Towards Improving Christofides Algorithm on Fundamental Classes by Gluing Convex Combinations of Tours (Journal Version)

Arash Haddadan, Alantha Newman

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Arash Haddadan[†] Alantha Newman[‡]

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Abstract

We present a new approach for gluing tours over certain tight, 3-edge cuts. Gluing over 3-edge cuts has been used in algorithms for finding Hamilton cycles in special graph classes and in proving bounds for 2-edge-connected subgraph problem, but not much was known in this direction for gluing connected multigraphs. We apply this approach to the traveling salesman problem (TSP) in the case when the objective function of the subtour elimination relaxation is minimized by a *θ-cyclic point*: $x_e \in \{0, \theta, 1-\theta, 1\}$, where the support graph is subcubic and each vertex is incident to at least one edge with x-value 1. Such points are sufficient to resolve TSP in general. For these points, we construct a convex combination of tours in which we can reduce the usage of edges with x-value 1 from the $\frac{3}{2}$ of Christofides algorithm to $\frac{3}{2} - \frac{\theta}{10}$ while keeping the usage of edges with fractional x-value the same as Christofides algorithm. A direct consequence of this result is for the Uniform Cover Problem for TSP: In the case when the objective function of the subtour elimination relaxation is minimized by a $\frac{2}{3}$-*uniform point*: $x_e \in \{0, \frac{2}{3}\}$, we give a $\frac{17}{12}$-approximation algorithm for TSP. For such points, this lands us halfway between the approximation ratios of $\frac{3}{2}$ of Christofides algorithm and $\frac{4}{3}$ implied by the famous "four-thirds conjecture".

1 Introduction

In the Traveling Salesperson Problem (TSP) we are given an integer $n \geq 3$ as the number of vertices and a non-negative cost vector c defined on the edges of the complete graph $K_n = (V_n = \{1,\ldots,n\}, E_n = \binom{\{1,\ldots,n\}}{2})$. We wish to find the minimum cost Hamilton cycle in the graph K_n with respect to costs c. This problem is NP-hard and it is even NP-hard to approximate within any constant factor [WS11]. A natural assumption is that the cost vector c is metric: $c_{ij} + c_{jk} \geq c_{ik}$ for $i, j, k \in V_n$. This special case of TSP is called metric TSP. Metric

*A preliminary version of these results was published in the Proceedings of the 27th Annual European Symposium on Algorithms (ESA) 2019 [HN19].

[†]Carnegie Mellon University, Pittsburgh PA 15213, USA. ahaddada@andrew.cmu.edu.

[‡]CNRS and Université Grenoble Alpes, 38000, Grenoble, France. alantha.newman@grenoble-inp.fr.

TSP is NP-hard [GJ90]. In fact, metric TSP is APX-hard and NP-hard to approximate with a ratio better than 220/219 [PV06]. Since we never deal with non-metric TSP in this paper, we henceforth refer to metric TSP by TSP.

The following linear programming relaxation for the TSP is known as the subtour elimination relaxation.

$$\min\{cx : \sum_{j \in V_n \setminus \{i\}} x_{ij} = 2 \text{ for } i \in V_n, \sum_{i \in U, j \notin U} x_{ij} \geq 2 \text{ for } \emptyset \subset U \subset V_n, \ x \in [0,1]^{E_n}\}.$$

We let $\text{SEP}(K_n)$ denote the feasible region of this linear programming relaxation. Since vector c is metric, any spanning, connected Eulerian multi-subgraph of K_n (henceforth a *tour of* K_n) can be used to find a Hamilton cycle of K_n of no greater cost.[1] We define $\text{TSP}(K_n)$ to be the convex hull of incidence vectors of tours of K_n. The integrality gap of the subtour elimination relaxation for the TSP is[2]

$$g(\text{TSP}) = \max_{n \geq 3, c \in \mathbb{R}^n_{\geq 0}} \frac{\min_{x \in \text{TSP}(K_n)} cx}{\min_{x \in \text{SEP}(K_n)} cx}. \tag{1.1}$$

By the characterization of the integrality gap by Goemans [Goe95] (see also [CV04]), $g(\text{TSP})$ can also be defined as

$$g(\text{TSP}) = \min\{\alpha : \alpha \cdot x \in \text{TSP}(K_n) \ : \ n \geq 3, \ x \in \text{SEP}(K_n)\}. \tag{1.2}$$

It is well-known that $g(\text{TSP}) \geq \frac{4}{3}$. Based on the definition of $g(\text{TSP})$ in (1.2), we can interpret this lower bound as follows: for any $\epsilon > 0$, there is a point x such that $x \in \text{SEP}(K_n)$ and $(\frac{4}{3} - \epsilon)x \notin \text{TSP}(K_n)$. As for upper bounds, a polyhedral analysis of the classical algorithm of Christofides proves $g(\text{TSP}) \leq \frac{3}{2}$, as well as providing a $\frac{3}{2}$-approximation algorithm for the TSP [Chr76, Wol80].

Theorem 1.1 (Polyhedral proof of Christofides algorithm [Chr76, Wol80])**.** *If $x \in \text{SEP}(K_n)$, then $\frac{3}{2}x \in \text{TSP}(K_n)$.*

After more than four decades, there is no result that shows for all $x \in \text{SEP}(K_n)$, the vector $(\frac{3}{2} - \epsilon)x \in \text{TSP}(K_n)$ for some constant $\epsilon > 0$. Motivated by the lower bound of $\frac{4}{3}$ on $g(\text{TSP})$, the following has been conjectured and is wide open.

Conjecture 1 (The four-thirds conjecture)**.** *If $x \in \text{SEP}(K_n)$, then $\frac{4}{3}x \in \text{TSP}(K_n)$.*

Despite the lack of progress towards resolution of Conjecture 1, there has been great success in providing new bounds on $g(\text{TSP})$ for special cases in the past decade [OSS11, MS16, SV14]. Next we present two equivalent formulations of Conjecture 1 that are relevant for our results.

[1] For a graph $G = (V_n, E)$, we define a *tour of* G to be a tour of K_n that uses only edges in E. Notice that the incidence vector for such a tour lives in $\mathbb{R}^E$.

[2] We use $\mathbb{R}^p_{\geq 0}$ to denote $\{x \in \mathbb{R}^p, x_i \geq 0 \text{ for } i \in \{1, \ldots, p\}, x \neq 0\}$.

1.1 Fundamental Classes for TSP

One approach to the four-thirds conjecture is to consider fundamental classes for TSP. Fundamental classes of points were introduced by Carr and Ravi [CR98] and further developed by Boyd and Carr [BC11] and Carr and Vempala [CV04]. A set of vectors $\mathcal{X}$ is a *fundamental class for TSP* if (i) for every $x \in \mathcal{X}$ we have $x \in \text{SEP}(K_n)$ and (ii) proving $\alpha \cdot x \in \text{TSP}(K_n)$ for all $x \in \mathcal{X}$ implies $g(\text{TSP}) \leq \alpha$.

1.1.1 Cyclic Points

For $x \in \text{SEP}(K_n)$, we define $G_x = (V_n, E_x)$ to be the subgraph of K_n whose edge set corresponds to the support of x (i.e., $E_x = \{e : x_e > 0\}$).[3] The set of *cyclic points* form a fundamental class for TSP with a very simple structure.

Definition 1. *A point x is called a θ-cyclic point for some $0 < \theta \leq \frac{1}{2}$ if:*

- *Vector x is in $\text{SEP}(K_n) \cap \{0, \theta, 1-\theta, 1\}^{E_n}$.*
- *The support graph of x, $G_x = (V_n, E_x)$, is subcubic.*
- *For each $v \in V_n$ there is at least one edge $e \in \delta(v)$ with $x_e = 1$.*

Observe that for a θ-cyclic point x we have: (i) the set of 1-edges in G_x, $W_x = \{e : x_e = 1\}$, forms vertex-disjoint paths of G_x, (ii) the fractional edges in G_x, $H_x = \{e : x_e < 1\}$, form vertex-disjoint cycles of G_x. It is easy to see that if $\theta < \frac{1}{2}$ all the cycles in H_x have even length since $x \in \text{SEP}(K_n)$. Conjecture 1 can be restated as follows.

Conjecture 2. *Let $x \in \mathbb{R}^{E_n}$ be a θ-cyclic point. We have $\frac{4}{3}x \in \text{TSP}(K_n)$.*

Similarly, Theorem 1.1 can also be restated as follows.

Theorem 1.2. *Let $x \in \mathbb{R}^{E_n}$ be a θ-cyclic point. We have $\frac{3}{2}x \in \text{TSP}(K_n)$.*

The main result of this paper is to show that we can save on the 1-edges of θ-cyclic points.

Theorem 1.3. *Let $x \in \mathbb{R}^{E_n}$ be a θ-cyclic point. Define vector y as follows: $y_e = \frac{3}{2} - \frac{\theta}{10}$ for $e \in W_x$, $y_e = \frac{3}{2}x_e$ for $e \in H_x$ and $y_e = 0$ for $e \notin E_x$. Then $y \in \text{TSP}(K_n)$.*

In fact a bound on $g(\text{TSP})$ restricted to $\frac{1}{2}$-cyclic points would also provide a bound on $g(\text{TSP})$ when restricted to all half-integral points of the subtour elimination relaxation [CV04], a special case that has received some attention [CR98, BS19, KKO20], and was highlighted in a recent conjecture of Schalekamp, Williamson and van Zuylen stating that the maximum on $g(\text{TSP})$ is achieved for half-integral points of the subtour elimination polytope [SWvZ13]. For $\frac{1}{2}$-cyclic points Theorem 1.3 implies the following.

[3] We sometimes abuse notation and treat x as a vector in $\mathbb{R}^{E_x}$.

Corollary 1.4. *Let x be a $\frac{1}{2}$-cyclic point. Define vector y as follows: $y_e = \frac{3}{2} - \frac{1}{20}$ for $e \in W_x$, $y_e = \frac{3}{4}$ for $e \in H_x$ and $y_e = 0$ for $e \notin E_x$. Then $y \in \mathrm{TSP}(K_n)$.*

Theorem 1.3 gives an improved bound for recently studied special case of *uniform points*, which form another fundamental class for TSP.

1.1.2 Uniform Points

A point $x \in \mathrm{SEP}(K_n)$ is a $\frac{2}{k}$*-uniform point* if x_e is a multiple of $\frac{2}{k}$ for some $k \in \mathbb{Z}_{\geq 3}$. It is clear that $g(\mathrm{TSP}) \leq \alpha$ if and only if for all $k \in \mathbb{Z}_{\geq 3}$ and all $\frac{2}{k}$-uniform points x we have $\alpha \cdot x \in \mathrm{TSP}(K_n)$. Therefore, $\frac{2}{k}$-uniform points form a fundamental class for TSP. We can restate the four-thirds conjecture as follows.

Conjecture 3. *For any integer $k \geq 3$ and $\frac{2}{k}$-uniform point $x \in \mathbb{R}^{E_n}$ we have $\frac{4}{3}x \in \mathrm{TSP}(K_n)$.*

Sebő et al. considered a weakening of Conjecture 3 in the case when $k = 3$ [SBS14]. For a $\frac{2}{3}$-uniform point $x \in \mathbb{R}^{E_n}$, we have $\frac{3}{2}x \in \mathrm{TSP}(K_n)$ by Theorem 1.1. Thus, they asked if there is constant $\epsilon > 0$ such that the vector $(\frac{3}{2} - \epsilon) \cdot x \in \mathrm{TSP}(K_n)$. Of course, the four-thirds conjecture itself implies that the value of ϵ is at least $\frac{1}{6}$.

Conjecture 4. *If $x \in \mathbb{R}^{E_n}$ is a $\frac{2}{3}$-uniform point, then $\frac{4}{3}x \in \mathrm{TSP}(K_n)$.*

One application of Theorem 1.3 is to show that the value of ϵ is at least $\frac{1}{12}$, which brings us "halfway" towards resolving Conjecture 4.

Theorem 1.5. *Let $x \in \mathbb{R}^{E_n}$ be a $\frac{2}{3}$-regular point. Then $\frac{17}{12}x \in \mathrm{TSP}(K_n)$. If G_x is Hamiltonian, then $\frac{87}{68}x \in \mathrm{TSP}(K_n)$.*

The following observation was first made by Carr and Vempala [CV04].

Proposition 1.6. *Let $k \in \mathbb{Z}_{\geq 3}$. We have $\alpha \cdot x \in \mathrm{TSP}(K_n)$ for all $\frac{2}{k}$-uniform points x if an only if for all k-edge-connected k-regular multigraphs $G = (V, E)$, the point $\alpha \cdot (\frac{2}{k} \cdot \chi^E)$ can be written as a convex combination of tours of G.*

Now consider approximating TSP on $\frac{2}{4}$-uniform points: if for any 4-edge-connected 4-regular graph G the vector $\alpha \cdot (\frac{1}{2} \cdot \chi^{E(G)})$ dominates a convex combination of incidence vectors of tours of G, then $g(\mathrm{TSP})$ restricted to half integral instances in at most α.

