form $z_j^i = \nabla\Psi_j^i + g_j^i$.

Step 1. We build a function Ψ_j^i such that $\nabla\Psi_j^i \in \left[H^{2,\mathrm{per}}(Q)\right]^3 + \left[H^2_{\mathrm{loc}}(\mathbb{R}^3)\right]^3$ and

$$-\Delta\Psi_j^i = w_j^i - \chi A_j^i \quad \text{on} \quad \mathbb{R}^3.$$

The periodic part of Ψ_j^i is defined by solving the problem

$$\begin{cases} -\Delta\Psi_j^{i,\mathrm{per}} = w_j^{i,\mathrm{per}} - \chi A_j^i \quad \text{on} \quad Q \\ \quad \Psi_j^{i,\mathrm{per}} \in H^{1,\mathrm{per}}(Q). \end{cases} \tag{3.28}$$

Since $\int_Q \left(w_j^{i,\mathrm{per}} - \chi A_j^i\right) = 0$, Problem (3.28) is well posed in $H^{1,\mathrm{per}}(Q)/\mathbb{R}$. We choose $\Psi_j^{i,\mathrm{per}}$ such that $\int_Q \Psi_j^{i,\mathrm{per}} = 0$. Because $w_j^{i,\mathrm{per}} - \chi A_j^i \in H^{1,\mathrm{per}}(Q)$, standard elliptic regularity results state that $\nabla\Psi_j^{i,\mathrm{per}} \in \left[H^{2,\mathrm{per}}(Q)\right]^3$. Besides, there exists a constant C such that

$$\|\nabla\Psi_j^{i,\mathrm{per}}\|_{[H^2(Q)]^3} \leq C\|w_j^{i,\mathrm{per}} - \chi A_j^i\|_{[H^1(Q)]^3} \leq C\|w_j^{i,\mathrm{per}}\|_{[H^1(Q)]^3}. \tag{3.29}$$

We now build the non-periodic part of Ψ_j^i. We extend $\widetilde{w_j^i}$ by $-w_j^{i,\mathrm{per}}$ in $\mathcal{O}$. We note that, with this extension, $\widetilde{w_j^i} \in \left[H^1(\mathbb{R}^3)\right]^3$. We consider the problem

$$-\Delta\widetilde{\Psi_j^i} = \widetilde{w_j^i} \quad \text{on} \quad \mathbb{R}^3, \quad \widetilde{\Psi_i^j} \underset{|x|\to+\infty}{\longrightarrow} 0.$$

The solution is given by the Green function:

$$\widetilde{\Psi_j^i} = C_3 \frac{1}{|\cdot|} \underset{\mathbb{R}^3}{*} \widetilde{w_j^i}.$$

Thanks to the remarks after the proof of [12, Theorem 9.9] (see [12, p.235]), we have that $D^2\widetilde{\Psi_j^i} \in \left[H^1(\mathbb{R}^3)\right]^{3\times 3}$ and

$$\|D^2\widetilde{\Psi_j^i}\|_{[L^2(\mathbb{R}^3)]^{3\times 3}} = \|\widetilde{w_j^i}\|_{L^2(\mathbb{R}^3)} \quad \text{and} \quad \|D^3\widetilde{\Psi_j^i}\|_{[L^2(\mathbb{R}^3)^3]^{3\times 3}} = \|\nabla\widetilde{w_j^i}\|_{[L^2(\mathbb{R}^3)]^3}. \tag{3.30}$$

Using the Sobolev injection $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ for $\nabla\widetilde{\Psi_j^i}$, we deduce that $\nabla\widetilde{\Psi_j^i} \in \left[L^6(\mathbb{R}^3)\right]^3$ and, using (3.30), that the estimate

$$\|\nabla\widetilde{\Psi_j^i}\|_{[L^6(\mathbb{R}^3)]^3} \leq C\|\widetilde{w_j^i}\|_{L^2(\mathbb{R}^3)}$$

holds true. In particular, $\nabla\widetilde{\Psi^i_j} \in \left[L^2_{\rm loc}(\mathbb{R}^3)\right]^3$ and, thanks to Hölder inequality, we have

$$\|\nabla\widetilde{\Psi^i_j}\|_{[L^2(\Omega_\varepsilon/\varepsilon)]^3} \le \frac{C}{\varepsilon}\|\nabla\widetilde{\Psi^i_j}\|_{[L^6(\mathbb{R}^3)]^3}.$$

We deduce that

$$\|\nabla\widetilde{\Psi^i_j}\|_{[L^2(\Omega_\varepsilon/\varepsilon)]^3} \le \frac{C}{\varepsilon}\|\nabla\widetilde{\Psi^i_j}\|_{[L^6(\mathbb{R}^3)]^3} \le \frac{C}{\varepsilon}\|\widetilde{w^i_j}\|_{[L^2(\mathbb{R}^3)]^3}. \tag{3.31}$$

Finally, collecting (3.30) and (3.31), we get

$$\|\nabla\widetilde{\Psi^i_j}\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3} \le \frac{C}{\varepsilon}\|\widetilde{w^i_j}\|_{[H^1(\mathbb{R}^3)]^3}. \tag{3.32}$$

We define $\Psi^i_j := \Psi^{i,\rm per}_j + \widetilde{\Psi^i_j}$ and verify that

$$-\Delta\Psi^i_j = w^{i,\rm per}_j - \chi A^i_j + \widetilde{w^i_j} = w^i_j - \chi A^i_j \quad \text{on} \quad \mathbb{R}^3.$$

We use the periodicity of $\nabla\Psi^{i,\rm per}_j$ and write that

$$\begin{aligned}\|\nabla\Psi^i_j\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3} &\le \|\nabla\Psi^{i,\rm per}_j\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3} + \|\nabla\widetilde{\Psi^i_j}\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3} \\ &\le C\varepsilon^{-\frac{3}{2}}\|\nabla\Psi^{i,\rm per}_j\|_{[H^2(Q)]^3} + \|\nabla\widetilde{\Psi^i_j}\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3},\end{aligned} \tag{3.33}$$

where the constant C is independent of ε. We make use of (3.32) and (3.29) and deduce that

$$\|\nabla\Psi^i_j\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3} \le C\varepsilon^{-\frac{3}{2}}\|w^{i,\rm per}_j\|_{[H^1(Q)]^3} + C\varepsilon^{-1}\|\widetilde{w^i_j}\|_{[H^1(\mathbb{R}^3)]^3}. \tag{3.34}$$

Step 2. We introduce a cut-off function χ_1 such that $\chi_1 = 1$ in Q' and $\chi_1 = 0$ out of Q (see Figure 2). We fix $k \in \mathbb{Z}^3$ and define $\chi^k_1 := \chi_1(\cdot + k)$. The goal of this step is to solve the following problem:

$$\begin{cases}\operatorname{div}(g^{i,k}_j) = 0 \quad \text{in} \quad Q_k \setminus \overline{\mathcal{O}_k} \\ g^{i,k}_j = -\nabla\Psi^i_j \quad \text{on} \quad \partial\mathcal{O}_k \\ g^{i,k}_j = 0 \quad \text{on} \quad \partial Q_k.\end{cases} \tag{3.35}$$

We first solve

$$\begin{cases}\operatorname{div}(h^{i,k}_j) = \operatorname{div}(\chi^k_1\nabla\Psi^i_j) \quad \text{on} \quad Q_k \setminus \overline{\mathcal{O}_k} \\ h^{i,k}_j \in \left[H^2_0(Q_k\setminus\overline{\mathcal{O}_k})\right]^3.\end{cases} \tag{3.36}$$

The compatibility condition (1.8) is satisfied:

$$\int_{Q_k\setminus\overline{\mathcal{O}_k}} \operatorname{div}(\chi^k_1\nabla\Psi^i_j) = \int_{\partial\mathcal{O}_k} \chi^k_1\nabla\Psi^i_j\cdot n + \int_{\partial Q_k} \chi^k_1\nabla\Psi^i_j\cdot n = \int_{\mathcal{O}_k} \Delta\Psi^i_j = 0.$$

Since $\operatorname{div}(\chi^k_1\nabla\Psi^i_j) \in H^1_0(Q_k \setminus \overline{\mathcal{O}_k})$, we obtain by Assumption **(A4)**$_1$ a solution $h^{i,k}_j \in \left[H^2_0(Q_k \setminus \overline{\mathcal{O}_k})\right]^3$ to (3.36) which satisfies the estimate

$$\|h^{i,k}_j\|_{\left[H^2(Q_k\setminus\overline{\mathcal{O}_k})\right]^3} \le C\|\operatorname{div}(\chi^k_1\nabla\Psi^i_j)\|_{H^1(Q_k\setminus\overline{\mathcal{O}_k})} \le C\|\nabla\Psi^i_j\|_{\left[H^2(Q_k\setminus\overline{\mathcal{O}_k})\right]^3} \le C\|\nabla\Psi^i_j\|_{[H^2(Q_k)]^3}.$$