For graph $G = (V, E)$, a cut $U \subset V$ is *proper* if $|U| \geq 2$ and $|V \setminus U| \geq 2$. If we assume that the 4-regular 4-edge-connected graph G does not contain proper 4-edge cuts, then the following theorem is relevant.

Theorem 1.7. *Let $G = (V, E)$ be a 4-edge-connected 4-regular graph G with even number of vertices and no proper 4-edge cuts. Then the vector $(\frac{3}{2} - \frac{1}{42}) \cdot (\frac{1}{2} \cdot \chi^{E(G)})$ dominates a convex combination of incidence vectors of tours of G.*

Theorem 1.7 could serve as the base case if we could glue over proper 4-edge cuts of G. However, the gluing arguments we present for θ-cyclic points can not easily be extended to this case due to the increased complexity of the distribution of patterns over 4-edge cuts.

1.2 Previous Work on Fundamental Points

Fundamental points were introduced in a series of papers for the TSP [CR98] and for the minimum cost 2-edge-connected multigraph problem (2ECM) [CR98, BC11, CV04]. Let $2\text{ECM}(K_n)$ be the convex hull of incidence vectors of 2-edge-connected multigraphs of K_n. Clearly, $\text{TSP}(K_n) \subseteq 2\text{ECM}(K_n)$.

Consider a $\frac{1}{2}$-cyclic point $x \in \mathbb{R}^{E_n}$. If H_x is a collection of 3-cycles, then Boyd and Carr showed $\frac{4}{3}x \in \text{TSP}(K_n)$ [BC11] and Boyd and Legault showed $\frac{6}{5}x \in 2\text{ECM}(K_n)$ [BL15]. Boyd and Sebő presented a polynomial time algorithm proving that if H_x is a collection of 4-cycles, then $\frac{10}{7}x \in \text{TSP}(K_n)$ [BS19]. For the same class, we gave an efficient algorithm proving $\frac{9}{7}x \in 2\text{ECM}(K_n)$ [HN18]. Recently, Karlin, Klein and Oveis Gharan showed that for any $\frac{1}{2}$-cyclic point $x \in \mathbb{R}^{E_n}$, we have $(\frac{3}{2} - \epsilon)x \in \text{TSP}(K_n)$ for some constant $\epsilon > 0$ [KKO20]. The interest in $\frac{1}{2}$-cyclic points stems in part from the aforementioned conjecture that the maximum value of $g(\text{TSP})$ is achieved for instances of the TSP where the optimal solution to $\min\{cx : x \in \text{SEP}(K_n)\}$ is $\frac{1}{2}$-cyclic [SWvZ13]. In fact, in each of the classes above there is a family of instances that achieves the largest known lower bound on $g(\text{TSP})$ and the integrality gap of the subtour elimination relaxation for 2ECM [CR98, ABE06, BC11, BS19].

Now we review the results on uniform points. Let $x \in \mathbb{R}^{E_n}$ be a $\frac{2}{k}$-uniform point. Carr and Ravi showed if $k = 4$, then $\frac{4}{3}x \in 2\text{ECM}(K_n)$ [CR98]. Boyd and Legault showed that if $k = 3$, then $\frac{6}{5}x \in 2\text{ECM}(K_n)$ [BL15]. Legault later improved the factor $\frac{6}{5}$ to $\frac{7}{6}$ [Leg17]. Haddadan, Newman and Ravi proved that for $k = 3$, we have $\frac{27}{19}x \in \text{TSP}(K_n)$ [HNR19]. Boyd and Sebő showed that if G_x is additionally Hamiltonian, then $\frac{9}{7}x \in \text{TSP}(K_n)$ [BS19]. Finally, the recent result of Karlin et al. also implies that if $k = 4$, then $(\frac{3}{2} - \epsilon)x \in \text{TSP}(K_n)$ for some constant $\epsilon > 0$ [KKO20].

1.3 Gluing Convex Combinations Over 3-edge Cuts

A key part of our proof of Theorem 1.3 is gluing solutions over certain 3-edge cuts, thereby reducing to instances without such cuts, which are easier to solve. This approach of gluing solutions over 3-edge cuts, and thereby reducing to a problem on graphs without proper 3-edge cuts was first introduced by Cornuéjols, Naddef and Pulleyblank [CNP85]. For a graph $G = (V, E)$, let $U \subset V$ and denote by G^U the graph obtained by contracting U (i.e., identifying all vertices in U to a single vertex and removing the resulting loops). Cornuéjols et al. defined a class of 3-edge-connected graphs $\mathcal{A}$ as *fully reducible* if

- If $G \in \mathcal{A}$ has a proper 3-edge cut U, then both G^U and $G^{\overline{U}}$ are in $\mathcal{A}$.

- The minimum cost Hamilton cycle of G can be found in polynomial time for the graphs in $\mathcal{A}$ that do not have a proper 3-edge cut.

Cornuéjols showed that TSP can be solved in polynomial time for fully reducible graphs [CNP85]. An example of such a fully reducible class are the Halin graphs [CNP83].

For many fully reducible classes $\mathcal{A}$, they showed that for $G \in \mathcal{A}$, if G does not have a proper 3-edge cut, then the convex hull of incidence vectors of Hamilton cycles of G coincides with a system of linear inequalities with polynomial separation [CNP85]. For example, they show if $G = (V, E)$ does not contain any disjoint cycles, then $P_G = \{x \in [0,1]^E \ : \ x(\delta(v)) = 2 \text{ for } v \in V\}$ is the convex hull of incidence vectors of Hamilton cycles of G. Let us describe this result in more detail. Suppose for any graph G with no proper 3-edge cuts that does not contain any disjoint cycles, we have $\text{TSP}(G) = P_G$. Now consider a graph $G = (V, E)$ with no disjoint cycles that has a 3-edge cut U. In the graph G^U, we say the vertex corresponding to the contracted set U is a *pseudovertex*. Suppose that the graphs G^U and $G^{\overline{U}}$ contain no proper 3-edge cuts and suppose we can write y restricted to the edge set of each graph as a convex combination of Hamilton cycles of the respective graph. Let us consider the patterns around the pseudovertices; if the edges adjacent to the pseudovertices are $\{a, b, c\}$ then each vertex can be adjacent to two edges in a Hamilton cycle and therefore, there are only three possible patterns around a vertex: $\{\{a, b\}, \{a, c\}, \{b, c\}\}$. Moreover, since each pattern appears the same percentage of time (in the respective convex combinations) for each pseudovertex, tours with corresponding patterns can be *glued* over the 3-edge cut. In this case, the gluing procedure is quite straightforward. This reduction has also proven quite useful for the minimum cost 2-edge-connected subgraph problem (2EC) [CR98, BL15, Leg17].

In contrast, it appears such a reduction is not known for TSP tours. Indeed, gluing proofs cannot be easily extended to tours for several reasons: (1) As just shown, they are often used for gluing subgraphs (no doubled edges). In TSP, we must allow edges to be doubled, so there are too many possible patterns around a vertex. For example, if we allow each possible pattern corresponding to an even degree, there are 13 possible patterns. (2) Gluing tours over a 3-edge cut might result in disconnected Eulerian multigraphs. Finally, (3) many of the algorithms based on gluing are not proven to run in polynomial time [CR98, BL15, Leg17].

1.4 Our Approach

The main technical contribution of this paper is to show that for a carefully chosen set of tours, we can design a gluing procedure over *critical cuts*, which, roughly speaking are proper 3-edge cuts that are *tight*: the x-values of the three edges crossing the cut sum to 2. As we will see, a tour generated via a polyhedral version of Christofides algorithm on a θ-cyclic point can have eight (rather than 13) possible patterns around a vertex. While this is still alot, we identify certain conditions for the convex combination, under which controlling the frequency of a single one of these patterns allows us to control the frequency of the other seven. Moreover, we show the frequency of one of the patterns (around an arbitrary vertex v) depends on the fraction of times in the convex combination of connectors that vertex v is a leaf. Thus, we can fix a critical cut $U \subset V_n$ in G_x and find a convex combination of tours for G_x^U. Then we can find a set of tours for $G_x^{\overline{U}}$ such that the distribution of patterns around the pseudovertex corresponding to U matches that of the pseudovertex corresponding to $\overline{U}$ in G_x^U, which enables us to glue over critical cuts.

Applying this gluing procedure, we can reduce an instance of TSP on a θ-cyclic point to base-case instances, which contain no critical cuts. On a high level, our proof of Theorem 1.3 for such instances is based on Christofides algorithm: We show that a θ-cyclic point x can be written as a convex combination of connected, spanning subgraphs of G_x with no doubled edges (henceforth a *connector of* G_x) with certain properties and then we show that the vector z, where $z_e = \frac{x_e}{2}$ for $e \in H_x$ and $z_e = \frac{1}{2} - \frac{\theta}{10}$ for $e \in W_x$, can be written as a convex combination of subgraphs, each of which can be used for parity correction of a connector (henceforth a *parity corrector*). Tight cuts are generally difficult to handle using an approach based on Christofides algorithm, since $(\frac{1}{2} - \epsilon)x$ is insufficient for parity correction of a tight cut if it is crossed by an odd number of edges in the connector. However, in our base-case instances, there are only two types of tight 3-edge cuts. The first type of cut is a *degenerate tight cut.* These cuts are easy to handle and we defer their formal definition to Section 2.1. The second type of cut is a *vertex cut*, which we show are also easy to handle. In particular, the parity of vertex cuts can be addressed with a key tool used by Boyd and Sebő [BS19] called *rainbow v-trees* (see Theorem 2.5). Using this in combination with a decomposition of the 1-edges into few *induced matchings*, which have some additional required properties, we can prove Theorem 1.3 for the base case.

1.5 Organization

In Section 2, we introduce notation and some definitions relevant to cut structure in cyclic points. We also review some well-known polyhedral tools. In Section 3, we present our main ideas and tools for gluing tours over critical cuts of G_x, thereby reducing to TSP on base cases that only contain certain types of tight cuts. We then show in Section 4 that we are able to handle these remaining tight cuts via an approach similar—on a high-level—to Christofides algorithm. For our gluing approach to work, we need to choose the connectors to have certain properties and to save on the 1-edges, we need to show that a vector with less than half on each 1-edge belongs to the O-join polytope. Both of these key technical ingredients can be found in Section 4. Sections 3 and 4 contain a complete proof of Theorem 1.3. In Section 5, we present an application of Theorem 1.3 to approximating TSP on $\frac{2}{3}$-uniform points. We also give an algorithm for TSP on $\frac{2}{4}$-uniform points under certain assumptions. Finally, in Section 6, we make some concluding remarks and present some problems for future research.

2 Notation and Tools

Let $G = (V, E)$ be a graph. For a subset $U \subset V$ of vertices, let $\delta_G(U) = \{uv \in E : u \in U, v \notin U\}$. (We use $\delta(U) = \delta_G(U)$ when the graph G is clear from the context.) Let $E[U] = \{uv \in E : u \in U, v \in U\}$.

A multi-subset (henceforth *multiset* for brevity) of edges of E is a set that can contain multiple copies of edges in E. A multi-subgraph (henceforth *multigraph* for brevity) of G is the graph on vertex set V whose edge set is a multiset of E (i.e., a multigraph can contain multiple copies of an edge). We sometimes consider a multigraph F of G to be a multiset of edges of G.

The incidence vector of multigraph F of G, denoted by χ^F is a vector in $\mathbb{R}^E$ where χ^F_e is the multiplicity of e in F. Let F and F' be two multigraphs of G, then $F + F'$ is the multigraph that contains $\chi^F_e + \chi^{F'}_e$ copies of edge e for $e \in E$. If e is an edge in F, then $F - e$ is the multigraph with incidence vector $\chi^{F-e} = \chi^F - \chi^{\{e\}}$. For a vector $x \in \mathbb{R}^E$ and a multigraph F of G, we denote $\sum_{e \in F} x_e \cdot \chi^F_e$ by $x(F)$. We use $\delta_F(U)$ to refer to the multiset of edges in F that have exactly one endpoint in U. The degree of a vertex $v \in V$ in F is the number of edges in F that are incident on v.

2.1 Cuts in Cyclic Points

Let x be a θ-cyclic point with support graph $G_x = (V_n, E_x)$. The graph G_x contains three types of tight cuts, by which we mean a cut $U \subset V_n$ such that $x(\delta(U)) = 2$. A *vertex cut* is a cut $U = \{u\}$. Notice that for all $u \in V_n$, $x(\delta(u)) = 2$. The second type of cut is a *critical cut.*

Definition 2. *A proper cut $U \subset V_n$ in G_x is called a* critical cut *if $|\delta(U)| = 3$ and $\delta(U)$ contains exactly one edge e with $x_e = 1$. Moreover, for each pair of edges in $\delta(U)$, their endpoints in U (and in $V \setminus U$) are distinct.*

We refer to the third type of cut as a *degenerate tight cut.*

Definition 3. *A proper cut $U \subset V_n$ in G_x is called a* degenerate tight cut *if $|\delta(U)| = 3$, $|U| > 3$ and $|V \setminus U| > 3$ and the two fractional edges in $\delta(U)$ share an endpoint in either U or $V \setminus U$.*

For a degenerate tight cut U, let $\delta(U) = \{e, f, g\}$, such that f and g are the fractional edges that share an endpoint v. Let e_v be the unique 1-edge incident on v. Observe that $\{e, e_v\}$ forms a 2-edge cut in G_x.

Let x be a θ-cyclic point and $U \subset V_n$ be a cut in G_x. Also, let $\overline{U} = V_n \setminus U$. We can obtain θ-cyclic point x^U by contracting set $\overline{U}$ in G_x to a single vertex. (Respectively, we can obtain a θ-cyclic point $x^{\overline{U}}$ by contracting set U in G_x to a single vertex.) We let $v_{\overline{U}}$ denote the vertex corresponding to set $\overline{U}$ in G^U_x (respectively, v_U corresponds to U in $G^{\overline{U}}_x$).

Observation 2.1. *Let $U \subseteq V_n$ be a minimal critical cut in G_x (i.e., for $S \subset U$, the cut defined by S is not critical in G_x). Then, G^U_x does not contain any critical cuts.*

Proof. Suppose for contradiction that there is $S \subset V(G_x^U)$ that is a critical cut of G_x^U. We can assume that $v_{\overline{U}} \notin S$. Moreover, $S \subsetneq U$. This is a contradiction to minimality of U since S constitutes a critical cut in G_x as well. □

Observation 2.2. *Suppose that G_x has k critical cuts. Let U be a critical cut of G_x. The number of critical cuts in G_x^U is at most $k-1$.*

Proof. Clearly, U is not a critical cut of G_x^U. We show that there is correspondence between the critical cuts of G_x^U and G_x. This implies that G_x^U can have at most $k-1$ critical cuts. If S is a critical cut of G_x^U we can assume without loss of generality that $v_{\overline{U}} \notin S$. Hence, S is also a critical cut of G_x. □

2.2 Polyhedral Basics

Let $G = (V, E)$ and let x be a vector in $\mathbb{R}^E$. Consider a collection of multigraphs $\mathcal{F}$ of G. We say $\lambda = \{\lambda_F\}_{F \in \mathcal{F}}$ are convex multipliers for $\mathcal{F}$ if $\sum_{F \in \mathcal{F}} \lambda_F = 1$ and $\lambda_F \geq 0$ for $F \in \mathcal{F}$. We say $\{\lambda, \mathcal{F}\}$ *is a convex combination for* x if $\lambda = \{\lambda_F\}_{F \in \mathcal{F}}$ are the convex multipliers for $\mathcal{F}$ and $x = \sum_{F \in \mathcal{F}} \lambda_F$. We say x *can be written as convex combination of multigraphs in* $\mathcal{F}$ if can find such $\mathcal{F}$ and λ in polynomial time in the size of x. Here by the size of x we refer to $|E|$ (i.e., the number of edges in the support of x).

2.2.1 The v-Tree Polytope and Rainbow v-Trees

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, a v-tree is a subgraph F of G such that $|F \cap \delta(v)| = 2$ and $F \setminus \delta(v)$ induces a spanning tree of $V \setminus \{v\}$. Denote by v-$\textsc{Tree}(G)$ the convex hull of incidence vectors of v-trees of G. The v-$\textsc{Tree}(G)$ is characterized by the following linear inequalities.

$$\begin{aligned} v\text{-}\textsc{Tree}(G) =& \{x \in [0,1]^E : x(\delta(v)) = 2, \\ & x(E[U]) \leq |U| - 1 \text{ for all } \emptyset \subset U \subseteq V \setminus \{v\},\ x(E) = |V|\}. \end{aligned} \tag{2.1}$$

Observation 2.3. *We have* $\text{SEP}(K_n) \subseteq v\text{-}\textsc{Tree}(K_n)$ *for all* $v \in K_n$.

Observation 2.4. *Let $x \in \text{SEP}(K_n)$ be such G_x is 3-edge-connected and cubic. Let $\mathcal{C}$ be any 2-factor in G_x. Define vector y to have $y_e = \frac{1}{2}$ for $e \in \mathcal{C}$, $y_e = 1$ for $e \in E_x \setminus \{C\}$ and $y_e = 0$ otherwise. Then $y \in \text{SEP}(K_n) \subseteq v\text{-}\textsc{Tree}(K_n)$.*

Proof. Take $\emptyset \subset U \subset V_n$. If $U = \{v\}$ for some $v \in V_n$, then $x(\delta(U) = 2$ as $\mathcal{C} \cap \delta(v) = 2$.

If $|\delta(U)| \geq 4$, then clearly $x(\delta(U)) \geq 2$. Otherwise, $|\delta(U)| = 3$. Since at most two edges in $\delta(U)$ belong to $\mathcal{C}$, there is at least one edge $e \in \delta(U)$ with $x_e = 1$. Hence, $x(\delta(U)) \geq 2$. Therefore, $x \in \text{SEP}(K_n)$. We have $x \in v\text{-}\textsc{Tree}(K_n)$ by Observation 2.3. □

It can be deduced from the discussion above that a vector x in the subtour elimination relaxation can be written as a convex combination of v-trees for any vertex v in G_x. In fact, the v-trees in this convex combination can satisfy some additional properties.

Definition 4. *Let $G = (V, E)$ and v be a vertex of G. Let $\mathcal{P}$ be a collection of disjoint subsets of E. A $\mathcal{P}$-rainbow v-tree, namely T, is a v-tree of G such that $|T \cap P| = 1$ for $P \in \mathcal{P}$.*

The following theorem can be proved via the matroid intersection theorem [Edm70] and Observation 2.3.

Theorem 2.5 ([BL95],[BS19])**.** *Let $x \in \text{SEP}(K_n)$ and $\mathcal{P}$ be a collection of disjoint subsets of E_x such that $x(P) = 1$ for $P \in \mathcal{P}$. Then x can be written as a convex combination of $\mathcal{P}$-rainbow v-trees of K_n for any $v \in V_n$.*

Grötchel and Padberg [GP85] observed that v-trees of a connected graph $G = (V, E)$ satisfy the basis axioms of a matroid. For $x \in \text{SEP}(K_n)$ we have $x \in v\text{-Tree}(K_n)$ by Observation 2.3. Also, $\mathcal{P}$ defines a partition matroid where each base intersect each part of $\mathcal{P}$ exactly once. Therefore, vector x is in the convex hull of incidence vector of common basis of the partition matroid defined by $\mathcal{P}$ and the matroid whose basis are the v-trees of K_n.

2.2.2 The O-join Polytope

Let $G = (V, E)$ be a graph and $O \subseteq V$ where $|O|$ is even. An O-join of G is a subgraph J of G where a vertex $v \in V$ has odd degree in J if and only if $v \in O$. Let $O\text{-Join}(G)$ be the convex hull of incidence vectors of O-joins of G. Edmonds and Johnson [EJ73] showed the following description for the $O\text{-Join}(G)$.

$$O\text{-Join}(G) = \{z \in [0,1]^E \ : z(\delta(U) \setminus A) - z(A) \geq 1 - |A| \quad \text{for } U \subseteq V, A \subseteq \delta(U), |U \cap O| + |A| \text{ odd}\}. \tag{2.2}$$

Observation 2.6. *If $x \in \text{SEP}(K_n)$, then $\frac{x}{2} \in O\text{-Join}(K_n)$ for any $O \subseteq V_n$ with $|O|$ odd.*

Proof. Let $z = \frac{x}{2}$. For $U \subseteq V_n$, we have $z(\delta(U)) \geq 1$. Moreover, for $e \in E_n$, we have $z_e \leq \frac{1}{2}$ since $x_e \leq 1$. Therefore, for $A \subseteq \delta(U)$ we have $z(A) \leq \frac{|A|}{2}$. This implies $z(\delta(U)) - 2z(A) \geq 1 - |A|$. □

The following observation shows that a convex combination of O-joins in a graph has the property that for each vertex $u \in O$, exactly one edge incident to u belongs to an O-join that we obtain from 2.2.

Observation 2.7. *Let $G = (V, E)$ be a graph, and let $O \subseteq V$ be a subset of vertices such that $|O|$ is even. Let $z \in O\text{-Join}(G)$, and $z(\delta(u)) \leq 1$ for all $u \in V$. Then z can be written as convex combination of O-joins of G denoted by $\{\psi, \mathcal{J}\}$ such that for $u \in O$ we have $|J \cap \delta(u)| = 1$ for $J \in \mathcal{J}$.*

Proof. By [EJ73] since $z \in O\text{-}\textsc{Join}(G)$ it can be written as a convex combination of O-joins of G denoted by $\{\psi, \mathcal{J}\}$. Let u be a vertex in O. We have $z(\delta(u)) \leq 1$. On the other hand, for every $J \in \mathcal{J}$ we have $|J \cap \delta(u)| \geq 1$. Therefore, $|J \cap \delta(u)| = 1$. □

2.2.3 Proof of Theorem 1.1: Polyhedral Analysis of Christofides

Now, we are ready to prove Theorem 1.1.

Theorem 1.1 (Polyhedral proof of Christofides algorithm [Chr76, Wol80])**.** *If* $x \in \text{SEP}(K_n)$, *then* $\frac{3}{2}x \in \text{TSP}(K_n)$.

Proof. Let $x \in \text{SEP}(K_n)$, then by Observation 2.3, $x \in v\text{-}\textsc{Tree}(K_n)$ for some $v \in V_n$. Hence, we can find v-trees $\mathcal{T}$ and convex multipliers λ for $\mathcal{T}$ such that $x \leq \sum_{T \in \mathcal{T}} \lambda_T \chi^T$. For each $T \in \mathcal{T}$, let O_T be the set of odd degree vertices of T. Notice that $\frac{x}{2} \in O_T\text{-}\textsc{Join}(K_n)$ for all $T \in \mathcal{T}$. This implies that $\frac{x}{2}$ can be written as a convex combination of O_T-joins $\mathcal{J}^T$ of K_n with convex multipliers $\theta = \{\theta_J\}_{J \in \mathcal{J}^T}$. Notice that for $T \in \mathcal{T}$ and $J \in \mathcal{J}^T$, multigraph $T + J$ is a tour of K_n. Hence, $\sum_{T \in \mathcal{T}} \lambda_T \sum_{J \in \mathcal{J}^T} \theta_J \chi^{T+J} \in \text{TSP}(K_n)$. Therefore, $\frac{3}{2}x \in \text{TSP}(K_n)$ polyhedron. □

3 Matching Patterns: Gluing Tours Over Critical Cuts

Let x be a θ-cyclic point and $G_x = (V_n, E_x)$ be the support of x. In this section, we present an approach for gluing tours over critical cuts of G_x. One property of the tours we construct, which is crucial to enable this gluing procedure, is that every tour contains at least one copy of each 1-edge. This allows us to assume that x belongs to a subclass of θ-cyclic points in which (i) G_x is cubic, (ii) W_x, the 1-edges of G_x, form a perfect matching, and (iii) H_x, the fractional edges of G_x, form a 2-factor. We can make this assumption, because we can contract a path of 1-edges to a single 1-edge; the tour of the new cubic graph yields a tour for the original subcubic graph. We work under this assumption throughout this section and in Section 4.

For a vertex $u \in V$, denote by e_u the unique 1-edge in G_x that is incident on u. Let $\delta(u) = \{e_u, f_u, g_u\}$ where f_u and g_u are the two fractional edges incident on u and $x_{f_u} = \theta$ and $x_{g_u} = 1 - \theta$. In each tour of G_x, a multiset of edges from $\delta(u)$ belongs to the tour. We call this multiset *the pattern around* u. Denote by $\mathbb{P}_u$ the set of possible patterns around a vertex u in a tour of G_x that contain at least one copy of the 1-edge e_u and in which u has degree either two or four (see Figure 3.1).

$$\mathbb{P}_u = \{\{2e_u\}, \{e_u, f_u\}, \{e_u, g_u\}, \{2e_u, 2f_u\}, \{2e_u, 2g_u\}, \{2e_u, f_u, g_u\}, \{e_u, 2f_u, g_u\}, \{e_u, f_u, 2g_u\}\}.$$

In every tour we construct, the pattern around each vertex $u \in V$ will be some pattern from $\mathbb{P}_u$. There are other multisets of $\delta(u)$ that can be valid patterns around u in a tour. For example, the pattern $\{f_u, g_u\}$ could be the pattern around u in some tour. However, in our construction this pattern will never be the pattern around u as we always include at least one copy of e_u in a tour. Formally, we write $\delta_F(u) = p$ if the pattern around u in the tour F is $p \in \mathbb{P}_u$.

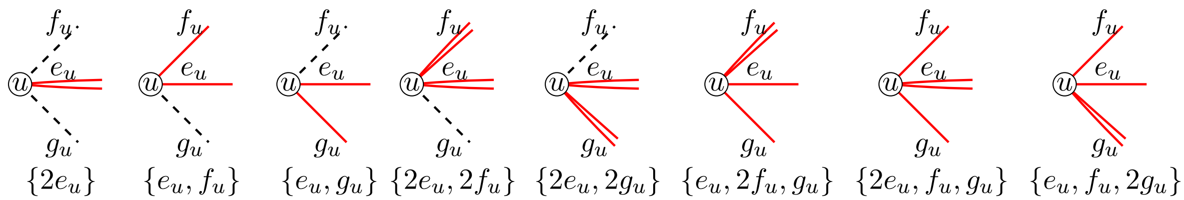

Figure 3.1: The different patterns in $\mathbb{P}_u$. The red solid edges are in the tour and black dashed edges are not used in the handpicked tour.