We extend $h^{i,k}_j$ by 0 to $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. We then define $g^{i,k}_j := h^{i,k}_j - \chi^k_1\nabla\Psi^i_j$. We note that $g^{i,k}_j = 0$ out of Q_k and that $g^{i,k}_j \in \left[H^2(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$. Besides, $g^{i,k}_j$ solves Problem (3.35) and satisfies the estimate

$$\|g^{i,k}_j\|_{\left[H^2(Q_k\setminus\overline{\mathcal{O}_k})\right]^3} \le C\|\nabla\Psi^i_j\|_{[H^2(Q_k)]^3}. \tag{3.37}$$

Step 3. We set

$$g^i_j(x) := g^{i,k}_j(x) \quad \text{if} \quad x \in Q_k.$$

Then we have

$$\begin{cases} \operatorname{div}(g_j^i) = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ g_j^i = -\nabla \Psi_j^i & \text{on} \quad \partial\mathcal{O}. \end{cases}$$

Besides, $g_j^i \in \left[H^2_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3$ and summing (3.37) over $k \in Y_\varepsilon$ yields the estimate

$$\|g_j^i\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3} \leq C \|\nabla \Psi_j^i\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3}. \tag{3.38}$$

We define $z_j^i := \nabla \Psi_j^i + g_j^i$. We have $z_j^i \in \left[H^2_{\text{loc}}(\mathbb{R}^3 \backslash \overline{\mathcal{O}})\right]^3$. Besides, z_j^i is a solution of (3.26) and, collecting (3.34) and (3.38), we prove the estimate (3.27):

$$\|z_j^i\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3} \leq C \|\nabla \Psi_j^i\|_{[H^2(\Omega_\varepsilon/\varepsilon)]^3} \leq C\varepsilon^{-\frac{3}{2}} \|w_j^{i,\text{per}}\|_{[H^1(Q)]^3} + C\varepsilon^{-1} \|\widetilde{w_j^i}\|_{[H^1(\mathbb{R}^3)]^3}. \tag{3.39}$$

It remains to prove that $z_j^i \in \left[H^2_0(\mathbb{R}^3 \backslash \mathcal{O})\right]^3$. For that, we fix $k \in \mathbb{Z}^3$ and we notice that in a neighbourhood of the perforation $\partial\mathcal{O}_k$, the equality $z_i^j = h_j^{i,k} + (1 - \chi_1^k)\nabla \Psi_j^i = h_j^{i,k}$ is satsified. Since $h_j^{i,k} \in \left[H^2_0(Q_k \backslash \overline{\mathcal{O}_k})\right]^3$, it proves that $z_j^i \in \left[H^2_{0,\text{loc}}(\mathbb{R}^3 \backslash \mathcal{O})\right]^3$. This ends the proof. $\square$

3.3.3 Proof of convergence Theorem 2.3

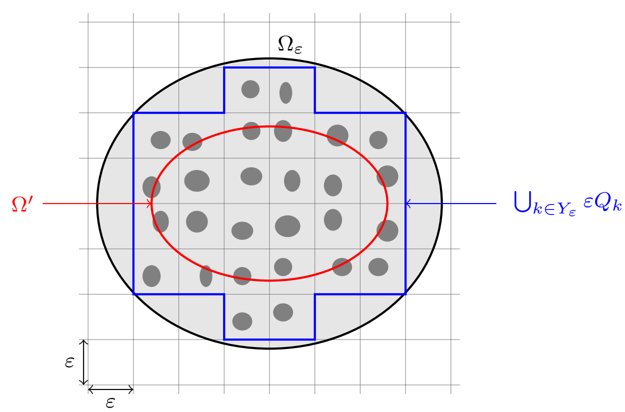


Figure 5: Proof of Theorem 2.3

Proof. We choose $\varepsilon > 0$ small enough such that

$$\operatorname{supp}(f) \subset \bigcup_{k \in Y_\varepsilon} \varepsilon Q_k.$$

We define (see Figure 5) $\Omega' := \{x \in \Omega \text{ s.t. } f(x) \neq 0\}$. We now set

$$u_\varepsilon^1 := \varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right) f_j + \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_i f_j$$

and

$$p_\varepsilon^1 := \varepsilon \left[p_j\left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j\right] f_j, \quad \lambda_\varepsilon^j := \frac{1}{|\frac{1}{\varepsilon}\Omega_\varepsilon|} \int_{\frac{1}{\varepsilon}\Omega_\varepsilon} p_j.$$

We have $u_\varepsilon^1 \in \left[H_0^1(\Omega_\varepsilon)\right]^3$ and $p_\varepsilon^1 \in L^2(\Omega_\varepsilon)$ and thus

$$-\Delta u_\varepsilon^1 + \nabla p_\varepsilon^1 \in \left[H^{-1}(\Omega_\varepsilon)\right]^3.$$

Since (see Figure 5) $f = 0$ in $\Omega \setminus \Omega'$, we have that u_ε^1 and p_ε^1 are compactly supported in Ω. It is thus sufficient to compute $-\Delta u_\varepsilon^1 + \nabla p_\varepsilon^1$ in $\Omega_\varepsilon \cap \Omega'$. We notice that $\Omega_\varepsilon \cap \Omega' = \Omega' \setminus \varepsilon\overline{\mathcal{O}}$. Besides, thanks to Lemma 3.5, we have $z_j^i(\cdot/\varepsilon) \in \left[H^2(\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3$. We compute in $\Omega' \setminus \varepsilon\overline{\mathcal{O}}$:

$$\begin{aligned}\Delta u_\varepsilon^1 = \Delta w_j \left(\frac{\cdot}{\varepsilon}\right) f_j + 2\varepsilon \partial_k w_j \left(\frac{\cdot}{\varepsilon}\right) \partial_k f_j + \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon}\right) \Delta f_j + \varepsilon \Delta z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_i f_j \\ + 2\varepsilon^2 \partial_k z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_k \partial_i f_j + \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \Delta \partial_i f_j.\end{aligned}$$

and

$$\nabla p_\varepsilon^1 = \nabla p_j \left(\frac{\cdot}{\varepsilon}\right) f_j + \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j \right\} \nabla f_j.$$

Thus,

$$\begin{aligned}\Delta u_\varepsilon^1 - \nabla p_\varepsilon^1 &= \Delta w_j \left(\frac{\cdot}{\varepsilon}\right) f_j + 2\varepsilon \partial_k w_j \left(\frac{\cdot}{\varepsilon}\right) \partial_k f_j + \varepsilon^2 w_j \left(\frac{\cdot}{\varepsilon}\right) \Delta f_j + \varepsilon \Delta z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_i f_j \\ &+ 2\varepsilon^2 \partial_k z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_k \partial_i f_j + \varepsilon^3 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \Delta \partial_i f_j - \nabla p_j \left(\frac{\cdot}{\varepsilon}\right) f_j - \varepsilon \left\{ p_j \left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j \right\} \nabla f_j \\ &= -f_j e_j + \varepsilon f_\varepsilon = -f + \varepsilon f_\varepsilon,\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}f_\varepsilon := 2\partial_k w_j \left(\frac{\cdot}{\varepsilon}\right) \partial_k f_j + \varepsilon w_j \left(\frac{\cdot}{\varepsilon}\right) \Delta f_j + \Delta z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_i f_j + 2\varepsilon \partial_k z_j^i \left(\frac{\cdot}{\varepsilon}\right) \partial_k \partial_i f_j \\ + \varepsilon^2 z_j^i \left(\frac{\cdot}{\varepsilon}\right) \Delta \partial_i f_j - \left\{ p_j \left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j \right\} \nabla f_j.\end{aligned}$$

Equation (3.40) is still valid in $\Omega_\varepsilon \setminus \Omega'$ (the LHS and RHS vanish). We define

$$R_\varepsilon := u_\varepsilon - u_\varepsilon^1 \quad \text{and} \quad \pi_\varepsilon := p_\varepsilon - p_\varepsilon^1.$$

Thus $(R_\varepsilon, \pi_\varepsilon) \in \left[H_0^1(\Omega_\varepsilon)\right]^3 \times L^2(\Omega_\varepsilon)$ and

$$-\Delta R_\varepsilon + \nabla \pi_\varepsilon = \varepsilon f_\varepsilon \quad \text{in} \quad \Omega_\varepsilon, \quad f_\varepsilon \in \left[L^2(\Omega_\varepsilon)\right]^3.$$

Using that $f \in \left[W^{3,\infty}(\Omega)\right]^3$, we infer

$$\begin{aligned}\|f_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3} &\le \left\| \nabla w_j \left(\frac{\cdot}{\varepsilon}\right) \right\|_{\left[L^2(\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^{3\times 3}} \|\nabla f_j\|_{[L^\infty(\Omega)]^3} + \varepsilon \left\| w_j \left(\frac{\cdot}{\varepsilon}\right) \right\|_{\left[L^2(\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} \|\Delta f_j\|_{L^\infty(\Omega)} \\ &+ \left\| \Delta z_j^i \left(\frac{\cdot}{\varepsilon}\right) \right\|_{\left[L^2(\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} \|\partial_i f_j\|_{L^\infty(\Omega)} + \varepsilon \left\| \nabla z_j^i \left(\frac{\cdot}{\varepsilon}\right) \right\|_{\left[L^2(\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^{3\times 3}} \|\nabla \partial_i f_j\|_{[L^\infty(\Omega)]^3} \\ &+ \varepsilon^2 \left\| z_j^i \left(\frac{\cdot}{\varepsilon}\right) \right\|_{\left[L^2(\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} \|\Delta \partial_i f_j\|_{L^\infty(\Omega)} + \left\| p_j \left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j \right\|_{L^2(\Omega' \setminus \varepsilon\overline{\mathcal{O}})} \|\nabla f_j\|_{[L^\infty(\Omega)]^3}. \\ &\le C\varepsilon^{\frac{3}{2}} \left[\|w_j\|_{\left[H^1(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} + \|z_j^i\|_{\left[H^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} + \|p_j - \lambda_\varepsilon^j\|_{L^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon\overline{\mathcal{O}})} \right] \\ &= C\varepsilon^{\frac{3}{2}} \left[(A) + (B) + (C) \right].\end{aligned} \tag{3.41}$$