Definition 5. *A tour F of $G_x = (V_n, E_x)$ is a* handpicked tour of G_x *if for all $u \in V_n$, the pattern around u in F belongs to $\mathbb{P}_u$ (i.e., if $\delta_F(u) \in \mathbb{P}_u$ for all $u \in V_n$).*

If $\mathscr{F}$ is a set of tours and $\phi = \{\phi_F\}_{F \in \mathscr{F}}$ is a set of convex multipliers, then we use $\phi(p) = \sum_{F \in \mathscr{F}:\delta_F(u)=p} \phi_F$ to denote the *pattern frequency* of $p \in \mathbb{P}_u$ in the convex combination $\{\phi, \mathscr{F}\}$. Notice that $\phi(p) \in [0, 1]$. Moreover, if $\mathscr{F}$ is a set of handpicked tour of G_x, then for all $u \in V_n$, we have $\sum_{p \in \mathbb{P}_u} \phi(p) = 1$. For each $u \in V_n$, we define the *pattern profile* of vertex u in the convex combination $\{\phi, \mathscr{F}\}$ to be the eight values $\{\phi(p)\}$ for all $p \in \mathbb{P}_u$.

Another key parameter of a convex combination is the frequency of doubled edges. For the convex combination $\{\phi, \mathscr{F}\}$, define $\phi_2(e) = \sum_{F \in \mathscr{F}:\chi^F_e=2} \phi_F$ for all $e \in E_x$. The pattern profile of a vertex u turns out to be directly related to the occurence of doubled edges from $\delta(u)$. This dependence is formalized in the next observation, which states that for a convex combination of handpicked tours, if the parameters $\phi_2(e)$ are fixed for $e \in \delta(u)$, then the pattern profile for each vertex u depends only on the pattern frequency of the pattern $\{2e_u\}$.

Observation 3.1. *Let y and q be vectors in $\mathbb{R}^{E_x}_{\geq 0}$. Suppose y can be written as a convex combination of tours of G_x denoted by $\{\phi, \mathscr{F}\}$ such that for all $e \in E_x$, we have $\phi_2(e) = q_e$.*

Then for each vertex $u \in V_n$ and for each pattern $p \in \mathbb{P}_u$, the frequency of pattern p, $\phi(p)$, in this convex combination is uniquely determined by the frequency of pattern $\{2e_u\}$.

Proof. Suppose $\phi(\{2e_u\}) = \zeta_u$ for some $\zeta_u \in [0, 1]$. Then for each $u \in V_n$, the following identities hold with respect to the convex combination $\{\phi, \mathscr{F}\}$.

$$\sum_{p \in \mathbb{P}_u:\chi^p_e=2} \phi(p) = q_e \qquad \text{for } e \in \{e_u, f_u, g_u\},$$

$$\sum_{p \in \mathbb{P}_u:\chi^p_e=1} \phi(p) = y_e - 2q_e \qquad \text{for } e \in \{e_u, f_u, g_u\},$$

$$\sum_{p \in \mathbb{P}_u} \phi(p) = 1,$$

$$\phi(\{2e_u\}) = \zeta_u.$$

Since the above system of eight equations has eight variables (i.e., the variables are $\phi(p)$ for $p \in \mathbb{P}_u$), it has a unique solution. Therefore, $\phi(p)$ is a function of ζ_u for all $p \in \mathbb{P}_u$. $\square$

We apply Observation 3.1 to control the pattern profile of a pseudovertex u by constructing tours in which the pattern frequency of $\{2e_u\}$ can be set arbitrarily. This enables us to prove Theorem 1.3 with an inductive (gluing) approach. For such an approach to work, we need to prove a stronger statement. Let α be a constant in $(0, 1]$ that we will fix later.

Proposition 3.2. *Define $y \in \mathbb{R}^{E_x}_{\geq 0}$ as follows: $y_e = \frac{3}{2} - \frac{\alpha\theta}{2}$ for $e \in W_x$ and $y_e = \frac{3}{2}x_e$ for $e \in H_x$. Then y can be written as a convex combination of handpicked tours of G_x denoted by $\{\phi, \mathscr{F}\}$ such that*

(i) $\phi_2(e) = \frac{1}{2} - \frac{\alpha\theta}{2}$, for $e \in W_x$, and

(ii) $\phi_2(e) = \frac{x_e^2}{2}$ for all $e \in H_x$.

Observe that Proposition 3.2 implies Theorem 1.3. As mentioned previously, it is in fact stronger; we construct tours for the base cases (θ-cyclic points whose support graphs have no critical cuts) and the additional properties in Proposition 3.2 enable us to "glue" these tours together over the critical cuts of G_x. Hence, our induction is on the number of critical cuts in G_x.

Observation 3.3 gives sufficient conditions under which we can glue tours of G_x^U and $G_x^{\overline{U}}$ together over the critical cut to obtain tours for G_x that preserve key properties.

Observation 3.3. *Let $U \subset V_n$ be a critical cut of G_x. Suppose x^U can be written as a convex combination of handpicked tours of G_x^U denoted by $\{\phi^U, \mathscr{F}^U\}$ with the following properties. (And suppose the same holds for $x^{\overline{U}}$, $G_x^{\overline{U}}$ $\{\phi^{\overline{U}}, \mathscr{F}^{\overline{U}}\}$, respectively.)*

(i) The pattern profiles of vertices v_U and $v_{\overline{U}}$ are the same in their respective convex combinations.

(ii) For every tour $F \in \mathscr{F}^U$, $F \setminus \delta(v_{\overline{U}})$ induces a connected multigraph on U.

Then x can be written as a combination of handpicked tour of G_x denoted by $\{\phi, \mathscr{F}\}$ such that

(a) $\phi_2(e) = \phi_2^U(e)$ for $e \in E(G_x^U)$,

(b) $\phi_2(e) = \phi_2^{\overline{U}}(e)$ for $e \in E(G_x^{\overline{U}})$, and

(c) $|\mathscr{F}| \leq |\mathscr{F}^U| + |\mathscr{F}^{\overline{U}}|$.

Proof. First, we prove the following simple claim.

Claim 1. *Consider a graph $G = (V, E)$ and nonempty $U \subset V$ such that U is a 3-edge-cut in $G = (V, E)$. Let F_U be a tour in G^U and let $F_{\overline{U}}$ be a tour in $G^{\overline{U}}$ such that $\chi_e^{F_U} = \chi_e^{F_{\overline{U}}}$ for $e \in \delta(U)$. Moreover, assume that $F_U \setminus \delta(v_{\overline{U}})$ induces a connected multigraph on U. Then the multiset of edges F defined as $\chi_e^F = \chi_e^{F_U}$ for $e \in E(G^U)$ and $\chi_e^F = \chi_e^{F_{\overline{U}}}$ for $e \in E(G^{\overline{U}})$ is a tour of G.*

Proof. It is clear that F induces an Eulerian spanning multigraph on G, but we need to ensure that F is connected. For example, the tour induced on $F_{\overline{U}} \setminus \delta(v_U)$ might not be connected. However, since the subgraph of F_U induced on the vertex set U is connected, the tour F is connected: each vertex in $\overline{U}$ is connected to some vertex in U. $\diamondsuit$

We observe that if the pattern profiles of v_U and $v_{\overline{U}}$ with respect to the convex combinations $\{\phi^U, \mathscr{F}^U\}$ and $\{\phi^{\overline{U}}, \mathscr{F}^{\overline{U}}\}$, respectively, are the same, then we can always find two tours $F_U \in \mathscr{F}$ and $F_{\overline{U}} \in \mathscr{F}^{\overline{U}}$ such that the pattern around $v_{\overline{U}}$ in F_U is the same as the pattern around v_U in $F_{\overline{U}}$. We can apply Claim 1 to obtain a new tour F, to which we assign convex multiplier $\phi_F = \min\{\phi^U_{F_U}, \phi^{\overline{U}}_{F_{\overline{U}}}\}$ and add to set $\mathscr{F}$. Then we subtract ϕ_F from each of these convex multipliers, remove tours with convex multipliers zero from $\mathscr{F}^U$ and $\mathscr{F}^{\overline{U}}$, and repeat. Observe the total number of tours in $\mathscr{F}$ is at most $|\mathscr{F}^U| + |\mathscr{F}^{\overline{U}}|$.

We need to show that each tour $F \in \mathscr{F}$ is handpicked. This follows from the fact that the pattern around each vertex in U in F is the same as the pattern around it in F_U. Moreover, each edge $e \in E_x \cap \delta(U)$ is doubled in a tour F of G_x iff it is doubled in both F_U and in $F_{\overline{U}}$. For $e \in E(G^U_x) \setminus \delta(U)$, edge e is doubled iff it is doubled in F_U. Analogously, each vertex in $\overline{U}$ has the same pattern in F as it has in $F_{\overline{U}}$, and each edge $e \in E(G^{\overline{U}}_x) \setminus \delta(U)$ is doubled iff it is doubled in $F_{\overline{U}}$. Thus, properties (a) and (b) hold for the convex combination $\{\phi, \mathscr{F}\}$. $\square$

In the base case (where graphs $G_x = (V_n, E_x)$ have no critical cuts), each tour in the convex combination that we construct consists of a connector plus a parity correction (as we will describe in Section 4). For each $u \in V_n$, the previously introduced (and yet to be fixed) parameter α is a lower bound on the fraction of connectors in this convex combination in which u has degree two. Let the parameter η_u denote the (exact) fraction of connectors in which vertex u has degree one. (Note that η_u can be different for every vertex.) In our construction of a convex combination of tours $\{\phi, \mathscr{F}\}$ for the base case, it will be the case that the frequency of pattern $\{2e_u\}$ will be equal to $\frac{\eta_u}{2}$. Thus, controlling the value of η_u allows us to control $\phi(\{2e_u\})$. A key technical tool is that when we construct a convex combination of tours $\{\phi, \mathscr{F}\}$ for graph G_x with no critical cuts (i.e., a base case graph), we can always ensure that for one arbitrarily chosen vertex v, the value of η_v (and hence $\phi(\{2e_v\})$) can be chosen arbitrarily. This allows us to ensure that the pattern frequency of $\{2e_{v_{\overline{U}}}\})$ equals the pattern frequency of $\{2e_{v_U}\})$ in the given convex combination for $G^{\overline{U}}_x$ (i.e., Condition (i) in Observation 3.3) when G^U_x is a base case.

Lemma 3.4. *Suppose G_x contains no critical cuts. Fix any vertex $v \in V$ and fix constant ζ with $0 \le \zeta \le \frac{(1-\alpha)\theta}{2}$. Define $y \in \mathbb{R}^{E_x}_{\ge 0}$ as follows: $y_e = \frac{3}{2} - \frac{\alpha\theta}{2}$ for $e \in W_x$ and $y_e = \frac{3}{2}x_e$ for $e \in H_x$. Then y can be written as a convex combination of handpicked tours of G_x denoted by $\{\phi, \mathscr{F}\}$ with the following properties.*

(i) $\phi_2(e) = \frac{1}{2} - \frac{\alpha\theta}{2}$, *for* $e \in W_x$,

(ii) $\phi_2(e) = \frac{x_e^2}{2}$ *for all* $e \in H_x$,

(iii) $\phi(\{2e_v\}) = \zeta$, *and*

(iv) $F \setminus \delta_F(v)$ *induces a connected multigraph on* $V \setminus v$ *for each* $F \in \mathscr{F}$.

Notice that Lemma 3.4 implies Proposition 3.2 for θ-cyclic points whose support graphs have no critical cuts. We prove Lemma 3.4 in the next section. In the remainder of this section, we show how Lemma 3.4 implies Proposition 3.2.

Proof of Proposition 3.2. Suppose G_x contains t critical cuts. We prove the statement by induction on t. In fact we show that the running time of our algorithm is polynomial in n and t. To this end, we show that the convex combination in our construction contains at most btn^d trees, where b and d are constants. Notice that t itself is a polynomial bounded by n.

If $t = 0$, then G_x does not contain a critical cut, then apply Lemma 3.4. Otherwise, find the minimal critical cut U of G_x. By Observations 2.1 and 2.2, graph G_x^U does not contain any critical cuts and $G_x^{\overline{U}}$ contains at most $t-1$ critical cuts.

Define $y^{\overline{U}}$ as follows: $y_e^{\overline{U}} = \frac{3}{2} - \frac{\alpha\theta}{2}$ for $e \in W_{x^{\overline{U}}}$ and $y_e^{\overline{U}} = \frac{x_e^{\overline{U}}}{2}$ for $e \in H_{x^{\overline{U}}}$. We apply the induction hypothesis on $G_x^{\overline{U}}$ to write $y^{\overline{U}}$ as convex combination of handpicked tours of $G_x^{\overline{U}}$ denoted by $\{\phi^{\overline{U}}, \mathscr{F}^{\overline{U}}\}$ such that (i) $\phi_2^{\overline{U}}(e) = \frac{1}{2} - \frac{\alpha\theta}{2}$ for $e \in W_{x^{\overline{U}}}$, and (ii) $\phi_2(e) = \frac{(x_e^{\overline{U}})^2}{2}$ for all $e \in H_{x^{\overline{U}}}$. By induction $|\mathscr{F}^{\overline{U}}| \leq b(t-1)(|\overline{U}|+1)^d$.

Let $\zeta^* = \phi^{\overline{U}}(\{e_{v_U}\})$. Define $y_e^U = \frac{3}{2} - \frac{\alpha\theta}{2}$ for $e \in W_{x^U}$ and $y_e^U = \frac{x_e^U}{2}$ for $e \in H_{x^U}$. Applying Lemma 3.4 to θ-cyclic point x^U with $\zeta = \zeta^*$ we can write y^U as convex combination of handpicked tours of G_x^U denoted by $\{\phi^U, \mathscr{F}^U\}$ such that (i) $\phi_2^U(e) = \frac{1}{2} - \frac{\alpha\theta}{2}$ for $e \in W_{x^U}$, (ii) $\phi_2^U(e) = \frac{(x_e^U)^2}{2}$ for all $e \in H_{x^U}$, (iii) $\phi^U(\{2e_v\}) = \zeta^*$, and (iv) $F \setminus \delta_F(v_{\overline{U}})$ induces a connected multigraph on U for each $F \in \mathscr{F}^U$. Moreover $|\mathscr{F}^U| \leq b(|U|+1)^d$.

Now we apply Observation 3.3 to write y as the desired convex combination. This convex combination contains at most $|\mathscr{F}^U| + |\mathscr{F}^{\overline{U}}| \leq b(|U|+1)^d + b(t-1)(|\overline{U}|+1)^d$. Note that for $d \geq 2$ this number is at most btn^d, as $|U| + |\overline{U}| = n$, $|U| \geq 3$, and $|\overline{U}| \geq 3$. □

4 Finding Tours in the Base Case: Proof of Lemma 3.4

In this section we present the proof of Lemma 3.4. We fix $G_x = (V_n, E_x)$ to be the support graph of a θ-cyclic point x. In the base case, G_x contains no critical cuts. (The lemmas in this section will be applied to this base case, but some hold even when G_x is not a base case.) Moreover, we remind the reader that we assume the support of G_x is cubic. (See discussion in the beginning of Section 3.)

A key tool we will use is that the 1-edges of G_x can be partitioned into five induced matchings in G_x. A set $M \subset W_x$ is an *induced matching of* G_x if M is a vertex induced subgraph of G_x and M is a matching. For each induced matching M, we find a set of connectors $\mathcal{T}$ of G_x where for each 1-edge e in M, both endpoints of e have degree two in every $T \in \mathcal{T}$. Thus, when G_x has no critical cuts, a 1-edge e in M does not belong to any odd cuts in T that are tight cuts in G_x.

For each 1-edge e in M, we can therefore reduce usage of e in the parity correction from $\frac{1}{2}$ to $\frac{1-\theta}{2}$; each 1-edge saves $\frac{\theta}{2}$ exactly $\frac{1}{5}$ of the times. This yields the saving of $\frac{\theta}{10}$ on the 1-edges as stated in Lemma 3.4 with $\alpha = \frac{1}{5}$.

The induced matchings require some additional properties that we need for technical reasons as we will see later. Recall that for a vertex u in G_x we denote by e_u the unique 1-edge incident on u. Let $N_2(u)$ denote the two vertices that are the other endpoints of the fractional-edges incident on u. In other words, suppose $\delta(u) = \{e_u, f_u, g_u\}$ and suppose that w_1 and w_2 are the other endpoints of f_u and g_u, respectively. Then $N_2(u) = \{w_1, w_2\}$. The proof of Lemma 4.1 is deferred to Section 4.5.

Lemma 4.1. *Suppose G_x has no critical cuts. Let v be a vertex in V_n and let $N_2(v) = \{w_1, w_2\}$. The set of 1-edges in G_x, W_x, can be partitioned into five induced matchings $\{M_1, \dots, M_5\}$ such that for $i \in [5]$, the following properties hold.*

(i) $|M_i \cap \{e_v, e_{w_1}, e_{w_2}\}| \leq 1$.

(ii) For $U \subseteq V_n$ such that $|\delta(U)| = 3$, $|\delta(U) \cap M_i| \leq 1$.

(iii) For $U \subseteq V_n$ such that $|\delta(U)| = 2$, $|\delta(U) \cap M_i|$ is even.

For the rest of this section, let v be a fixed vertex in V_n, $N_2(v) = \{w_1, w_2\}$ and let $\{M_1, \dots, M_h\}$ denote the partition of W_x into induced matchings with the additional properties enumerated in Lemma 4.1. These properties will be used to ensure that we can save on the edges in W_x when augmenting connectors of G_x (with parity correctors) into tours. While Lemma 4.1 implies that $h = 5$, we will use h throughout this section, since if Lemma 4.1 could be proved with, say, four matchings, it would allow a larger value of $\alpha = \frac{1}{h}$ and hence directly yield a better bound in the statement of Lemma 3.4.

The proof of Lemma 3.4 consists of two main parts. First we show there is a convex combination of connectors of G_x that satisfy certain properties. Second, we show that for each connector, we can find parity correctors such that the union of a connector and a parity corrector is a tour.

4.1 Constructing Connectors

Now we will show how to construct connectors for G_x. We will apply this when G_x is a base case (i.e., G_x has no critical cuts), but Definition 6 and Lemmas 4.2 and 4.3 apply even when this is not the case. Recall that v is fixed vertex in V_n and that $N_2(v) = \{w_1, w_2\}$.

Definition 6. *Suppose $M \subset W_x$ is a subset of 1-edges of G_x. Let Λ be a constant such that $0 \leq \Lambda \leq \theta$. Suppose x can be written as a convex combination of connectors of G_x denoted by $\{\lambda, \mathcal{T}\}$. Then we say $P(v, M, \Lambda)$ holds for $\{\lambda, \mathcal{T}\}$ if it has the following properties.*

1. $\sum_{T \in \mathcal{T}: |\delta_T(v)|=1} \lambda_T = \sum_{T \in \mathcal{T}: |\delta_T(v)|=3} \lambda_T = \Lambda$ *and* $\sum_{T \in \mathcal{T}: |\delta_T(v)|=2} \lambda_T = 1 - 2\Lambda$.

2. *For each edge $st \in M$, $|\delta_T(s)| = |\delta_T(t)| = 2$ for all $T \in \mathcal{T}$.*

3. *$T \setminus \delta_T(v)$ induces a connector on $V \setminus \{v\}$.*

Let us explain why the properties described above are useful in our construction. The first property allows us to control the fraction of time vertex v has degree one in a connector in the convex combination $\{\lambda, \mathcal{T}\}$, which in turn will allow us to control the fraction of time a tour has the pattern $\{2e_v\}$ around v. This flexibility is required to perform the gluing procedure; it allows us to manipulate the convex combination of connectors to have the desired pattern profile for the pseudovertex (which will be v). The second condition ensures that no 1-edge in M is part of a tight cut that is crossed an odd number of times in a connector $T \in \mathcal{T}$. Lastly, the third property guarantees that we maintain connectivity of the tours after gluing them together over critical cuts.

We defer the proofs of the next two lemmas to Section 4.3.

Lemma 4.2. *Suppose $M \subset W_x$ forms an induced matching in G_x and edge $e_v \in M$. Then x can be written as a convex combination of connectors of G_x denoted by $\{\lambda, \mathcal{T}\}$ for which $P(v, M, 0)$ holds.*

Lemma 4.3. *Let Λ be any constant such that $0 \leq \Lambda \leq \theta$. Suppose $M \subset W_x$ forms an induced matching in G_x, $e_v \notin M$ and $|M \cap \{e_{w_1}, e_{w_2}\}| \leq 1$. Then x can be written as a convex combination of connectors of G_x denoted by $\{\lambda, \mathcal{T}\}$ for which $P(v, M, \Lambda)$ holds.*

Recall that $\{M_1, \ldots, M_h\}$ is the partition of W_x into induced matchings obtained via Lemma 4.1. Assume without loss of generality that $e_v \in M_1$. For $i = 1$, let $\mathcal{T}_1$ be a set of connectors of G_x and let $\{\vartheta, \mathcal{T}_1\}$ be a convex combination for x for which $P(v, M_1, 0)$ holds (by Lemma 4.2). For $i \in \{2, \ldots, h\}$, let $\mathcal{T}_i$ be a set of connectors of G_x and let $\{\vartheta, \mathcal{T}_i\}$ be a convex combination for x for which $P(v, M_i, \frac{\Lambda}{1-\alpha})$ holds (by Lemma 4.3). Notice that $\frac{\Lambda}{1-\alpha} \leq \theta$ since $\Lambda \leq (1-\alpha)\theta$.

We can write x as a convex combination of connectors from $\mathscr{T}$, by weighting each set $\mathcal{T}_i$ by α. In particular, we have $x = \alpha \sum_{i=1}^{h} \sum_{T \in \mathcal{T}_i} \vartheta_T \chi^T$. For each $T \in \mathscr{T}$, let $\sigma_T = \alpha \cdot \vartheta_T$. Then $\{\sigma, \mathscr{T}\}$ is a convex combination for x. Observe that since $x_e = 1$ for $e \in W_x$, we have $W_x \subseteq T$ for $T \in \mathscr{T}$. From Definition 6 and Lemmas 4.2 and 4.3, we observe the following.

Claim 2. *For each $T \in \mathscr{T}$, $T \setminus \delta(v)$ induces a connected, spanning subgraph on $V \setminus \{v\}$.*

4.2 Constructing Parity Correctors

For each $T \in \mathscr{T}$, let O_T be the set of odd degree vertices of T. In the second part of the proof we show that each connected subgraph $T \in \mathscr{T}$ has a "cheap" convex combination of O_T-joins.

Lemma 4.4. *Suppose G_x has no critical cuts. Let $M \subset W_x$ be a subset of 1-edges of G_x such that each 3-edge cut in G_x contains at most one edge from M. Let $O \subseteq V$ be a subset of vertices* *such that $|O|$ is even and for all $e = st \in M$, neither s nor t is in O. Also suppose for any set $U \subseteq V$ such that $|\delta(U)| = 2$, both $|U \cap O|$ and $|\delta(U) \cap M|$ are even. Define vector z as follows: $z_e = \frac{1}{2}$ if $e \in W_x$ and $e \notin M$, $z_e = \frac{1-\theta}{2}$ if $e \in M$, and $z_e = \frac{x_e}{2}$ if $e \in H_x$. Then vector $z \in O$-*Join*(G_x).*

For each $i \in [h]$, define $z_e^i = \frac{1-\theta}{2}$ if $e \in M_i$ and $z_e^i = \frac{x_e}{2}$ otherwise. For each $T \in \mathcal{T}_i$, let $O_T \subseteq V$ be the set of odd-degree vertices of T. By construction, we have $V(M_i) \cap O_T = \emptyset$. By Lemma 4.4, we have $z^i \in O_T$-Join(G_x). So by Observation 2.7 we can write z^i as a convex combination of O_T-joins of G_x denoted by $\{\psi^T, \mathcal{J}_T\}$ where $|J \cap \delta(u)| = 1$ for $u \in O_T$ and $J \in \mathcal{J}_T$. This implies that $x + z^i$ can be written as a convex combination of tours of G_x. We denote this set of tours by $\mathcal{F}_i$ and we let $\mathscr{F} = \{F \in \mathcal{F}_i \ : \ i \in \{1, \ldots, h\}\}$. Now for $F \in \mathscr{F}$ we have $F = T + J$ for some $T \in \mathscr{T}$ and $J \in \mathcal{J}_T$. Define $\phi_F = \sigma_T \cdot \psi_J^T$. The vector $\sum_{i=1}^h \alpha(x + z^i)$ can be written as $\{\phi, \mathscr{F}\}$.

Claim 3. *Every tour in $\mathscr{F}$ is handpicked.*

Proof. Let $F \in \mathscr{F}$. By construction, $F = T + J$, where $T \in \mathscr{T}$ and $J \in \mathcal{J}_T$. Let $u \in V_n$ and $\delta(u) = \{e_u, f_u, g_u\}$. Notice that $\chi_{e_u}^F \geq 1$, since $e_u \in T$ for all $T \in \mathscr{T}$. Hence, we only need to show that $|\delta_F(u)| < 6$. Suppose for contradiction that $|\delta_F(u)| \geq 6$. This implies that $|\delta_T(u)| = 3$ and $|\delta_J(u)| = 3$. However, if $|\delta_T(u)| = 3$, then $u \in O_T$. From Observation 2.7, if $u \in O_T$ and $z(\delta(u)) \leq 1$, then $|J \cap \delta(u)| = 1$ which is a contradiction. $\Diamond$

Claim 4. *Suppose G_x contains no critical cuts. Define vector $y \in \mathbb{R}^{E_x}$ as $y_e = \frac{3}{2} - \frac{\alpha\theta}{2}$ for $e \in W_x$ and $y_e = \frac{3}{2}x_e$ for $e \in H_x$. Then $\{\phi, \mathscr{F}\}$ is a convex combination for y.*

Proof. We need to show that $y = \sum_{i=1}^h \alpha(x + z^i)$. First, let e be a 1-edge of G_x and M_j be the induced matching that contains e. Then, $x_e = 1$, $z_e^i = \frac{1}{2}$ for $i \in [h] \setminus \{j\}$ and $z_e^j = \frac{1-\theta}{2}$. Hence,

$$\sum_{i=1}^h \alpha(x_e + z_e^i) = \sum_{\ell=1}^h \alpha \cdot \frac{3}{2} - \alpha \cdot \frac{\theta}{2} = \frac{3}{2} - \frac{\alpha\theta}{2}.$$

For a fractional e of G_x, we have $z_e^i = \frac{x_e}{2}$ for $i \in [h]$, so $\sum_{i=1}^h \alpha(x_e + z_e^i) = \frac{3}{2}x_e$. $\Diamond$

Now we prove some additional useful properties of the convex combination $\{\phi, \mathscr{F}\}$ for θ-cyclic point x.