We treat each term separetely. For (A), we have

$$\begin{aligned}\|w_j\|_{\left[H^1(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} &\le \|w_j^{\text{per}}\|_{\left[H^1(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} + \|\widetilde{w_j}\|_{\left[H^1(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} \\ &\le C\varepsilon^{-\frac{3}{2}} \|\nabla w_j^{\text{per}}\|_{H^1(Q)} + \|\widetilde{w_j}\|_{\left[H^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})\right]^3}.\end{aligned} \tag{3.42}$$

For (B), we apply Lemma 3.5 (and especially (3.27)):

$$\|z_j^i\|_{\left[H^2(\frac{1}{\varepsilon}\Omega' \setminus \varepsilon\overline{\mathcal{O}})\right]^3} \le C\varepsilon^{-\frac{3}{2}} \|w_j^{i,\text{per}}\|_{[H^1(Q)]^3} + C\varepsilon^{-1} \|\widetilde{w_j^i}\|_{[H^1(\mathbb{R}^3)]^3} \tag{3.43}$$

For (C), Theorem 2.1 gives

$$\begin{aligned}\|p_j-\lambda_\varepsilon^j\|_{L^2(\frac{1}{\varepsilon}\Omega'\setminus\varepsilon\overline{\mathcal{O}})} &\le \|p_j^{\mathrm{per}}-\lambda_\varepsilon^{j,\mathrm{per}}\|_{L^2(\frac{1}{\varepsilon}\Omega'\setminus\varepsilon\overline{\mathcal{O}})}+\|\widetilde{p_j}-\widetilde{\lambda_\varepsilon^j}\|_{L^2(\frac{1}{\varepsilon}\Omega'\setminus\varepsilon\overline{\mathcal{O}})}\\ &\le C\varepsilon^{-\frac{3}{2}}\|p_j^{\mathrm{per}}\|_{L^2(Q)}+C\varepsilon^{-1}.\end{aligned} \tag{3.44}$$

Collecting (3.42),(3.43) and (3.44), we conclude that there exists a constant $C>0$ independent of ε such that

$$\|f_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^3}\le C.$$

We now study $\mathrm{div}(R_\varepsilon)$. Using Lemma 3.5, we have in Ω_ε:

$$\mathrm{div}(R_\varepsilon)=-\varepsilon^2\chi\left(\frac{\cdot}{\varepsilon}\right)A_j^i\partial_i f_j-\varepsilon^3 z_j^i\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla\partial_i f_j.$$

We recall that $\mathrm{div}(Af)=A_j^i\partial_i f_j=0$. Thus,

$$-\mathrm{div}(R_\varepsilon)=\varepsilon^3 z_j^i\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla\partial_i f_j.$$

We have that $\varepsilon^3 z_j^i\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla\partial_i f_j\in\left[H_0^1(\Omega_\varepsilon)\right]^3$ and $\int_{\Omega_\varepsilon}\varepsilon^3 z_j^i\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla\partial_i f_j=0$. By Lemma A.5 stated in the appendix, there exists $S_\varepsilon\in\left[H_0^2(\Omega_\varepsilon)\right]^3$ such that

$$\mathrm{div}(S_\varepsilon)=\varepsilon^3 z_j^i\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla\partial_i f_j\quad\text{and}\quad\|S_\varepsilon\|_{H^2(\Omega_\varepsilon)}\le C\varepsilon^2\left\|z_j^i\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla\partial_i f_j\right\|_{H_0^1(\Omega_\varepsilon)}.$$

Using that $f\in\left[W^{2,\infty}(\Omega)\right]^3$ and Lemma 3.5, we get

$$\left\|z_j^i\left(\frac{\cdot}{\varepsilon}\right)\cdot\nabla\partial_i f_j\right\|_{[H^1(\Omega_\varepsilon)]^3}\le\frac{C}{\varepsilon}.$$

Thus

$$\|S_\varepsilon\|_{H^2(\Omega_\varepsilon)}\le C\varepsilon. \tag{3.45}$$

We now define $\widehat{R_\varepsilon}:=R_\varepsilon+S_\varepsilon$. The pair $(\widehat{R_\varepsilon},\pi_\varepsilon)\in\left[H_0^1(\Omega_\varepsilon)\right]^3\times L^2(\Omega_\varepsilon)$ is solution to the following Stokes sytem:

$$\begin{cases}-\Delta\widehat{R_\varepsilon}+\nabla\pi_\varepsilon=\varepsilon f_\varepsilon-\Delta S_\varepsilon\\ \mathrm{div}(\widehat{R_\varepsilon})=0\\ \widehat{R_\varepsilon}_{|\partial\Omega_\varepsilon}=0.\end{cases} \tag{3.46}$$

We notice that $\varepsilon f_\varepsilon-\Delta S_\varepsilon\in\left[L^2(\Omega_\varepsilon)\right]^3$ thus we may apply Theorem 2.2: for all $\Omega''\subset\Omega$, we have for $\varepsilon<\varepsilon_0(\Omega'')$,

$$\|D^2\widehat{R_\varepsilon}\|_{L^2(\Omega\cap\Omega''_\varepsilon)}\le C\|\varepsilon f_\varepsilon-\Delta S_\varepsilon\|_{L^2(\Omega_\varepsilon)}\le C\varepsilon\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}+\|S_\varepsilon\|_{H^2(\Omega_\varepsilon)}\le C\varepsilon,$$

and

$$\|\nabla\pi_\varepsilon\|_{L^2(\Omega''\cap\Omega_\varepsilon)}\le C\varepsilon.$$

By the triangle inequality and (3.45), we conclude that

$$\left\|D^2\left[u_\varepsilon-\varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right)f_j\right]\right\|_{L^2(\Omega''\cap\Omega_\varepsilon)}\le C\varepsilon\quad\text{and}\quad\left\|\nabla\left[p_\varepsilon-\varepsilon\left\{p_j\left(\frac{\cdot}{\varepsilon}\right)-\lambda_\varepsilon^j\right\}f_j\right]\right\|_{L^2(\Omega''\cap\Omega_\varepsilon)}\le C\varepsilon.$$

□

Acknowledgments

I am very grateful to my PhD advisor Xavier Blanc for many fruitful discussions and for careful reading of the manuscript. I also thank Claude le Bris for suggesting this subject to me and supporting this project.

A Technical Lemmas

We recall that if $R > 0$, we define

$$\Omega^R := R\Omega \setminus \bigcup_{k,\ Q_k \subset R\Omega} \mathcal{O}_k. \tag{A.1}$$

Lemma A.1 (Divergence Lemma on Ω^R)**.** *Suppose that Assumption* ***(A4)***$_0$ *is satisfied. Let* $1 < q < +\infty$ *and* $R > 0$*. Let* $f \in L^q(\Omega^R)$ *be such that*

$$\int_{\Omega^R} f = 0.$$

The problem

$$\begin{cases} -\mathrm{div}(v) = f & \text{in} \quad \Omega_R \\ \qquad\quad v = 0 & \text{on} \quad \partial\Omega_R \end{cases} \tag{A.2}$$

admits a solution $v \in \left[W^{1,q}(\Omega_R)\right]^3$ *such that*

$$\|v\|_{[W^{1,q}(\Omega_R)]^3} \leq CR\|f\|_{L^q(\Omega_R)} \tag{A.3}$$

where $C > 0$ *is a constant independent of* f *and* R*.*

Proof. We first extend f by 0 in the perforations. We then solve the problem

$$\begin{cases} -\mathrm{div}(v_1) = f \text{ in } R\Omega \\ \qquad\quad v_1 \in \left[W_0^{1,q}(R\Omega)\right]^3. \end{cases} \tag{A.4}$$

By Lemma [11, Theorem III.3.1] and a simple scaling argument, Problem (A.4) admits a solution v_1 such that

$$\|v_1\|_{W^{1,q}(R\Omega)} \leq CR\|f\|_{L^q(R\Omega)}$$

with the constant C being independent of R. For $k \in \mathbb{Z}^3$ such that $Q_k \subset R\Omega$, we consider the problem

$$\begin{cases} \mathrm{div}(v_2^k) = 0 & \text{in} \quad Q_k \setminus \overline{\mathcal{O}_k} \\ \qquad v_2^k = 0 & \text{on} \quad \partial Q_k \\ \qquad v_2^k = -v_1 & \text{on} \quad \partial\mathcal{O}_k. \end{cases} \tag{A.5}$$