Claim 5. *For convex combination $\{\phi, \mathscr{F}\}$, we have $\phi_2(e) = \frac{1}{2} - \frac{\alpha\theta}{2}$ for $e \in W_x$ and $\phi_2(e) = \frac{(x_e)^2}{2}$.*

Proof. Notice that ϕ defines a probability distribution on $\mathscr{F}$. We sample F from $\mathscr{F}$ with the probabilities defined by ϕ. Recall that $\phi_2(e) = \Pr[e \text{ is doubled in } F]$. Moreover, recall that $F = T + J$ where $T \in \mathscr{T}$ and $J \in \mathcal{J}_T$ and T is associated with some matching $M \in \{M_1, \ldots, M_h\}$ (i.e., $T \in \mathcal{T}_i$ for $i \in [h]\}$). For $e \in E_x$, we have

$$\Pr[e_u \text{ is doubled in } F\ ] = \Pr[e_u \in J \text{ and } e_u \in T] = \Pr[e_u \in T] \cdot \Pr[e_u \in J].$$

For $e \in H_x$, this implies that $\phi_2(e) = x_e \cdot \frac{x_e}{2} = \frac{(x_e)^2}{2}$. Also

$$\begin{aligned}\Pr[e_u \text{ is doubled in } F\,] &= \Pr[e_u \in J] \\ &= \Pr[e_u \in J | e_u \in M] \cdot \Pr[e_u \in M] + \Pr[e_u \in J | e_u \notin M] \cdot \Pr[e_u \notin M] \\ &= \frac{1-\theta}{2} \cdot \alpha + \frac{1}{2} \cdot (1-\alpha) \\ &= \frac{1}{2} - \frac{\alpha\theta}{2}.\end{aligned}$$

$\Diamond$

Claims 2, 4 and 5 yield Lemma 3.4. It remains to prove Lemmas 4.2, 4.3 and 4.4.

4.3 Proofs of Lemmas 4.2 and 4.3 for Constructing Connectors

In this section, we prove Lemmas 4.2 and 4.3, which are necessary in order to write a θ-cyclic point x as a convex combination of connectors with property P described in Definition 6.

Lemma 4.2. *Suppose $M \subset W_x$ forms an induced matching in G_x and edge $e_v \in M$. Then x can be written as a convex combination of connectors of G_x denoted by $\{\lambda, \mathcal{T}\}$ for which $P(v, M, 0)$ holds.*

Proof. For each $st \in M$, pair the half-edges incident on s and pair those incident on t to obtain disjoint subsets of edges $\mathcal{P}$. Decompose x into a convex combination of $\mathcal{P}$-rainbow v-trees $\mathcal{T}$ (i.e., $x = \sum_{T \in \mathcal{T}} \lambda_T \chi^T$) via Theorem 2.5. This is the desired convex combination since for all $T \in \mathcal{T}$, we have $|\delta_T(v)| = 2$ and $|\delta_T(u)| = 2$ for all endpoints u of edges in M. Thus, the first and second conditions are satisfied. The third condition holds by definition of v-trees. $\square$

Lemma 4.3. *Let Λ be any constant such that $0 \leq \Lambda \leq \theta$. Suppose $M \subset W_x$ forms an induced matching in G_x, $e_v \notin M$ and $|M \cap \{e_{w_1}, e_{w_2}\}| \leq 1$. Then x can be written as a convex combination of connectors of G_x denoted by $\{\lambda, \mathcal{T}\}$ for which $P(v, M, \Lambda)$ holds.*

Proof. As in the proof of Lemma 4.2, for each $st \in M$, pair the half-edges incident on s and pair those incident on t to obtain a collection of disjoint subsets of edges $\mathcal{P}$. Apply Theorem 2.5 to obtain $\{\lambda, \mathcal{T}\}$ which is a convex combination for x, where $\mathcal{T}$ is a set of $\mathcal{P}$-rainbow v-trees (i.e., $x = \sum_{T \in \mathcal{T}} \lambda_T \chi^T$). Notice that this convex combination clearly satisfies the second requirement in Definition 6.

Now let $\delta(v) = \{e_v, f, g\}$, where w_1 and w_2 are the other endpoints of f and g, respectively. Assume $x_f = \theta$ and $x_g = 1 - \theta$. Since $x = \sum_{T \in \mathcal{T}} \lambda_T \chi^T$ and $x_{e_v} = 1$, we have $e_v \in T$ for $T \in \mathcal{T}$. In addition, we have $|\delta_T(v)| = 2$ for all $T \in \mathcal{T}$ by the definition of v-trees. Hence, $\sum_{T \in \mathcal{T}: f \in T, g \notin T} \lambda_T = \theta$ and $\sum_{T \in \mathcal{T}: f \notin T, g \in T} \lambda_T = x_f = 1 - \theta$. Define

$$\mathcal{T}_f = \{T \in \mathcal{T} : f \in T \text{ and } g \notin T\} \text{ and } \mathcal{T}_g = \{T \in \mathcal{T} : g \in T \text{ and } f \notin T\},$$

where $\mathcal{T}_f \cup \mathcal{T}_g = \mathcal{T}$ and $\mathcal{T}_f \cap \mathcal{T}_g = \emptyset$. We can also assume that there are subsets $\mathcal{T}_f^1 \subseteq \mathcal{T}_f$ and $\mathcal{T}_g^1 \subseteq \mathcal{T}_g$ such that $\sum_{T\in\mathcal{T}_f^1} \lambda_T = \Lambda$ and $\sum_{T\in\mathcal{T}_g^1} \lambda_T = \Lambda$, since $\Lambda \leq \theta$. Now we consider two cases

1. If $e_{w_1} \notin M$: For $T \in \mathcal{T}_f^1$, replace T with $T - f$. For $T \in \mathcal{T}_g^1$, replace T with $T + f$.
2. If $e_{w_1} \in M$: For $T \in \mathcal{T}_f^1$, replace T with $T + g$. For $T \in \mathcal{T}_g^1$, replace T with $T - g$.

For all $T \in \mathcal{T} \setminus (\mathcal{T}_f^1 \cup \mathcal{T}_g^1)$, keep T as is. Observe that $T \in \mathcal{T}$ is still a connector of G_x: for every $T \in \mathcal{T}_f$, $T - f$ is a spanning tree of G_x and for every $T \in \mathcal{T}_g$, $T - g$ is spanning tree of G_x. Observe that in the first case, the degree of e_{w_2} is preserved for every T, and in the second case the degree of e_{w_1} is preserved for every T. We want to show that the new convex combination $\{\lambda, \mathcal{T}\}$ is the desired convex combination for x. Notice that in the first case,

$$\begin{aligned}\sum_{T\in\mathcal{T}} \lambda_T \chi_f^T &= \sum_{T\in\mathcal{T}_f^1} \lambda_T \chi_f^T + \sum_{T\in\mathcal{T}_f\setminus\mathcal{T}_f^1} \lambda_T \chi_f^T + \sum_{T\in\mathcal{T}_g^1} \lambda_T \chi_f^T + \sum_{T\in\mathcal{T}_g\setminus\mathcal{T}_g^1} \lambda_T \chi_f^T \\ &= 0 + (\theta - \Lambda) + \Lambda + 0 = x_f.\end{aligned}$$

In the second case,

$$\begin{aligned}\sum_{T\in\mathcal{T}} \lambda_T \chi_g^T &= \sum_{T\in\mathcal{T}_f^1} \lambda_T \chi_g^T + \sum_{T\in\mathcal{T}_f\setminus\mathcal{T}_f^1} \lambda_T \chi_g^T + \sum_{T\in\mathcal{T}_g^1} \lambda_T \chi_g^T + \sum_{T\in\mathcal{T}_g\setminus\mathcal{T}_g^1} \lambda_T \chi_f^T \\ &= \Lambda + 0 + 0 + (1 - \theta - \Lambda) = x_g.\end{aligned}$$

So $x = \sum_{T\in\mathcal{T}} \lambda_T \chi^T$. Moreover, notice that for $T \in \mathcal{T}$, $T \setminus \delta_T(v)$ still induces a connector on $V \setminus \{v\}$ since we did not remove any edge in $T \setminus \delta(v)$ from the v-tree T. Finally, for each vertex s with $e_s \in M$, we have $|\delta_T(s)| = 2$ for all $T \in \mathcal{T}$. To observe this, notice that the initial convex combination satisfies this property for vertex s (since the convex combination is obtained via Theorem 2.5). In the transformation of the convex combination we only change edges incident on w_1 and w_2, so if $s \neq w_1, w_2$ the property clearly still holds after the transformation. If $s \in \{w_1, w_2\}$, then as noted previously, we do not remove or add an edge incident on s if $e_s \in M$. □

4.4 Proof of Lemma 4.4 for Constructing Parity Correctors

We use O-joins as parity correctors for each $T \in \mathscr{T}$. We now give the complete proof of Lemma 4.4.

Proof of Lemma 4.4. Our goal is to show that z belongs to O-JOIN(G_x). By definition, $z \in [0,1]^{E_x}$. Now we will show that z satisfies the constraint (2.2). First, we state three useful claims.

Claim 6. *If $z(\delta(U)) \geq 1$ for $U \subset V_n$, then $z(\delta(U) \setminus A) - z(A) \geq 1 - |A|$.*

Proof. We have $z(\delta(U) \setminus A) - z(A) = z(\delta(U)) - 2z(A)$. Since $z_e \leq \frac{1}{2}$ for all $e \in E_x$, we have $z(\delta(U)) - 2z(A) \geq 1 - |A|$. $\Diamond$

Claim 7. *If $\delta(U) \cap M = \emptyset$, we have $z(\delta(U)) \geq 1$.*

Proof. This follows from the fact that for every edge $e \notin M$, we have $z_e = \frac{x_e}{2}$. $\Diamond$

Claim 8. *For all $U \subset V_n$, $|\delta(U) \cap W_x|$ and $|\delta(U)|$ have same parity.*

Proof. This follows from the fact that $|\delta(U) \cap H_x|$ is always even since H_x is a 2-factor of G_x. $\Diamond$

We consider the following cases. **Case 1:** $|\delta(U) \cap W_x| \geq 3$, **Case 2:** $|\delta(U) \cap W_x| = 2$, and **Case 3:** $|\delta(U) \cap W_x| = 1$.

Case 1: If $|\delta(U) \cap W_x| \geq 4$, then $z(\delta(U)) \geq 2 - 2\theta \geq 1$. Thus we may assume $|\delta(U) \cap W_x| = 3$. If $|\delta(U) \geq 4$, then $z(\delta(U)) \geq \frac{3}{2} - \frac{3}{2}\theta + \frac{\theta}{2} \geq 1$, since $\theta \leq \frac{1}{2}$. If $|\delta(U)| = 3$, then by assumption we have $|\delta(U) \cap M| \leq 1$. Thus, $z(\delta(U)) \geq 1$. Thus, Claim 6 applies in each subcase.

Case 2: In this case, if $|\delta(U)| \geq 4$, then $z(\delta(U)) \geq (1 - \theta) + \theta \geq 1$. By Claim 8, the only remaining subcase to consider is when $|\delta(U)| = 2$.

By assumption, $|U \cap O|$ is even. Hence, $|A|$ must be odd, which implies that $|A| = 1$. Let $\delta(U) = \{e', e''\}$. Since $|\delta(U)| = 2$, we have either $|\delta(U) \cap M| = 2$ or $|\delta(U) \cap M| = 0$. In both cases $z_{e'} = z_{e''}$. Hence, $z(A) = z(\delta(U) \setminus A)$. Therefore, $z(\delta(U) \setminus A) - z(A) = 0 = 1 - |A|$.

Case 3: If $\delta(U) \cap M = \emptyset$, then by Claim 7 we have $z(\delta(U)) \geq 1$. Hence, we assume $|\delta(U) \cap M| = 1$. By Claim 8, we only need to consider the following cases: *Case 3i:* $|\delta(U)| = 3$, and *Case 3ii:* $|\delta(U)| \geq 5$.