The compatibility condition for (A.5) is satisfied:

$$-\int_{\partial\mathcal{O}_k} v_1 \cdot n = -\int_{\mathcal{O}_k} \mathrm{div}(v_1) = \int_{\mathcal{O}_k} f = 0.$$

Arguing as for Problem (3.35), we show that Problem (A.5) admits a solution $v_2^k \in \left[W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})\right]^3$ such that (the constant C is independent of k thanks to Assumption **(A4)**$_0$):

$$\|v_2^k\|_{\left[W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})\right]^3} \leq C\|v_1\|_{\left[W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})\right]^3}. \tag{A.6}$$

We extend v_2^k by zero to $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. We define the function

$$v_2 := \sum_{k, Q_k \subset R\Omega} v_2^k 1_{Q_k \setminus \overline{\mathcal{O}_k}},$$

Summing (A.6) over k such that $Q_k \subset R\Omega$ yields

$$\|v_2\|_{W^{1,q}(\Omega_R)} \leq C\|v_1\|_{W^{1,q}(R\Omega)} \leq CR\|f\|_{L^q(Q_R)}.$$

We set $v = v_1 + v_2$ and notice that v satisfies the conclusion of Lemma A.1. $\square$

Lemma A.2. *Suppose that Assumption **(A4)**$_0$ is satisfied. Let $1 < q < +\infty$ and $R > 0$. Let $f \in \mathcal{D}'(\Omega^R)$ be such that $\nabla f \in \left[W^{-1,q}(\Omega^R)\right]^3$. Then $f \in L^q(\Omega^R)/\mathbb{R}$ and*

$$\|f\|_{L^q(\Omega^R)/\mathbb{R}} \leq CR\|\nabla f\|_{[W^{-1,q}(\Omega^R)]^3} \tag{A.7}$$

where C is a constant independent of f and R.

Proof. The fact that $f \in L^q(\Omega_R)/\mathbb{R}$ follows from [4, Lemma 2.7]. We now show the estimate (A.7). For $u \in L^1(\Omega^R)$, we denote $\lambda_u := \frac{1}{|\Omega^R|}\int_{\Omega^R} u$. We prove that there exists a constant C independent of R such that

$$\|f - \lambda_f\|_{L^q(\Omega_R)} \leq CR\|\nabla f\|_{[W^{-1,q}(\Omega^R)]^3}. \tag{A.8}$$

We argue by duality. We set $q' = q/(q-1)$. We fix a function $g \in L^{q'}(\Omega^R)$ and we define $\overline{g} := g - \lambda_g$. We apply Lemma A.1 to $\overline{g}$: there exists a function $v_g \in \left[W_0^{1,q'}(\Omega^R)\right]^3$ such that

$$\begin{cases} -\mathrm{div}(v_g) = \overline{g} \\ \|v_g\|_{\left[W^{1,q'}(\Omega^R)\right]^3} \leq CR\|\overline{g}\|_{L^{q'}(\Omega^R)}. \end{cases}$$

Since $\|\overline{g}\|_{L^{q'}(\Omega_R)} \leq 2\|g\|_{L^{q'}(\Omega_R)}$, we have $\|v_g\|_{\left[W^{1,q'}(\Omega_R)\right]^3} \leq CR\|g\|_{L^{q'}(\Omega_R)}$. We now write :

$$\langle \nabla f, v_g\rangle_{\left[W^{-1,q}\times W_0^{1,q'}(\Omega^R)\right]^3} = -\int_{\Omega^R}(f-\lambda_f)\mathrm{div}(v_g) = -\int_{\Omega^R}(f-\lambda_f)(g-\lambda_g) = -\int_{\Omega^R}(f-\lambda_f)g.$$

Thus

$$\left|\int_{\Omega^R}(f-\lambda_f)g\right| \leq \|\nabla f\|_{[W^{-1,q}(\Omega^R)]^3}\|v_g\|_{\left[W_0^{1,q'}(\Omega^R)\right]^3} \leq CR\|\nabla f\|_{[W^{-1,q}(\Omega^R)]^3}\|g\|_{L^{q'}(\Omega^R)}.$$

Taking the supremum over g, we conclude the proof of the Lemma. ☐

Lemma A.3 (Scaling)**.** *Suppose that Assumption **(A4)**$_0$ is satisfied. Let $1 < q < +\infty$. Let $\varepsilon > 0$ and Ω_ε be defined by (1.15). There exists a constant $C > 0$ independent of ε such that for all $f \in \mathcal{D}'(\Omega_\varepsilon)$ such that $\nabla f \in W^{-1,q}(\Omega_\varepsilon)$, we have $f \in L^q(\Omega_\varepsilon)/\mathbb{R}$ and the estimate*

$$\|f\|_{L^q(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon^{-1}\|\nabla f\|_{[W^{-1,q}(\Omega_\varepsilon)]^3}.$$

Proof. We apply Lemma A.2 with $R = 1/\varepsilon$ and use a scaling argument ☐

Lemma A.4. *Suppose that Assumption **(A4)**$_0$ is satisfied. Let $1 < q < +\infty$ and $F \in \left[W^{1,q}(\mathbb{R}^3)\right]^3$. Suppose that for all $k \in \mathbb{Z}^3$,*

$$\int_{\partial\mathcal{O}_k} F\cdot n = 0. \tag{A.9}$$

The problem

$$\begin{cases} -\mathrm{div}(v) = \mathrm{div}(F) \quad \text{in} \quad \mathbb{R}^3\setminus\overline{\mathcal{O}} \\ \qquad v = 0 \quad \text{on} \quad \partial\mathcal{O} \end{cases} \tag{A.10}$$

admits a solution $v \in \left[W^{1,q}(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^3$ such that

$$\|v\|_{\left[W^{1,q}(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^3} \leq C\|F\|_{\left[W^{1,q}(\mathbb{R}^3\setminus\overline{\mathcal{O}})\right]^3}$$

where C is a constant independent of F.

Proof. As in the proof of Lemma 3.5, we search the function v under the form $v = \nabla\Psi + v_1$ where

$$-\Delta\Psi = \mathrm{div}(F) \quad \text{on} \quad \mathbb{R}^3, \quad \text{that is} \quad \Psi(x) = C\int_{\mathbb{R}^3} \frac{F(y)\cdot(x-y)}{|x-y|^3}\mathrm{d}y.$$

and

$$\begin{cases} \mathrm{div}(v_1) = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\mathcal{O}} \\ \qquad v_1 = -\nabla\Psi & \text{on} \quad \partial\mathcal{O}. \end{cases}$$

Since $F \in \left[L^q(\mathbb{R}^3)\right]^3$, we know that $\nabla\Psi \in \left[L^q(\mathbb{R}^3)\right]^3$ and that there exists a constant $C > 0$ such that $\|\nabla\Psi\|_{[L^q(\mathbb{R}^3)]^3} \le C\|F\|_{[L^q(\mathbb{R}^3)]^3}$ (see e.g. [11, Exercice II.11.9]). Besides, since $\mathrm{div}(F) \in L^q(\mathbb{R}^3)$, the estimate $\|D^2\Psi\|_{[L^q(\mathbb{R}^3)]^{3\times 3}} \le C\|\mathrm{div}(F)\|_{L^q(\mathbb{R}^3)}$ holds true (see e.g. [12, Theorem 9.9 & p. 235]). Thus,

$$\nabla\Psi \in \left[W^{1,q}(\mathbb{R}^3)\right]^3 \quad \text{and} \quad \|\nabla\Psi\|_{[W^{1,q}(\mathbb{R}^3)]^3} \le C\|F\|_{[W^{1,q}(\mathbb{R}^3)]^3}.$$

We define the function v_1 on each cell $Q_k \setminus \overline{\mathcal{O}_k}$ as a solution of

$$\begin{cases} -\mathrm{div}(v_1^k) = 0 & \text{in } Q_k \setminus \overline{\mathcal{O}_k} \\ \qquad v_1^k = 0 & \text{on} \quad \partial Q_k \\ \qquad v_1^k = -\nabla\Psi & \text{on} \quad \partial\mathcal{O}_k. \end{cases} \tag{A.11}$$

Assumption $\mathbf{(A4)}_0$ together with (A.9) guarantee that Problem (A.11) admits a solution that satisfies the estimate $\|v_1^k\|_{\left[W^{1,q}(Q_k\setminus\overline{\mathcal{O}_k})\right]^3} \le C\|\nabla\Psi\|_{[W^{1,q}(Q_k)]^3}$. This proves the Lemma. □

Lemma A.5. *Suppose that Assumption **(A4)**$_1$ is satisfied. Let $g \in H_0^1(\Omega_\varepsilon)$ be such that*

$$\int_{\Omega_\varepsilon} g = 0.$$

The problem

$$\begin{cases} -\mathrm{div}(u) = g & \text{in} \quad \Omega_\varepsilon \\ \qquad u = 0 & \text{on} \quad \partial\Omega_\varepsilon \end{cases} \tag{A.12}$$

admits a solution $u \in \left[H_0^2(\Omega_\varepsilon)\right]^3$ such that

$$\|u\|_{H^2(\Omega_\varepsilon)} \le \frac{C}{\varepsilon}\|g\|_{H^1(\Omega_\varepsilon)}, \tag{A.13}$$

where the constant C is independent of ε.

Proof. The proof is very similar to the proof of Lemma 3.5. We explain here only the main lines and refer to Subsection 3.3.2 for details. We first extend g by 0 in the perforations. We notice that

$$g \in H_0^1(\Omega) \quad \text{and} \quad \int_\Omega g = 0.$$

We consider the problem

$$\begin{cases} -\mathrm{div}(v) = g & \text{in} \quad \Omega \\ \qquad v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{A.14}$$

Thanks to [11, Theorem III.3.3], Problem (A.14) admits a solution $v \in \left[H_0^2(\Omega)\right]^3$ such that

$$\|\nabla v\|_{H^1(\Omega)} \le C(\Omega)\|g\|_{H^1(\Omega)} \quad \text{and} \quad \|v\|_{H^1(\Omega)} \le C(\Omega)\|g\|_{L^2(\Omega)}. \tag{A.15}$$

We fix a cell Q_k such that $\varepsilon Q_k \subset \Omega$. We build a function $v_1^k \in \left[H^2(\varepsilon\left[Q_k \setminus \overline{\mathcal{O}_k}\right])\right]^3$ such that

$$\begin{cases} \operatorname{div}(v_1^k) = 0 \quad \text{in} \quad \varepsilon\left[Q_k \setminus \overline{\mathcal{O}_k}\right] \\ \quad v_1^k = -v \quad \text{on} \quad \varepsilon\partial\mathcal{O}_k \\ \quad \nabla v_1^k = -\nabla v \quad \text{on} \quad \varepsilon\partial\mathcal{O}_k. \end{cases} \tag{A.16}$$

For that, we use a cut-off function $\chi_\varepsilon^k := \chi\left(\varepsilon[\cdot + k]\right)$ as in **Step 2** of the proof of Lemma 3.5. We solve

$$\begin{cases} \operatorname{div}(w^k) = \operatorname{div}(\chi_k v) \quad \text{in} \quad \varepsilon\left[Q_k \setminus \overline{\mathcal{O}_k}\right] \\ \quad w^k = 0 \quad \text{on} \quad \varepsilon\partial\left[Q_k \setminus \overline{\mathcal{O}_k}\right] \end{cases} \tag{A.17}$$

and then set $v_1^k := w^k - \chi_\varepsilon^k v$. [11, Theorem III.3.3] together with Assumption **(A4)**$_1$ and a standard scaling argument show that Problem (A.16) admits a solution such that

$$\begin{aligned} \varepsilon^2\|D^2 v_1^k\|_{L^2(\varepsilon Q_k\setminus\overline{\mathcal{O}_k})} + \varepsilon\|\nabla v_1^k\|_{L^2(\varepsilon Q_k\setminus\overline{\mathcal{O}_k})} + \|v_1^k\|_{L^2(\varepsilon Q_k\setminus\overline{\mathcal{O}_k})} \\ \le C(\varepsilon^2\|D^2 v\|_{L^2(\varepsilon Q_k)} + \varepsilon\|\nabla v\|_{L^2(\varepsilon Q_k)} + \|v\|_{L^2(\varepsilon Q_k)}), \end{aligned} \tag{A.18}$$

where the constant C is independent of k and ε. We extend v_1^k by zero to Ω_ε. We define

$$v_1 := \sum_{k\in Y_\varepsilon} v_1^k.$$

Then, after summation of (A.18) over k, the estimate

$$\begin{aligned} \varepsilon^2\|D^2 v_1\|_{[L^2(\Omega_\varepsilon)^3]^{3\times 3}} + \varepsilon\|\nabla v_1\|_{[L^2(\Omega_\varepsilon)]^{3\times 3}} + \|v_1\|_{[L^2(\Omega_\varepsilon)]^3} \\ \le C\left[\varepsilon^2\|D^2 v\|_{[L^2(\Omega)^3]^{3\times 3}} + \varepsilon\|\nabla v\|_{[L^2(\Omega)]^{3\times 3}} + \|v\|_{[L^2(\Omega)]^3}\right] \end{aligned} \tag{A.19}$$

holds true. We note that the function $u := v + v_1$ satisfies the conclusion of Lemma A.5. Furthermore, using (A.15) and (A.19), we get

$$\begin{aligned} &\varepsilon^2\|D^2 u\|_{[L^2(\Omega_\varepsilon)^3]^{3\times 3}} + \varepsilon\|\nabla u\|_{[L^2(\Omega_\varepsilon)]^{3\times 3}} + \|u\|_{[L^2(\Omega_\varepsilon)]^3} \\ &\quad \le C\left[\varepsilon^2\|D^2 v\|_{[L^2(\Omega)^3]^{3\times 3}} + \varepsilon\|\nabla v\|_{[L^2(\Omega)]^{3\times 3}} + \|v\|_{[L^2(\Omega)]^3}\right] \\ &\quad \le C\left[\varepsilon^2\|g\|_{H^1(\Omega)} + \|v\|_{[H^1(\Omega)]^3}\right] \le C\left[\varepsilon^2\|g\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega_\varepsilon)}\right] \\ &\quad \le C\left[\varepsilon^2\|g\|_{H^1(\Omega)} + \varepsilon\|g\|_{H^1(\Omega_\varepsilon)}\right] \le C\varepsilon\|g\|_{H^1(\Omega)}. \end{aligned} \tag{A.20}$$

where we used Lemma 3.1 on g in the last inequality. Thus (A.13) is proved. □

B Geometric assumptions

We prove in this section that Assumptions **(A3)** and **(A4)'** imply Assumption **(A4)** and that Assumptions **(A3)** and **(A5)'** imply Assumption **(A5)**. Appendix B follows the proofs of [11, Theorem III.3.1] and [11, Theorem IV.5.1] and makes precise the dependance of the constants appearing in these arguments. We begin by a covering Lemma.

Lemma B.1. *Suppose that Assumption **(A3)** is satisfied. Let $0 < \rho < d(\partial\mathcal{O}_0^{\text{per}}, \partial Q)$. There exists $N \in \mathbb{N}^*$ such that for all $k \in \mathbb{Z}^3$, there exist $2N$ balls $B_i^k, i = 1, ..., 2N$ such that*

(i) for all $i = 1, ..., N$, we have that $B_i^k = B(\xi_i^k, \rho)$, $\xi_i^k \in \partial\mathcal{O}_k$ and $\{x \in Q_k \setminus \overline{\mathcal{O}_k},\ d(x, \partial\mathcal{O}_k) < 3\rho/16\} \subset \bigcup_{i=1}^N B_i^k$;

(ii) for all $i = N+1, ..., 2N$, we have that $B_i^k = B(\xi_i^k, \rho/32)$, $\xi_i^k \in \{x \in Q_k \setminus \overline{\mathcal{O}_k},\ d(x, \partial\mathcal{O}_k) > \rho/16\}$ and $\{x \in Q_k \setminus \overline{\mathcal{O}_k},\ d(x, \partial\mathcal{O}_k) \ge 3\rho/16\} \subset \bigcup_{i=N+1}^{2N} B_i^k$.

Moreover, there exist $2N$ balls $B_i^{0,\mathrm{per}}$, $i = 1, ..., 2N$ and $\eta = \eta(\rho) > 0$ such that

(iii) for all $i = 1, ..., 2N$, $B_i^{0,\mathrm{per}} \subset Q$ and $\left\{x \text{ s.t. } d\left(x, Q \setminus \overline{\mathcal{O}_0^{\mathrm{per}}}\right) < \eta\right\} \subset \bigcup_{i=1}^{2N} B_i^{0,\mathrm{per}}$.

(iv) there exists a bijection $\sigma : \{1, ..., 2N\} \to \{1, ..., 2N\}$ such that for all $i \in \{1, ..., 2N-1\}$, we have that

$$\Omega_{\sigma(i)}^{0,\mathrm{per}} \cap \left(\bigcup_{s=i+1}^{2N} \Omega_{\sigma(s)}^{0,\mathrm{per}}\right) \neq \emptyset \quad \text{and} \quad \Omega_j^{0,\mathrm{per}} := B_j^{0,\mathrm{per}} \cap \left(Q \setminus \overline{\mathcal{O}_0^{\mathrm{per}}}\right).$$

(v) for all but a finite number of $k \in \mathbb{Z}^3$, we have that $B_i^{k,\mathrm{per}} \subset B_i^k$ for all $i = 1, ..., 2N$ and $\left\{x \text{ s.t. } d\left(x, Q_k \setminus \overline{\mathcal{O}_k}\right) < \eta/2\right\} \subset \bigcup_{i=1}^{2N} B_i^{k,\mathrm{per}}$, where $B_i^{k,\mathrm{per}} := B_i^{0,\mathrm{per}} + k$.