Case 3i: Notice that $x(\delta(U)) \geq 2$. In this case, $\delta(U)$ is either a critical cut, a vertex cut, or a degenerate tight cut. We assumed that G_x has no critical cuts. So U is either a vertex cut or a degenerate tight cut. We prove in both cases that $|U \cap O|$ is even. Then we only need to consider $|A|$ odd. If $|A| = 1$, then $z(\delta(U) \setminus A) - z(A) \geq \frac{\theta}{2} \geq 0 = 1 - |A|$. If $|A| = 3$, then $z(\delta(U) \setminus A) - z(A) \geq -1 + \frac{\theta}{2} \geq -2 = 1 - |A|$.

If $U = \{u\}$ for some $u \in V_n$, then $u \notin O$ by assumption. Otherwise, U is a degenerate tight cut. Let $\delta(U) = \{e_u, f_v, g_v\}$ where $\{f_v, g_v\} = \delta(v) \cap H_x$ for some $v \in V_n$. Notice that $\delta(U \setminus \{v\}) = \{e_u, e_v\}$ and $e_u \in M$. This implies by assumption that $e_v \in M$, which implies that $v \notin O$. Since $|(U \setminus \{v\}) \cap O|$ is even, hence $|U \cap O|$ is even.

Case 3ii: Since $|\delta(U)| \geq 5$, if there is an edge $e \in \delta(U) \cap H_x$ with $z_e = \frac{1-\theta}{2}$, then $z(\delta(U)) \geq 1 - \theta + 3 \cdot \frac{\theta}{2} = 1 + \frac{\theta}{2}$. Therefore, $\theta < \frac{1}{2}$ and for all edges e in $\delta(U) \cap H_x$, we have $z_e = \frac{\theta}{2}$.

Let $\mathcal{C}$ be the collection of cycles in H_x. Since $\theta < \frac{1}{2}$, every cycle in $\mathcal{C}$ is even length. Clearly, any cut crosses every cycle $C \in \mathcal{C}$ an even number of times. If all the edges in $\delta(U) \cap C$ have the same x value, then $|U \cap V(C)|$ is even. We have

$$|U| = \sum_{C \in \mathcal{C}: V(C) \subseteq U} |V(C) \cap U| + \sum_{C \in \mathcal{C}: V(C) \cap U \neq \emptyset} |V(C) \cap U|. \tag{4.1}$$

By the argument above and the fact that $|V(C)|$ is even for all $C \in \mathcal{C}$, we conclude that $|U|$ is even. Since G_x is a cubic graph, this implies that $|\delta(U)|$ is even. However, by Claim 8, we know that $|\delta(U)|$ is odd.

This concludes the case analysis and the proof. □

4.5 Proof of Lemma 4.1: Partitioning 1-edges into Induced Matchings

The goal of this section is to prove the following lemma.

Lemma 4.1. *Suppose G_x has no critical cuts. Let v be a vertex in V_n and let $N_2(v) = \{w_1, w_2\}$. The set of 1-edges in G_x, W_x, can be partitioned into five induced matchings $\{M_1, \dots, M_5\}$ such that for $i \in [5]$, the following properties hold.*

(i) $|M_i \cap \{e_v, e_{w_1}, e_{w_2}\}| \leq 1$.

(ii) For $U \subseteq V_n$ such that $|\delta(U)| = 3$, $|\delta(U) \cap M_i| \leq 1$.

(iii) For $U \subseteq V_n$ such that $|\delta(U)| = 2$, $|\delta(U) \cap M_i|$ is even.

We say $\delta(U)$ is a triangular 3-cut if $|U| = 3$ or $|V \setminus U| = 3$, and $|\delta(U)| = 3$. A bad 3-edge cut is a proper 3-edge cut that is not triangular. We construct the desired partition of W_x into induced matchings by gluing over the bad cuts of G_x and perform induction on the number of bad 3-edge cuts. We prove Lemma 4.1 using a two-phase induction. Claim 9 is the base case and Claims 10 and 11 are the first and second inductive steps.

Claim 9. *Suppose G_x is 3-edge-connected and contains no bad 3-edge cuts. Then Lemma 4.1 holds.*

Proof. In G_x, contract every edge in W_x. We get a connected 4-regular graph $H = (W_x, H_x)$. An independent set in H corresponds to a set of edges in W_x that forms an induced matching in G_x. We consider two cases. If H is the complete graph on five vertices, then partition the vertex set into five independent sets, which corresponds to five induced matchings in G_x. Notice that the condition (i) from Lemma 4.1 is satisfied since each induced matching contains one edge.

If H is not the complete graph on five vertices, by Brook's Theorem (see Theorem 8.4 in [BM08]) we can partition the vertices of H into four independent sets where each independent set corresponds to an induced matching $\{M_1, \dots, M_4\}$ in G_x and these four induced matchings partition W_x. If $|M_i \cap \{e_r, e_{w_1}, e_{w_2}\}| \leq 1$ for $i \in [4]$, then we are done. Otherwise, assume without loss of generality that $\{e_{w_1}, e_{w_2}\} \in M_4$. Then let $M'_4 = M_4 \setminus \{e_{w_1}\}$. The desired partition is $\{M_1, M_2, M_3, M'_4, \{e_{w_1}\}\}$. Thus, condition (i) is satisfied.

Now we prove condition (ii). First, consider a vertex $u \in V$ and the cut $\delta(u)$ in G_x. Clearly $|\delta(u) \cap M_i| \leq |\delta(u) \cap W_x| \leq 1$. For a triangular 3-cut, $\delta(U) = \{e_1, e_2, e_3\}$, we cannot have $|\delta(U) \cap \{e_1, e_2, e_3\}| \geq 2$, since $\delta(U) \subseteq W_x$ and no pair of edges from $\delta(U)$ can belong to an induced matching. Since condition (iii) does not apply, this completes the proof of the claim. $\diamondsuit$

Claim 10. *Suppose G_x is 3-edge-connected. Then Lemma 4.1 holds.*

Proof. Now let us consider a bad cut. In particular, consider graph G_x with 3-edge-cut $\delta(U) = \{e_1, e_2, e_3\}$, and assume without loss of generality that $r \in U$. Let s_1, s_2 and s_3 be the endpoints of e_1, e_2 and e_3 that are in U, and t_1, t_2 and t_3 be the other endpoints. Notice that s_1, s_2, s_3 (and analogously t_1, t_2, t_3) are distinct vertices since G_x is 3-edge-connected. Construct graph $G_1 = G_x[(V \setminus U) \cup \{s_1, s_2, s_3\}] + \{s_1s_2, s_1s_3, s_2s_3\}$ and, symmetrically, graph $G_2 = G_x[U \cup \{t_1, t_2, t_3\}] + \{t_1t_2, t_1t_3, t_2t_3\}$. If both G_1 and G_2 have no bad 3-edge cuts, then we can apply Claim 9 to both G_1 and G_2. For G_1, we find induced matchings $\{M_1^1, \ldots, M_5^1\}$ such that conditions (i) and (ii) hold. Similarly, for G_2, we find induced matchings $\{M_1^2, \ldots, M_5^2\}$ such that (i) and (ii) hold.

Notice that for each edge $e \in \{e_1, e_2, e_3\}$, there is exactly one induced matching in $\{M_1^1, \ldots, M_5^1\}$ and in $\{M_1^2, \ldots, M_5^2\}$ that contains e_1. Without loss of generality, suppose M_i^1 and M_i^2 each contain edge e_i for $i \in [3]$. Then let $M_i = M_i^1 \cup M_i^2$ for $i \in [5]$ and notice that M_i is an induced matching in G_x. We conclude by induction on the number of bad cuts in G_x, since both G_1 and G_2 have fewer bad 3-edge cuts than does G_x. $\diamondsuit$

Claim 11. *Suppose G_x is 2-edge-connected. Then Lemma 4.1 holds.*

Proof. We proceed by induction on the number of 2-edge cuts of G_x. If G_x does not contain any 2-edge cuts then G_x is 3-edge-connected, so by Claim 10 the claim follows.

For the induction step, consider 2-edge cut $\delta(U) = \{e_1, e_2\}$. Since x is a half-cycle point, note that $e_1, e_2 \in W_x$. Let s_1 and s_2 be the endpoints of e_1 and e_2 that are in U and let t_1 and t_2 be the other endpoints. (Observe that neither s_1s_2 nor t_1t_2 is an edge in G_x; otherwise G_x would contain a cut of x-value less than 2.) Consider graphs $G_1 = G[U] + s_1s_2$ and $G_2 = G[V \setminus U] + t_1t_2$. The set of 1-edges of G_1 is $\{W_x \cap E(G_1)\} \cup \{s_1s_2\}$, and the set of 1-edges of G_2 is $\{W_x \cap E(G_2)\} \cup \{t_1t_2\}$.

Without loss of generality, assume $r \in S$. Apply induction on G_1 to find induced matchings $\{M_1^1, \ldots, M_5^1\}$ where $s_1s_2 \in M_1^1$, and on G_2 to obtain induced matchings $\{M_1^2, \ldots, M_5^2\}$ where $t_1t_2 \in M_1^2$. Set $M_1 = \{M_1^1 \cup M_1^2 \cup \{e_1, e_2\}\} \setminus \{s_1s_2, t_1t_2\}$ and set $M_i = M_i^1 \cup M_i^2$ for $i \in \{2, \ldots, 5\}$. Then $\{M_1, \ldots, M_5\}$ partition W_x into induced matchings and satisfy conditions (i), (ii) and (iii). $\diamondsuit$

The proof of Lemma 4.1 follows from Claim 11.

5 Construction of Tours for Uniform Points

Recall the definition of $\frac{2}{k}$-uniform point from Section 1.1.2. In this section we prove Theorem 1.5 regarding $\frac{2}{3}$-uniform points and then we prove Theorem 1.7 concerning $\frac{2}{4}$-uniform points.

5.1 TSP on $\frac{2}{3}$-Uniform Points

We start by the following lemma reducing TSP on $\frac{2}{3}$-uniform points to TSP on $\frac{1}{2}$-cyclic points.

Lemma 5.1. *If for any $\frac{1}{2}$-cyclic point x the vector y defined as: $y_e = \frac{3}{2} - \epsilon$ for $e \in W_x$ and $y_e = \frac{3}{4} - \delta$ for $e \in H_x$ and $y_e = 0$ for $e \notin E_x$ for constants $\epsilon, \delta \geq 0$ belongs to* $\text{TSP}(K_n)$*, then for any $\frac{2}{3}$-uniform point z we have $(\frac{3}{2} - \frac{\epsilon}{2} - \delta)z \in \text{TSP}(K_n)$.*

Proof. Let z be a $\frac{2}{3}$-uniform point, and let $G_z = (V_n, E_z)$ be its support. Notice that $z \in \emptyset\text{-}\textsc{Join}(G_z)$. Hence z can be written as a convex combination of $\emptyset$-joins of G_z denoted by $\{\lambda, \mathscr{C}\}$. Observe that each $\emptyset$-join $\mathcal{C} \in \mathscr{C}$ is in fact a 2-factor of G_z since $z(\delta(u)) = 2$ and $|\mathcal{C} \cap \delta(u)| \leq 2$ for $u \in V_n$. For $\mathcal{C} \in \mathscr{C}$, we define $p^{\mathcal{C}}$ to be such that $p^{\mathcal{C}}_e = 1$ for $e \in \mathcal{C}$ and $p^{\mathcal{C}}_e = \frac{1}{2}$ for $e \in E_z \setminus \mathcal{C}$ and $p^{\mathcal{C}}_e = 0$ for $e \in E_n \setminus E_z$. Notice that $p^{\mathcal{C}}$ is a $\frac{1}{2}$-cyclic point. Define $y^{\mathcal{C}}$ as follows: for $e \in E_n$ let $y^{\mathcal{C}}_e = \frac{3}{2} - \epsilon$ if $e \in W_{p^{\mathcal{C}}}$, and $y^{\mathcal{C}}_e = \frac{3}{4} - \delta$ if $e \in H_{p^{\mathcal{C}}}$ and $y_e = 0$ otherwise. By assumption, we have $y^{\mathcal{C}} \in \text{TSP}(K_n)$. Therefore,

$$\hat{z} = \sum_{\mathcal{C} \in \mathscr{C}} \lambda_{\mathcal{C}} y^{\mathcal{C}} \in \text{TSP}(K_n).$$

Observe that for $e \in E_x$

$$\begin{aligned}\hat{z}_e =& \frac{1}{3} \cdot (\frac{3}{2} - \epsilon) + \frac{2}{3} \cdot (\frac{3}{4} - \delta)\\ =& 1 - \frac{\epsilon}{3} - \frac{2\delta}{3}\\ =& (\frac{3}{2} - \frac{\epsilon}{2} - \delta) \cdot \frac{2}{3} = (\frac{3}{2} - \frac{\epsilon}{2} - \delta) \cdot x_e.\end{aligned}$$

Finally, for $e \in E_n \setminus E_x$, we have $\hat{z}_e = 0$. □

A consequence of Theorem 1.3 is that for $\frac{2}{3}$-uniform point $x \in \mathbb{R}^{E_n}$, we have $(\frac{3}{2} - \frac{1}{40})x \in \text{TSP}(K_n)$.