Remark B.2. *Lemma B.1.(iv) means that we can relabel the family $B_i^{0,\mathrm{per}}, i = 1, ..., 2N$ such that for all $i \in \{1, ..., 2N-1\}$, we have that $\Omega_i^{0,\mathrm{per}} \cap \left(\Omega_{i+1}^{0,\mathrm{per}} \cup \cdots \cup \Omega_{2N}^{0,\mathrm{per}}\right) \neq \emptyset$.*

Proof. The proof of Lemma B.1 relies on the periodic structure and on Assumption **(A3)**. We first fix by compactness N_0 balls $B_i^{0,\mathrm{per}} = B(x_i, \rho/2)$, $i = 1, ..., N_0$ such that

$$\{x \in Q, \quad d(x, \partial\mathcal{O}_0^{\mathrm{per}}) \leq \rho/4\} \subset \bigcup_{i=1}^{N_0} B_i^{0,\mathrm{per}} \quad \text{and} \quad x_i \in \partial\mathcal{O}_0^{\mathrm{per}}. \tag{B.1}$$

We note that there exists $\widehat{\rho} > 0$ such that for all $i \in \{1, ..., N_0\}$, there exist two points $y_i \in B_i^{0,\mathrm{per}} \cap \mathcal{O}_0^{\mathrm{per}}$ and $z_i \in B_i^{0,\mathrm{per}} \setminus \mathcal{O}_0^{\mathrm{per}}$ satisfying $d(y_i, \partial\mathcal{O}_0^{\mathrm{per}}) > \widehat{\rho}$ and $d(z_i, \partial\mathcal{O}_0^{\mathrm{per}}) > \widehat{\rho}$. We define for each $k \in \mathbb{Z}^3$, $x_i^k := x_i + k$, $y_i^k := y_i + k$, $z_i^k := z_i + k$ and $B_i^{k,\mathrm{per}} := B_i^{0,\mathrm{per}} + k = B(x_i^k, \rho/2)$. By translation invariance, we obviously have (B.1) with 0 replaced by any $k \in \mathbb{Z}^3$.

We consider $k \in \mathbb{Z}^3$ such that $\alpha_k < \min(\rho/16, \widehat{\rho})$ (where we recall that α_k is introduced in **(A3)**). Then, by Assumption **(A3)** and (B.1), we have that

$$\{x \in Q_k, \quad d(x, \partial\mathcal{O}_k) < 3\rho/16\} \subset \{x \in Q_k, \quad d(x, \partial\mathcal{O}_k^{\mathrm{per}}) < \rho/4\} \subset \bigcup_{i=1}^{N_0} B_i^{k,\mathrm{per}}. \tag{B.2}$$

We next claim that each ball $B_i^{k,\mathrm{per}}$, $i = 1, ..., N_0$ intersects $\partial\mathcal{O}_k$. By definition, we have that $y_i^k \in B_i^{k,\mathrm{per}} \cap \mathcal{O}_k^{\mathrm{per}}$ and that $d(y_i^k, \partial\mathcal{O}_k^{\mathrm{per}}) > \widehat{\rho} > \alpha_k$. Thus, by **(A3)**, we get that $y_i^k \in B_i^{k,\mathrm{per}} \cap \mathcal{O}_k$. Similarly, we have that $z_i^k \in B_i^{k,\mathrm{per}} \setminus \mathcal{O}_k$. Thus, there exists $\xi_i^k \in [y_i^k, z_i^k] \cap \partial\mathcal{O}_k$, proving that $\partial\mathcal{O}_k \cap B_i^{k,\mathrm{per}} \neq \emptyset$. We fix an arbitrary point $\xi_i^k \in \partial\mathcal{O}_k \cap B_i^{k,\mathrm{per}}$ and we notice that $B_i^{k,\mathrm{per}} \subset B(\xi_i^k, \rho)$. By (B.2), we conclude that

$$\{x \in Q_k, \quad d(x, \partial\mathcal{O}_k) < 3\rho/16\} \subset \bigcup_{i=1}^{N_0} B_i^k. \tag{B.3}$$

It remains to cover $\{x \in Q_k \setminus \overline{\mathcal{O}_k},\ d(x, \partial\mathcal{O}_k) \geq 3\rho/16\}$. By **(A3)**, we have that

$$\{x \in Q_k \setminus \overline{\mathcal{O}_k},\ d(x, \partial\mathcal{O}_k) \geq 3\rho/16\} \subset \{x \in \overline{Q_k \setminus \mathcal{O}_k^{\mathrm{per}}},\ d(x, \partial\mathcal{O}_k^{\mathrm{per}}) \geq \rho/8\}. \tag{B.4}$$

By compactness and translation invariance, we can cover the right hand side of (B.4) by N_1 balls $B_i^{k,\mathrm{per}} = B(x_i^k, \rho/32)$, $i = N_0 + 1, ..., N_0 + N_1$ where x_i^k is of the form $x_i^k = x_i + k$ and $x_i \in \{x \in Q \setminus \overline{\mathcal{O}_0^{\mathrm{per}}},\ d(x, \partial\mathcal{O}_0^{\mathrm{per}}) \geq \rho/8\}$. We set $B_i^k := B_i^{k,\mathrm{per}}$ and $\xi_i^k := x_i^k$. By **(A3)**, we get that $\xi_i^k \in \{Q_k \setminus \overline{\mathcal{O}_k}, \quad d(x, \partial\mathcal{O}_k) > \rho/16\}$. With N to be fixed later, we have proved (i)-(ii) for $k \in \mathbb{Z}^3$ such that $\alpha_k < \min(\rho/16, \widehat{\rho})$.

We fix $k \in \mathbb{Z}^3$ such that $\alpha_k \geq \min(\rho/16, \widehat{\rho})$. We take any covering of $\{x \in Q_k,\ d(x, \partial\mathcal{O}_k) < 3\rho/16\}$ with balls $B_i^k = B(\xi_i^k, \rho)$, $\xi_i^k \in \partial\mathcal{O}_k$ and $i \in \{1, ..., N_0^k\}$. We then take any covering of

$\{x \in Q_k \backslash \overline{\mathcal{O}_k},\ d(x, \partial \mathcal{O}_k) \geq 3\rho/16\}$ with balls $B_i^k = B(\xi_i^k, \rho/32)$, $\xi_i^k \in \{x \in Q_k \backslash \overline{\mathcal{O}_k},\ d(x, \partial\mathcal{O}_k) \geq 3\rho/16\}$ and $i \in \{N_0^k + 1, ..., N_0^k + N_1^k\}$.

We set $N := \max_{k\in\mathbb{Z}^3}\{N_0^k, N_1^k\}$ where $N_0^k = N_0$ and $N_1^k = N_1$ if $\alpha_k < \min(\rho/16, \widehat{\rho})$. Note that because of **(A3)**, we have that $N < +\infty$. If $N_0^k < N$ or $N_1^k < N$, we duplicate one of the balls in order to define $2N$ balls $B_i^k, i = 1, ..., 2N$. We proceed similarly for $B_i^{0,\text{per}}$, $i = 1, ..., N_0 + N_1$. Assertions (i), (ii), (iii) and (v) are proved. We prove easily (iv) by connectedness of $Q \setminus \overline{\mathcal{O}_0^{\text{per}}}$. □

Assumptions (A3) and (A4)' imply (A4)

Proof that $(\mathbf{A4})_0$ is satisfied

Let $k \in \mathbb{Z}^3$. We formulate [11, Theorem III.3.1] in our particular setting: suppose that there exists $\Omega_i^k, i = 1, ..., N_k$ such that

$$Q_k \setminus \overline{\mathcal{O}_k} = \bigcup_{i=1}^{N_k} \Omega_i^k, \tag{B.5}$$

where Ω_i^k is star-shaped with respect to a ball B_i^k of radius ρ_i^k such that $B_i^k \subset\subset \Omega_i^k$. We define for $i = 1, ..., N^k - 1$

$$F_i^k := \Omega_i^k \cap \left(\bigcup_{s=i+1}^{N_k} \Omega_s^k \right)$$

and we assume that $F_i^k \neq \emptyset$ for all $i \in \{1, ..., N_k - 1\}$. Then Problem (1.7) with $f \in L^q(Q_k \setminus \overline{\mathcal{O}_k})$ admits a solution v satisfying (1.9) with

$$C_q^0(k) \leq C(q) \left(\frac{\text{diam}(Q_k \setminus \overline{\mathcal{O}_k})}{\min_{i=1}^{N_k} \rho_i^k} \right)^3 \left(1 + \frac{\text{diam}(Q_k \setminus \overline{\mathcal{O}_k})}{\min_{i=1}^{N_k} \rho_i^k} \right) \left(1 + \frac{|Q_k \setminus \overline{\mathcal{O}_k}|^{1-1/q}}{\min_{i=1}^{N_k-1} |F_i|^{1-1/q}} \right)^{N_k}. \tag{B.6}$$

To bound $C_q^0(k)$ uniformly in k, it is sufficient to show that $Q_k \setminus \overline{\mathcal{O}_k}$ admits a decomposition of the form (B.5) where N_k is independent of k, ρ_i^k and $|F_i^k|$ are uniformly bounded from below in k and i. We first explain how to find such a decomposition with N_k and ρ_i^k independent of k and i. By making precise the dependance on the geometry of $\partial\mathcal{O}_k$ at each step of the proof of [11, Lemma II.1.3], we can show that **(A4)'** implies that there exists $\rho > 0$ such that for all $k \in \mathbb{Z}^3$ and $\xi^k \in \partial\mathcal{O}_k$, there exists an open set G_{ξ^k} such that $\Omega_{\xi^k} := G_{\xi^k} \cap \left(Q_k \setminus \overline{\mathcal{O}_k}\right)$ is star-shaped with respect to a ball of radius ρ strictly included in Ω_{ξ^k} and $B(\xi^k, \rho) \subset G_{\xi^k}$.