Haddadan, Newman and Ravi [HNR19] used the following theorem in [BIT13] to obtain the first factor below $\frac{3}{2}$ for approximating TSP on $\frac{2}{3}$-uniform point.

Theorem 5.2 ([BIT13])**.** *Let $G = (V, E)$ be a bridgeless cubic graph. Then G has a 2-factor that covers all 3-edge cuts and 4-edge cuts of G.*

We can combine the ideas in the proof of Theorem 1 of [HNR19] with Theorem 1.3 to prove the following.

Theorem 5.3. *Let x be a $\frac{2}{3}$-uniform point and $G_x = (V_n, E_x)$ its support graph. Then $\frac{17}{12}x \in \text{TSP}(K_n)$. If G_x is Hamiltonian, then $\frac{87}{68}x \in \text{TSP}(K_n)$.*

Proof. By Theorem 5.2, G_x has a 2-factor $\mathcal{C}$ that covers all 3-edge cuts and 4-edge cuts of G_x. Define vector y^1 as follows: $y^1_e = 1$ for $e \in \mathcal{C}$ and $y^1_e = \frac{4}{5}$ for $e \in E_x \setminus \mathcal{C}$ and $y^1_e = 0$ for $e \in E_n \setminus E_x$.

Claim 12. *We have $y^1 \in \text{TSP}(K_n)$.*

Proof. Notice graph $G' = G_x/\mathcal{C}$ is a 5-edge-connected graph. We can assume without loss of generality that G' is also 5-regular[4]. For any vertex r of G' we have $\frac{2}{5}\chi^{E(G')} \in r\text{-TREE}(G')$. So the vector $\frac{2}{5}\chi^{E(G')}$ can be written as a convex combination of r-trees of G' denoted by $\{\lambda, \mathcal{T}\}$. For $T \in \mathcal{T}$ the multigraph $F_T = \mathcal{C} + 2T$ is a tour of G_x and therefore K_n. Finally, $y^1 = \sum_{T \in \mathcal{T}} \lambda_T \chi^{F_T}$. $\diamondsuit$

On the other hand, we can define z where $z_e = \frac{1}{2}$ for $e \in \mathcal{C}$ and $z_e = 1$ for $e \in E_x \setminus \mathcal{C}$ for $z_e = 0$ for $e \in E_n \setminus E_x$. Vector z is a $\frac{1}{2}$-cyclic point, hence we can apply Theorem 1.3 to obtain vector $y^2 \in \text{TSP}(K_n)$ such that $y^2_e = \frac{3}{4}$ for $e \in \mathcal{C}$, $y^2_e = \frac{3}{2} - \frac{1}{20}$ for $e \in E_x \setminus \mathcal{C}$ and $y^2_e = 0$ for $e \in E_n \setminus E_x$. Notice that $\frac{7}{9}y^1 + \frac{2}{9}y^2 \in \text{TSP}(G)$ and is equal to $\frac{17}{12}x$.

If G_x is Hamiltonian, we can assume $\mathcal{C}$ is the Hamiltonian cycle of G_x. Hence $\chi^{\mathcal{C}} \in \text{TSP}(K_n)$. In this case $\frac{7}{17} \cdot \chi^{\mathcal{C}} + \frac{10}{17} \cdot y^2 \in \text{TSP}(K_n)$ and is equal to $\frac{87}{68}x$. $\square$

5.2 A Base Case for $\frac{2}{4}$-Uniform Points

Due to the fact that we can glue over critical cuts, we observed that TSP on a θ-cyclic point x is essentially equivalent to the problem with the assumption that G_x contains no critical cuts. Analogously, in the case of a $\frac{2}{4}$-uniform point x, Theorem 1.7 could serve as the base case if we were able to glue over the proper minimum cuts of G_x. However, the difference here is that (1) the gluing arguments we presented for θ-cyclic points can not easily be extended to this case (due to the increased complexity of the distribution of patterns), and (2) we require an even number of vertices for our arguments.

Theorem 1.7. *Let $G = (V, E)$ be a 4-edge-connected 4-regular graph G with even number of vertices and no proper 4-edge cuts. Then the vector $(\frac{3}{2} - \frac{1}{42}) \cdot (\frac{1}{2} \cdot \chi^{E(G)})$ dominates a convex combination of incidence vectors of tours of G.*

Proof. We prove the claim by showing that there is a distribution of tours that satisfies the properties. It is easy to see that the proof yields a convex combination of tours of G. Since G does not have a proper 4-edge cut and since it is Eulerian, a proper cut of G has at least 6 edges.

[4]Replace every vertex v of degree more than 5 with a doubled cycle of length $|\delta(v)|$ and connect each vertex in the cycle to a neighbor of v in G'.

Define $y_e = \frac{1}{4}$ for all $e \in E$. Vector y is in the perfect matching polytope of G and can be written as a convex combination of perfect matchings of G. Choose a perfect matching M at random from the distribution defined be the convex multipliers of this convex combination.

Let $r \in V$. Define vector z as follows: $z_e = 1$ if $z \in M$ and $z_e = \frac{1}{3}$ for $z \in E \setminus M$. Observe that $z \in r\text{-}\textsc{Tree}(G_x)$ for any $v \in V$: $z(\delta(U)) \geq \frac{1}{3} \cdot |\delta(U)| \geq 2$ if $|U| \geq 2$ and $|V \setminus U| \geq 2$ and $z(\delta(r)) = 2$.

Applying Brook's theorem (similar to the proof of Lemma 4.1) we can find collection $\{M_1, \ldots, M_7\}$ of induced matchings of G that partition M. Choose $i \in [7]$ uniformly at random. For each $e = st \in M_i$, include the three edges incident on s in one set and the three edges incident to t in another set. Notice all six edge are distinct since G has no proper 4-edge cuts. Apply Theorem 2.5 to decompose z into a convex combination of rainbow r-trees of G with respect to this partition. Take a random r-tree T from this convex combination using the distribution defined by the convex multipliers. Let O be the set of odd degree vertices of T. Note that for each $e = st \in M_i$, $s, t \notin O$ by construction. Define vector p to be such that $p_e = \frac{1}{2}$ for $e \in M \setminus \{M_i\}$ and $p_e = \frac{1}{6}$ otherwise. We have $p \in O\text{-}\textsc{Join}(G)$. Therefore, we can write p as convex combination of O-joins of G. Choose one of the O-joins at random from the convex combination and label it J. Note that $F = T + J$ is a tour of G. For an edge $e \in M$ we have

$$\begin{aligned}\Pr[e \in J | e \in M] &= \Pr[e \in J | e \in M_i] \Pr[e \in M_i] + \Pr[e \in J | e \in M \setminus M_i] \Pr[e \in M \setminus M_i] \\ &= \frac{1}{6} \cdot \frac{1}{7} + \frac{1}{2} \cdot \frac{6}{7} = \frac{19}{42}.\end{aligned}$$

If $e \notin M$, then we have $\Pr[e \in J | e \notin M] = \frac{1}{6}$. Hence,

$$\begin{aligned}\Pr[e \in J] &= \Pr[e \in J | e \in M] \Pr[e \in M] + \Pr[e \in J | e \notin M] \Pr[e \notin M] \\ &= \frac{19}{42} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{3}{4} = \frac{5}{21}.\end{aligned}$$

Observe that $\mathbb{E}[z_e] = 1 \cdot \Pr[e \in M] + \frac{1}{3} \cdot Pr[e \notin M] = \frac{1}{2}$. Therefore, $\Pr[e \in T] = \Pr[e \notin T] = \frac{1}{2}$.

$$\begin{aligned}\mathbb{E}[\chi_e^F] &= 2 \cdot \Pr[e \in T \text{ and } e \in J] + \Pr[e \in T \text{ and } e \notin J] + \Pr[e \notin T \text{ and } e \in J] \\ &= 2 \cdot \frac{1}{2} \cdot \frac{5}{21} + \frac{1}{2} \cdot \frac{16}{21} + \frac{1}{2} \cdot \frac{5}{21} = \frac{3}{4} - \frac{1}{84}.\end{aligned}$$

Thus, each edge $e \in E$ is used to an extent $(\frac{3}{2} - \frac{1}{42}) \cdot \frac{1}{2}$. This concludes the proof. □

6 Concluding Remarks

In this paper, we showed how to improve the multiplicative approximation factor of Christofides algorithm on the 1-edges of θ-cyclic points from $\frac{3}{2}$ to $\frac{3}{2} - \frac{\theta}{10}$. Approaching Conjecture 1 from this angle, we propose the following open problem, which is implied by the four-thirds conjecture.

Open Problem 1. *Let $x \in \mathbb{R}^{E_n}$ be a θ-cyclic point. Define vector y as follows: $y_e = \frac{4}{3}$ for $e \in W_x$, $y_e = \frac{3}{2}x_e$ for $e \in H_x$ and $y_e = 0$ for $e \notin E_x$. Can we show $y \in \mathrm{TSP}(K_n)$?*

In fact, the bound above is tight: for $\epsilon > 0$, there exists a $\frac{1}{2}$-cyclic point x^ϵ such that vector y^ϵ defined as $y^\epsilon_e = \frac{4}{3} - \epsilon$ for $e \in W_{x^\epsilon}$, $y_e = \frac{3}{2}x^\epsilon_e$ for $e \in H_{x^\epsilon}$ and $y_e = 0$ for $e \notin E_{x^\epsilon}$. Then $y^\epsilon \notin \mathrm{TSP}(K_n)$ (See Figure 6.1).

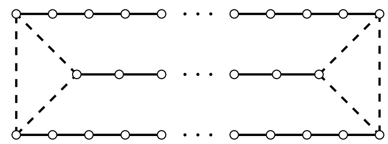

Figure 6.1: The support of x^ϵ: In the figure above each of the three paths of solid edges contain $\lceil \frac{1}{\epsilon} + 1 \rceil$ vertices. We have $x^\epsilon_e = 1$ for solid edge e, $x^\epsilon_e = \frac{1}{2}$ for dashed edge e, and $x^\epsilon_e = 0$ for edge e not depicted.

This makes the problem above intriguing even when restricted to $\frac{1}{2}$-cyclic points. For a $\frac{1}{2}$-cyclic point x where H_x is disjoint union of 3-cycles, Boyd and Carr [BC11] achieved this bound. Notice that this class includes x^ϵ in Figure 6.1. Interestingly, a construction very similar to that of Boyd and Sebő [BS19] implies that for a $\frac{1}{2}$-cyclic point x where H_x is a union of vertex-disjoint 4-cycles, we can go beyond this factor.

Theorem 6.1. *Let $x \in \mathbb{R}^{E_n}$ be a $\frac{1}{2}$-cyclic point where H_x is a union of vertex-disjoint 4-cycles. Define vector y as follows: $y_e = \frac{5}{4}$ for $e \in W_x$, $y_e = \frac{3}{2}x_e$ for $e \in H_x$ and $y_e = 0$ for $e \notin E_x$. We have $y \in \mathrm{TSP}(K_n)$.*

Proof Sketch. In this case, Boyd and Sebő [BS19] showed that G_x has a Hamilton cycle H such that $H \subset W_x$ and H intersects each 4-cycle of H_x at opposite edges.

They also show that x can be written as convex combination of connectors of G_x denoted by $\{\lambda, \mathcal{T}\}$ such that for $T \in \mathcal{T}$ we have $|T \cap C| = 2$ and $|T \cap (C \cap H)| = 1$ for each 4-cycle $C \in H_x$ [BS19].

For a $T \in \mathcal{T}$ define O_T be the odd degree vertices of T and define vector z^T as follows: $z^T_e = \frac{1}{3}$ for $e \in W_x$, $z^T_e = \frac{1}{6}$ for $e \in H_x \cap H$ and $z^T_e = \frac{1}{2}$ for $e \in H_x \setminus H$. For $T \in \mathcal{T}$, we can show $z^T \in O_T\text{-}\textsc{Join}(G_x)$ by following essentially the same arguments as Boyd and Sebő. This implies that $\chi^T + z^T \in \mathrm{TSP}(K_n)$. Therefore, $p = \sum_{T \in \mathcal{T}} \lambda_T(\chi^T + z^T) \in \mathrm{TSP}(K_n)$. In addition, we have $\chi^H \in \mathrm{TSP}(K_n)$. We conclude that $y = \frac{1}{4}\chi^H + \frac{3}{4}p \in \mathrm{TSP}(K_n)$. □

Since the proof of Theorem 6.1 is essentially the same as that in [BS19], it does not seem to extend to θ-cyclic points in which the fractional edges form 4-cycles.

For $\frac{2}{4}$-uniform points it would be interesting to find improvements over Christofides for TSP using the gluing approach. Such an approach might yield improved factors to the one presented in [KKO20]. For $\frac{2}{k}$-uniform points with $k \geq 5$ nothing is known for TSP or 2EC beyond Christofides $\frac{3}{2}$-approximation.

Open Problem 2. *Let* $x \in \mathbb{R}^{E_n}$ *be a* $\frac{2}{5}$*-uniform point. Can we find a constant* $\epsilon > 0$ *such that* $(\frac{3}{2} - \epsilon)x \in \text{TSP}(K_n)$*?*

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