We next apply Lemma B.1 with ρ given before and we denote by B_i^k, $i = 1, ..., 2N$ the family of balls that we obtain. For $i = 1, ..., N$, we define $G_i^k := G_{\xi_i^k}$ and $\Omega_i^k := \Omega_{\xi_i^k}$. For $i = N+1, ..., 2N$, we define $\Omega_i^k := B_i^k$. Since $B_i^k \subset G_i^k$ for $i = 1, ..., N$ and because $B_i^k, i = 1, ..., 2N$ covers $Q_k \setminus \overline{\mathcal{O}_k}$, we have that (B.5) is satisfied with $\rho_i^k \geq \rho/32$ and $N_k = N$.

It remains to check that there exists a relabeling of the Ω_i^k's such that we have that $\min_{i=1}^{2N-1} |F_i^k| \geq C$ where $C > 0$ is independent of k. We use Lemma B.1.(iii)-(v). According to Remark B.2, we relabel the $\Omega_i^{k,\text{per}}$ (note that this also implies a relabeling of the Ω_i^k's) such that

$$\forall i \in \{1, ..., 2N-1\},\ F_i^{k,\text{per}} := \Omega_i^{k,\text{per}} \cap \left(\Omega_{i+1}^{k,\text{per}} \cup \cdots \cup \Omega_{2N}^{k,\text{per}} \right) \neq \emptyset.$$

We then fix $\rho' > 0$ such that for all $i \in \{1, ..., 2N-1\}$, we have that $F_i^{k,\text{per}}$ contains a ball $\left(B_i^{k,\text{per}}\right)'$ of radius ρ' such that $\left(B_i^{k,\text{per}}\right)' \subset\subset Q_k \setminus \overline{\mathcal{O}_k^{\text{per}}}$. We fix $k \in \mathbb{Z}^3$ such that Lemma B.1.(v) is satisfied and such that

$$\alpha_k < \min_{i=1}^{2N-1} d\left(\left(B_i^{k,\text{per}}\right)', \partial\mathcal{O}_k^{\text{per}} \right). \tag{B.7}$$

Then, for all $i \in \{1, ..., 2N-1\}$, we have that

$$\left(B_i^{k,\mathrm{per}}\right)' \subset\subset Q_k \setminus \overline{\mathcal{O}_k}. \tag{B.8}$$

We then recall that

$$F_i^k = \Omega_i^k \cap \left(\bigcup_{s=i+1}^{2N} \Omega_i^k \right) = \left[G_i^k \cap \left(\bigcup_{s=i+1}^{2N} G_i^k \right) \right] \cap \left(Q_k \setminus \overline{\mathcal{O}_k} \right).$$

By Lemma B.1.(v), we have that $B_j^{k,\mathrm{per}} \subset G_j^k$ for all $j \in \{1, ..., 2N\}$. Together with (B.8), this yields that $\left(B_i^{k,\mathrm{per}}\right)' \subset F_i^k$ for all $i \in \{1, ..., 2N-1\}$. Thus, $\min_{i=1}^{2N-1} |F_i^k| \geq \frac{4}{3}\pi\rho'^3$. Since by **(A3)** there are only a finite number of indices k such that (B.7) is not satisfied, we conclude that, after eventually relabeling the F_i^k's, we have that $\min_{i=1}^{2N-1} |F_i^k| \geq C > 0$.

Proof that $(\mathbf{A4})_1$ is satisfied

We briefly sketch the proof of $(\mathbf{A4})_1$ and we refer to the proof of $(\mathbf{A4})_0$ for some ingredients. Let $k \in \mathbb{Z}^3$ and $f \in W_0^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})$. To solve Problem (1.7), we use a decomposition of the form (B.5) with N_k uniform in k $(= N)$ and Ω_i^k that is star-shaped with respect to a ball of radius ρ uniformly bounded from below in k and i, as constructed in the proof of $(\mathbf{A4})_0$. We then write $f = f_1 + \cdots + f_N$ where $f_i \in W_0^{1,q}(\Omega_i)$, $\int_{\Omega_i} f_i = 0$, $\|f_i\|_{W^{1,q}(\Omega_i^k)} \leq C_i^k \|f\|_{W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})}$ and we solve the Problem:

$$\begin{cases} \mathrm{div}\ v_i = f_i \quad \text{in} \quad \Omega_i^k \\ \qquad v_i \in \left[W_0^{2,q}(\Omega_i^k) \right]^3. \end{cases}$$

Thanks to the estimate (III.3.23) of [11, p. 168], we have that

$$\|v_i\|_{[W^{2,q}(\Omega_i)]^3} \leq C(q,\rho) \|f_i\|_{W^{1,q}(\Omega_i^k)} \leq C(q,\rho) C_i^k \|f\|_{W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})}.$$

Extending v_i by zero to $Q_k \setminus \overline{\mathcal{O}_k}$ and setting $v := v_1 + \cdots + v_N$, we have that v solves Problem (1.7) with the estimate

$$\|v\|_{\left[W^{2,q}(Q_k \setminus \overline{\mathcal{O}_k})\right]^3} \leq C(q,\rho) N C_i^k \|f\|_{W^{1,q}(Q_k \setminus \overline{\mathcal{O}_k})}.$$

We can conclude that $(\mathbf{A4})_1$ is satisfied if C_i^k is uniformly bounded in i and k. To prove that, we make precise the dependance of the constant controlling $\|f_i\|_{W^{1,q}(\Omega_i^k)}$ in the proof of [11, Lemma III.3.4.(vii)-(viii)]. This constant depends on N and on the maximum of the $W^{1,\infty}$−norms of the functions Ψ_i^k, $i = 1, ..., 2N$ and χ_i^k, $i = 1, ..., 2N-1$ where $\{\Psi_1^k, ..., \Psi_{2N}^k\}$ is a partition of unity associated to $\{G_1^k, ..., G_{2N}^k\}$ and $\chi_i^k \in \mathcal{D}(F_i^k)$ satisfies $\int_{F_i^k} \chi_i^k = 1$. Because of Lemma B.1.(v), the family $\{\Psi_1^k, ..., \Psi_{2N}^k\}$ may be chosen independently of k (by using the periodic balls), except for a finite number of indices k. Besides, still after the exclusion of a finite number of indices k, we have shown in the proof of $(\mathbf{A4})_0$ that F_i^k contains a ball of radius ρ' which is uniformly bounded in i and k. Thus, χ_i^k may be chosen as the translation of a reference function χ satisfying $\chi \in \mathcal{D}(B(0,\rho'))$ and $\int_{B(0,\rho')} \chi = 1$. This proves that $\max_{i=1}^{2N} C_i^k \leq C$ for all but a finite number of $k \in \mathbb{Z}^3$. Applying [11, Lemma III.3.4] for the remaining indices k, we conclude that $\max_{i=1}^{2N} C_i^k \leq C$ for all $k \in \mathbb{Z}^3$. This concludes the proof of $(\mathbf{A4})_1$.

Assumptions (A3) and (A5)' imply (A5)

We fix $f \in \left[L^q(Q_k'' \setminus \overline{\mathcal{O}_k})\right]^3$ and we consider the pair (v,p) solution to (1.11). We want to prove the regularity estimate (1.12). The interior regularity property is given by the following result (see [11, Theorem IV.4.1]):

$$\|D^2 v\|_{L^q(\Omega_k)^{3\times3\times3}} + \|\nabla p\|_{L^q(\Omega_k)^3} \leq C\big[\|v\|_{W^{1,q}(\Omega_k')^3} + \|p\|_{L^q(\Omega_k')} + \|f\|_{L^q(\Omega_k')^3}\big], \tag{B.9}$$

where $\Omega_k \subset\subset \Omega_k' \subset\subset Q_k''\backslash\overline{\mathcal{O}_k}$ and C depends only on q and on the distance between Ω_k and $(\Omega_k')^c$. The regularity up to the boundary follows from the discussion [11, pp.271-274]. By tracing the dependance of the constants in these arguments, we can show that, under Assumption **(A5)'**, there exist a radius $\rho > 0$, a constant $d > 1$ and a constant $C > 0$ such that $d\rho < d(Q, \partial Q'')$ and for all $k \in \mathbb{Z}^3$ and $x \in \partial\mathcal{O}_k$, we have that

$$\begin{aligned}\|D^2 v\|_{L^q\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(x,\rho)\right)^{3\times3\times3}} + \|\nabla p\|_{L^q\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(x,\rho)\right)^3} &\leq C\big[\|v\|_{W^{1,q}\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(x,d\rho)\right)^3} \\ &+ \|p\|_{L^q\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(x,d\rho)\right)} + \|f\|_{L^q\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(x,d\rho)\right)^3}\big].\end{aligned} \tag{B.10}$$

We combine estimates (B.9) and (B.10). We fix $k \in \mathbb{Z}^3$. Let $\left(B_i^k\right)_{i=1,\dots,2N}$ be the family of balls given by Lemma B.1 (applied with ρ defined by (B.10)). Thanks to (B.10) and the inequality

$$\forall\ a_1, ..., a_p > 0, \quad a_1^q + \dots + a_p^q \leq (a_1 + \dots + a_p)^q \leq C_{p,q}(a_1^q + \dots + a_p^q), \tag{B.11}$$

we have for all $i \in \{1, ..., N\}$,

$$\begin{aligned}\|D^2 v\|^q_{L^q\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(\xi_i^k,\rho)\right)^{3\times3\times3}} + \|\nabla p\|^q_{L^q\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(\xi_i^k,\rho)\right)^3} &\leq C\big[\|v\|^q_{W^{1,q}\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(\xi_i^k,d\rho)\right)^3} \\ &+ \|p\|^q_{L^q\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(\xi_i^k,d\rho)\right)} + \|f\|^q_{L^q\left(\left(Q_k\backslash\overline{\mathcal{O}_k}\right)\cap B(\xi_i^k,d\rho)\right)^3}\big].\end{aligned} \tag{B.12}$$

Summing (B.12) over $i \in \{1, ..., N\}$ and using that

$$U_k := \{x \in Q_k \setminus \overline{\mathcal{O}_k},\ d(x, \partial\mathcal{O}_k) < 3\rho/16\} \subset \bigcup_{i=1}^N B(\xi_i^k, \rho) \quad \text{and} \quad \left(Q_k \setminus \overline{\mathcal{O}_k}\right) \cap B(\xi_i^k, d\rho) \subset Q_k'' \setminus \overline{\mathcal{O}_k}$$

yield

$$\|D^2 v\|^q_{L^q(U_k)^{3\times3\times3}} + \|\nabla p\|^q_{L^q(U_k)^3} \leq CN\left[\|v\|^q_{W^{1,q}\left(Q_k''\backslash\overline{\mathcal{O}_k}\right)^3} + \|p\|^q_{L^q\left(Q_k''\backslash\overline{\mathcal{O}_k}\right)} + \|f\|^q_{L^q\left(Q_k''\backslash\overline{\mathcal{O}_k}\right)^3}\right]. \tag{B.13}$$

We now apply (B.9) to $\Omega_k = \{x \in Q_k \setminus \overline{\mathcal{O}_k},\ d(x, \partial\mathcal{O}_k) > \rho/8\}$ and $\Omega_k' = Q_k'' \setminus \overline{\mathcal{O}_k}$. We have that $d\big(\Omega_k, (\Omega_k')^c\big) = \min\big(d(Q, \partial Q''), \rho/8\big)$ is independent of k. Thus, using (B.9) and (B.11) yield

$$\|D^2 v\|^q_{L^q(\Omega_k)^{3\times3\times3}} + \|\nabla p\|^q_{L^q(\Omega_k)^3} \leq C\left[\|v\|^q_{W^{1,q}(Q_k''\backslash\overline{\mathcal{O}_k})^3} + \|p\|^q_{L^q(Q_k''\backslash\overline{\mathcal{O}_k})} + \|f\|^q_{L^q(Q_k''\backslash\overline{\mathcal{O}_k})^3}\right], \tag{B.14}$$

where C is independent of k. Summing (B.12) and (B.14) and using that $U_k \cup \Omega_k = Q_k \setminus \overline{\mathcal{O}_k}$ together with (B.11) proves **(A5)**.

Counter-examples to the geometric assumptions

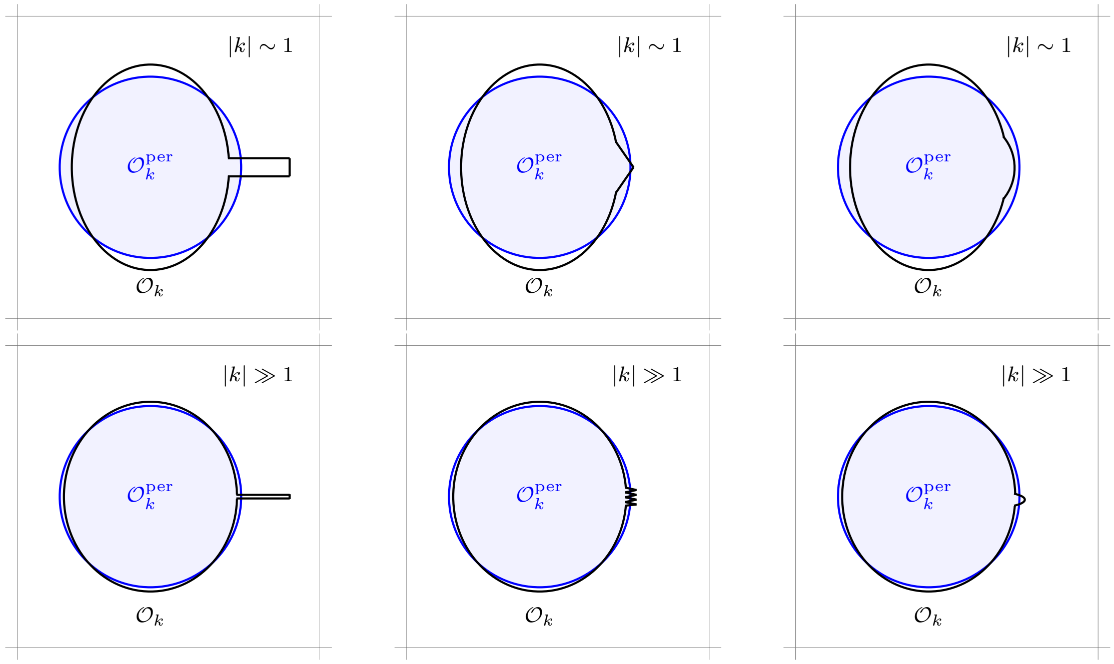
References

[1] Grégoire Allaire. Homogenization of the Stokes flow in a connected porous medium. Asymptotic Analysis, 2(3):203–222, 1989.

[2] Grégoire Allaire. Continuity of the Darcy's law in the low-volume fraction limit. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 18(4):475–499, 1991.

[3] Grégoire Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes I. Abstract framework, a volume distribution of holes. Archive for Rational Mechanics and Analysis, 113(3):209–259, 1991.

[4] Chérif Amrouche and Vivette Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. Czechoslovak Mathematical Journal, 44(1):109–140, 1994.



(a) Counter-example to **(A3)**, close to the origin and at infinity

(b) Counter-example to **(A4)'**, close to the origin and at infinity

(c) Counter-example to **(A5)'**, close to the origin and at infinity

Figure 6: Counter-examples to Assumptions **(A3)-(A4)'-(A5)'**. For each assumption, the picture above expresses the fact that there is no restriction on the perforation $\mathcal{O}_k$ when k remains bounded. The picture below shows a perforation $\mathcal{O}_k$ that is not allowed by Assumption **(A3)**, **(A4)'** or **(A5)'** when $|k| \to +\infty$.

[5] Xavier Blanc, Claude Le Bris, and Pierre-Louis Lions. A possible homogenization approach for the numerical simulation of periodic microstructures with defects. Milan J. Math., 80(2):351–367, 2012.

[6] Xavier Blanc, Claude Le Bris, and Pierre-Louis Lions. Local profiles for elliptic problems at different scales: defects in, and interfaces between periodic structures. Comm. Partial Differential Equations, 40(12):2173–2236, 2015.

[7] Xavier Blanc, Claude Le Bris, and Pierre-Louis Lions. On correctors for linear elliptic homogenization in the presence of local defects. Comm. Partial Differential Equations, 2018. To appear.

[8] Xavier Blanc and Sylvain Wolf. Homogenization of the Poisson equation in a non periodically perforated domain. Accepted for publication in Asymptotic Analysis, 2020.

[9] Lawrence C. Evans. Partial differential equations, volume 19 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, second edition, 2010.

[10] Florian Feppon. High order homogenization of the Stokes system in a periodic porous medium. Preprint Arxiv, 2020.

[11] Giovanni Galdi. An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems. Springer Science & Business Media, 2011.

[12] David Gilbarg and Neil S. Trudinger. Elliptic partial differential equations of second order. Springer, 2015.

[13] Willi Jäger and Andro Mikelic. On the flow conditions at the boundary between a porous medium and an impervious solid. 1994.

[14] Eduard Marušić-Paloka. Asymptotic expansion for a flow in a periodic porous medium. Comptes Rendus de l'Academie des Sciences-Series IIB-Mechanics-Physics-Chemistry-Astronomy, 325(7):369–374, 1997.

[15] Eduard Marusic-Paloka and Andro Mikelic. An error estimate for correctors in the homogenization of the Stokes and Navier-Stokes equations in a porous medium. Boll. Unione Mat. Ital, 10(3):661–671, 1996.

[16] Nader Masmoudi. Some uniform elliptic estimates in a porous medium. Comptes Rendus Mathematiques, 339(12):849–854, 2004.

[17] Enrique Sanchez-Palencia. Fluid flow in porous media. Non-homogeneous media and vibration theory, pages 129–157, 1980.

[18] Zhongwei Shen. Sharp convergence rates for Darcy's law. Preprint Arxiv, 2011.

[19] Luc Tartar. Incompressible fluid flow in a porous medium. Convergence of the homogenization process. Appendix in E. Sanchez-Palencia, Nonhomogeneous media and vibration theory. Lecture Notes in Phys., 127, 1980.